2) a Green's function $G_0(\tau, \varphi)$ for the invariant torus problem exists which satisfies estimate (3);

3) the inequality $\gamma - |l|\alpha > 0$ holds, where $\alpha \geqslant \max_{\|\eta\|=1} \left\| \frac{\partial a}{\partial \varphi} \eta \right\|$.

Then estimate (26) holds for the partial derivatives of order l of Green's function $G_0(\tau, \varphi)$ with respect to the variables φ.

Remark 6. The hypotheses of Theorem 4 give prerequisites for obtaining analogous results for the smoothness of an invariant torus in systems of nonlinear differential equations.

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FINITELY GENERATED GROUPS WITH COMMUTATOR GROUP OF PRIME ORDER

V. V. Sergeichuk

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In [1] defining relations and a system of invariants are obtained for finite groups whose commutator subgroup has prime order $p \neq 2$. In this paper, we obtain defining relations and a system of invariants for finitely generated groups with commutator of prime order and for finitely generated groups with a complementable cyclic commutator of order p^ν, $p \neq 2$. This result was announced in part in [2].

We recall (cf. [3]) that a product $G = G_1G_2 \ldots G_n$ is called a direct product with common subgroup A if $g_ig_j = g_jg_i$, $G_i \cap G_j = A$ for all $g_i \in G_i$, $g_j \in G_j$, $i \neq j$. As follows from Lemma 1 of [4], a finite p-group with commutator A of order p is a direct product with common subgroup A of groups with at most two generators. However, this decomposition is nonunique and does not therefore give a complete classification.

In this paper we obtain the following description of finite p-groups with commutator of order p. A finite p-group with two generators and commutator of order p is defined by the following relations:

$$[g,h]=a,\ g^{p^\alpha}=a^\lambda,\ h^{p^\beta}=a^\mu,\ a^p=1,\ [g,a]=[h,a]=1,$$

and is called uninull if $\lambda = 1$, $\mu = 0$, except for the case $p = 2$, $\alpha = \beta = 1$; if $\lambda = \mu = 1$ it is called uninull in the case $p = 2$, $\alpha = \beta = 1$; it is called binull if $\lambda = \mu = 0$. Let G be a finite p-group with commutator A of order p, and assume G is indecomposable as a direct product. Up to an isomorphism of the factors, the group G is uniquely decomposable as a direct product with common group A:

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$$G = G_1 G_2 \dots G_n,$$

where G_1 is either a cyclic p-group with subgroup A or else a uninull or binull group with commutator A; $G_2, \dots, G_n$ are binull groups with commutator A.

The following question arises: Is it possible to classify all finite p-groups with cyclic commutator? In [5], matrix methods are used to prove that there exists no good system of invariants even for finite p-groups with cyclic central commutator of order p^2. A system of invariants is obtained only for the metacyclic p-groups [6] and finite p-groups ($p \neq 2$) with cyclic commutator and two generators [7].

1. Description of the Method Applied. In this paper we apply matrix methods: We define a group using a larger set of numerical parameters than is necessary, which we write in the form of matrices and vectors, and then get rid of the superfluous parameters by performing admissible transformations. The remaining parameters comprise a complete system of invariants for the group.

We give some information on the matrix problem. G denotes a finitely generated group with commutator G' of prime order p. Let

$$G' = \langle a \rangle_p, \; G/G' = \langle b_1 \rangle_{n_1} \times \dots \times \langle b_m \rangle_{n_m}, \tag{1}$$

where $\langle b_i \rangle_{n_i}$ is a cyclic group of order $n_i = \infty$ or $n_i = p_i^{\nu_i}$ (p_i a prime), and let $g_i \in b_i$,

$$g_i^{-1} a g_i = a^{t_i}, \; g_i^{n_i} = a^{r_i} \; (n_i < \infty), \; [g_i, g_j] = a^{s_{ij}}, \; a^p = 1, \tag{2}$$

where we put $r_i = 0$ for $n_i = \infty$.

Equalities (2) constitute a set of defining relations for G. Indeed, for any natural numbers α, β it is easy to obtain from them that

$$a^\alpha g_i^\beta = g_i^\beta a^{\alpha t_i^\beta}, \; g_i^\alpha g_j^\beta = g_j^\beta g_i^\alpha a^{s_{ij}(1+t_i+\dots+t_i^{\alpha-1})(1+t_j+\dots+t_j^{\beta-1})} \tag{3}$$

and then to completely determine the multiplication in G of elements which are uniquely expressible in the form $g_1^{\alpha_1} g_2^{\alpha_2} \dots g_m^{\alpha_m} a^\beta$, where $0 \leq \alpha_i < n_i$, β is defined modulo p. Since t_i, r_i, s_{ij} are defined modulo p, we assume that they belong to the field of p elements $\mathbf{Z}/p\mathbf{Z}$.

