CLASSIFICATION OF LINEAR OPERATORS IN A FINITE-DIMENSIONAL UNITARY SPACE

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UDC 512.863

In this article, we suggest an algorithm, which is simpler than that in [1, 2], for the reduction of a matrix to canonical form by transformations of unitary similarity and describe matrices of canonical form.

This problem is equivalent, in the sense of [3], to the following more general problem, which is also solved by the suggested algorithm. In analogy with the definition of the Kolchan representation [4], we will say that a unitary representation of an oriented graph (loops and multiple arcs are admitted) is given if a finite-dimensional unitary space $U_v$ is associated with each vertex v of it and a linear mapping $\varphi_\lambda: U_v \to U_w$ is associated with each arc from v into w. It is required to classify all unitary representations.

Selecting orthonormal bases in all the spaces $U_v$, $U_w$, ..., we get a system of the matrices $A_\lambda$ of mappings $\varphi_\lambda: U_v \to U_w$. Under reselection of orthonormal bases, the matrix $A_\lambda$ transforms into $B_\lambda = S_w^{-1}A_\lambda S_v$, where $S_v$ and $S_w$ are unitary matrices of transition to new bases. Therefore, the classification of the unitary representations of an oriented graph reduces to the classification of the systems of the matrices $A_\lambda$ up to these transformations. It will be convenient for us to give the system of the matrices $A_\lambda$ in the form of the blocks of a block matrix. In this article, certain square blocks of the considered block matrices may be distinguished by shading.

<u>Definition 1.</u> The following matrix problem is called a unitary problem. A block matrix $A = (A_{ij})$ is equivalent to the matrix $B = (S_i^{-1}A_{ij}R_j)$ with the same arrangement of shaded blocks, where $S_i$ and $R_j$ are unitary matrices such that $S_i = R_j$ whenever the block $A_{ij}$ is shaded. It is required to select one "canonical" matrix in each class of equivalent block matrices.

The passage to an equivalent matrix will be called an admissible transformation. In particular, a matrix that consists of a single block is reduced by transformations of unitary equivalence if the block is not shaded, and by transformations of unitary similarity if the block is shaded.

The matrices of unitary representation of an oriented graph can be accommodated in a block matrix such that the admissible transformations from it correspond to a reselection of the bases in a unitary representation; e.g., Fig. 1.

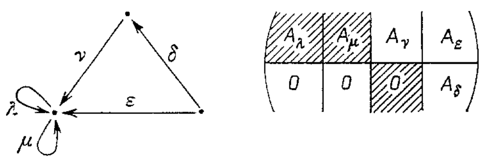



Fig. 1

Kiev State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 18, No. 3, pp. 57-62, July-September, 1984. Original article submitted March 17, 1983.

 0016-2663/84/1803-0224$08.50 

## 1. AN ALGORITHM FOR THE SOLUTION OF A UNITARY PROBLEM

We will call the matrix $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ the direct sum of the matrices A and B. In particular, $A \oplus 0_{m0} = \begin{pmatrix} A \\ 0 \end{pmatrix}$, and $A \oplus 0_{0n} = (A0)$, where $0_{m0}$ and $0_{0n}$ are "empty" null matrices of order $m \times 0$ and $0 \times n$, respectively; they give linear mappings of spaces, one of which is zero-dimensional.

We need two lemmas. The first lemma is well known and the second one strengthens the Schur lemma and is contained in [2].

LEMMA 1. a) Each matrix A is unitarily equivalent to the matrix

$$D = a_1E_1 \oplus \ldots \oplus a_{k-1}E_{k-1} \oplus 0, \tag{1}$$

where $a_1 > \ldots > a_{k-1} > 0$ are real numbers; $E_1, \ldots, E_{k-1}$ are unit matrices; and 0 is the null matrix of order $p \times q$, $p, q \geqslant 0$.

b) If $S^{-1}DR = D$, where R and S are unitary matrices and D is a matrix of the form (1), then $S = S_1 \oplus \ldots \oplus S_{k-1} \oplus S'$, and $R = S_1 \oplus \ldots \oplus S_{k-1} \oplus R'$, where $S_i$ has the same order as $E_i$.

