A Banach Principle for Semifinite von Neumann Algebras

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Abstract. Utilizing the notion of uniform equicontinuity for sequences of functions with the values in the space of measurable operators, we present a non-commutative version of the Banach Principle for L^∞.

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1 Introduction

Let (Ω, Σ, μ) be a probability space. Denote by $\mathcal{L} = \mathcal{L}(\Omega, \mu)$ the set of all (classes of) complex-valued measurable functions on Ω. Let τ_μ stand for the measure topology in $\mathcal{L}$. The classical Banach Principle may be stated as follows.

Classical Banach Principle. *Let $(X, \|\cdot\|)$ be a Banach space, and let $a_n : (X, \|\cdot\|) \to (\mathcal{L}, \tau_\mu)$ be a sequence of continuous linear maps. Consider the following properties:*

(I) *the sequence $\{a_n(x)\}$ converges almost everywhere (a.e.) for every $x \in X$;*

(II) *$a^\star(x)(\omega) = \sup_n |a_n(x)(\omega)| < \infty$ a.e. for every $x \in X$;*

(III) (II) *holds, and the maximal operator $a^\star : (X, \|\cdot\|) \to (\mathcal{L}, \tau_\mu)$ is continuous at 0;*

(IV) *the set $\{x \in X : \{a_n(x)\}$ converges a.e.$\}$ is closed in X.*

Implications (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) *always hold. If, in addition, there is a set $D \subset X$, $\overline{D} = X$, such that the sequence $\{a_n(x)\}$ converges a.e. for every $x \in D$, then all four conditions* (I)–(IV) *are equivalent.*

The Banach Principle is most often and successfully applied in the context $X = (L^p, \|\cdot\|_p)$, $1 \le p < \infty$. At the same moment, in the case $p = \infty$ the uniform topology in L^∞ appears to be too strong for the "classical" Banach Principle to be effective in L^∞. For example, continuous functions are not uniformly dense in L^∞.

In [1], employing the fact that the unit ball $L_1^\infty = \{x \in L^\infty : \|x\|_\infty \le 1\}$ is complete in τ_μ, the authors suggest to consider the measure topology in L^∞ replacing $(X, \|\cdot\|)$ by (L_1^∞, τ_μ). Note that, since L_1^∞ is not a linear space, geometrical complications occur, which in [1] are treated with the help of the following lemma.

Lemma 1. *If $N(x, \delta) = \{y \in L_1^\infty : \|y - x\|_1 \le \delta\}$, $x \in L_1^\infty$, $\delta > 0$, then $N(0, \delta) \subset N(x, \delta) - N(x, \delta)$ for any $x \in L_1^\infty$, $\delta > 0$.*

An application of the Baire category theorem yields the following replacement of (I) $\Rightarrow$ (II).

Theorem 1 ([1]). *Let $a_n : L^\infty \to \mathcal{L}$ be a sequence of τ_μ-continuous linear maps such that the sequence $\{a_n(x)\}$ converges a.e. for all $x \in L^\infty$. Then the maximal operator $a^\star(x)(\omega) = \sup_n |a_n(x)(\omega)|$, $x \in L^\infty$, is τ_μ-continuous at 0 on L_1^∞.*

At the same time, as it is known [1], even for a sequence $a_n : L^\infty \to L^\infty$ of contractions, in which case condition (II) is clearly satisfied, the maximal operator $a^\star : L_1^\infty \to L_1^\infty$ may be not τ_μ-continuous at 0, i.e., (II) does not necessarily imply (III), whereas a replacement of the implication (III) $\Rightarrow$ (IV) does hold:

Theorem 2 ([1]). *Assume that each $a_n : L^\infty \to \mathcal{L}$ is linear, condition* (II) *holds with $X = L^\infty$, and the maximal operator $a^\star : L^\infty \to \mathcal{L}$ is τ_μ-continuous at 0 on L_1^∞. Then the set $\{x \in L_1^\infty : \{a_n(x)\}$ converges a.e.$\}$ is closed in (L_1^∞, τ_μ).*

A non-commutative Banach Principle for measurable operators affiliated with a semifinite von Neumann algebra was established in [5]. Then it was refined and applied in [7, 4, 3]. In [3] the notion of uniform equicontinuity of a sequence of functions into $L(M, \tau)$ was introduced. The aim of this study is to present a non-commutative extension of the Banach Principle for L^∞ that was suggested in [1]. We were unable to prove a verbatim operator version of Lemma 1. Instead, we deal with the mentioned geometrical obstacles via essentially non-commutative techniques, which helps us to get rid of some restrictions in [1]. First, proof of Lemma 1 essentially depends on the assumption that the functions in $\mathcal{L}$ be real-valued while the argument of the present article does not employ this condition. Also, our approach eliminates the assumption of the finiteness of measure.

