PENCILS

PSEUDOLINEAR MATRIX ~~BUNDLES~~ AND SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

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We multiply the system $Ay' + By = f(t)$ (primes indicate d/dt) of linear differential equations, with constant coefficients, by R^{-1} and make the substitution $y = Sz$ (R and S are invertible constant matrices), thus obtaining the system $(R^{-1}AS)z' + (R^{-1}BS)z = R^{-1}f(t)$. The canonical form of a matrix pair under transformations $(A, B) \to (R^{-1}AS, R^{-1}BS)$ was obtained by Kronecker (in work on a matrix bundle; see [1], and [2] Chap. XII).

We multiply the linear differential equation system

$$A(t)y' + B(t)y = f(t) \tag{1}$$

with meromorphic coefficients, by the matrix $R(t)^{-1}$ and make the substitution $y = S(t)z$ [$R(t)$ and $S(t)$ are invertible meromorphic matrices], thus obtaining the system $A_0(t)z' + B_0(t)z = R(t)^{-1}f(t)$, where

$$A_0(t) = R(t)^{-1}A(t)S(t), \quad B_0(t) = R(t)^{-1}(B(t)S(t) + A(t)S'(t)). \tag{2}$$

In this work we establish the canonical form of the matrix pair $[A(t), B(t)]$ with respect to transformations (2).

The direct sum of pairs $[A_1(t), B_1(t)]$ and $[A_2(t), B_2(t)]$ is defined to be the pair

$$\begin{pmatrix} A_1(t) & O \\ O & A_2(t) \end{pmatrix}, \quad \begin{pmatrix} B_1(t) & O \\ O & B_2(t) \end{pmatrix}.$$

In Sec. 1 we prove that each pair of meromorphic matrices of the same order is reduced, by the transformations (2), to direct sums of the following matrix pairs:

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}; \tag{3}$$

$$\begin{pmatrix} 0 & & 0 \\ 1 & \ddots & \\ & \ddots & 0 \\ 0 & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \ddots & 0 \\ 0 & \ddots & \\ & \ddots & 1 \\ 0 & & 0 \end{pmatrix}; \tag{4}$$

$$\begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}; \tag{5}$$

$$\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad \begin{pmatrix} -\alpha_1 & \dots & \dots & -\alpha_n \\ 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}. \tag{6}$$

In Sec. 2, we impose conditions on $\alpha_1 = \alpha_1(t), \dots, \alpha_n = \alpha_n(t)$, ensuring that the direct sum over the original pair is uniquely determined.

It will follow that each system (1), after the substitution $y = S(t)z$, splits into subsystems with matrix pairs of the form (3)-(6). Systems with pairs (3)-(5) can be solved by differentiation. A system with a pair (6) can be reduced to a single equation $z^{(n)} - \alpha_1(t)z^{(n-1)} + \dots + (-1)^n\alpha_n(t)z = h(t)$.

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Author's remark to the English translation:
the word "bundle" must be replaced by "pencil";
the word "body" must be replaced by "skew field".

1. An Algorithm for Direct-Sum Representation. We describe how a pair (A, B) of meromorphic matrices of the same dimension (we omit t) can be reduced, by transformations (2), to a direct sum of pairs of the form (3)-(6). The reduction method was suggested in [3, 4]. The third step of the algorithm recalls the Danilevskii method of finding coefficients of characteristic polynomials ([5], Sec. 46). The rows and columns of A and B are interpreted to be vectors over the field F of meromorphic functions.

First Step. Calculation of Direct Terms of the Form (4), (5).

We reduce a pair to the form

$$\begin{pmatrix} 0 \\ \hline A_2 \end{pmatrix}, \quad \begin{pmatrix} B_1 \\ \hline B_2 \end{pmatrix}, \qquad (7)$$

where the sector A_2 is row-nonsingular (i.e., its rows are linearly independent over F). We carry out transformations (2) conserving the 0 in (7) by applying the following operations:

a) elementary transformations of rows of B_1;

b) elementary transformations of rows simultaneously in A_2 and B_2;

c) addition of a linear combination of rows of B_1 to a row of B_2;

d) replacement of A_2, B_1, and B_2 by A_2S, B_1S, and $B_2S + A_2S'$, respectively, where S is an invertible meromorphic matrix.