Proof. a) Let $A = (a_{ij})$, $A^* = (\overline{a}_{ji})$, where $a \to \overline{a}$ is the complex conjugation. The matrix $A^*A$ is Hermitian. Therefore, there exists a unitary matrix U such that $U^*A^*AU = \operatorname{diag}(b_1^2, \ldots, b_r^2, 0, \ldots, 0)$, where $b_1 \geqslant \ldots \geqslant b_r > 0$ are real numbers. The first r columns of the matrix AU are pairwise orthogonal and the remaining columns are null. Let V denote an arbitrary unitary matrix, the first r columns of which are the same as those of the matrix $AU \operatorname{diag}(b_1^{-1}, \ldots, b_r^{-1}, 0, \ldots, 0)$. Then $V^*AU$ has the form (1).

b) Since $S^{-1}DR = D$, $S^* = S^{-1}$, $R^* = R^{-1}$, and $D^* = D$, it follows that $R^*D^*S^{*-1} = D^*$ and $R^{-1}DS = D$. Hence $D^2S = SD^2$ and $D^2R = R^2D$. Therefore, R and S have the block-diagonal form.

We will assume the complex numbers to be lexicographically ordered: $a + bi \geqslant c + di$ if $a > c$, or $a = c$ and $b \geqslant d$.

LEMMA 2. a) Each square matrix A is unitarily similar to the block-triangular matrix

$$F = (F_{ij}), \quad F_{ij} = 0 \quad \text{for} \quad i < j, \quad F_{ii} = \lambda_i E_i, \tag{2}$$

where $i, j = 1, \ldots, k$; $E_i$ is the unit matrix, and $\lambda_1 \geqslant \ldots \geqslant \lambda_k$; in addition, $F_{i+1,i}$ is nonsingular with respect to rows for $\lambda_i = \lambda_{i+1}$.

b) Let $S^{-1}FS = F'$, where S is a unitary matrix and $F = (F_{ij})$ and $F' = (F'_{ij})$ have the form (2), and, in addition, $F_{ij} = F'_{ij}$ for $i \leqslant j$. Then $S = S_1 \oplus \ldots \oplus S_k$, where $S_i$ has the same order as $F_{ii}$.

Proof. a) Let $\varphi$ be the operator in the unitary space V given in a certain orthonormal basis by the matrix A, $(x - \lambda_1)\ldots(x - \lambda_k)$ be its minimal polynomial, and $\lambda_1 \geqslant \ldots \geqslant \lambda_k$. We choose an orthonormal basis $e_1, \ldots, e_n$ in V such that $e_{s_i+1}, \ldots, e_n$ is a basis of the space $(\varphi - \lambda_1 I)\ldots(\varphi - \lambda_i I)V$, where I is the identity operator, $1 \leqslant i < k$. The matrix of the operator $\varphi$ in this basis has the form (2), divisions into blocks pass between the $s_i$-th and the $(s_i + 1)$-th rows and between the $s_i$-th and the $(s_i + 1)$-th columns, $1 \leqslant i < k$.

b) Let $S = (S_{ij})$ be partitioned into blocks of the same size as $F = (F_{ij})$. Successively considering the blocks with the indices $(1, k), (1, k-1), \ldots, (1, 1); (2, k), (2, k-1), \ldots, (2, 2); (3, k), (3, k-1), \ldots, (3, 3); \ldots$ in the matrix $FS = SF'$ and using nonsingularity with respect to rows of the blocks $F'_{i+1,i}$ for $\lambda_{i+1} = \lambda_i$, we get $S_{ij} = 0$ for all $i < j$. Since S is unitary, it follows that $S_{ij} = 0$ for $i \neq j$. The lemma is proved.