Consequently, the group G is completely determined by the numbers $n_1, \dots, n_m$, p, as well as by the vectors $T = (t_1, \dots, t_m)$, $R = (r_1, \dots, r_m)$ and the skew-symmetric (since $[g_i, g_j]^{-1} = [g_j, g_i]$) matrix $S = (s_{ij})$ over the field $\mathbf{Z}/p\mathbf{Z}$. But for another choice of generators $g_1, \dots, g_m$, a, of G we obtain another triple T, R, S. The purpose of this article is to distinguish among all triples T, R, S obtained from a single group one such triple which we call canonical. This canonical triple (together with n_i, p) will be a complete set of invariants for the group G.

It is obvious that we can go from the generators $g_1, \dots, g_m$, a to any other possible set of generators $\bar{g}_1, \dots, \bar{g}_m$, $\bar{a}$ by applying the following changes of generators several times (α is an integer, i, j are fixed indices):

I) $g_i' = g_j$, $g_j' = g_i$, $g_k' = g_k$ ($k \neq i, j$), $a' = a$;

II) $g_i' = g_i^\alpha$, $g_k' = g_k$ ($k \neq i$), $a' = a$, where $(\alpha, n_i) = 1$, $\alpha = -1$ for $n_i = \infty$;

III) $g_i' = g_i g_j^\alpha$, $g_k' = g_k$ ($k \neq i$), $a' = a$, where either $p_i = p_j$, $p_i^{\nu_j - \nu_i} \mid \alpha$ for $\nu_j \geq \nu_i$, or $n_i = \infty$;

IV) $g_i' = a^\alpha g_i$, $g_k' = g_k$ ($k \neq i$), $a' = a$;

V) $a' = a^\alpha$, $g_k' = g_k$ for all k, where $0 < \alpha < p$.

The substitutions I)-III) amount to rechoosing the generators b_i in the decomposition of G/G' (1), (IV), to rechoosing $g_i \in b_i$, and V) to rechoosing $a \in G'$. We note that in substitution I), the set of n_i changes: $n_i' = n_j$, $n_j' = n_k$, $n_k' = n_k$ ($k \neq i, j$). To each change of generators I)-V) there corresponds some transformation of the triple T, R, S which we call elementary. Thus we need to find the canonical form of the triple T, R, S with respect to elementary transformations I)-V).

A triple T, R, S is called decomposable if by applying the transformations I)-V) we can arrange that for some i

$$t_i = 1, \; r_i = s_{i1} = s_{i2} = \dots = s_{im} = 0. \tag{4}$$

Decomposable triples correspond to groups G which can be decomposed into direct products. Indeed, if (4) holds, then by (2) $\langle g_i \rangle$ is a direct factor in G. Conversely, if G is decomposable then it has the form $G = G_1 \times \langle g \rangle$;

we obtain (4) by taking the generator g_i to be the element g.

Since the group G is uniquely decomposable up to isomorphism as a direct product $G = G_1 \times A$, where G_1 is indecomposable with commutator of prime order p and A is Abelian, it is sufficient to classify indecomposable groups G. We will therefore assume that the group G and hence also the triple T, R, S are indecomposable.

2. Groups with Central Commutator. THEOREM 1. A finitely generated indecomposable group whose commutator subgroup is central and has prime order p is determined by the set of invariants:

$$\left(\delta \begin{matrix} \alpha_0\alpha_1 \ldots \alpha_k \\ \beta_0\beta_1 \ldots \beta_k \end{matrix} \gamma_1\gamma_2 \ldots \gamma_l\right)_p, \qquad (5)$$

where $\gamma_1, \ldots, \gamma_l$ is an unordered collection of nonzero elements of the field $\mathbf{Z}/p\mathbf{Z}$ defined up to sign and a common nonzero factor, $l \geq 0$; α_i is a natural number, β_i is a natural number of the symbol ∞, $\alpha_i \leq \beta_i$ for $i > 0$, the set of columns $(\alpha_1, \beta_1), \ldots, (\alpha_k, \beta_k)$ is unordered, $k \geq 0$; for $k > 0$ or $l > 0$ the inequality $\alpha_0 \geq \beta_0 = 0$ is admissible; δ is a natural number, $\delta = 1$ for $\beta_0 < \infty$, $\delta \leq p/2$ for $\beta_0 = \infty$.

