Ukrainian Mathematical Journal, Vol. 50, No. 5, 1998

ON SUBGROUPS LIFTING MODULO CENTRAL COMMUTANT

V. V. Sergeichuk

UDC 512.544

We consider a finitely generated group G with the commutant of odd order $p_1^{n_1} \dots p_s^{n_s}$ located at the center and prove that there exists a decomposition of G/G' into the direct product of indecomposable cyclic groups such that each factor except at most $n_1 + \dots + n_s$ factors lifts modulo commutant.

Let G be a group with commutant G'. We call a commutant *central* if it is contained in the center of a group. We say that a subgroup $\overline{H} \subset G/G'$ *lifts modulo commutant* if there exists a set of representatives $h \in \overline{h}$ (one for every $\overline{h} \in \overline{H}$) that forms a subgroup of the group G (obviously, it is isomorphic to $\overline{H}$).

Our principal result is the following theorem:

Theorem 1. *Let A be a finite Abelian group of odd order $p_1^{n_1} \dots p_s^{n_s}$. Then, for any finitely generated group G with central commutant $G' \simeq A$, there exists a decomposition G/G' into a direct product of indecomposable cyclic groups in which the number of factors not lifting modulo commutant is less than or equal to $n_1 + \dots + n_s$. Moreover, there exists G for which this number cannot be made less than $n_1 + \dots + n_s$.*

Proof. Let

$$G/G' = \overline{G}_\infty \times \overline{G}_{q_1} \times \dots \times \overline{G}_{q_k},$$

where $\overline{G}_\infty$ is a group without torsion and $\overline{G}_{q_i}$ are Sylow q_i-subgroups.

Indecomposable direct factors of the group $\overline{G}_\infty$ lift modulo commutant.

Let $p \in \{q_1, \dots, q_k\}$,

$$\overline{G}_p = \langle \overline{g}_1 \rangle_{p^{m_1}} \times \dots \times \langle \overline{g}_r \rangle_{p^{m_r}}, \qquad m_1 \geq \dots \geq m_r \geq 1, \tag{1}$$

and $g_i \in \overline{g}_i$, $g_i^{p^{m_i}} = a \in G'$, $|a| = lp^n$, $p \nmid l$, $n \geq 0$. If we take g_i^l and $\overline{g}_i^l$ as new g_i and $\overline{g}_i$, we get $|a| = p^n$. If $p \notin \{p_1, \dots, p_s\}$, then $a = 1$ and $\langle \overline{g}_i \rangle_{p^{m_i}}$ lifts to the subgroup $\langle g_i \rangle_{p^{m_i}}$.

In view of this fact, we take $p \in \{p_1, \dots, p_s\}$. Then $g_i^{p^{m_i}} = a_1^{\alpha_{1i}} \dots a_t^{\alpha_{ti}}$, where

$$A_p = \langle a_1 \rangle_{p^{k_1}} \times \dots \times \langle a_t \rangle_{p^{k_t}} \tag{2}$$

is the Sylow p-subgroup of the group G'. Consider the integer-valued matrix

$$S_p = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{t1} & \alpha_{t2} & \cdots & \alpha_{tr} \end{pmatrix} \tag{3}$$

the ith row of which is determined modulo p^{k_i}.

Institute of Mathematics, Ukrainian Academy of Sciences, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 50, No. 5, pp. 742–745, May, 1998. Original article submitted June 9, 1997.

 0041-5995/98/5005-0842 $20.00

Let us show that, by choosing again the direct factors and elements generating these factors in decomposition (1), we can transform the matrix S_p as follows:

(i) multiply the i th column by an integer number $\gamma \not\equiv 0 \pmod{p}$;

(ii) add to the i th column the j th column $(j<i)$ multiplied by an integer number δ.

(iii) add to the i th column the j th column $(j>i)$ multiplied by an integer number $\delta p^{m_i-m_j}$.

Indeed, transformation (i) is obtained by replacing $\langle \bar{g}_i \rangle_{p^{m_i}}$ by $\langle \bar{g}_i^{\gamma} \rangle_{p^{m_i}}$ and, respectively, g_i by $g_i' = g_i^{\gamma}$ because $g_i'^{p^{m_i}} = a_1^{\gamma\alpha_{1i}} \ldots a_t^{\gamma\alpha_{ti}}$.

