CANONICAL FORM OF THE MATRIX OF A BILINEAR FORM OVER AN ALGEBRAICALLY CLOSED FIELD OF CHARACTERISTIC 2

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The field K is always assumed to be algebraically closed and of characteristic 2. For a nondegenerate Jordan block Φ with eigenvalue λ we denote by Φ^- a Jordan block of the same size with eigenvalue λ^{-1}, by Φ^+ the matrix $\begin{pmatrix} 0 & \Phi \\ E & 0 \end{pmatrix}$, by the cosquare root $\hat{\Phi}$ of Φ we mean a fixed solution of the equation $XX^\vee = \Phi$, where $X^\vee = (X^T)^{-1}$, X^T is the transposed matrix (we show in Lemma 1 that a solution exists only if Φ is of odd size with $\lambda = 1$, and we find the form of $\hat{\Phi}$). By the direct sum we mean the matrix $A \oplus B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

THEOREM. In a finite-dimensional vector space over a field K, for each bilinear form one can find a basis in which its matrix has the form

$$A = \Phi_1^+ \oplus \ldots \oplus \Phi_p^+ \oplus \hat{\Psi}_1 \oplus \ldots \oplus \hat{\Psi}_q \oplus F_1 \oplus \ldots \oplus F_r, \tag{1}$$

where Φ_i, Ψ_j are nondegenerate Jordan cells, $\Phi_i \neq \Psi_j$ for all i, j, F_k is a nondegenerate Jordan cell. The matrix A is determined uniquely by the bilinear form up to permutation of the summands and replacement of Φ_i by Φ_i^-.

Under a new choice of basis the matrix A of the bilinear form is replaced by a congruent matrix SAS^T (S being a nondegenerate matrix), so the theorem establishes the canonical form for a matrix with respect to congruences. We call a matrix congruently indecomposable if it is not congruent to a matrix of the form $A \oplus B$, where A and B are square matrices. The matrices Φ_i^+, $\hat{\Psi}_j$, F_k in the sum (1) are congruently indecomposable.

The problem of classification of a bilinear form over an arbitrary field was considered in [1-3], over a field of characteristic ≠2 in [4-6]. If the field K in the formulation of the theorem is replaced by an algebraically closed field L of characteristic ≠2, then the phrase "$\Phi_i \neq \Psi_j$ for all i and j" should be replaced by the phrase "there does not exist a $\hat{\Phi}_i$" (cf. [5, 6]). We note that over the field L each matrix is congruent to a direct sum of congruently indecomposable matrices, uniquely defined up to congruence of the direct summands. Over the field K even the number of summands of such a direct sum is not uniquely determined: the matrices $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus (1)$ and $(1) \oplus (1) \oplus (1)$ are congruent, although the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is congruently indecomposable.

To prove the theorem, we establish what form one can reduce a nondegenerate matrix A to by congruence transformations. Since its cosquare $\bar{A} = AA^\vee$ can be reduced by similarity transformations $S\bar{A}S^{-1} = (SAS^T)(SAS^T)^\vee$, it can be reduced to Jordan normal form:

$$\bar{A} = \Phi_1 \oplus \ldots \oplus \Phi_t, \tag{2}$$

T. G. Shevchenko Kiev State University. Translated from Matematicheskie Zametki, Vol. 41, No. 6, pp. 781-788, June, 1987. Original article submitted December 25, 1985.

0001-4346/87/4156-0441$12.50

where Φ_i is a nondegenerate Jordan block. We shall only make congruence transformations for the matrix A which do not change its cosquare (2) (an analogous method was used in [2]). We divided the matrix A into blocks A_{ij} such that the sizes of the blocks A_{ii} and Φ_i coincide; then $A = \bar{A}A^T$,

$$A_{ij} = \Phi_i A_{ji}^T = \Phi_i A_{ij} \Phi_j^T. \tag{3}$$

We clarify the form of the block A_{ij}. By E_{mn} $(m \leqslant n)$ we denote the matrix obtained from the identity of size n × n by crossing out its first (n − m) rows. We define the matrix $M_n = (a_{ij})$ of size n × n, where $a_{ij} = 0$ for $i \leqslant n/2$, $j \leqslant (n+1)/2$ and for $i + j > n + 1$, $a_{ij} = 1$ for i + j = n + 1, and the other a_{ij} are found from the condition

$$a_{i,\,j+1} + a_{i+1,\,j+1} + a_{i+1,\,j} = 0. \tag{4}$$

We define the matrix $N_n = M_n$ for odd n, $N_n = FM_n$ for even n, where F is a degenerate Jordan block. We always locate ones in a Jordan block over the eigenvalues.

LEMMA 1. Let Φ, Ψ be Jordan blocks of sizes $m \times m$, $n \times n$ with eigenvalues λ, μ.