A block that remains invariant under admissible transformations is said to be reduced. We assume that the blocks are lexicographically ordered with respect to indices: $A_{11}$, $A_{12}$, $A_{13}$, ... . The unitary problem belongs to the class of the matrix problems which A. V. Roiter proposed to call self-reproducing. If in a block matrix the first nonreduced block is reduced to the form (1) or (2) and, further, we restrict ourselves to those admissible transformations that "do not spoil" the form of this block, then the obtained problem will again be unitary. It is defined in the following manner.

Definition 2. Let $A_{pq}$ be the first nonreduced block of the matrix $A = (A_{ij})$. Depending on the arrangement of the shaded blocks, it is reduced by the transformations of unitary equivalence or similarity. Reduce $A = (A_{ij})$ to the matrix $\bar{A} = (\bar{A}_{ij})$, where, in the first case, $\bar{A}_{pq} = a_1E_1 \oplus \ldots \oplus a_{k-1}E_{k-1} \oplus 0$ is of the form (1) and the subblocks $a_1E_1, \ldots, a_{k-1}E_{k-1}$ are shaded and, in the second case, $\bar{A}_{pq} = (F_{\alpha\beta})$, $\alpha, \beta = 1, \ldots, k$, is of the form (2), and if $A_{pq}$ was shaded, then the subblocks $F_{11}$, $F_{22}$, ..., $F_{kk}$ are also shaded. The block $\bar{A}_{pq}$ is partitioned into k horizontal and k vertical strips; we extend this partition to the whole p-th horizontal and the whole q-th vertical strips of the matrix $\bar{A}$. If the new divisions pass through the shaded block $\bar{A}_{ij}$, then we carry out more divisions perpendicular to them such that it is partitioned into subblocks $\bar{A}_{ij} = (B_{\alpha\beta})$ with the square subblocks $B_{11}$, $B_{22}$, ..., $B_{kk}$, and we remove the shading from the subblocks $B_{\alpha\beta}$, $\alpha \neq \beta$. We do this with all the shaded blocks through which new divisions pass. We call the obtained new block matrix the derived matrix of the block matrix $A = (A_{ij})$ and denote it by $A'$.

Let us observe that $A'$ is defined up to equivalence.

LEMMA 3. If two block matrices A and B are equivalent, then their derived matrices $A'$ and $B'$ are also equivalent.

Proof. To construct $A'$, at first we pass to the equivalent matrix $\bar{A} = (\bar{A}_{ij})$, reducing the block $A_{pq}$ to the form (1) or (2). Obviously, in the construction of $B'$ the block $B_{pq}$ with the same p and q is reduced. Since A and B are equivalent, $\bar{A}$ and $\bar{B}$ are also equivalent, i.e., $\bar{B} = (S_i^{-1}\bar{A}_{ij}R_j)$, and $\bar{B}_{pq} = S_p^{-1}A_{pq}R_q$. By virtue of Lemmas 1, b) and 2, b), $S_p$ and $R_q$ have block-diagonal form, consistent with the additional division of $A'$ and $B'$ into blocks. The lemma is proved.

Let us consider the sequence of derived matrices $A, A', A'', \ldots, A^{(t)}, \ldots$ . Since the derived matrix contains more shaded blocks, this sequence ends with a certain matrix $A^{(s)}$, $s \geqslant 0$, for which the derived matrix is not defined. This means that the admissible transformations with the matrix $A^{(s)}$ do not change any of its blocks, i.e., $A^{(s)}$ is equivalent only to itself. If in addition A is equivalent to B, then, by Lemma 3, $A^{(s)}$ is equivalent to $B^{(s)}$ and consequently $A^{(s)} = B^{(s)}$.

Definition 3. Let A be a block matrix, $A^{(s)}$ be its s-th derived matrix, and the (s + 1)-th derived matrix be not defined. Enlarge the blocks in $A^{(s)}$ to the sizes of the blocks of the matrix A and shade the blocks that are arranged in the same manner as the shaded blocks of A. The obtained block matrix will be called a canonical matrix and is denoted by $A^{\infty}$.