The group is defined by the following generators a, g_0, $h_0, \ldots, g_{k+l}$, h_{k+l} and defining relations:

1) $[g_i, g_j] = [h_i, h_j] = [g_i, h_j] = [g_i, a] = [h_i, a] = a^p = 1$ $(i \neq j)$;

2) $g_0^{p^{\alpha_0}} = a^{\delta}$, $h_0^{p^{\beta_0}} = 1$ $(h_0^{p^{\beta_0}} = a$ for $p = 2$, $\alpha_0 = \beta_0 = 1)$, $[g_0, h_0] = a$ for $\beta_0 > 0$;

3) $g_i^{p^{\alpha_i}} = h_i^{p^{\beta_i}} = 1$, $[g_i, h_i] = a$ $(i = 1, 2, \ldots, k)$;

4) $[g_i, h_i] = a^{\gamma_{i-k}}$ $(i = k + 1, \ldots, k + l$; g_i, h_i are of infinite order).

It follows from the defining relations 1)-4) that the group decomposes into a direct product with common subgroup $\langle a \rangle_p$ of the subgroups G_i with two generators g_i, h_i. The commutator $G' = \langle a \rangle_p$ and the center equals $\langle g_0^p \rangle \times \langle h_0^p \rangle \times \ldots \times \langle g_{k+l}^p \rangle \times \langle h_{k+l}^p \rangle$ (for $\alpha_0 = 0$ we must replace $\langle g_0^p \rangle$ by $\langle a \rangle$). The group is finite if and only if all $\beta_i < \infty$ and $l = 0$ [i.e., $\gamma_1, \ldots, \gamma_l$ do not appear in (5)], and in this case the order of G is equal to p^n, where $n = 1 + \Sigma(\alpha_i + \beta_i)$.

The proof of Theorem 1 occupies the rest of this section. We impose an extra condition on the group G in Sec. 1: The commutator G' is contained in the center. Then all the $t_i = 1$ in (2), i.e., G is completely determined by the pair R, S. If $n_i = p_i^{\nu_i}$, $p_i \neq p$, then after a substitution II) $g_i' = g_i^p$ we get $g_i'^{n_i} = a^{pr_j} = 1$, $[g_i', g_j'] = [g_i^p, g_j] = a^{s_{ij}p} = 1$, i.e., we obtain (4) $t_j' = 1$, $r_j' = s_{i1}' = \ldots = s_{im}' = 0$, which contradicts the indecomposability of G.

Hence each $n_i = p^{\nu_i}$ or ∞. We put $\nu_i = \infty$ for $n_i = \infty$. We call ν_i the weight of the i-th row and i-th column of the matrix S; we also call it the weight of the i-th element of R. Using (3) it is easy to show that the pair R, S can be transformed using I)-V) as follows ($\alpha \in \mathbf{Z}/p\mathbf{Z}$):

I. The elements of R change places "together with their weight," i.e., the rows and columns of S with indices i, j are interchanged.

II. The element of R, the row and column of S with index i, are multiplied by $\alpha \neq 0$; $\alpha = -1$ for $\nu_i = \infty$.

III. a) If $\nu_i = \nu_j$, then the j-th column in S multiplied by α is added to the i-th column, and the j-th row multiplied by α is added to the i-th row. In R, α times the j-th element is added to the i-th element, and for $p = 2$, $\nu_i = 1$ the term αs_{ij} is also added.

b) An element of strictly greater weight multiplied by α is added to any element in R; S is not changed.

c) A column of strictly smaller weight multiplied by α is added to any column in S, rows with the same indices being transformed in the same way; R is not changed.

Indeed, by (3) the substitution III gives $g_i'^{p^{\nu_i}} = (g_i g_j^{\alpha})^{p^{\nu_i}} = g_j^{\alpha p^{\nu_i}} g_i^{p^{\nu_i}} a^{s_{ij}\alpha t} = a^{r_i + r_j \alpha p^{\nu_i - \nu_j} + s_{ji}\alpha t}$, where $t = 1 + 2 + \ldots + p^{\nu_i} = {}^1/_2 p^{\nu_i}(p^{\nu_i} + 1)$, $t \equiv 0 \pmod p$ for $p > 2$ or $\nu_i > 1$, $t \equiv 1 \pmod p$ for $p = 2$ and $\nu_i = 1$; $[g_i', g_k] = a^{s_{ik} + \alpha s_{jk}}$, i.e., $r_i' = r_i + r_j \alpha p^{\nu_i - \nu_j} + s_{ij}\alpha t$, $s_{ik}' = s_{ik} + \alpha s_{jk}$.