If, by the transformation a), we can make the first row of B_1 the null vector, then we obtain from the pair a direct term of the form (4), of dimension 1×0.* Removing all such terms, we obtain a pair (7) with B_1 row-nonsingular. By the transformations d) and c), we make B_1 equal to $(E|0)$, and all elements under E nonzero; then the transformation b) leads to a pair

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline A_3 & 0 \\ 0 & A_4 \end{array}\right), \quad \left(\begin{array}{c|c} E & 0 \\ \hline 0 & B_3 \\ 0 & B_4 \end{array}\right) \qquad (8)$$

with row-nonsingular A_3 and A_4 [identically located sectors of the matrices (8) are of the same order]. The transformation d) yields $A_3 = (0_{mn}E)$, and the left vertical strip of the second matrix is spoiled by the application of the transformations a) and c). By removing n direct terms of the form (5) of order 1×1, we obtain the pair (8) with $A_3 = E$.

Now consider the fragment $\bar{A} = \begin{pmatrix} 0 \\ A_4 \end{pmatrix}$, $\bar{B} = \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}$. A linear combination of rows of B_3 can be added to a row of B_4 [transformation b)]. In the pair (8), there will no longer be a null sector below $A_3 = E$; we make it zero again by applying the null transformation d). There will then not be null sectors to the left of B_3 and B_4; they are made null by applying transformation c).

Hence, with the fragment $\bar{A}$, $\bar{B}$, we can make transformations similar to a)-d); in them the only difference is that A_2, B_1, and B_2 must be replaced by A_4, B_3, and B_4, respectively [this is true for the transformation c)].

With the fragment $(\bar{A}, \bar{B})$ we perform the same transformations as with the whole pair; we remove the 2×1 direct term of the form (4) and the 2×2 terms of the form (5), and obtain the pair (see top of following page) with a row-nonsingular A_6.

*As is customary, we admit the existence of null matrices $0_{m,n}$, in which the number of rows is $m = 0$ or the number of columns is $n = 0$. In particular, the pairs (3) and (4) can be $(0_{01}, 0_{01})$ and $(0_{10}, 0_{10})$, respectively. The direct sum of pairs (M, N) and $(0_{mn}, 0_{mn})$ is obtained by attributing m null rows and n null columns to M and N, respectively ($m \geq 0$, $n \geq 0$).

$$\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline E & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & A_6 \end{array}\right), \quad \left(\begin{array}{c|cc} E & 0 & 0 \\ \hline 0 & E & 0 \\ 0 & 0 & B_5 \\ 0 & 0 & B_6 \end{array}\right)$$

The same transformations are applied to the fragment $\begin{pmatrix} 0 \\ A_6 \end{pmatrix}$, $\begin{pmatrix} B_5 \\ B_6 \end{pmatrix}$.

These transformations are repeated until the dimension of E becomes 0 × 0 [the dimension of E is decreased by each removal of direct terms of the form (4), (5)]. We thus obtain a matrix pair with a row-nonsingular first matrix.

Second Step. Separation of Direct Terms of the Form (3). Let (A, B) be a pair with a row-nonsingular first matrix. This pair is reduced to

$$(0 \mid A_2), \quad (B_1 \mid B_2) \tag{9}$$

with a nonsingular A_2, and the following transformations (2) with $S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}$, are applied, conserving the 0 in (9):

a) elementary transformations of rows, simultaneously in A_2, B_1, and B_2;

b) elementary transformations of columns of B_1;

c) addition, to a column of B_2, of a linear combination of columns of B_1;

d) replacement of A_2 and B_2 by A_2S_3 and $B_2S_3 + A_2S_3$, respectively, where S_3 is an invertible meromorphic matrix.

If, by a transformation b), we can produce a null column in B_1, then we separate a direct term of the form (3) of dimension 0 × 1 from the pair. Removing all such terms, we apply a transformation a) to reduce B_1 to the form $\left(\begin{array}{c} E \\ \hline 0 \end{array}\right)$, and then transformations d) and c) to reduce the pair to

$$\left(\begin{array}{c|cc} 0 & E & 0 \\ \hline 0 & 0 & A_4 \end{array}\right), \quad \left(\begin{array}{c|cc} E & 0 & 0 \\ \hline 0 & B_3 & B_4 \end{array}\right)$$

with a nonsingular A_4.

Making the same transformations with the fragment $\bar{A} = (0\ A_4)$, $\bar{B} = (B_3 B_4)$, we remove direct terms of the form (3) of dimension 1 × 2 from the pair, and reduce it to

$$\left(\begin{array}{c|ccc} 0 & E & 0 & 0 \\ \hline 0 & 0 & E & 0 \\ 0 & 0 & 0 & A_6 \end{array}\right), \quad \left(\begin{array}{c|ccc} E & 0 & 0 & 0 \\ \hline 0 & E & 0 & 0 \\ 0 & 0 & B_5 & B_6 \end{array}\right)$$

with a nonsingular A_6.