2 Preliminaries

Let M be a semifinite von Neumann algebra acting on a Hilbert space H, and let $B(H)$ denote the algebra of all bounded linear operators on H. A densely-defined closed operator x in H is said to be *affiliated* with M if $y'x \subset xy'$ for every $y' \in B(H)$ with $y'z = zy'$, $z \in M$. We denote by $P(M)$ the complete lattice of all projections in M. Let τ be a faithful normal semifinite trace on M. If I is the identity of M, denote $e^\perp = I - e$, $e \in P(M)$. An operator x affiliated with M is said to be *τ-measurable* if for each $\epsilon > 0$ there exists a projection $e \in P(M)$ with $\tau(e^\perp) \le \epsilon$ such that eH lies in the domain of the operator x. Let $L = L(M, \tau)$ stand for the set of all τ-measurable operators affiliated with M. Denote $\|\cdot\|$ the uniform norm in $B(H)$. If for any given $\epsilon > 0$ and $\delta > 0$ one sets

$$V(\epsilon, \delta) = \{x \in L : \|xe\| \le \delta \text{ for some } e \in P(M) \text{ with } \tau(e^\perp) \le \epsilon\},$$

then the topology t_τ in L defined by the family $\{V(\epsilon, \delta) : \epsilon > 0, \delta > 0\}$ of neighborhoods of zero is called a *measure topology*.

Theorem 3 ([9], see also [8]). *(L, t_τ) is a complete metrizable topological $*$-algebra.*

Proposition 1. *For any $d > 0$, the sets $M_d = \{x \in M : \|x\| \le d\}$ and $M_d^h = \{x \in M_d : x^* = x\}$ are t_τ-complete.*

Proof. Because (L, t_τ) is a complete metric space, it is enough to show that M_d and M_d^h are (sequentially) closed in (L, t_τ). If $M_d \ni x_n \to_{t_\tau} x \in L$, then $0 \le x_n^* x_n \le d \cdot I$ and, due to Theorem 3, $x_n x_n^* \to_{t_\tau} x^* x$. Since $\{x \in L : x \ge 0\}$ is t_τ-complete, we have $0 \le x^* x \le d \cdot I$, which implies that $x \in M_d$. Therefore, M_d is closed in (L, t_τ). Similarly, it can be checked that M_d^h is closed in (L, t_τ). ■

A sequence $\{y_n\} \subset L$ is said to converge *almost uniformly* (a.u.) to $y \in L$ if for any given $\epsilon > 0$ there exists a projection $e \in P(M)$ with $\tau(e^{\perp}) \le \epsilon$ satisfying $\|(y - y_n)e\| \to 0$.

Proposition 2. *If $\{y_n\} \subset L$, then the conditions*

(i) *$\{y_n\}$ converges a.u. in L;*

(ii) *for every $\epsilon > 0$ there exists $e \in P(M)$ with $\tau(e^{\perp}) \le \epsilon$ such that $\|(y_m - y_n)e\| \to 0$ as $m, n \to \infty$*

are equivalent.

Proof. Implication (i) ⇒ (ii) is trivial. (ii) ⇒ (i): Condition (ii) implies that the sequence $\{y_n\}$ is fundamental in measure. Therefore, by Theorem 3, one can find $y \in L$ such that $y_n \to y$ in t_τ. Fix $\epsilon > 0$, and let $p \in P(M)$ be such that $\tau(p^{\perp}) \le \epsilon/2$ and $\|(y_m - y_n)p\| \to 0$ as $m, n \to \infty$. Because the operators y_n, $n \ge 1$, are measurable, it is possible to construct such a projection $q \in P(M)$ with $\tau(q^{\perp}) \le \epsilon/2$ that $\{y_n q\} \subset M$. Defining $e = p \wedge q$, we obtain $\tau(e^{\perp}) \le \epsilon$, $y_n e = y_n q e \in M$, and

$$\|y_m e - y_n e\| = \|(y_m - y_n)pe\| \le \|(y_m - y_n)p\| \longrightarrow 0,$$

$m, n \to \infty$. Thus, there exists $y(e) \in M$ satisfying $\|y_n e - y(e)\| \to 0$. In particular, $y_n e \to y(e)$ in t_τ. On the other hand, $y_n e \to ye$ in t_τ, which implies that $y(e) = ye$. Hence, $\|(y_n - y)e\| \to 0$, i.e. $y_n \to y$ a.u. ■

The following is a non-commutative Riesz's theorem [9]; see also [5].

Theorem 4. *If $\{y_n\} \subset L$ and $y = t_\tau - \lim_{n\to\infty} y_n$, then $y = a.u. - \lim_{k\to\infty} y_{n_k}$ for some subsequence $\{y_{n_k}\} \subset \{y_n\}$.*

3 Uniform equicontinuity for sequences of maps into $L(M, \tau)$

Let E be any set. If $a_n : E \to L$, $x \in E$, and $b \in M$ are such that $\{a_n(x)b\} \subset M$, then we denote

$$S(x, b) = S(\{a_n\}, x, b) = \sup_n \|a_n(x)b\|.$$

Definition below is in part due to the following fact.