The following theorem follows from what we have said above.

THEOREM 1. Each block matrix A is equivalent to the canonical matrix $A^{\infty}$. If A and B are equivalent, then $A^{\infty} = B^{\infty}$.

## 2. SCHEME OF A CANONICAL MATRIX WITH RESPECT TO THE TRANSFORMATIONS OF UNITARY SIMILARITY

We will study the structure of the canonical matrix $A^{\infty}$. For simplicity, we restrict ourselves to the case where the initial matrix A consists of a single shaded block, i.e., is reduced by transformations of unitary similarity.

A rectangle of size m × n, partitioned into unit cells, is called an (m × n)-rectangle (an n-square for m = n). We give to the cell, situated in the i-th horizontal line and the j-th vertical line, counting downwards from above and from left to right, the pair (i, j), $1 \leqslant i \leqslant m$, and $1 \leqslant j \leqslant n$.

We will represent the canonical matrix $A^{\infty}$ of order n × n schematically by an n-square, partitioned into zones, in which some of the cells are marked. To this end, we associate the cell (i, j) with each element $a_{ij}$ of the matrix. The matrix $A^{\infty}$ is constructed with respect to A by successive reduction of blocks. Let $A_{pq}$ be one of the blocks to be reduced. If the new block $\bar{A}_{pq}$ has the form (1), then the cells associated with its elements form a zone; each cell, associated with a nonzero element, is marked with a circle situated at its center; in addition, the circles corresponding to equal elements of $\bar{A}_{pq}$ are joined by a line. But if $\bar{A}_{pq} = (F_{\alpha\beta})$ is of the form (2), then the union of the cells associated with the elements of all the blocks $F_{\alpha\beta}$, $\alpha \leqslant \beta$, form a zone and the cells, situated on the principal diagonal of the zone, are marked with stars.

An example of a canonical matrix and its schematic representation are given in Fig. 2.

We suggest the following description of the set of the canonical matrices $A^{\infty}$. At first, we give an explicit definition of the set of all schemes of the form of Fig. 2. Then for each scheme we describe all the canonical matrices with this scheme.

We call an (m × n)-rectangle, which either has no marked cells or has cells (1, 1), (2, 2), ..., (t, t) marked by circles for a certain $t \leqslant \min(m, n)$, an equivalence zone of size m × n. Two adjacent circles can be joined by a line. The number of circles below the one with which a circle is joined is called the height of this circle.

We call an n-square, in which all the cells of the principal diagonal are marked by stars and, possibly, all the cells (i, j), $i > t \geqslant j$, are removed for certain natural numbers $t < n$, a similarity zone of size $n \times n$. We call the number of cells of the zone situated under a star the height of this star.

We give an example of an equivalence zone and a similarity zone (Fig. 3). The heights of their markings form the sequence 0, 2, 1, 0, 1, 0.

<u>Definition 4.</u> An n-square, partitioned into equivalence and similarity zones such that the <u>similarity zone</u> is of size n × n and the following condition is fulfilled for every other zone Z, is called a scheme of order n × n:

(A) Let (α, β) be the upper left cell of the zone Z. Let us take away markings from the zone Z and the zones, all the cells (i, j) of which follow (α, β) lexicographically:

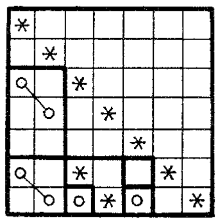
$$\begin{pmatrix} 3+i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3+i & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 5 & 0 & i & 0 & 0 & 2 & 0 \\ 0 & 5 & 8 & i & 2 & 0 & 2 \end{pmatrix},$$ 