The same transformations are applied to the fragment $(0\ A_6)$, (B_5B_6).

Repeating these transformations until the dimension of E is 0 × 0, we obtain a pair with a nonsingular first matrix.

Third Step. Separation of Direct Terms of the Form (6). Suppose that there is a pair with a nonsingular first matrix. We reduce the pair to the form (E, B), and apply transformations (2) with R = S (conserving the first matrix E). Taking S to be an elementary matrix, we obtain the following set of elementary transformations with the matrix B:

a) permutation of the i-th and j-th columns, and then permutation of the i-th and j-th rows;

b) multiplication of the i-th column by $\alpha \neq 0$, addition of α' to the (i, i)-th element, and then division of the i-th row by α;

c) addition of the i-th column multiplied by α to the j-th column, addition of α' to the (i, j)-th element, and subtraction of the j-th row multiplied by α from the i-th row.

We reduce the matrix B by induction. Suppose that it has been reduced to one of the following forms:

$$B_1 = \left(\begin{array}{ccc|cccc} & C_1 & & & D_1 & & \\ \hline \alpha_1 & \dots & \alpha_m & \beta_1 & \dots & & \beta_n \\ & & & 1 & 0 & & 0 \\ & 0 & & & \ddots & \ddots & \\ & & & 0 & & 1 & 0 \end{array}\right), \quad B_2 = \left(\begin{array}{c|cccc} & \gamma_1 & & & \\ C_2 & \vdots & & 0 & \\ & \gamma_m & & & \\ \hline & \delta_1 & 1 & & 0 \\ D_2 & \vdots & 0 & \ddots & \\ & \vdots & & \ddots & 1 \\ & \delta_n & 0 & & 0 \end{array}\right)$$

(the start of the induction is for n = 1). It is sufficient to show either that an $n \times n$ pair of the form (6) is separated from the pair by direct terms (in which case we remove this pair and begin the reduction again), or that the matrix can be reduced to the form B_1 or the form B_2, but with larger n.

Suppose, for example, that B is of the form B_1.

If $(\alpha_1, \dots, \alpha_m) \neq (0, \dots, 0)$, then transformations a) and b) make $\alpha_m = 1$, then c) makes $(\alpha_1, \dots | \dots \beta n) = (0 \dots 01 | 0 \dots 0)$, and we obtain a matrix of the form B_1 with larger n.

If $\alpha_1 = \dots = \alpha_m = 0$, then in the right vertical strip, we interchange the first and last columns, the second and second-last columns, etc.; the same interchange of rows is made in the lower horizontal strip [transformation a)]. This yields

$$\left(\begin{array}{c|cccc} C_1 & & D_3 & & \\ \hline & 0 & 1 & & 0 \\ 0 & & \ddots & \ddots & \\ & 0 & & 0 & 1 \\ & \beta_n & \dots & & \beta_1 \end{array}\right).$$

By adding the second-last row to the remaining rows [transformation c)], we reduce the last column to the form $(0 \dots 0 | 0 \dots 010)$. We next reduce the second-last column to the form $(0 \dots 0 | 0 \dots 100)$, etc. After the reduction of the second column of the right vertical strip, we obtain a matrix of the form B_2 with $D_2 = 0$. If $\gamma_1 = \dots = \gamma_m = 0$, then we separate a direct term from the pair, thus obtaining a pair of the form (6) of dimension $n \times n$. If $(\gamma_1, \dots, \gamma_m) \neq (0, \dots, 0)$, then by making $(\gamma_1 \dots | \dots \delta_n) = (0 \dots 01 | 0 \dots 0)$ we obtain a matrix of the form B_2, with larger n.

2. Algebraic Theory of Pseudolinear ~~Bundles~~ Pencils. Here we describe how to reduce a pair of meromorphic matrices to a unique direct sum of pairs of the form (3)-(6). We give the result in the most general form possible, which includes a classification of pseudolinear operators (see [6], Sec. 8.4), and a classification of pairs consisting of linear and semilinear mappings [7, 8].

Let K be a ~~body~~ skew field in which an automorphism $\varphi: K \to K$ and Ψ-differentiation $\delta: K \to K$ are fixed; here the differentiation is such that $(\alpha + \beta)^\delta = \alpha^\delta + \beta^\delta$ and $(\alpha\beta)^\delta = \alpha^\delta\beta + \alpha\beta^\delta$ (for example K is the field of meromorphic functions with the identity automorphism and the customary differentiation (see [6], Sec. 0.5).

