CLASSIFICATION OF PAIRS OF SUBSPACES IN SCALAR PRODUCT SPACES

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Up to the classification of Hermitian forms a classification has been given of triples $\mathcal{P} = (V_F;\ U_1,\ U_2)$, consisting of a finite dimensional vector space V over a field of characteristic $\neq 2$ with a symmetric, or a skew-symmetric, or Hermitian form F and two subspaces U_1, U_2. Two triples $\mathcal{P}$ and $\mathcal{P}'$ are identified with each other if there exists an isometry $\varphi:\ V_F \to V'_{F'}$ such that $\varphi(U_i) = U_i'$, $i = 1, 2$.

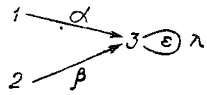
The classification problem for quadruples of subspaces in finite dimensional vector spaces has been solved by Nazarova [1, 2] and independently by Gel'fand and Ponomarev [3, 4]. In this paper we consider a classification problem for pairs of subspaces in scalar product spaces. We will solve it over a field of characteristic $\neq 2$ up to the classification of Hermitian forms over the field. The result has been partially announced in [5].

Let us strictly define the problem. Denote by $\mathcal{P} = (V_F;\ U_1,\ U_2)$ a triple consisting of a finite dimensional vector space V with a symmetric, or skew-symmetric, or Hermitian form and two subspaces U_1, U_2. Two triples $\mathcal{P}$ and $\mathcal{P}'$ will be called isomorphic if there exists a nondegenerate linear map $\varphi:\ V \to V'$ preserving the scalar product and the subspaces U_1, U_2, i.e., $F(x, y) = F'(\varphi(x), \varphi(y))$, $\varphi(U_1) = U_1'$, $\varphi(U_2) = U_2'$. The aim of this article is to characterize triples $\mathcal{P}$ up to an isomorphism.

1. Main Result. To characterize triples $\mathcal{P} = (V_f;\ U_1,\ U_2)$ we will use a method presented in [5, 6, 7].

Let K be a field of characteristic $\neq 2$ with an involution $a \to \bar{a}$ (possibly trivial). Let us fix a number $\varepsilon \in \{-1, 1\}$ equal to 1 for nontrivial involution in the field K.

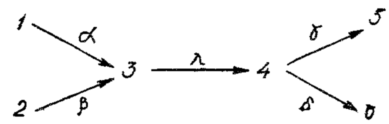
According to [5, 7], a representation A of an oriented graph

G: 

is given if to its vertices 1, 2, and 3 there correspond finite dimensional vector spaces A_1, A_2, A_3; and to its arrows α, β linear mappings $A_\alpha:\ A_1 \to A_3$, $A_\beta:\ A_2 \to A_3$; to its loop λ ε-Hermitian form $A_\lambda(x, y) = \overline{\varepsilon A_\lambda(y, x)}$ on space A_3 (i.e., a symmetric, or skew-symmetric, or Hermitian form on A_3). Two representations A and B are isomorphic if there exist nondegenerate linear mappings $\varphi_i:\ A_i \to B_i$, $i = 1, 2, 3$ such that $\varphi_3 A_\alpha = B_\alpha \varphi_1$, $\varphi_3 A_\beta = B_\beta \varphi_2$, $A_\lambda(x, y) = B_\lambda(\varphi_3(x), \varphi_3(y))$. The direct sum of the representations A and B is the representation $C = A \oplus B$, where $C_i = A_i \oplus B_i$, $i \in \{1, 2, 3, \alpha, \beta, \lambda\}$.

Obviously, every representation A determines a triple $\mathcal{P} = ((A_3)_{A_\lambda};\ \mathrm{Im}(A_\alpha),\ \mathrm{Im}(A_\beta))$ where isomorphic representations correspond to isomorphic triples [for the sake of mutual unique correspondence one can assume that $\mathrm{Ker}(A_\alpha) = \mathrm{Ker}(A_\beta) = 0$].