IV. R and S do not change.

V. R, S are divided by $\alpha \neq 0$.

We begin by simplifying the pair R, S using the transformations I-V. If $S \neq 0$ we use transformation I to make the nonzero row with smallest weight the first row, and then we replace the second column by a column whose first element has minimal weight. Using the element $s_{12} \neq 0$ obtained, we successively make the elements $s_{13}, s_{13}, \ldots, s_{1m}; s_{32}, s_{42}, \ldots, s_{m2}$ zero using transformations III, a), c). If $\nu_1 < \infty$ or $\nu_2 < \infty$, we make $s_{12} = 1$. We obtain

$$S=\left(\begin{array}{cc|ccc} 0 & s_{12} & 0 & \ldots & 0 \\ -s_{12} & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & S_1 & \\ 0 & 0 & & & \end{array}\right).$$

If $S_1 \neq 0$, we reduce S_1 analogously. Repeating this reduction sufficiently many times, we obtain the matrix

$$S=\left(\begin{array}{cccccccccccc} \multicolumn{6}{c}{\overbrace{\qquad\qquad\qquad}^{n \text{ blocks}}} \\ 0 & 1 & & & & & & & & & & \\ -1 & 0 & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & 0 & 1 & & & & & & & \\ & & & -1 & 0 & & & & 0 & & & \\ & & & & & 0 & \gamma_1 & & & & & \\ & & & & & -\gamma_1 & 0 & & & & & \\ & & & & & & & \ddots & & & & \\ & & & & & & & & 0 & \gamma_l & & \\ & & & 0 & & & & & -\gamma_l & 0 & 0 & \\ & & & & & & & & & & & \ddots \\ & & & & & & & & & & & 0 \end{array}\right), \qquad (6)$$

where $\nu_{2i-1} < \infty$ or $\nu_{2i} < \infty$ $(i \leq n)$, $\nu_{2n+1} = \nu_{2n+2} = \ldots = \nu_{2n+2l} = \infty$.

We determine the form to which the vector R can be reduced by transformations I-V in such a way that the form of the matrix (6) is preserved. In particular, the following transformations are also possible with R:

α) interchange the pairs (r_{2i-1}, r_{2i}) and (r_{2j-1}, r_{2j}) "together with their weight" (transformation I), $i, j \leq n$;

β) the pair (r_{2i-1}, r_{2i}) with weight (ν_{2i-1}, ν_{2i}) can be replaced by the pair $(r_{2i}, -r_{2i-1})$ with weight (ν_{2i}, ν_{2i-1}) (using transformation II, we multiply r_{2i} by -1 and then interchange $-r_{2i}$ and r_{2i-1}, with (6) being preserved), $i, j \leq n$.

Let $R \neq 0$. Using transformation III b) and a nonzero element of maximal weight, we make all the elements of R of smaller weight zero. We obtain a vector R in which all nonzero elements have the same weight $<\infty$ (since $r_i = 0$ if $\nu_i = \infty$).

If $m > 2(n + l)$, then $r_{2n+2l+1} \neq 0$; otherwise (4) would hold and the group G would be decomposable. By transformation II we arrange that $r_{2n+2l+1} = 1$, and then use transformation III a) and this element to make all the r_i with $i \neq 2(n + l) + 1$ equal to zero. Then since G is indecomposable, we obtain $m = 2(n + l) + 1$, $R = (0, \ldots, 0, 1)$.

Let $m = 2(n + l)$. Among all the pairs $(r_{2i-1}, r_{2i}) \neq (0, 0)$, we choose a pair such that the sum of the weights $\nu_{2i-1} + \nu_{2i}$ is maximal and use transformation α) to interchange it with the pair (r_1, r_2). If the new $r_1 = 0$ then $r_2 \neq 0$; we make $r_1 \neq 0$ by transformation β).

If $r_{2i-1} \neq 0$, $i > 1$, we add r_1 multiplied by $\alpha = -r_1^{-1} r_{2i-1}$ to it [transformation III a)] and obtain $r_{2i-1} = 0$. But this spoils the form of (6). We recover (6) by transformations III a) or b), by adding to the second row and column of S the 2i-th row and column multiplied by $-\alpha$. This transformation is admissible, since $\nu_1 + \nu_2 \geq \nu_{2i-1} + \nu_{2i}$ and $\nu_1 = \nu_{2i-1}$ (since $r_1 \neq 0$, $r_{2i-1} \neq 0$) implies $\nu_2 \geq \nu_{2i}$. Similarly, we make all the r_{2i} with $i > 1$ equal to zero.