Lemma 2. *Let $(X, +)$ be a semigroup, $a_n : X \to L$ be a sequence of additive maps. Assume that $\bar{x} \in X$ is such that for every $\epsilon > 0$ there exist a sequence $\{x_k\} \subset X$ and a projection $p \in P(M)$ with $\tau(p^{\perp}) \le \epsilon$ such that*

(i) *$\{a_n(\bar{x} + x_k)\}$ converges a.u. as $n \to \infty$ for every k;*

(ii) *$S(x_k, p) \to 0$, $k \to \infty$.*

Then the sequence $\{a_n(\bar{x})\}$ converges a.u. in L.

Proof. Fix $\epsilon > 0$, and let $\{x_k\} \subset X$ and $p \in P(M)$, $\tau(p^{\perp}) \le \epsilon/2$, be such that conditions (i) and (ii) hold. Pick $\delta > 0$ and let $k_0 = k_0(\delta)$ be such that $S(x_{k_0}, p) \le \delta/3$. By Proposition 2, there is a projection $q \in P(M)$ with $\tau(q^{\perp}) \le \epsilon/2$ and a positive integer N for which the inequality

$$\|(a_m(\bar{x} + x_{k_0}) - a_n(\bar{x} + x_{k_0}))q\| \le \frac{\delta}{3}$$

holds whenever $m, n \ge N$. If one defines $e = p \wedge q$, then $\tau(e^{\perp}) \le \epsilon$ and

$$\begin{aligned}\|(a_m(\bar{x}) - a_n(\bar{x}))e\| &\le \|(a_m(\bar{x} + x_{k_0}) - a_n(\bar{x} + x_{k_0}))e\| \\ &\quad + \|a_m(x_{k_0})e\| + \|a_n(x_{k_0})e\| \le \delta\end{aligned}$$

for all $m, n \ge N$. Therefore, by Proposition 2, the sequence $\{a_n(\bar{x})\}$ converges a.u. in L. ■

Let (X,t) be a topological space, and let $a_n : X \to L$ and $x_0 \in X$ be such that $a_n(x_0) = 0$, $n = 1, 2, \ldots$. Recall that the sequence $\{a_n\}$ is *equicontinuous* at x_0 if, given $\epsilon > 0$ and $\delta > 0$, there is a neighborhood U of x_0 in (X,t) such that $a_n U \subset V(\epsilon, \delta), n = 1, 2, \ldots$, i.e., for every $x \in U$ and every n one can find a projection $e = e(x,n) \in P(M)$ with $\tau(e^{\perp}) \leq \epsilon$ satisfying $\|a_n(x)e\| \leq \delta$.

Definition. Let (X,t), $a_n : X \to L$, and $x_0 \in X$ be as above. Let $x_0 \in E \subset X$. The sequence $\{a_n\}$ will be called *uniformly equicontinuous* at x_0 on E if, given $\epsilon > 0, \delta > 0$, there is a neighborhood U of x_0 in (X,t) such that for every $x \in E \cap U$ there exists a projection $e = e(x) \in P(M)$, $\tau(e^{\perp}) \leq \epsilon$, satisfying $S(x,e) \leq \delta$.

As it can be easily checked, the uniform equicontinuity is a non-commutative generalization of the continuity of the maximal operator, a number of equivalent forms of which are presented in [1].

Let ρ be an invariant metric in L compatible with t_τ (see Theorem 3).

Lemma 3. *Let $d > 0$. If a sequence $a_n : M \to L$ of additive maps is uniformly equicontinuous at 0 on M_d^h, then it is also uniformly equicontinuous at 0 on M_d.*

Proof. Fix $\epsilon > 0$, $\delta > 0$. Let $\gamma > 0$ be such that, given $x \in M_d^h$, $\rho(0,x) < \gamma$, there is $e = e(x) \in P(M)$ for which $\tau(e^{\perp}) \leq \epsilon/2$ and $S(x,e) \leq \delta/2$ hold. Pick $x \in M_d$ with $\rho(0,x) < \gamma$. We have $x = \mathrm{Re}\,(x) + i\,\mathrm{Im}\,(x)$, where $\mathrm{Re}\,(x) = \frac{x+x^*}{2}$, $\mathrm{Im}\,(x) = \frac{x-x^*}{2i}$. Clearly, $\mathrm{Re}\,(x), \mathrm{Im}\,(x) \in M_d^h$ and $\rho(0, \mathrm{Re}\,(x)) < \gamma$, $\rho(0, \mathrm{Im}\,(x)) < \gamma$. Therefore, one can find such $p, q \in P(M)$ with $\tau(p^{\perp}) \leq \epsilon/2$ and $\tau(q^{\perp}) \leq \epsilon/2$ that $S(\mathrm{Re}\,(x), p) \leq \delta/2$ and $S(\mathrm{Im}\,(x), q) \leq \delta/2$. Defining $r = p \wedge q$, we get $\tau(r^{\perp}) \leq \epsilon$ and