It has been proved in [5, 7] that classification of representations of a graph G can be obtained from a classification of representations of the quiver

Q: 

We recall that a representation of quiver Q associates with a vertex a finite dimensional space, with an arrow a linear mapping. A homomorphism $\varphi:\ M \to N$ of representations is called a collection of linear mappings $\varphi_i:\ M_i \to N_i$, $1 \leq i \leq 6$ satisfying the conditions

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$\varphi_3 M_\alpha = N_\alpha \varphi_1$, $\varphi_3 M_\beta = N_\beta \varphi_2$, $\varphi_4 M_\lambda = N_\lambda \varphi_3$, $\varphi_5 M_\gamma = N_\gamma \varphi_4$, $\varphi_6 M_\delta = N_\delta \varphi_4$. The dimension of representation M is called the vector $(m_1, \ldots, m_6)$, where $m_i = \dim(M_i)$.

Representations of quiver Q are characterized in [2] (see also Sec. 2). If there exists only one, up to an isomorphism, representation, which is not decomposable into a direct sum, of dimension $(m_1, \ldots, m_6)$, then it will be denoted by $[m_1, \ldots, m_6]$.

Representations of graph G and quiver Q we define by collections of matrices $A = [A_\alpha, A_\beta, A_\lambda]$ and $M = [M_\alpha, M_\beta, M_\lambda, M_\gamma, M_\delta]$, while assuming that some bases in the spaces have been chosen.

For representation M of quiver Q we will define representation M^+ of the graph G: $M^+ = [M_\alpha \oplus M_\gamma^*, M_\beta \oplus M_\delta^*, M_\lambda \backslash \varepsilon M_\lambda^*]$, where $P^* = \bar{P}^T = (\bar{a}_{ji})$ is the matrix adjoint to the matrix $P = (a_{ij})$.

$$P \oplus R = \begin{pmatrix} P & O \\ O & R \end{pmatrix}, \quad P \backslash R = \begin{pmatrix} O & R \\ P & O \end{pmatrix}.$$

We will introduce the notation: if $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_n \in K[x]$, then $\bar{f}(x) = \bar{a}_0 x^n + \bar{a}_1 x^{n-1} + \ldots + \bar{a}_n$, O_{mn} is the null matrix of dimension $m \times n$, $O_n = O_{nn}$, E_n is the unit matrix of dimension $n \times n$, F_n is a matrix obtained from E_n by the reversed ordering of columns (i.e., the unities are situated on the side diagonal), Φ_n is the Frobenius box with unities under the main diagonal and the characteristic polynomial $x^n + \lambda_1 x^{n-1} + \ldots + \lambda_n \in K[x]$ which is a power of an irreducible polynomial $p_{\Phi_n}(x)$. As in [7, Theorem 8], in the case of $\Phi_n = \bar{\Phi}_n$ we will define a matrix Φ_n' of dimension $n \times n$: $\Phi_n' = F_n$ for degenerate Φ_n, $\Phi_n' = (a_{i+j})$ for nondegenerate Φ_n, where $a_2 = 1$, $a_3 = \ldots = a_{n+1} = 0$, $a_{l+n} = -\lambda_1 a_{l+n-1} - \ldots - \lambda_n a_l$, $\ell \geq 2$.

The following theorem is the main result of this paper.

<u>THEOREM 1.</u> Over field K of characteristic $\neq 2$ for every representation A of a graph G in spaces A_1, A_2, A_3 it is possible to choose bases in such a way that the triple $(A_\alpha, A_\beta, A_\lambda)$ be given by a direct sum of collections of matrices of the following forms:

1) $[n, n, 2n; 2n, n, n \pm 1]^+$, $[n, n, 2n; 2n, n \pm 1, n]^+$, $[n, n, 2n + 1; 2n + 1, n + 1, n + 1]^+$, $[n, n + 1, 2n + 1; 2n + 1, n + 1, n]^+$, $[n, n + 1, 2n + 1; 2n + 1, n + i, n + i]^+$, $[n + 1, n, 2n + 1; 2n + 1, n + i, n + i]^+$, where $i \in \{0, 1\}$;