(A) If $X = \Phi X \Psi^T$, $\lambda\mu \neq 1$, then X = 0.

(B) If $X = \Phi X \Psi^T$, $\lambda = \mu = 1$, $m \leqslant n$, then $X = f(\Phi) E_{mn} M_n$ $(f(x) \in K[x])$, its elements $x_{1m} = x_{2,\,m-1} = \ldots = x_{m1}$, $x_{ij} = 0$ for $i + j > m + 1$.

(C) If $X = \Phi X^T$, $\lambda = 1$, then $X = f(\Phi + \Phi^{-1}) N_m$, $f(x) \in K[x]$. The cosquare root $\hat{\Phi}$ exists only for $\lambda = 1$ and odd m, and in this case one can take $\hat{\Phi} = M_m$.

Proof. (A) By the s-th diagonal of the matrix $A = (a_{ij})$ we mean the collection of elements a_{ij}, $i + j = s + 1$. Let $A = \Phi A \Psi^T$; then $a_{ij} = \lambda\mu a_{ij} + \lambda a_{i,\,j+1} + \mu a_{i+1,\,j} + a_{i+1,\,j+1}$ (we assume $a_{i,\,n+1} = a_{m+1,\,j} = 0$). If $\lambda\mu \neq 1$, then $a_{mn} = 0$, and provided all diagonals below the s-th are zero, then the s-th diagonal is also zero, so A = 0.

(B) Let $\lambda = \mu = 1$, $m \leqslant n$. Then (4) holds so the (s + 1)-st diagonal and any element of the s-th diagonal determine the whole s-th diagonal. Since $a_{m+1,\,1} = \ldots = a_{m+1,\,n+1} = 0$, all the diagonals below the m-th are zero, the matrix A is completely determined by representatives of the 1st, 2nd, ..., m-th diagonals. Consequently, the set of matrices $A = \Phi A \Psi^T$ forms an m-dimensional space. The elements of the matrix $E_{mn} M_n$ satisfy (4) so it is a solution of the equation $X = \Phi X \Psi^T$. The matrices $f(\Phi) E_{mn} M_n$, where $f(x) \in K[x]$, are all its solutions, since they form a space of dimension m.

(C) For the elements of the matrix N_m (4) holds and $a_{i,\,j+1} = a_{j,\,i+1}$ (we assume $0 \leqslant i \leqslant m$, $1 \leqslant j \leqslant m$, setting $a_{0,\,j+1} = a_{j1}$), so $a_{j,\,i+1} = a_{i+1,\,j} + a_{i+1,\,j+1}$, $N_m^T = N_m \Phi^T$. Consequently, $N_m^T = \Phi N_m \Phi^T$, $N_f = f(\Phi + \Phi^{-1}) N_m = f(\Phi + \Phi^{-1}) \Phi N_m^T = \Phi N_f^T$, where $f(x) \in K[x]$.

Let $m = 2k - \alpha$, $\alpha \in \{0, 1\}$, $g(x) \in K[x]$ be a polynomial of degree k such that $(x+1)^{2k} = x^k g(x + x^{-1})$. Since $(x+1)^m$ is the characteristic polynomial of Φ and the matrix M_m is nondegenerate, one has $N_f = f(\Phi + \Phi^{-1})(E + \Phi)^{1-\alpha} M_m = 0$ only if g(x) divides f(x). Hence the dimension of the space of matrices $N_f = \Phi N_f^T$ is equal to k. On the other hand, if $A = \Phi A^T$, then $a_{ii} = a_{ii} + a_{i,\,i+1}$, $a_{i,\,i+1} = 0$, $A = \Phi A \Phi^T$, so the matrix A is completely determined by representatives of the 1st, 3rd, ..., (2k − 1)-st diagonals [point (B) of the proof], the dimension of the space of such matrices does not exceed k. Consequently, $A = N_f$.

The matrix $\hat{\Phi} = \Phi\hat{\Phi}^T = \Phi\hat{\Phi}\Phi^T$, so $\lambda = 1$ [Lemma 1 (A)] and $\hat{\Phi} = f(\Phi + \Phi^{-1}) N_m$. Since $\hat{\Phi}$ is nondegenerate, N_m is also nondegenerate and m is odd.

LEMMA 2. Let A be a congruently indecomposable nondegenerate matrix. Then A is congruent to $\hat{\Phi}$ or Φ^+, where Φ is a nondegenerate Jordan block.