$$S(x,r) \leq S(\mathrm{Re}\,(x), r) + S(\mathrm{Im}\,(x), r) \leq S(\mathrm{Re}\,(x), p) + S(\mathrm{Im}\,(x), q) \leq \delta,$$

implying that the sequence $\{a_n\}$ is uniformly equicontinuous at 0 on M_d. ■

Lemma 4. *Let a sequence $a_n : M \to L$ of additive maps be uniformly equicontinuous at 0 on M_d for some $0 < d \in \mathbb{Q}$. Then $\{a_n\}$ is also uniformly equicontinuous at 0 on M_s for every $0 < s \in \mathbb{Q}$.*

Proof. Pick $0 < s \in \mathbb{Q}$, and let $r = d/s$. Given $\epsilon > 0$, $\delta > 0$, one can present such $\gamma > 0$ that for every $x \in M_d$ with $\rho(0,x) < \gamma r$ there is a projection $e = e(x) \in P(M)$, $\tau(e^{\perp}) \leq \epsilon$, satisfying $S(x,e) \leq \delta r$. Since a_n is additive and $d, s \in \mathbb{Q}$, we have $a_n(rx) = r a_n(x)$. Also, $rx \in M_d$ and $\rho(0, rx) < \gamma r$ is equivalent to $x \in M_s$ and $\rho(0,x) < \gamma$. Thus, given $x \in M_s$ with $\rho(0,x) < \gamma$, we have

$$\|a_n(x)e\| = \frac{1}{r} \cdot \|a_n(rx)e\| \leq \delta,$$

meaning that the sequence $\{a_n\}$ is uniformly equicontinuous at 0 on M_s. ■

4 Main results

Let $0 \in E \subset M$. For a sequence of functions $a_n : (M, t_\tau) \to L$, consider the following conditions

(CNV(E)) almost uniform convergence of $\{a_n(x)\}$ for every $x \in E$;

(CNT(E)) uniform equicontinuity at 0 on E;

(CLS(E)) closedness in (E, t_τ) of the set $C(E) = \{x \in E : \{a_n(x)\} \text{ converges a.u.}\}$.

In this section we will study relationships among the conditions (CNV(M_1)), (CNT(M_1)), and (CLS(M_1)).

Remarks. 1. Following the classical scheme (see Introduction), one more condition can be added to this list, namely, a non-commutative counterpart of the existence of the maximal operator, which can be stated as [5]:

(BND(E)) given $x \in E$ and $\epsilon > 0$, there is $e \in P(M)$, $\tau(e^\perp) \le \epsilon$, with $S(x,e) < \infty$.

This condition can be called a *pointwise uniform boundedness* of $\{a_n\}$ on E. It can be easily verified that (CNV(E)) implies (BND(E)). But, as it was mentioned in Introduction, even in the commutative setting, (BND(M_1)) does not guarantee (CNT(M_1)).

2. If a_n is additive for every n, then (CNV(M)) follows from (CNV(M_1)).

3. If E is closed in (M, t_τ) (for instance, if $E = M_d$, or $E = M_d^h$; see Proposition 1), then (CLS(E)) is equivalent to the closedness of $C(E)$ in (L, t_τ).

In order to show that (CNV(M_1)) entails (CNT(M_1)), we will provide some auxiliary facts.

Lemma 5. *For any* $0 \le x \in L$ *and* $e \in P(M)$, $x \le 2(exe + e^\perp x e^\perp)$.

Proof. If $a = e - e^\perp$, then $a^* = a$, which implies that

$$0 \le axa = exe - exe^\perp - e^\perp xe + e^\perp xe^\perp.$$

Therefore, $exe^\perp + e^\perp xe \le exe + e^\perp xe^\perp$, and we obtain

$$x = (e + e^\perp)x(e + e^\perp) \le 2(exe + e^\perp xe^\perp).$$ ■

For $y \in M$, denote $l(y)$ the projection on $\overline{yH}$, and let $r(y) = I - n(y)$, where $n(y)$ denotes the projection on $\{\xi \in H : y\xi = 0\}$. It is easily checked that $l(y^*) = r(y)$, so, if $y^* = y$, one can define $s(y) = l(y) = r(y)$. The projections $l(y)$, $r(y)$, and $s(y)$ are called, respectively, a *left support* of y, a *right support* of y, and a *support* of $y = y^*$. It is well-known that $l(y)$ and $r(y)$ are equivalent projections, in which case one writes $l(y) \sim r(y)$. In particular, $\tau(l(y)) = \tau(r(y))$, $y \in M$. If $y^* = y \in M$, $y_+ = \int_0^\infty \lambda dE_\lambda$, and $y_- = -\int_{-\infty}^0 \lambda dE_\lambda$, where $\{E_\lambda\}$ is the spectral family of y, then we have $y = y_+ - y_-$, $y_+ = s(y_+)ys(y_+)$, and $y_- = -s(y_+)^\perp ys(y_+)^\perp$.
The next lemma is, in a sense, a non-commutative replacement of Lemma 0.1.