2) $[n + i, n + j, 2n + 1; 2n, n, n]^+$, $[n - i, n - j, 2n - 1; 2n, n, n]^+$, where $i, j \in \{0, 1\}$;

3) $\left[\begin{pmatrix} E_n \\ \Phi_n \end{pmatrix}, \begin{pmatrix} E_n \\ E_n \end{pmatrix}, E_{2n}, (E_n O_n), (O_n E_n)\right]^+$, where $p_{\Phi_n}(x)$ equals x or x − 1;

4) $\left[\begin{pmatrix} E_n \\ O_n \end{pmatrix}, \begin{pmatrix} O_n \\ E_n \end{pmatrix}, \begin{pmatrix} E_n & E_n \\ E_n & \Phi_n \end{pmatrix}, (E_n O_n), (O_n E_n)\right]^+$ if $\varepsilon = -1$ or $\Phi_n \neq \bar{\Phi}_n$;

4') $\mathcal{P}(\Phi_n, f) = \left[\begin{pmatrix} E_n \\ O_n \end{pmatrix}, \begin{pmatrix} O_n \\ E_n \end{pmatrix}, \begin{pmatrix} (\Phi_n')^{-1} & E_n \\ E_n & (\Phi_n' \Phi_n) \end{pmatrix} f(\Phi_n^* \oplus \Phi_n)\right]$, if $\varepsilon = 1$ and $\Phi_n = \bar{\Phi}_n$, where $0 \neq f(x) = \bar{f}(x) \in K[x]$, $\deg f(x) < \deg p_{\Phi_n}(x)$;

5) $\left[\begin{pmatrix} E_n \\ O_n \end{pmatrix}, \begin{pmatrix} O_n \\ E_n \end{pmatrix}, \begin{pmatrix} \Phi_n & E_n \\ E_n & E_n \end{pmatrix}, (E_n O_n), (O_n E_n)\right]^+$ for $\varepsilon = -1$ and degenerate Φ_n;

5') $\mathcal{a}(n, a) = \left[\begin{pmatrix} E_n \\ O_n \end{pmatrix}, \begin{pmatrix} O_n \\ E_n \end{pmatrix}, a \begin{pmatrix} F_{n-1} \oplus O_1 & E_n \\ E_n & F_n \end{pmatrix}\right]$ for $\varepsilon = 1$, where $0 \neq a = \bar{a} \in K$;

6) $[A_i, B_j, C, A_i^T, B_j^T]^+$ for $\varepsilon = -1$, where $i, j \in \{0, 1\}$, $A_0 = \begin{pmatrix} E_n \\ O_{n+1,n} \end{pmatrix}$, $A_1 = \begin{pmatrix} O_{n,n+1} \\ E_{n+1} \end{pmatrix}$, $B_0 = \begin{pmatrix} E_n \\ E_n \\ O_{1n} \end{pmatrix}$, $B_1 = \begin{pmatrix} E_n O_{n1} \\ E_{n+1} \end{pmatrix}$, $C = \begin{pmatrix} F_n & O_{n,n+1} \\ O_{n+1,n} & F_{n+1} \end{pmatrix}$;

6') $\mathcal{R}(n, a) = [A_i, B_j, aC]$ for $\varepsilon = 1$, where $i, j \in \{0, 1\}$, $0 \neq a = \bar{a} \in K$, the matrices A_i, B_j, C are from 6).