A pseudolinear bundle over the body K is understood to be a set

$$\mathcal{P} = (V, W; \mathcal{A}, \mathcal{B}). \tag{10}$$

We use the following notation: V and W are finite-dimensional right vector spaces over the body K; $\mathcal{A}: V \to W$ is a linear mapping; $\mathcal{B}: V \to W$ is a mapping such that, for v, $v_1 \in V$ and

$\alpha \in K$, we have $(v+v_1)\mathscr{B} = v\mathscr{B}+v_1\mathscr{B}$, and $(v\alpha)\mathscr{B} = (v\mathscr{B})\alpha^{\varphi} + (v\mathscr{A})\alpha^{\delta}$ [when V = W and $\mathscr{A}$ = 1, a mapping $\mathscr{B}$ is called pseudolinear ([6], Sec. 8.4); for φ = 1 and δ = 0, Gabriel [9] calls the set (10) a Kronecker module]. A bundle morphism $(\mathscr{S}, \mathscr{R}) : (V_0, W_0; \mathscr{A}_0, \mathscr{B}_0) \to (V, W; \mathscr{A}, \mathscr{B})$ is understood to be a pair of linear mappings $\mathscr{S}: V_0 \to V$ and $\mathscr{R}: W_0 \to W$, for which $\mathscr{A}_0\mathscr{R} = \mathscr{S}\mathscr{A}$, and $\mathscr{B}_0\mathscr{R} = \mathscr{S}\mathscr{B}$. Pseudolinear bundles form an Abel category with the direct sum $\mathscr{P} \oplus \mathscr{P}_1 = (V \oplus V_1, W \oplus W_1; \mathscr{A} \oplus \mathscr{A}_1, \mathscr{B} \oplus \mathscr{B}_1)$. Isomorphic bundles will be called equivalent.

Each bundle is equivalent to a bundle of the form

$$(K^n, K^m; \mathscr{A}, \mathscr{B}), \tag{11}$$

where K^n is a right space of column vectors $(\lambda_1, \ldots, \lambda_n)^T$, $\lambda_i \in K$, $K^0 = 0$. A bundle (11) will be identified with the pair of matrices (A, B) of dimension m × n, whose columns are the images of the basis vectors $e_1 = (1, 0, \ldots, 0)^T, \ldots, e_n = (0, \ldots, 0, 1)^T$, and $A = B = 0_{mn}$ when m = 0 or n = 0. Then $(\lambda_1, \ldots, \lambda_n)^T \mathscr{A} = A(\lambda_1, \ldots, \lambda_n)^T$ and $(\lambda_1, \ldots, \lambda_n)^T \mathscr{B} = B(\lambda_1^{\varphi}, \ldots, \lambda_n^{\varphi})^T + A(\lambda_1^{\delta}, \ldots, \lambda_n^{\delta})^T$. Bundles (A, B) and (A_0, B_0) are equivalent if and only if $A_0 = R^{-1}AS$ and $B_0 = R^{-1}(BS^{\varphi}+AS^{\delta})$, where R and S are nonsingular matrices [see (2)].

A skew-polynomial ring $\Lambda = K[x; \varphi, \delta]$ ([6], Sec. 0.8) is understood to be the ring of polynomials K[x], in which ordinary multiplication is replaced by multiplication defined as follows: $\alpha x = x\alpha^{\varphi} + \alpha^{\delta}$, $\alpha \in K$. A polynomial $a \in \Lambda$, $a \notin K$, is called unsplittable ([6], Sec. 3.2), if it cannot be expressed as $a = a_1b_1 = a_2b_2$, where the polynomials $a_1 \notin K$, and $a_2 \notin K$ are mutually left-prime (i.e., $a_1\Lambda + a_2\Lambda = \Lambda$) and the sum of their degrees is equal to the degree of the polynomial a. Polynomials a, b $\in \Lambda$ are similar ([6], Sec. 3.3), if their degrees are equal and au = vb for some u,v $\in \Lambda$ such that a and v are mutually left-prime (it can be required that the degrees of u and v are, respectively, lower than the degrees of b and a).

The following theorem was proved by Kronecker for φ = 1 and δ = 0 [1] (see also [2]), and for $\varphi \neq 1$ and δ = 0 in [7, 8].

THEOREM. Each pseudolinear bundle over K is equivalent to a direct sum of bundles of the form (3)-(6), uniquely determined to within a permutation of terms; the elements $\alpha_1, \ldots, \alpha_n$ in the bundle (6) are the coefficients of an unsplittable skew polynomial $x^n + x^{n-1}\alpha_1 + \ldots + \alpha_n \in K[x; \varphi, \delta]$, determined to within a similarity.