Lemma 6. *Let* $y^* = y \in M$, $-I \le y \le I$. *Denote* $e_+ = s(y_+)$. *If* $x \in M$ *is such that* $0 \le x \le I$, *then*

$$-I \le y - e_+xe_+ \le I \qquad \text{and} \qquad -I \le y + e_+^\perp xe_+^\perp \le I.$$

Proof. Because $e_+xe_+ \ge 0$, we have $y - e_+xe_+ \le y \le I$; analogously, $-I \le y + e_+^\perp xe_+^\perp$. On the other hand, since we obviously have $e_+xe_+ \le e_+$, $e_+^\perp xe_+^\perp \le e_+^\perp$, $e_+ye_+ \le e_+$, and $e_+^\perp ye_+^\perp \ge -e_+^\perp$, one can write

$$y - e_+xe_+ = y_+ - y_- - e_+xe_+ = y_+ + e_+^\perp ye_+^\perp - e_+xe_+ \ge y_+ - e_+^\perp - e_+ = y_+ - I \ge -I$$

and

$$y + e_+^\perp xe_+^\perp = e_+ye_+ - y_- + e_+^\perp xe_+^\perp \le e_+ - y_- + e_+^\perp = I - y_- \le I,$$

which finishes the proof. ■

Lemma 7. $aV(\epsilon,\delta)b \subset V(2\epsilon,\delta)$ *for all* $\epsilon > 0$, $\delta > 0$, *and* $a, b \in M_1$.

Proof. Let $x \in V(\epsilon,\delta)$. There exists $e \in P(M)$ such that $\tau(e^\perp) \le \epsilon$ and $\|xe\| \le \delta$. If we denote $q = n(e^\perp b)$, then

$$bq = (e + e^\perp)bq = ebq + e^\perp bn(e^\perp b) = ebq.$$

Besides, we have $q^\perp = r(e^\perp b) \sim l(e^\perp b) \le e^\perp$, which implies that $\tau(q^\perp) \le \epsilon$. Now, if one defines $p = e \wedge q$, then $\tau(p^\perp) \le 2\epsilon$ and

$$\|axbp\| = \|axbqp\| = \|axebqp\| \le \|axeb\| \le \|a\| \cdot \|xe\| \cdot \|b\| \le \delta.$$

Therefore, $axb \in V(2\epsilon,\delta)$. ■

Lemma 8 ([5]). *Let f be the spectral projection of $b \in M$, $0 \le b \le I$, corresponding to the interval $[1/2, 1]$. Then*

(i) $\tau(f^\perp) \le 2 \cdot \tau(I - b)$;

(ii) *$f = bc$ for some $c \in M$ with $0 \le c \le 2 \cdot I$.*

We will also need the following fundamental result.

Theorem 5 ([6]). *Let $a : M \to M$ be a positive linear map such that $a(I) \le I$. Then $a(x)^2 \le a(x^2)$ for every $x^* = x \in M$.*

The next theorem represents a non-commutative extension of Theorem 1.

Theorem 6. *Let $a_n : M \to L$ be a (CNV(M_1)) sequence of positive t_τ-continuous linear maps such that $a_n(I) \le I$, $n = 1, 2, \dots$. Then the sequence $\{a_n\}$ is also (CNT(M_1)).*

Proof. Fix $\epsilon > 0$ and $\delta > 0$. For $N \in \mathbb{N}$ define

$$F_N = \left\{ x \in M_1^h : \sup_{n \ge N} \|(a_N(x) - a_n(x))b\| \le \delta \text{ for some } b \in M, 0 \le b \le I, \tau(I - b) \le \epsilon \right\}.$$

Show that the set F_N is closed in (M_1^h, ρ). Let $\{y_m\} \subset F_N$ and $\rho(y_m, \bar{x}) \to 0$ for some $\bar{x} \in L$. It follows from Proposition'1 that $\bar{x} \in M_1^h$. We have $a_1(y_m) \to a_1(\bar{x})$ in t_τ, which, by Theorems 3 and 4, implies that there is a subsequence $\{y_m^{(1)}\} \subset \{y_m\}$ such that $a_1(y_m^{(1)})^* \to a_1(\bar{x})^*$ a.u. Similarly, there is a subsequence $\{y_m^{(2)}\} \subset \{y_m^{(1)}\}$ for which $a_2(y_m^{(2)})^* \to a_2(\bar{x})^*$ a.u. Repeating this process and defining $x_m = y_m^{(m)} \in F_N$, $m = 1, 2, \dots$, we obtain

$$a_n(x_m)^* \longrightarrow a_n(\bar{x})^* \ \text{ a.u.}, \quad m \to \infty, \quad n = 1, 2, \dots.$$