Proof. Let the cosquare $\bar{A}$ have the form (2), where Φ_i is a Jordan block of size $n_i \times n_i$ with eigenvalue λ_i. We can assume that $\lambda_i = 1$. Assume that $\lambda_1 = \ldots = \lambda_q \neq 1$, $\lambda_1^{-1} = \lambda_{q+1} = \ldots = \lambda_r$, $\lambda_1 \neq \lambda_i \neq \lambda_1^{-1}$ for $i > r$. By (3) and Lemma 1 (A), $A_{ij} = 0$ in the following four cases: $i, j \in \{1, \ldots, q\}$; $i, j \in \{q+1, \ldots, r\}$; $i \leqslant r < j$; $i > r \geqslant j$. Hence, in view of the nondegeneracy and congruent indecomposability, $A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$, where B and C are nondegenerate blocks. Taking $S = R \oplus (R^\vee C^{-1})$, we get a matrix $A' = SAS^T$ with blocks $B' = R(BC^\vee)R^{-1}$, $C' = E$. Since A' is congruently indecomposable, we can make B' a Jordan block Ψ; then $A' = \Psi^+$.

In what follows we shall assume $\lambda_1 = \ldots = \lambda_t = 1$, $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_t$. We set $n = n_1$, $\Phi = \Phi_1$. Let us assume first that the block A_{11} is nondegenerate. In view of (3) and Lemma 1 (b), $A_{i1} = f_i(\Phi_i) E_{n_i n} M_n$. We apply the transformation $A' = SAS^T$ with block matrix S, in which $S_{ii} = E_{n_i n_i}$ $(1 \leqslant i \leqslant t)$, $S_{i1} = g_i(\Phi_i) E_{n_i n}$ $(i \geqslant 2,\ g_i(x) \in K[x])$, the remaining blocks being zero. We get $A_{i1}' = [g_i(\Phi_i) f_1(\Phi_i) + f_i(\Phi_i)] E_{n_i n} M_n$. In view of the nondegeneracy of the matrix $f_1(\Phi_i)$ one can choose $g_i(\Phi_i)$ so that $A_{i1}' = 0$ $(i \geqslant 2)$. By (3) and the congruent indecomposability, $A = A_{11}$.

In view of (3), Lemma 1 (C), and the nondegeneracy of $A = f(N)\hat{\Phi}$, where $N = \Phi + \Phi^{-1}$ is a nilpotent matrix, $f(x) \in K[x]$, $f(0) \neq 0$. Let $S = aE + H(E + \Phi)$, where $a^2 = f(0)$, $H = b_0 E + b_1 N + b_2 N^2 + \ldots$ Since $\hat{\Phi} = \Phi\hat{\Phi}\Phi^T$, one has $S\hat{\Phi}S^T = [aE + H(E + \Phi)][aE + H(E + \Phi^{-1})]\hat{\Phi} = [a^2 E + aHN + H^2 N]\hat{\Phi} = [a^2 E + (ab_0 + b_0^2) N + ab_1 N^2 + (ab_2 + b_1^2) N^3 + \ldots]\hat{\Phi}$. One can choose b_0, b_1, ... so that $S\hat{\Phi}S^T = f(N)\Phi$, so $\hat{\Phi}$ is congruent to A.

Let the block A_{11} be degenerate. The matrix A is nondegenerate so one can find a block A_{i1} with nonzero last column. By virtue of the relations $n_i \leqslant n$ (3), and Lemma 1 (B), such a block is nondegenerate. We shall assume i = 2. Then $n_1 = n_2 = n$, $A_{ij} = f_{ij}(\Phi_i) E_{n_i n} M_n$ $(j \leq 2)$. We apply the transformation $A = SAS^T$ with block matrix S, in which $S_{ii} = E_{n_i n_i}$ $(1 \leqslant i \leqslant t)$, $S_{ij} = g_{ij}(\Phi_i) E_{n_i n}$ $(i \geqslant 3,\ j \leqslant 2,\ g_{ij}(x) \in K[x])$, the remaining blocks being zero. We get $A_{ij} = [g_{i1}(\Phi_i) f_{1j}(\Phi_i) + g_{i2}(\Phi_i) f_{2j}(\Phi_i) + f_{ij}(\Phi_i)] E_{n_i n} M_n$ $(i \geqslant 3,\ j \leqslant 2)$. The matrix $(f_{\alpha\beta}(\Phi_i))_{\alpha,\beta=1,2}$ is nondegenerate, so one can choose $g_{ij}(\Phi_i)$ so that $A_{i1}' = A_{i2}' = 0$ $(i \geqslant 3)$. By (3), Lemma 1 (C), and the congruent indecomposability, $A = (A_{ij})_{i,j=1,2}$, $A_{ii} = f_i(\Phi + \Phi^{-1}) N_n$. By the transformation SAS^T, $S = E_{nn} \oplus f_{21}(\Phi)^{-1}$, we make $A_{21} = M_n$, $A_{12} = \Phi A_{21}^T = \Phi M_n^T$.