We have obtained $R = (r_1, r_2, 0, \ldots, 0)$, $r_1 \neq 0$. If $p = 2$, $\nu_1 = \nu_2 = 1$, then in the case $R = (1, 0, \ldots, 0)$ we add $r_2 + s_{12} = 1$ to $r_1 = 1$ [transformation III a)] and obtain $R = 0$. In the case $R = (1, 1, 0, \ldots, 0)$, this transformation does not alter R since then $r_2 + s_{12} = 0$.

Assume the condition $p = 2$, $\nu_1 = \nu_2 = 1$ does not hold. We then use the element $r_1 \neq 0$ to make $r_2 = 0$ [transformation III a)]. If $\nu_2 \neq \infty$, we divide r_1 by r_1 (transformation II), and in order to recover the form (6) of S we simultaneously multiply the second row and second column of S by r_1; we get $R = (1, 0, \ldots, 0)$. If on the other hand $\nu_2 = \infty$ then the second row and column of S can only be multiplied by -1, so that it is possible without changing the form of S in (6) to multiply r_1 only by ± 1 and make $R = (\delta, 0, \ldots, 0)$, where $0 < \delta \leq p/2$.

We have obtained a pair R, S where S has the form (6) and R has one of the forms:

1) $R = (0, \ldots, 0)$, $m = 2(n + l)$;

2) $R = (0, \ldots, 0, 1)$, $m = 2(n + l) + 1$;

3) $R = (\delta, 0, \ldots, 0)$ ($R = (1, 1, 0, \ldots, 0)$ for $p = 2$, $\nu_1 = \nu_2 = 1$), $m = 2(n + l)$, where $\delta = 1$ for $\nu_2 < \infty$, $0 < \delta \leq p/2$ for $\nu_2 = \infty$.

It is easy to see that the pair R, S is indecomposable (cf. Sec. 1). The weight of the i-th ($i \leq n$) block of S is denoted by (α_i, β_i) in cases 1), 2), and by $(\alpha_{i-1}, \beta_{i-1})$ in case 3). We introduce the notation $\alpha_0 = \beta_0 = 0$, $k = n$ for 1); α_0 is the weight of the last column, $\beta_0 = 0$, $k = n$ for 2); $k = n - 1$ for 3). Using transformation β), we can make $\alpha_i \leq \beta_i$ $(1 \leq i \leq k)$.

The elements $\gamma_1, \ldots, \gamma_l$ in (6) are determined up to order of succession, sign (in place of any γ_i we can take $-\gamma_i$ by applying transformation II), and up to a common nonzero multiple (by applying transformation V we obtain $\gamma_1' = \lambda\gamma_1, \ldots, \gamma_l' = \lambda\gamma_l$, $\lambda \neq 0$, which spoils the form of the first n blocks of S in (6) and the vector R; we correct them by transformation II). It is easy to see that in the definition of $\gamma_1, \ldots, \gamma_l$ with this degree of freedom, distinct pairs R, S of the above form cannot be taken into one another by the transformations I-V, i.e., we have obtained a canonical form for the indecomposable pair R, S with weight relative to the transformations I-V. The parameters appearing in R, S can be written down in the form of the set (5).

As is shown in Sec. 1, the group G is completely defined by the indecomposable triple $T = 0$, R, S (and the set of invariants $n_1, \ldots, n_m$, p) by means of the defining relations (2), distinct triples defining the same group if and only if we can go from one triple to the other using transformations I-V. Therefore, (5) is a complete system of invariants for G. Writing (2) in the new notation, we obtain the defining relations 1)-4) in Theorem 1. The theorem is proved.

3. Groups with Complementable Commutator. It remains to describe the finitely generated groups with commutator of order p which is not central. It is easy to show that the commutator in such groups is complementable.