By definition of F_N, there exists a sequence $\{b_m\} \subset M$, $0 \le b_m \le I$, $\tau(I - b_m) \le \epsilon$, such that $\sup_{n \ge N} \|(a_N(x_m) - a_n(x_m))b_m\| \le \delta$ for every m. Because M_1 is weakly compact, there are a subnet $\{b_\alpha\} \subset \{b_m\}$ and $b \in M$ such that $b_\alpha \to b$ weakly, i.e. $(b_\alpha \xi, \xi) \to (b\xi, \xi)$ for all $\xi \in H$. Clearly $0 \le b \le I$. Besides, by the well-known inequality (see, for example [2]),

$$\tau(I - b) \le \liminf_\alpha \tau(I - b_\alpha) \le \epsilon.$$

We shall show that $\sup_{n \ge N} \|(a_N(\bar{x}) - a_n(\bar{x}))b\| \le \delta$. Fix $n \ge N$. Since $a_k(x_m)^* \to a_k(\bar{x})^*$ a.u., $k = n, N$, given $\sigma > 0$, there exists a projection $e \in P(M)$ with $\tau(e^\perp) \le \sigma$ satisfying

$$\|e\,(a_k(x_m) - a_k(\bar{x}))\| = \|(a_k(x_m)^* - a_k(\bar{x})^*)\,e\| \longrightarrow 0, \qquad m \to \infty, \quad k = n, N.$$

Show first that $\|e(a_N(\bar{x}) - a_n(\bar{x}))b\| \le \delta$. For every $\xi, \eta \in H$ we have

$$|(e((a_N(x_m) - a_n(x_m))b_m - (a_N(\bar{x}) - a_n(\bar{x}))b)\xi, \eta)|$$

$$\leq |(e(a_N(x_m) - a_n(x_m) - a_N(\bar{x}) + a_n(\bar{x}))b_m\xi, \eta)| \\ + |((b_m - b)\xi, (a_N(\bar{x})^* - a_n(\bar{x})^*)e\eta)|. \tag{1}$$

Fix $\gamma > 0$ and choose m_0 be such that

$$\|e\,(a_k(x_m) - a_k(\bar{x}))\| < \gamma, \qquad k = n, N \tag{2}$$

whenever $m \geq m_0$. Since $b_\alpha \to b$ weakly, one can find such an index $\alpha(\gamma)$ that

$$|((b_\alpha - b)\xi, (a_N(\bar{x})^* - a_n(\bar{x})^*)e\eta)| < \gamma \tag{3}$$

as soon as $\alpha \geq \alpha(\gamma)$. Because $\{b_\alpha\}$ is a subnet of $\{b_m\}$, there is such an index $\alpha(m_0)$ that $\{b_\alpha\}_{\alpha \geq \alpha(m_0)} \subset \{b_m\}_{m \geq m_0}$. In particular, if $\alpha_0 \geq \max\{\alpha(\gamma), \alpha(m_0)\}$, then $b_{\alpha_0} = b_{m_1}$ for some $m_1 \geq m_0$. It follows now from (1)–(3) that, for all $\xi, \eta \in H$ with $\|\xi\| = \|\eta\| = 1$, we have

$$\begin{aligned} |(e(a_N(\bar{x}) - a_n(\bar{x}))b\xi, \eta)| &\leq |(e(a_N(x_{m_1}) - a_n(x_{m_1}))b_{m_1}\xi, \eta)| \\ &+ |(e(a_N(x_{m_1}) - a_n(x_{m_1}) - a_N(\bar{x}) + a_n(\bar{x}))b_{m_1}\xi, \eta)| \\ &+ |((b_{m_1} - b)\xi, (a_N(\bar{x})^* - a_n(\bar{x})^*)e\eta)| \\ &\leq \delta + \|e\,(a_N(x_{m_1}) - a_N(\bar{x}))\| + \|e(a_n(x_{m_1}) - a_n(\bar{x}))\| + \gamma < \delta + 3\gamma. \end{aligned}$$

Due to the arbitrariness of $\gamma > 0$, we get

$$\|e(a_N(\bar{x}) - a_n(\bar{x}))b\| = \sup_{\|\xi\| = \|\eta\| = 1} |(e(a_N(\bar{x}) - a_n(\bar{x}))b\xi, \eta)| \leq \delta.$$

Next, we choose $e_j \in P(M)$ such that $\tau(e_j^\perp) \leq \frac{1}{j}$ and

$$\|e_j\,(a_k(x_m) - a_k(\bar{x}))\| \longrightarrow 0 \;\text{ as }\; m \to \infty, \qquad k = n, N; \quad j = 1, 2, \ldots.$$

Since $e_j \to I$ weakly, $e_j(a_N(\bar{x}) - a_n(\bar{x}))b \to (a_N(\bar{x}) - a_n(\bar{x}))b$ weakly, therefore,

$$\|(a_N(\bar{x}) - a_n(\bar{x}))b\| \leq \limsup_{j \to \infty} \|e_j(a_N(\bar{x}) - a_n(\bar{x}))b\| \leq \delta.$$

Thus, for every $n \geq N$ the inequality $\|(a_N(\bar{x}) - a_n(\bar{x}))b\| \leq \delta$ holds, which implies that $\bar{x} \in F_N$ and $\overline{F_N} = F_N$.