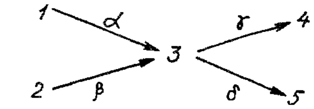
The components with respect to the initial representation A are determined as follows: for 1, 2, 3, 5, and 6 uniquely; for 4 up to exchange of Φ_n by the box $\bar{\Phi}_n$; for 4' up to exchange of the whole group of components $\bigoplus_i \mathcal{P}(\Phi_n, f_i)$ with the same box Φ_n on $\bigoplus_i \mathcal{P}(\Phi_n, g_i)$, where $\sum_i f_i(\omega) x_i^\circ x_i$ and $\sum_i g_i(\omega) x_i^\circ x_i$ are equivalent Hermitian forms over the field $K(\omega) = K[x]/p_{\Phi_n}(x)$ with the involution $f(\omega)^\circ = \bar{f}(\omega)$; for 5' and 6' up to the replacement of the whole group of components $\bigoplus_i \mathcal{C}(n, a_i)$ [$\bigoplus_i \mathcal{R}(n, a_i)$ respectively] by the same number n on $\bigoplus_i \mathcal{C}(n, b_i)$ [on $\bigoplus_i \mathcal{R}(n, b_i)$ respectively], where $\sum_i a_i \bar{x}_i x_i$ and $\sum_i b_i \bar{x}_i x_i$ are equivalent Hermitian forms over the field K.

2. Classification of Representations of Quiver Q. Theorem 1 assumes that the classification of representations of quiver Q is known. This classification has been obtained in [2]. We will present it in the form suggested by [4].

We will introduce the notation: $\Phi_n(\lambda)$ is the Frobenius box with the characteristic polynomial $(x - \lambda)^n$, $E_{n+1,n}{\uparrow}$, $E_{n+1,n}{\downarrow}$, $E_{n,n+1}{\leftarrow}$, $E_{n,n+1}{\rightarrow}$ are matrices obtained from E_n by adding a null row or a null column from above, from below, from the left, and from the right, respectively.

2.1 A complete system ind(Q') of nonisomorphic indecomposable into a direct sum representations of the quiver

Q':



contains exactly one representation for each dimension (n, n, 2n, n, n ± 1), (n, n, 2n, n ± 1, n), (n, n ± 1, 2n, n, n), (n ± 1, n, 2n, n, n), x_1, x_2, 2n + 1, x_4, x_5), where $x_1, x_2, x_4, x_5 \in \{n, n + 1\}$. These representations can be obtained from the following indecomposable representations $M = [M_\alpha, M_\beta, M_\gamma, M_\delta]$:

1) $\begin{pmatrix} E_n \\ O_n \end{pmatrix}, \begin{pmatrix} O_n \\ E_n \end{pmatrix}, \begin{pmatrix} E_n \\ E_n \end{pmatrix}^T, \begin{pmatrix} E^{\uparrow}_{n,n-1} \\ E^{\downarrow}_{n,n-1} \end{pmatrix}^T$ or $\begin{pmatrix} E^{\leftarrow}_{n,n+1} \\ E^{\rightarrow}_{n,n+1} \end{pmatrix}^T$;

2) $\begin{pmatrix} E_{n+1} \\ O_{n,n+1} \end{pmatrix}, \begin{pmatrix} O_{n+1,n} \\ E_n \end{pmatrix}, \begin{pmatrix} E^{\uparrow}_{n+1,n} \\ E_n \end{pmatrix}^T, \begin{pmatrix} E^{\downarrow}_{n+1,n} \\ E_n \end{pmatrix}^T$ or $\begin{pmatrix} E_{n+1} \\ E^{\rightarrow}_{n,n+1} \end{pmatrix}^T$;

3) $\begin{pmatrix} E_{n+1} \\ O_{n,n+1} \end{pmatrix}, \begin{pmatrix} O_{n+1,n} \\ E_n \end{pmatrix}, \begin{pmatrix} E_{n+1} \\ E^{\leftarrow}_{n,n+1} \end{pmatrix}^T, \begin{pmatrix} E_{n+1} \\ E^{\rightarrow}_{n,n+1} \end{pmatrix}^T$;

4) $\begin{pmatrix} E_n \\ O_{n+1,n} \end{pmatrix}, \begin{pmatrix} O_{n+1,n} \\ E_n \end{pmatrix}, \begin{pmatrix} E_{n+1} \\ E^{\leftarrow}_{n,n+1} \end{pmatrix}^T, \begin{pmatrix} E^{\leftarrow}_{n-1,n} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ E^{\rightarrow}_{n,n+1} \end{pmatrix}^T$;