Indeed, let G be such a group. Then there exists a $t_i \neq 0$ in its defining relations (2). We make $t_1 \neq 0$ by an interchange I. We apply the replacement IV: $g_1' = g_1$, $g_i' = g_i a^{\alpha_i}$ $(i > 1)$, $a' = a$, where $\alpha_i = -s_{i1}(t_1 - 1)^{-1}$. Then by (3) $[g_i', g_1'] = a^{-\alpha_i} g_i^{-1} g_1^{-1} g_i g_1 a^{\alpha_i t_1} = a^{s_{i1} + \alpha_i(t_1 - 1)} = 1$, i.e., $s_{i1}' = 0$. We show that all the $s_{ij}' = r_i' = 0$, i.e., that G is a semidirect product of its commutator and the subgroup generated by the elements $g_1', \ldots, g_m'$. In order to do this, we substitute $x = g_1'$, $y = g_i'$, $z = g_j'^{-1}$ into the Jacobi identity (which is easily verified directly):

$$[[x, y^{-1}], z]^y [[y, z^{-1}], x]^z [[z, x^{-1}], y]^x = 1,$$

where $x^y = y^{-1}xy$. We get $[a^{s_{ij}'}, g_1] = a^{s_{ij}'(t_1 - 1)} = 1$, $s_{ij}' = 0$. Let $n_i < \infty$; then $a^{r_i'(t_1 - 1)} = g_1^{-1} a^{r_i'} g_1 a^{-r_i'} = g_1'^{-1} g_i'^{n_i} g_1' g_i'^{-n_i} = 1$, $r_i' = 0$.

The following result is proved analogously: A finite cyclic commutator subgroup having trivial intersection with the center of a finitely generated group is complementable.

THEOREM 2. Let G be a finitely generated group which cannot be decomposed into a direct product, and assume G has a complementable cyclic commutator subgroup of order p^ν ($\nu \leq 2$ for $p = 2$). Then G is given by the set of invariants

$$\left[\begin{pmatrix} \lambda_{11} \ldots \lambda_{1k_1} \\ \mu_{11} \ldots \mu_{1k_1} \end{pmatrix}_{p_1}, \ldots, \begin{pmatrix} \lambda_{s1} \ldots \lambda_{sk_s} \\ \mu_{s1} \ldots \mu_{sk_s} \end{pmatrix}_{p_s}, \tau \right]_{p^\nu}, \tag{7}$$

where the p_i are primes in the decomposition $(p - 1)p^{\nu - 1} = p_1^{\omega_1} \ldots p_l^{\omega_l}$ ($s \leq l$, $s > 0$ or $\tau \neq 0$), λ_{ij}, μ_{ij}, τ are nonnegative real numbers satisfying the conditions:

$$\omega_i \geqslant \lambda_{i1} > \lambda_{i2} > \ldots > \lambda_{ik_i} > 0, \quad \mu_{i1} > \mu_{i2} > \ldots > \mu_{ik_i},$$
$$\tau \leqslant \frac{1}{2}(p - 1)p^{\nu-1} p_1^{-\lambda_{11}} p_2^{-\lambda_{21}} \ldots p_s^{-\lambda_{s1}}, \quad k_i \geqslant 1. \tag{8}$$

G is given by the defining relations (a is a generator of the commutator subgroup)

$$g_{ij}^{-1}ag_{ij}=a^{t_{ij}},\ a^{p^{\nu}}=g_{ij}^{p_i^{\lambda_{ij}+\mu_{ij}}}=[g_{ij},\ g_{i'j'}]=1,$$

supplemented if $\tau \neq 0$ by the relations:

$$g^{-1}ag=a^{t},\ [g_{ij},\ g]=1,$$

i, i' = 1, 2, . . . , s; j, j' = 1, 2, . . . , k_i; g is an element of infinite order. Here $t_{ij}=\varepsilon^{(p-1)p^{\nu-1}p_i^{-\lambda_{ij}}}$, $t = \varepsilon^{\tau}$, ε is a smallest primitive root modulo p^{ν}, i.e., a generator of the cyclic multiplicative group of the ring $\mathbf{Z}/p^{\nu}\mathbf{Z}$ ($p^{\lambda_{ij}}$ is the order of t_{ij} in this group).