Transformation (ii) is obtained by replacing $\langle \bar{g}_i \rangle_{p^{m_i}}$ by $\left\langle \bar{g}_i \bar{g}_j^{\delta p^{m_j-m_i}} \right\rangle_{p^{m_i}}$ and, respectively, g_i by $g_i' = g_i g_j^{\delta p^{m_j-m_i}}$ because, for the central commutant, $p \neq 2$, and $g_i^{p^{m_i}} \in G'$, we have

$$g_i'^{p^{m_i}} = g_i^{p^{m_i}} g_j^{\delta p^{m_j}} \left[g_j^{\delta p^{m_j-m_i}}, g_i \right]^{1+2+\ldots+(p^{m_i}-1)} = g_i^{p^{m_i}} g_j^{\delta p^{m_j}} \left[g_j^{\delta p^{m_j-m_i}}, g_i \right]^{\frac{(p^{m_i}-1)p^{m_i}}{2}}$$

$$= g_i^{p^{m_i}} g_j^{\delta p^{m_j}} \left[g_j^{\delta p^{m_j}}, g_i \right]^{\frac{p^{m_i}-1}{2}} = g_i^{p^{m_i}} g_j^{\delta p^{m_j}} = a_1^{\alpha_{1i}+\delta\alpha_{1j}} \ldots a_t^{\alpha_{ti}+\delta\alpha_{tj}}.$$

Transformation (iii) is obtained by replacing $\langle \bar{g}_i \rangle_{p^{m_i}}$ by $\langle \bar{g}_i \bar{g}_j^{\delta} \rangle_{p^{m_i}}$ and, respectively, g_i by $g_i' = g_i g_j^{\delta}$ because

$$g_i'^{p^{m_i}} = g_i^{p^{m_i}} g_j^{\delta p^{m_i}} \left[g_j^{\delta}, g_i \right]^{1+2+\ldots+(p^{m_i}-1)} = g_i^{p^{m_i}} g_j^{(\delta p^{m_i-m_j}) p^{m_j}} = a_1^{\alpha_{1i}+\delta p^{m_i-m_j}\alpha_{1j}} \ldots a_t^{\alpha_{ti}+\delta p^{m_i-m_j}\alpha_{tj}}.$$

Obviously, if we take a new decomposition (1) and new representatives $g_i \in \bar{g}_i$, then we can pass from the matrix S_p to a new one by using transformations (i)–(iii).

Taking into account that the elements of the first row of the matrix S_p are determined modulo p^{k_1} and using transformation (i), we transform them to the form p^l, $0 \le l \le k_1$, or 0, and then, by transformation (ii) we transform the first row to the form

$$(0, \ldots, 0, p^{l_1}, 0, \ldots, 0, p^{l_2}, 0, \ldots\ldots, 0, p^{l_d}, 0, \ldots, 0), \tag{4}$$

where $k_1 > l_1 > \ldots > l_d \ge 0$. Columns with nonzero element of the first row are called *selected.* Their number is at most k_1.

Then, with the use of the same procedure, we transform elements of the second row, except elements from selected columns (in this case, the first row does not change) and call columns with nonzero elements of the first and second rows *selected;* their number is at most $k_1 + k_2$, and so on. After reduction of the last row, we obtain a matrix S_p that has at most $k_1 + \ldots + k_r = n(p)$ selected columns (where $p^{n(p)} = |A_p|$); the other columns are zero.

The zero column with number i corresponds to a (new) element $g_i \in \bar{g}_i$ whose order is equal to the order of $\bar{g}_i$. Therefore, the subgroup $\langle \bar{g}_i \rangle_{p^{m_i}}$ lifts modulo G' to the subgroup $\langle g_i \rangle_{p^{m_i}}$. Since $n(p_i) = n_i$, by performing this procedure for all $p \in \{q_1, \ldots, q_k\}$ we obtain the decomposition of G/G' into a direct product of indecomposable cyclic groups in which almost every factor, except at most $n_1 + \ldots + n_s$, lifts modulo commutant.

It remains to show that, for every A, there exists a group G_A for which the number of factors that do not lift is at least $n_1 + \ldots + n_s$. It is sufficient to construct G_{A_p} for p-groups A_p because, in this case, for any $A = A_{p_1} \times \ldots \times A_{p_s}$ $(3 \le p_1 \le \ldots \le p_s)$, we can set $G_A = G_{A_{p_1}} \times \ldots \times G_{A_{p_s}}$. Let A_p have the form (2) with $k_1 \ge \ldots \ge k_t$. Consider the integer-valued matrix $S_p = (\alpha_{ij}) = (M_{k_1} \mid M_{k_1 - 1} \mid \ldots \mid M_0)$ of dimension $t \times r$, where $r = (k_1+1)t$, $M_i = \mathrm{diag}\,(p^i, \ldots, p^i, 0, \ldots, 0)$ and nonzero elements of the ith row of the matrix S_p form the sequence $p^{k_i - 1}, p^{k_i - 2}, \ldots, p, 1$ (in particular, $M_{k_1} = 0$). Let G_p be a group with generators $a_1, \ldots, a_t, g_1, \ldots, g_r$ and determining relations $a_i^{p^{k_i}} = 1$, $g_i^{p^{m_i}} = a_1^{\alpha_{1i}} \ldots a_t^{\alpha_{ti}}$, $m_i = 2(r-i) + k_1$, $[g_i, g_{t+i}] = a_i$, $1 \le i \le t$; the other pairs of generators commute. Since the number of nonzero columns of the matrix S_p is equal to $n(p) = k_1 + \ldots + k_r$ and their number cannot be increased by transformations (i)-(iii), the number of factors that cannot lift cannot be made less than $n(p)$ by the choice of decomposition (1) and we may set $G_{A_p} = G_p$. The theorem is proved.