5) $\begin{pmatrix} E^{\downarrow}_{n+1,n} \\ E_n \end{pmatrix}, \begin{pmatrix} O_{n+1,n} \\ E_n \end{pmatrix}, \begin{pmatrix} O_{n+1,n} \\ E_n \end{pmatrix}^T, \begin{pmatrix} E^{\uparrow}_{n+1,n} \\ E_n \end{pmatrix}^T$;

6) $\begin{pmatrix} E_{n+1} \\ O_{n,n+1} \end{pmatrix}, \begin{pmatrix} E_{n+1} \\ E^{\leftarrow}_{n,n+1} \end{pmatrix}, \begin{pmatrix} E_{n+1} \\ O_{n,n+1} \end{pmatrix}^T, \begin{pmatrix} E_{n+1} \\ E^{\rightarrow}_{n,n+1} \end{pmatrix}^T$,

using transpositions of the matrices M_α, M_β, transpositions of matrices M_γ, M_δ, passage to the adjoint indecomposable representation $M^0 = [M_\gamma^*, M_\delta^*, M_\alpha^*, M_\beta^*]$ of dimension (m_4, m_5, m_3, m_1, m_2).

The set ind(Q') contains also the following representations of dimension (n, n, 2n, n, n):

$$\mathcal{M}_1(\Phi_n) = \left[\begin{pmatrix} E_n \\ O_n \end{pmatrix}, \begin{pmatrix} O_n \\ E_n \end{pmatrix}, (E_n E_n), (E_n \Phi_n)\right],$$

$$\mathcal{M}_2 = \left[\begin{pmatrix} E_n \\ O_n \end{pmatrix}, \begin{pmatrix} O_n \\ E_n \end{pmatrix}, (\Phi_n(0)\, E_n), (E_n E_n)\right],$$

$$\mathcal{M}_3(\lambda) = \left[\begin{pmatrix} E_n \\ \Phi_n(\lambda) \end{pmatrix}, \begin{pmatrix} E_n \\ E_n \end{pmatrix}, (E_n O_n), (O_n E_n)\right], \lambda \in \{0, 1\},$$

$$\mathcal{M}_4 = \mathcal{M}_3(0)^0.$$

The set ind(Q') does not contain any other representations.

2.2 A complete system ind(Q) of nonisomorphic indecomposable into a direct sum represensations of the quiver Q consists of the representations

a) $\mathcal{N}_1(A) = [A_\alpha, A_\beta, E, A_\gamma, A_\delta]$, where $A = [A_\alpha, A_\beta, A_\gamma, A_\delta] \in$ ind(Q') is a representation of dimension ≠ (n, n, 2n, n, n);

b) $\mathcal{N}_2(A) = \left[A_\alpha, A_\beta, \begin{pmatrix}A_\gamma\\ A_\delta\end{pmatrix}, (E_n O_n), (O_n E_n)\right]$, where A ∈ ind(Q') is a representation of dimension (n, n, 2n, n, n), or (n + i, n + j, 2n + 1, n, n) or (n − i, n − j, 2n − 1, n, n), i, j ∈ {0, 1};

c) $\mathcal{N}_3(A) = \left[\begin{pmatrix}E_n\\ O_n\end{pmatrix}, \begin{pmatrix}O_n\\ E_n\end{pmatrix}, (A_\gamma^\bullet A_\delta^\bullet), A_\alpha^\bullet, A_\beta^\bullet\right]$, where A = $\mathcal{M}_3(1)$ or A ∈ ind(Q') is a representation of dimension (n + i, n + j, 2n + 1, n, n) or (n − i, n − j, 2n − 1, n, n), i, j ∈ {0, 1}.