Further, as $\{a_n(x)\}$ converges a.u. for every $x \in M_1$, taking into account Proposition 2, we obtain

$$M_1^h = \bigcup_{N=1}^{\infty} F_N.$$

By Proposition 1, the metric space (M_1^h, ρ) is complete. Therefore, using the Baire category theorem, one can present such N_0 that F_{N_0} contains an open set. In other words, there exist $x_0 \in F_{N_0}$ and $\gamma_0 \geq 0$ such that for any $x \in M_1^h$ with $\rho(x_0, x) < \gamma_0$ it is possible to find $b_x \in M$, $0 \leq b_x \leq I$, satisfying $\tau(I - b_x) \leq \epsilon$ and

$$\sup_{n \geq N_0} \|(a_{N_0}(x) - a_n(x))b_x\| \leq \delta.$$

Let f_x be the spectral projection of b_x corresponding to the interval $[1/2, 1]$. Then, according to Lemma 8, $\tau(f_x^\perp) \leq 2\epsilon$ and

$$\sup_{n \geq N_0} \|(a_{n_0}(x) - a_n(x))f_x\| \leq 2\delta$$

whenever $x \in M_1^h$ and $\rho(x_0, x) < \gamma_0$. Since the multiplication in L is continuous with respect to the measure topology, Lemma 7 allows us to choose $0 < \gamma_1 < \gamma_0$ in such a way that $\rho(0, x) < \gamma_1$ would imply $\rho(0, ax^2b) < \gamma_0$ for every $a, b \in M_1$. Denote $e_+ = s(x_0^+)$. Because $a_i : (M, \rho) \to (L, t_\tau)$ is continuous for each i, there exists such $0 < \gamma_2 < \gamma_1$ that, given $x \in M$ with $\rho(0, x) < \gamma_2$, it is possible to find such a projection $p \in P(M)$, $\tau(p^\perp) \le \epsilon$, that

$$\|a_i(e_+ x^2 e_+)p\| \le \delta \qquad \text{and} \qquad \|a_i(e_+^\perp x^2 e_+^\perp)p\| \le \delta,$$

$i = 1, \dots, N_0$. Let $x \in M_1^h$ be such that $\rho(0, x) < \gamma_2$. Since $0 \le x^2 \le I$, Lemma 6 yields

$$-I \le x_0 - e_+ x^2 e_+ \le I \qquad \text{and} \qquad -I \le x_0 + e_+^\perp x^2 e_+^\perp \le I,$$

so, we have

$$y = x_0 - e_+ x^2 e_+ \in M_1^h \qquad \text{and} \qquad z = x_0 + e_+^\perp x^2 e_+^\perp \in M_1^h.$$

Besides, $\rho(x_0, y) = \rho(0, -e_+ x^2 e_+) < \gamma_0$, which implies that there is $f_1 \in P(M)$ such that $\tau(f_1^\perp) \le 2\epsilon$ and

$$\sup_{n \ge N_0} \|(a_{N_0}(y) - a_n(y))f_1\| \le 2\delta.$$

Analogously, one finds $f_2 \in P(M)$, $\tau(f_2^\perp) \le 2\epsilon$, satisfying

$$\sup_{n \ge N_0} \|(a_{N_0}(z) - a_n(z))f_2\| \le 2\delta.$$

As $\rho(0, x) < \gamma_2$, there is $p \in P(M)$ with $\tau(p^\perp) \le \epsilon$ such that the inequalities

$$\|a_i(e_+ x^2 e_+)p\| \le \delta \qquad \text{and} \qquad \|a_i(e_+^\perp x^2 e_+^\perp)p\| \le \delta$$

hold for all $i = 1, \dots, N_0$. Let $e = f_{x_0} \wedge f_1 \wedge f_2 \wedge p$. Then we have $\tau(e^\perp) \le 7\epsilon$ and, for $n > N_0$,

$$\begin{aligned}\|a_n(e_+ x^2 e_+)e\| &\le \|(a_{N_0}(x_0 - e_+ x^2 e_+) - a_n(x_0 - e_+ x^2 e_+) \\ &\quad + a_n(x_0) - a_{N_0}(x_0) + a_{N_0}(e_+ x^2 e_+))e\| \le \|(a_{N_0}(y) - a_n(y))f_1 e\| \\ &\quad + \|(a_{N_0}(x_0) - a_n(x_0))f_{x_0} e\| + \|a_{N_0}(e_+ x^2 e_+)pe\| \le 5\delta.\end{aligned}$$