Fig. 2

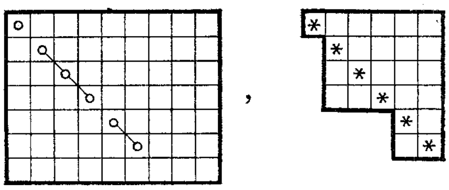


Fig. 3

$i > \alpha$, or $i = \alpha$ and $j > \beta$. We call the union of the cells $(i, 1), (i, 2), \ldots, (i, i), (i + 1, i), \ldots, (n, i)$ the $i$-th angle $(1 \leqslant i \leqslant n)$. We call two angles cohesive if they can be included in a sequence of angles in which any two adjacent angles have a common circle. Let $h_i$ be the least of the heights of the markings of the $i$-th angle and of the angles cohesive with it. Then $(h_\alpha + 1) \times (h_\beta + 1)$ is the size of the zone Z and Z is an equivalence zone if the $\alpha$-th and the $\beta$-th angles are not cohesive and is a similarity zone if these angles are cohesive.

We call the set of the stars $(\alpha, \beta), (\alpha + 1, \beta + 1), \ldots, (\alpha + k, \beta + k)$ of a zone such that the height of the star $(\alpha, \beta)$ is equal to $k$ and either $(\alpha, \beta)$ is the upper star of the zone or the star $(\alpha - 1, \beta - 1)$ is of height 0 a constellation of the scheme.

Definition 5. A matrix $A = (a_{ij})$ of the order of a scheme S which satisfies the following conditions is called a matrix with the scheme S:

1. If cell $(i, j)$ is not marked, then $a_{ij} = 0$.

2. If $(i, j)$ is a circle, then $a_{ij}$ is a positive real number. If $(i + 1, j + 1)$ is a circle from the same zone, then $a_{ij} \geqslant a_{i+1,\, j+1}$, and, in addition, $a_{ij} = a_{i+1,j+1}$ if and only if the circles are joined by a line.

3. If $(\alpha, \beta), \ldots, (\alpha + k, \beta + k)$ is a constellation, then $a_{\alpha\beta} = \ldots = a_{\alpha+k,\beta+k}$. If $(\alpha + k + 1, \beta + k + 1), \ldots, (\alpha + s, \beta + s)$ is a constellation from the same zone, then the complex numbers $a_{\alpha\beta} = a + bi$ and $a_{\alpha+k+1,\beta+k+1} = c + di$ are lexicographically ordered: $a > c$, or $a = c$ and $b \geqslant d$. For $a_{\alpha\beta} = a_{\alpha+k+1,\beta+k+1}$ block $(a_{ij})$, $\alpha + k + 1 \leqslant i \leqslant \alpha + s$, and $\beta \leqslant j \leqslant \beta + k$, is nonsingular with respect to rows.

An example of a matrix with a scheme is given in Fig. 2.

THEOREM 2. The set of the canonical matrices $A^\infty$ coincides with the set of the matrices with schemes.

Proof. By the scheme of a canonical matrix we have defined its schematic representation at the beginning of the section. Conversely, if A is a matrix with a scheme, then $A^\infty = A$.

Definition 6. For a scheme of order $n \times n$ the graph with the vertices $1, 2, \ldots, n$, in which $i$ and $j$ are joined by an edge if and only if the cell $(i, j)$ contains a circle, is called its graph.

For example, the scheme in Fig. 2 has the graph

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         5
         |
4—2—7—3—1—6
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LEMMA 4. The graph of each scheme is a union of trees.

Proof. The absence of cycles is explained by the condition (A), defining the type of a zone in a scheme.

We will write $A \equiv B$ if the matrix A can be obtained from the matrix B by a permutation of rows and the same permutation of columns (a transformation of unitary similarity).

LEMMA 5. If a matrix $A = A_1 \oplus A_2$, then $A^\infty \equiv A_1^\infty \oplus A_2^\infty$, and the relative arrangement of the rows and columns in the matrix $A_\alpha^\infty$ $(\alpha = 1, 2)$ is the same as in $A^\infty$.

Proof. It is sufficient to show that for the matrix A the sequence of derived matrices $A, A', A'', \ldots, A^{(t)} = (A_{ij}^{(t)}), \ldots$, satisfying the following conditions, exists:

1. $A_{ij}^{(t)} = X_{1ij}^{(t)} \oplus X_{2ij}^{(t)}$ , the empty terms $0_{0n}$ and $0_{m0}$ being admitted.