The case p = 2, $\nu > 2$ requires a special argument since the multiplicative group of the ring $\mathbf{Z}/2^{\nu}\mathbf{Z}$ is noncyclic for $\nu > 2$. We note that the group in Theorem 2 is finite if and only if $\tau = 0$, in which case its order is equal to $p^{\nu}\prod\limits_{ij}p_i^{\lambda_{ij}+\mu_{ij}}$ and the center $Z=\prod\limits_{i=1}\left[\prod\limits_{j=1}^{k_i-1}\langle g_{ij}^{p^{\lambda_{ij}-\lambda_{ij+1}}}g_{ij+1}^{-1}\rangle_{p_i^{\lambda_{ij+1}+\mu_{ij}}}\right]\times\langle g_{ik_i}^{p_i^{\lambda_{ik_i}}}\rangle_{p^{\mu_{ik_i}}}$. If $\tau \neq 0$, a direct factor $\langle g_{1i}^{\beta_1}\dots g_{si}^{\beta_s}g^{(p-1)p^{\nu-1}d^{-1}}\rangle_{\infty}$ is added, where a is the greatest common divisor of the numbers $(p-1)p^{\nu-1}$ and $\lambda\tau$; $\lambda = p_1^{\lambda_{11}}\dots p_s^{\lambda_{s1}}$, $\lambda p_i^{-\lambda_{i1}}\beta_i + \lambda\tau d^{-1} \equiv 0 \mod p_i^{\lambda_{i1}}$, $0 \leq \beta_i < p_i^{\lambda_{i1}}$. In the case s = 1, $p_1 = p$, $p-1 \mid \tau$ (including $\tau = 0$) we add another direct factor $\langle ap^{\alpha}\rangle_{p^{\nu-\alpha}}$, where α is the smallest natural number satisfying the conditions $\alpha \geq \lambda_{11}$, $p^{\nu-\alpha-1}\mid\tau$.

Proof of the Theorem. Let G be a finitely generated group which cannot be decomposed into a direct product, and let it have a complementable cyclic commutator subgroup of order p^{ν} ($\nu \leq 2$ for p = 2). The defining relations (2) for this group have the form

$$g_i^{-1}ag_i=a^{t_i},\ g_i^{n_i}=1\ (n_i<\infty),\ [g_i,\ g_j]=a^{p^{\nu}}=1, \tag{9}$$

i.e., G is completely determined by the set $t_1, \dots, t_m$ of elements of the ring $\mathbf{Z}/p^{\nu}\mathbf{Z}$. Since $g_i^{-1}ag_i = a^{t_i}$ is an automorphism, p does not divide t_i, and therefore there exist integers τ_i such that $t_i = \varepsilon^{\tau_i}$, where ε is a generator of the cyclic multiplicative group of the ring $\mathbf{Z}/p^{\nu}\mathbf{Z}$, the order of ε being equal to $(p-1)p^{\nu-1}$. Since $a=g_i^{-n_i}ag_i^{n_i}=a^{t_i^{n_i}}=a^{\varepsilon^{\tau_i n_i}}$, we have

$$\tau_i n_i \equiv 0 \mod (p-1)p^{\nu-1} \tag{10}$$

(it is more convenient to simplify τ_i and not t_i).

From the set of generators $g_1, \dots, g_m$, a of G we can go to any other set $\bar{g}_1, \dots, \bar{g}_m$, $\bar{a}$ by applying the replacements I)–V) in Sec. 1, where in V) we must replace the condition $0 < \alpha < p$ by $0 < \alpha < p^{\nu}$. It is easy to see that the replacements IV)–V) do not alter $\tau_1, \dots, \tau_m$, while under replacements I)–III) they transform as follows:

I. τ_i and τ_j are interchanged (along with n_i, n_j).

II. The element τ_i is multiplied by α and the remaining τ_k ($k \neq i$) are unchanged; $(\alpha, n_i) = 1$, $\alpha = -1$ for $n_i = \infty$.

III. We add τ_j multiplied by α to τ_i, the remaining τ_k ($k \neq i$) being unchanged; here either $p_i = p_j, p_i^{\nu_j-\nu_i}\mid\alpha$ for $\nu_j \geq \nu_i$ or else $n_i = \infty$.

Let $n_i = n_j$ and $0 < \tau_i \leq \tau_j$. We use transformation III to take a new τ_j' equal to the remainder obtained upon dividing τ_j by τ_i: $\tau_j = \tau_i\alpha + \tau_j'$, $0 < \tau_j' < \tau_i$; we then divide τ_i by the new τ_j': $\tau_i = \tau_j'\beta + \tau_i'$, $0 \leq \tau_i' < \tau_j'$, etc. Repeating this process a sufficient number of times, we get $\tau_i = 0$ or $\tau_j = 0$, which contradicts the indecomposability of G into a direct product.