3. <u>Proof of Theorem 1.</u> A representation adjoint to the representation $M = [M_\alpha, M_\beta, M_\lambda, M_\gamma, M_\delta]$ of quiver Q is the representation $M^0 = [M_\gamma^*, M_\delta^*, \varepsilon M_\lambda^*, M_\alpha^*, M_\beta^*]$ (ε is the same as in the quiver Q). An adjoint homomorphism to the homomorphism $\psi = (\Psi_1, \ldots, \Psi_6)$: M → N is the homomorphism $\psi^0 = (\Psi_5^*, \Psi_6^*, \Psi_4^*, \Psi_3^*, \Psi_1^*, \Psi_2^*) : N^0 \to M^0$.

We will replace each representation from ind(Q), isomorphic to an adjoint one, by a self-adjoint representation and we will denote their set by $\mathrm{ind}_0(Q)$. We will include into $\mathrm{ind}_1(Q)$ all representations from ind(Q) isomorphic with an adjoint but not self-adjoint one, and one from each pair {M, N} ⊂ ind(Q), where $M \simeq N \simeq M^0$.

The ring of endomorphisms Λ = End(N) of an indecomposable representation $N \in \mathrm{ind}_0(Q)$ is local, the set R of its irreversible elements is the radical; therefore T(N) = Λ/R is a field. By means of the representation $N \in \mathrm{ind}_0(Q)$ and its self-adjoint automorphism $\psi = \psi^0$ we will define the representation of the graph G: $N^\psi = [N_\alpha, N_\beta, N_\lambda \Psi_3]$.

The following theorem is a particular case of Theorem 1 [7].

<u>THEOREM 2.</u> Every representation of graph G over field K of characteristic ≠ 2 is decomposable into a direct sum of representations of the forms

a) M^+, where $M \in \mathrm{ind}_1(Q)$;

b) N^ψ, where $N \in \mathrm{ind}_0(Q)$, $\psi = \psi^0 \in \mathrm{Aut}(N)$.

The components are determined as follows: of the form a) uniquely; of the form b) up to the exchange of the whole group of components $\bigoplus_i N^{\psi_i}$ with the same N for $\bigoplus_i N^{\varphi_i}$, where $\sum_i (\psi_i + R)x_i{}^0 x_i$ and $\sum_i (\varphi_i + R)x_i{}^0 x_i$ are equivalent Hermitian forms over the field T(N) = Λ/R with the involution $(\psi + R)^0 = \psi^0 + R$.

We will use Theorem 2 to prove Theorem 1. The set ind(Q) has been introduced in subsection 2.2. If $z = (z_1, \ldots, z_6)$ is the dimension of the representation M, then $z^0 = (z_5, z_6, z_4, z_3, z_1, z_2)$ is the dimension of the adjoint representation M^0.

Representations of dimensions $z \neq z^0$ from ind(Q) are fully determined by their dimensions and they are not isomorphic to self-adjoint ones. We will divide them into pairs of representations of mutually adjoint dimensions z, z^0 and from each pair we will choose one representation. We will obtain all representations of M from $\mathrm{ind}_1(Q)$ of nonself-adjoint dimensions. Passing to representations of M^+, we will obtain all representations 1-2 in Theorem 1.

The representations $\mathcal{N}_2(\mathcal{M}_3(\lambda))$, λ ∈ {0, 1} (see 2.2) are not isomorphic to self-adjoint ones since $\mathcal{N}_2(\mathcal{M}_3(0))^0 \simeq \mathcal{N}_2(\mathcal{M}_4)$, $\mathcal{N}_2(\mathcal{M}_3(1))^0 \simeq \mathcal{N}_3(\mathcal{M}_3(1))$. We obtain representations 3 in Theorem 1.