At the same time, if $n \in \{1, \dots, N_0\}$, then $\|a_n(e_+ x^2 e_+)e\| = \|a_n(e_+ x^2 e_+)pe\| \le \delta$, so

$$\|a_n(e_+ x^2 e_+)e\| \le 5\delta, \qquad n = 1, 2, \dots.$$

Analogously,

$$\|a_n(e_+^\perp x^2 e_+^\perp)e\| \le 5\delta, \qquad n = 1, 2, \dots.$$

Next, by Lemma 5, we can write $0 \le x^2 \le 2(e_+ x^2 e_+ + e_+^\perp x^2 e_+^\perp)$. Since a_n is positive for every n, applying Theorem 5, we obtain

$$0 \le e a_n(x)^2 e \le e a_n(x^2) e \le 2(e a_n(e_+ x^2 e_+) e + e a_n(e_+^\perp x^2 e_+^\perp) e).$$

Therefore,

$$\|a_n(x)e\|^2 = \|e a_n(x)^2 e\| \le 20\delta, \qquad n = 1, 2, \dots.$$

Summarizing, given $\epsilon > 0$, $\delta > 0$, it is possible to find such $\gamma > 0$ that for every $x \in M_1^h$ with $\rho(0, x) < \gamma$ there is a projection $e = e(x) \in P(M)$ such that $\tau(\epsilon^\perp) \le 7\epsilon$ and

$$S(x, e) = \sup_n \|a_n(x)e\| \le \sqrt{20\delta}.$$

Thus, the sequence $\{a_n\}$ is (CNT(M_1^h)), hence, by Lemma 3, (CNT(M_1)). ∎

Now we shall present a non-commutative extension of Theorem 2.

Theorem 7. *A* (CNT(M_1)) *sequence* $a_n : M \to L$ *of additive maps is also* (CLS(M_1)).

Proof. Let $\bar{x}$ belong to the t_τ-closure of $C(M_1)$. By Proposition 1, $\bar{x} \in M_1$. Fix $\epsilon > 0$. Since, by Lemma 4, the sequence $\{a_n\}$ is (CNT(M_2)), for every $k \in \mathbb{N}$, there is $\gamma_k > 0$ such that, given $x \in M_2$ with $\rho(0, x) < \gamma_k$, one can find a projection $p_k = p_k(x) \in P(M)$, $\tau(p_k^\perp) \leq \epsilon/2^k$, satisfying $S(x, p_k) \leq 1/k$. Let a sequence $\{y_n\} \subset C(M_1)$ be such that $\rho(\bar{x}, y_k) < \gamma_k$. If we set $x_k = y_k - \bar{x}$, then $x_k \in M_2$, $\rho(0, x_k) = \rho(\bar{x}, x_k + \bar{x}) = \rho(\bar{x}, y_k) < \gamma_k$, and $\bar{x} + x_k = y_k \in C(M_1)$, $k = 1, 2, \ldots$. If $e_k = p_k(x_k)$, then $\tau(e_k^\perp) \leq \epsilon/2^k$ and also $S(x_k, e_k) \leq 1/k$. Defining $e = \wedge_{k=1}^\infty$, we obtain $\tau(e^\perp) \leq \epsilon$ and $S(x_k, e) \leq 1/k$. Therefore, by Lemma 2, the sequence $\{a_n(\bar{x})\}$ converges a.u., i.e. $\bar{x} \in C(M_1)$. ■

The following is an immediate consequence of the previous results of this section.

Theorem 8. *Let* $a_n : M \to L$ *be a sequence of positive* t_τ*-continuous linear maps such that* $a_n(I) \leq I$, $n = 1, 2, \ldots$. *If* $\{a_n\}$ *is* (CNV(D)) *with* D *being* t_τ*-dense in* M_1*, then conditions* (CNV(M_1)), (CNT(M_1)), *and* (CLS(M_1)) *are equivalent.*

5 Conclusion

First we would like to stress that, due to Theorem 6, when establishing the almost uniform convergence of a sequence $\{a_n(x)\}$ for all $x \in L^\infty(M, \tau) = M$, the uniform equicontinuity at 0 on M_1 of the sequence $\{a_n\}$ is assumed. Also, as it is noticed in [1], the above formulation is important because, for example, if $\{a_n\}$ are bounded operators in a non-commutative L^p-space, $1 \leq p < \infty$, one may want to show that not only do these operators fail to converge a.u., but they fail so badly that $\{a_n\}$ may fail to converge a.u. on any class of operators which is t_τ-dense in M.

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