Hence $n_i \neq n_j$ for $i \neq j$. We introduce a double system of enumeration by grouping the $n_i = p_i^{\nu_i}$ according to the prime p_i:

$$n_{r1}=p_r^{\nu_{r1}},\quad n_{r2}=p_r^{\nu_{r2}},\dots,n_{rk_r}=p_r^{\nu_{rk_r}}\quad (r=1,\dots,s),$$

$$\nu_{r1}>\nu_{r2}>\dots>\nu_{rk_r},\quad p_r\neq p_{r'}\ \text{for}\ r\neq r'.$$

We rewrite condition (10) in the form $\tau_{ri}p_r^{\nu_{ri}} \equiv 0 \mod (p-1)p^{\nu-1}$. Since $r_{ri} \not\equiv 0 \mod (p-1)p^{\nu-1}$, we have $p_r \mid (p-1)p^{\nu-1}$, i.e., $(p-1)p^{\nu-1} = p_r^{\omega_r}q_r$, where $\omega_r \geq 1$, p_r does not divide q_r. Then by (10) $\tau_{ri} = \pi_{ri}q_r$, where π_{ri} is defined mod $p_r^{\omega_r}$. Using transformation II we make $\pi_{ri} = p_r^{\delta_{ri}}$, $0 \leq \delta_{ri} < \omega_r$.

The δ_{ri} satisfy the inequalities

$$0 \leqslant \delta_{r1} < \delta_{r2} < \ldots < \delta_{rk_r} < \omega_r, \tag{11}$$
$$\nu_{r1} + \delta_{r1} > \nu_{r2} + \delta_{r2} > \ldots > \nu_{rk_r} + \delta_{rk_r} \geqslant \omega_r.$$

Indeed, if $\delta_{ri} \geq \delta_{rj}$ for $i < j$ then we use transformation III to arrange that $\tau'_{ri} = \tau_{ri} - \tau_{rj} p_r^{\delta_{ri}-\delta_{rj}} = 0$; if $\nu_{ri} + \delta_{ri} \leq \nu_{rj} + \delta_{rj}$ for $i < j$ then $\delta_{rj} - \delta_{ri} \leq \nu_{ri} - \nu_{rj}$, and we arrange that $\tau'_{rj} = \tau_{rj} - \tau_{ri} p_r^{\delta_{rj}-\delta_{ri}} = 0$. This contradicts the indecomposability of G. The inequality $\nu_{ri} + \delta_{ri} \geq \omega_r$ assures that condition (10) holds: $p_r^{\delta_{ri}} \times q_r p_r^{\nu_{ri}} \equiv 0 \bmod p_r^{\omega_r} q_r$. Clearly it is not possible to change δ_{ri} by a transformation III when (11) holds, i.e., the δ_{ri} are invariants of the group.

Assume that some $n_i = \infty$; then using transformation I we make $n_m = \infty$. Since $n_i \neq n_j$ for $i \neq j$, all $n_i < \infty$ for $i < m$, i.e., the $\tau_1, \ldots, \tau_{m-1}$ are already reduced and it remains to reduce τ_m. Any set of transformations II-III for τ_m can be written as a single formula: $\tau'_m = \pm\tau_m + \alpha_1\tau_1 + \ldots + \alpha_{m-1}\tau_{m-1} + \alpha_m(p-1)p^{\nu-1}$ [the summand $\alpha_m(p-1)p^{\nu-1}$ can be added since τ_m is defined mod $(p-1)p^{\nu-1}$]. Let d be the greatest common divisor $\tau_1, \ldots, \tau_{m-1}, (p-1)p^{\nu-1}$. Then $d = (p_1^{\delta_{11}}q_1, \ldots, p_s^{\delta_{s1}}q_s, (p-1)p^{\nu-1}) = (p-1)p^{\nu-1}p_1^{\delta_{11}-\omega_1} \ldots p_s^{\delta_{s1}-\omega_s}$. Then $\tau'_m = \pm\tau_m + \alpha d$; we can arrange that τ_m satisfies the condition

$$0 < \tau_m \leqslant \frac{1}{2} d = \frac{1}{2}(p-1)p^{\nu-1}p_1^{\delta_{11}-\omega_1} \ldots p_s^{\delta_{s1}-\omega_s}. \tag{12}$$

Then this τ_m is an invariant of G.

We introduce some new notation: $\lambda_{ij} = \omega_i - \delta_{ij}$, $\mu_{ij} = \nu_{ij} - \lambda_{ij}$, $\tau = \tau_m$ for $n_m = \infty$, $\tau = 0$ for $n_m < \infty$. Then conditions (11), (12) can be written in the simpler form (8). To each class of sets $(\tau_1, \ldots, \tau_m)$ going into one another under the transformations I-V (i.e., defining the same indecomposable group G) we have associated the set (7) satisfying conditions (8). Hence (7) is a complete set of invariants of G. Rewriting relations (9) in the new notation, we obtain the defining relations in Theorem 2. The theorem is proved.

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