2. $X_{\alpha ij}^{(t)}$ form a block matrix for fixed $\alpha$ and $t$. Let us denote it by $X_\alpha^{(t)} = (X_{\alpha ij}^{(t)})$ and shade the block $X_{\alpha ij}^{(t)}$ if the block $A_{ij}^{(t)}$ is shaded.

3. $X_\alpha^{(t)}$ is the derived matrix of $A_\alpha^{(l_\alpha)}$ for some $l_\alpha \leqslant t$, possibly augmented by empty strips.

This lemma can be proved by induction over t. Let $A^{(t)}$ have the desired form, $A_{pq}^{(t)} = X_{1pq}^{(t)} \oplus X_{2pq}^{(t)}$ be its first nonreduced block (see Definition 2). We can obviously reduce $X_{1pq}^{(t)}$ and $X_{2pq}^{(t)}$ separately. Then we permute the strip to obtain the reduction for $A_{pq}^{(t)}$. As a result, we get the desired form for $A^{(t+1)}$.

A square matrix is said to be indecomposable if it cannot be reduced to a direct sum of matrices of lesser order by unitary transformations of similarity.

THEOREM 3. Each square matrix A is unitarily similar to a unique, up to permutation of summands, direct sum of indecomposable canonical matrices. This sum can be obtained in the following manner. Let the graph G of the scheme of the canonical matrix $A^\infty$ be the union of the trees $G_i$ with the vertices $v_{i1} < v_{i2} < \ldots < v_{ir_i}$ $(1 \leqslant i \leqslant m)$. Rearrange the columns of the matrix $A^\infty$ such that their old numbers form the sequence

$$v_{11}, \ldots, v_{1r_1};\ v_{21}, \ldots, v_{2r_2};\ \ldots;\ v_{m1}, \ldots, v_{mr_m},$$

then rearrange the rows in the same manner. The obtained matrix will be a direct sum of indecomposable canonical matrices $A_i$ of order $r_i \times r_i$ $(1 \leqslant i \leqslant m)$.

Proof. Let the matrix A be unitarily similar to $B_1 \oplus \ldots \oplus B_s$, where $B_i = B_i^\infty$ is an indecomposable canonical matrix. By Theorem 1 and Lemma 5, $A^\infty = (B_1 \oplus \ldots \oplus B_s)^\infty \equiv B_1 \oplus \ldots \oplus B_s$ where the arrangement of rows and columns in $B_i$ is the same as in $A^\infty$. Let $w_{i1}, \ldots, w_{il_i}$ be the numbers of the rows of matrix $A^\infty$ that form the matrix $B_i$. It is sufficient to prove that this set of vertices is a connected component of the graph G. If the vertices $w_{i\alpha}$ and $w_{j\beta}$, $w_{i\alpha} > w_{j\beta}$, are joined by an edge, then the cell $(w_{i\alpha}, w_{j\beta})$ of the scheme is marked by a circle and a nonzero element of the matrix $A^\infty$ corresponds to it, which, by virtue of $A^\infty \equiv B_1 \oplus \ldots \oplus B_s$, is possible only when i = j. On the other hand, any two vertices $w_{i\alpha}$ and $w_{i\beta}$ are connected in the graph G, since otherwise we would have $B_i \equiv X \oplus Y$, which contradicts the indecomposability of $B_i$.

COROLLARY. The number of circles in the scheme of a canonical matrix of order n × n does not exceed n − 1. This number is equal to n − 1 if and only if the matrix is indecomposable.

Proof. The number of vertices of the graph G of the scheme of a canonical matrix of order n × n is equal to n, and the number of edges is equal to the number of circles in its scheme. The graph G is a union of trees. Therefore, the number of its edges is less than n, and is equal to n − 1 only when G is a tree, which, by virtue of Theorem 3, is equivalent to the indecomposability of the matrix.