A similar result was obtained by Thompson (see [1], Theorem 12.2); if G is a finite p-group $(p > 2)$ all elements of which of order p are contained in its center $Z(G)$, then the minimal number of generators G is not greater than the minimal number of generators $Z(G)$.

Remark 1. If the group G considered above has a finite order, then for its determination one should, for every $p \in \{q_1, \ldots, q_k\} \cap \{p_1, \ldots, p_s\}$, give not only the matrix S_p but also the set of skew-symmetric $r \times r$-matrices $C_p^{(1)} = (\beta_{ij}^{(1)}), \ldots, C_p^{(t)} = (\beta_{ij}^{(t)})$ determining the commutators $[g_i g_j] = a_1^{\beta_{ij}^{(1)}} \ldots a_t^{\beta_{ij}^{(t)}}$ (and, hence, orders of indecomposable direct factors of the groups G' and G/G'); in this paper we give only matrices S_p. Note that the problem of classification of finite p-groups with central commutant of the type (p, p) and the problem of classification of finite p-groups with central commutant of the type (p^2) is reduced to the classical unsolved problem of the canonical form of a pair of matrices with respect to transformations of simultaneous similarity (see [2]). A complete classification of finitely generated groups with commutant of order p was obtained in [3].

Remark 2. The theorem is also true for an infinitely generated group G for which G/G' is the (probably, infinite) direct product of indecomposable locally cyclic groups (i.e., the groups $\mathbb{Q}^+$, $\mathbb{Z}^+$, (p^∞), (p^n)); in this case, however, one should somewhat modify the proof. Namely, instead of (1), we consider the direct product $\bar{G}_p = \times_{i<r} H_i$ (r is an ordinal number), where H_i has the type (p^{m_i}), $\infty \ge m_1 \ge m_2 \ge \ldots$. For every i, we fix m_i elements $\bar{g}_{il} \in H_i$, $1 \le l \le m_i + 1$, for which $\bar{g}_{i1}^p = 1$, $\bar{g}_{i2}^p = \bar{g}_{i1}$, $\bar{g}_{i3}^p = \bar{g}_{i2}, \ldots$ and choose $g_{il} \in \bar{g}_{il}$ so that $g_{i2}^p = g_{i1}$, $g_{i3}^p = g_{i2}, \ldots$. As in the proof of the theorem, we can assume that the order of an element $g_{i1}^p \in G'$ is the power of p and, hence, it is sufficient to consider the case $p \in \{p_1, \ldots, p_s\}$. Then $g_{i1}^p \in A_p$ [see (2)], $g_{i1}^p = a_1^{\alpha_{1i}} \ldots a_t^{\alpha_{ti}}$, and we obtain an integer-valued matrix S_p of the form (3), but, probably, with infinitely many columns. This matrix can be transformed with the use of transformations (i) and (ii) and the substitutions $g'_{il} = g_{il}^\gamma$ and, respectively, $g'_{il} = g_{il} g_{jl}^\delta$ for all $l < m_i + 1$. By using transfinite induction, we reduce the first row to the form (4) and so on. As follows from [4] (Theorem 1), these results can be extended to the case of groups with (noncentral) commutant that are finite direct products of cyclic groups of prime order.

REFERENCES

1. B. Huppert, *Endlishe Gruppen,* Springer, Berlin (1967).
2. V. V. Sergeichuk, "On classification of metabelian groups," in: *Matrix Problems* [in Russian], Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1977), pp. 151–161.
3. V. V. Sergeichuk, "Finitely generated groups with commutant of prime order," *Ukr. Mat. Zh.,* **30**, No. 6, 789–795 (1978).
4. O. O. Mazurok, "Groups with elementary Abelian commutant of at most p^2th order," *Ukr. Mat. Zh.,* **50**, No. 4, 534–539 (1998).