Proof. The ring of endomorphisms of a bundle $\mathscr{P} = (V, W; \mathscr{A}, \mathscr{B})$ with no representation as a direct sum, is local; if φ, and ψ are noninvertible endomorphisms of the bundle, then $\varphi + \psi$ is not invertible. Suppose that $\varphi + \psi$ is invertible; assume that $\varphi + \psi = 1$, let $\varphi = (\mathscr{S}, \mathscr{R})$, and let m be a positive integer such that $\operatorname{Im} \mathscr{S}^m = \operatorname{Im} \mathscr{S}^{m+1}$, and $\operatorname{Im} \mathscr{R}^m = \operatorname{Im} \mathscr{R}^{m+1}$. Then $\mathscr{P} = (\operatorname{Im} \mathscr{S}^m, \operatorname{Im} \mathscr{R}^m; \mathscr{A}_1, \mathscr{B}_1) \oplus (\operatorname{Ker} \mathscr{S}^m, \operatorname{Ker} \mathscr{R}^m; \mathscr{A}_2, \mathscr{B}_2)$, where $\mathscr{A}_i$, and $\mathscr{B}_i$ are restrictions of $\mathscr{A}$, and $\mathscr{B}$. The bundle $\mathscr{P}$ cannot be expanded, and so $\varphi^m = 0$, and $1 + \varphi + \ldots + \varphi^{m-1} = (1-\varphi)^{-1} = \psi^{-1}$, and we have a contradiction.

Hence, by virtue of the Krull–Schmidt theorem for additive categories ([10], p. 31), each pseudolinear bundle is equivalent to a direct sum of nonexpandable bundles, to within an equivalence of direct terms.

The algorithms in Sec. 1 are easily converted to apply to pseudolinear bundles over a body (we assume that the rows of matrices are in the left vector space and the columns are in the right vector space).

Let (A, B) be a pseudolinear bundle over K with a singular matrix A, that is not expandable in a direct sum. The algorithm in Sec. 1 implies that this bundle is equivalent to one of the bundles (3)-(5), which cannot be expanded because they have local rings of endomorphisms ([10], p. 31), consisting of matrix pairs of the form

$$S = \begin{pmatrix} \alpha & & * \\ & \ddots & \\ 0 & & \alpha \end{pmatrix}, \quad R = \begin{pmatrix} \alpha & & * \\ & \ddots & \\ 0 & & \alpha \end{pmatrix}$$

for the bundle (3), and a matrix pair of the form (S^T, R^T) for bundles (4) and (5).

Let (A, B) be a nonexpandable pseudolinear bundle over K, with a nonsingular matrix A. It is equivalent to a bundle of the form $(V, V; 1, \mathscr{B})$. We follow [6] (Sec. 8.4), and convert the vector space V into a right module over the ring of skew polynomials $\Lambda = K[x; \varphi, \delta]$, putting

The ring Λ has left and right division algorithms; there is therefore a domain of principal left and principal right ideals. The module V cannot be expanded; hence $V = v\Lambda$, and the non-null skew polynomial $\chi_v(x) = x^n + x^{n-1}\alpha_1 + \ldots + \alpha_n$ of the lowest degree, such that $v\chi_v(x) = 0$, is not splittable, and $v\Lambda$ and $w\Lambda$ are isomorphic if and only if the similar polynomials $\chi_v(x)$ and $\chi_w(x)$ are similar ([6], Sec. 3.2 and 3.3). In the basis v, $v(x + \alpha_1)$, ..., $v(x^{n-1} + x^{n-2}\alpha_1 + \ldots + \alpha_{n-1})$, the matrix of the mapping $\mathscr{B}$ coincides with the second matrix of the bundle (6); hence $(V, V; 1, \mathscr{B})$ is equivalent to the bundle (6). This proves the theorem.

<u>COROLLARY.</u> A linear differential-equation system $A(t)y' + B(t)y = 0$, with coefficients from a function field F closed under differentiation (for example the field of meromorphic functions), when multiplied by $R(t)^{-1}$ and subject to the variable change $y = S(t)z$ [where R(t) and S(t) are invertible matrices with elements from F], can be split into subsystems with matrices of the form (3)-(6); the elements α_1, ..., α_n in (6) are the coefficients of a nonsplittable skew polynomial $\chi(x) = x^n + x^{n-1}\alpha_1 + \ldots + \alpha_m \in F[x; 1, d/dt]$. The subsystems with the matrices (3)-(5) are uniquely determined by the original system, and the subsystems with matrices (6) are uniquely determined by the original system to within the replacement of the skew polynomial $\chi(x)$ by a similar polynomial.

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