Definition 7. A scheme of order n × n is said to be simple if the number of its circles is equal to n − 1 and all the stars, situated on the principle diagonal, have height 0.

The scheme of a canonical matrix is simple if and only if the matrix is indecomposable and each of its characteristic roots has only one, up to a scalar factor, characteristic vector (e.g., it has no multiple characteristic roots).

THEOREM 4. A simple scheme is completely determined by its graph. An arbitrary tree with numbered vertices can be the graph of a simple scheme.

Proof. Let G be a tree with the vertices 1, ..., n. In the n-square we mark the cell (i, j) by a circle for each pair of vertices i and j, i > j, joined by an edge. We mark with a star each cell of the principal diagonal and also the cells $(i_1, i_m)$ for each sequence of the vertices $i_1, i_2, \ldots, i_m$ $(m \geqslant 3)$, in which the vertices $i_\alpha$ and $i_{+1\alpha}$ $(1 \leqslant \alpha < m)$ are joined by an edge and $i_2 < \min(i_1, i_m)$, $i_3 < i_1, \ldots, i_m < i_1$. We call the union of the cells (i, j), i ≤ j, and also each cell (i, j), i > j, a zone. As a result, we obtain a simple scheme with the graph G. It is obvious that the tree G cannot be the graph of any simple scheme.

Since the number of trees with the vertices 1, ..., n is equal to $n^{n-2}$ (see [5]), the number of simple schemes of order $n \times n$ is equal to $n^{n-2}$.

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# INVERSE SPECTRAL PROBLEM FOR LIMIT-PERIODIC SCHRÖDINGER OPERATORS


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UDC 517.43 + 517.9


The intensive development of the theory of nonlinear systems which are integrable by the method of the inverse problem of scattering theory, which began about 15 years ago, has served as a powerful new stimulus for the study of inverse spectral problems of the most diverse kinds. The work of recent years shows that the language of the inverse problem of scattering theory is adequate to describe important classes of operators both qualitatively and quantitatively. In this article, such an approach is applied to the study of one-dimensional Schrödinger operators with limit-periodic potential.

Examples have been given in recent articles [2-4] of limit-periodic Schrödinger operators whose spectra, from a set-theoretic point of view, are nowhere dense perfect sets of nonzero measure. It is not difficult, however, to show by the methods in the above articles that there exists an everywhere dense set of "gaps" in the spectrum of an operator only for a certain topologically massive (but implicitly describable) set of potentials, or even for potentials of a very special type. This difficulty is probably objective, and an approach using the inverse problem of scattering theory seems more natural. We obtain a result below which is, in a well-known sense, a converse of the results in [2-4]. Namely, it is established that any family of intervals on the real axis (bounded below, of course) whose endpoints satisfy certain analytical conditions and whose lengths decrease sufficiently rapidly (with respect to a natural indexing of the intervals) can be taken as the set of "gaps" in the spectrum of a limit-periodic Schrödinger operator. The Dubrovin equations and the trace formulas [5]-[6] lie at the foundation of the proof.

We recall the basic steps in the solution of the inverse problem of scattering theory for a periodic potential. Suppose that we are given

a) a sequence of nonnegative numbers $h = \{h_s\}$ indexed by the points of some one-dimensional lattice $\{s\colon Ns \in \mathbf{Z}\}$, $N \in \mathbf{N}$, where $h_0 = 0$, $h_{-s} = h_s$, $H = \sup_s h_s < \infty$;

b) a region $\Theta(h)$ in the complex plane of the form

$$\Theta(h) = \mathbf{C}_+ \setminus \bigcup_s \{\operatorname{Re} z = s,\ \operatorname{Im} z \in [0, h_s]\};$$

c) a conformal mapping $\theta\colon \mathbf{C}_+ \to \Theta(h)$ of the form


Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 18, No. 3, pp. 63-66, July-September, 1984. Original article submitted July 18, 1983.




0016-2663/84/1803-0230$08.50