Let us consider the representation

$$\mathcal{N}_2(\mathcal{M}_1(\Phi_n)) = \left[\begin{pmatrix}E_n\\ O_n\end{pmatrix}, \begin{pmatrix}O_n\\ E_n\end{pmatrix}, \begin{pmatrix}E_n & E_n\\ E_n & \Phi_n\end{pmatrix}, (E_n O_n), (O_n E_n)\right].$$

Obviously, $\mathcal{N}_2(\mathcal{M}_1(\Phi_n))^0 \simeq \mathcal{N}_2(\mathcal{M}_1(\overline{\Phi}_n))$.

Let $\varphi : \mathcal{N}_2(\mathcal{M}_1(\Phi_n)) \to B = B^0$ be an isomorphism into a self-adjoint representation. Replacing B by an isomorphic self-adjoint representation we can write

$$B = \left[\begin{pmatrix}E_n\\ O_n\end{pmatrix}, \begin{pmatrix}O_n\\ E_n\end{pmatrix}, \begin{pmatrix}M & E_n\\ \varepsilon E_n & N\end{pmatrix}, (E_n O_n), (O_n E_n)\right],$$

$M = \varepsilon M^*$, $N = \varepsilon N^*$. Then the isomorphism takes the form

$$\varphi = (S_1, S_2, S_1 \oplus S_2, S_2 \oplus \varepsilon S_1, S_2, \varepsilon S_1). \quad (1)$$

Replacing M, N, S_1, S_2 by $S_2^{-1}MS_2^{*-1}$, $S_2^*NS_2$, $S_2^*S_1$, E_n we obtain an isomorphism φ of the form (1) in which $S_2 = E_n$ and by the definition of an isomorphism $MS_1 = E_n$, $N = \varepsilon S_1\Phi_n$. Since $M = \varepsilon M^*$, $N = \varepsilon N^*$, then $S_1 = \varepsilon S_1^*$, $S_1\Phi_n = \varepsilon(S_1\Phi_n)^*$. By [7, Lemma 8, Theorem 8] $\varepsilon = 1$, $\Phi_n = \overline{\Phi}_n$ and, therefore, we can put $S_1 = \Phi_n'$, $M = (\Phi_n')^{-1}$, $N = \Phi_n'\Phi_n$ (Φ_n' has been defined in Sec. 1).

Let $\psi : B \to B$ be an endomorphism. Then $\eta = \varphi^{-1}\psi\varphi$ is an endomorphism of the representation $\mathcal{N}_2(\mathcal{M}_1(\Phi_n))$. By the definition of a homomorphism $\eta = (H, H, H \oplus H, H \oplus H, H, H)$. A matrix commuting with a Frobenius box is a polynomial with respect to this box and therefore $H = f(\Phi_n)$, $f \in K[x]$. Since $\Phi_n'H(\Phi_n')^{-1} = f(\Phi_n'\Phi_n(\Phi_n')^{-1}) = f(\Phi_n^*)$, then $\psi = \varphi\eta\varphi^{-1} = (f(\Phi_n'), f(\Phi_n), f(\Phi_n^* \oplus \Phi_n), f(\Phi_n \oplus \Phi_n^*), f(\Phi_n), f(\Phi_n^*))$, $\psi^0 = (\overline{f}(\Phi_n^*), \overline{f}(\Phi_n), \ldots)$, and the field $T(B) = \mathrm{End}(B)/R$ can be identified with the field $K(\omega) = K[x]/p_{\Phi_n}(x)$ with the involution $f(\omega)^0 = \overline{f}(\omega)$. By Theorem 2 we obtain the components 4 and 4' of Theorem 1. By representation $\mathcal{N}_2(\mathcal{M}_2)$ we obtain the components 5 and 5'.

In the set ind(Q) there are still not considered representations of the dimensions $(n + i, n + j, 2n + 1; 2n + 1, n + i, n + j)$, where $i, j \in \{0, 1\}$. It is easy to verify that these representations are isomorphic to the representations $[A_i, B_j, C, A_i^T, B_j^T]$ (see 6 in Theorem 1), which are self-adjoint for $\varepsilon = 1$. Thus, we obtain the components 6 and 6' of Theorem 1. An application of Theorem 2 concludes the proof of Theorem 1.

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