We show that by a congruence transformation one can make $A_{11} = A_{22} = 0$. Let $f_1(x) = a_0 + a_1 x + \ldots$, $f_2(x) = b_0 + b_1 x + \ldots$, $a_0 = \ldots = a_{r-1} = 0 \neq a_r$ $(r \geqslant 0)$. It suffices to prove that if the rank of A_{11} is not less than the rank of A_{22} (i.e., $b_0 = \ldots = b_{r-1} = 0$), then the rank of A_{11} can be lowered without changing A_{22}. We apply the transformation $1' = SAS^T$, $S = \begin{pmatrix} E & g(\Phi) \\ 0 & E \end{pmatrix}$; we get $A_{22}' = A_{22}$, $A_{11}' = f_1(\Phi + \Phi^{-1}) N_n + \Phi g(\Phi^{-1}) M_n^T + g(\Phi) M_n + g(\Phi) f_2(\Phi + \Phi^{-1}) g(\Phi^{-1}) N_n$. We introduce notation: $h(x) = g(x + 1)$, F is a degenerate Jordan block $G = E + F + F^2 + \ldots$, then $\Phi = E + F$, $\Phi^{-1} = G = E + FG$, $\Phi + \Phi^{-1} = F^2 G$. Since $N_n = F^{1-\alpha} M_n$, $M_n = \Phi^\alpha M_n^T$, where $n = 2k + \alpha$,

$\alpha \in \{0, 1\}$ one has $A'_{11} = [f_1(F^2G)F^{1-\alpha} + (E+F)^{1-\alpha}h(FG) + h(F) + \check{h}(F)f_2(F^2G)h(FG)F^{1-\alpha}]M_n$. Let $\beta \in K$ be such that $a_0 + \beta + \beta^2 b_0 = 0$; we take $h(x) = \beta$ for r = 0 (then α = 0 due to the degeneracy of A_{11}), $h(x) = a_r x^{2r-\alpha}$ for r > 0; we get $A_{11} = (c_1F^{2r+1} + c_2F^{2r+2} + \ldots)F^{1-\alpha}M_n$ $(c_i \in K)$, its rank is less than the rank of A_{11}.

Thus, one can make $A = \begin{pmatrix} 0 & \Phi M_n^T \\ M_n & 0 \end{pmatrix}$. Taking $S = E \oplus M_n^{-1}$, we get $SAS^T = \Phi^+$. The lemma is proved.

Proof of the Theorem. To classify bilinear forms, Gabriel [1] (cf. also [4-6]) proposed using the following result of Kronecker. The pair of matrices (A, B) of size $m \times n$ is called equivalent to the pair (SAR, SBR), where S and R are nondegenerate matrices of sizes $m \times m$, $n \times n$. By the direct sum one means the pair $(A, B) \oplus (C, D) = (A \oplus C, B \oplus D)$. By Kronecker's theorem (the bundle of matrices problem; cf. [7, Chap. XII]) a pair of matrices of the same size is equivalent to a direct sum of pairs of the form (Φ, E), (E, J_n), (G_n, H_n), (G_n^T, H_n^T) uniquely determined up to permutation of the summands, where Φ is a Jordan block, J_n is a degenerate Jordan block of size $n \times n$, G_n and H_n are gotten from the identity matrix of size $n \times n$ by crossing out the last and, respectively, first rows.

It follows from the equivalence of the pairs (A, B) and (C, D) that the matrices $(A, B)^+$, $(C, D)^+$ are congruent, where $(X, Y)^+ = \begin{pmatrix} 0 & X \\ Y^T & 0 \end{pmatrix}$. According to [1], a congruently indecomposable degenerate matrix is congruent to $(J_n, E)^+$ or $(G_n, H_n)^+$. But $S(J_n, E)^+S^T = J_{2n}$, $R(G_n, H_n)^+R^T = J_{2n-1}$, where $S = (s_{ij})$, $R = (r_{ij})$, $s_{2\alpha-1, n+\alpha} = s_{2\alpha, \alpha} = r_{2\alpha-1, 2n-\alpha} = r_{2\beta, n-\beta} = 1$ $(1 \leqslant \alpha \leqslant n,\ 1 \leqslant \beta \leqslant n-1)$, the other $s_{ij} = r_{ij} = 0$. It follows from this and Lemma 2 that each square matrix is congruent to a matrix of the form (1). One can impose the condition $\Phi_i \neq \Psi_j$ on its summands, since

$$S(\Psi^+ \oplus \hat{\Psi})S^T = \hat{\Psi} \oplus \hat{\Psi} \oplus \hat{\Psi}, \text{ where } S = \begin{pmatrix} E & \hat{\Psi} & E \\ E & 0 & E \\ 0 & \hat{\Psi} & E \end{pmatrix}.$$

It follows from the congruence of the matrices A and B that the pairs (A, A^T), (B, B^T) are equivalent (the converse is also true over an algebraically closed field of characteristic ≠2; cf. [4]). The pair (A, A^T) for the matrix (1) is equivalent to the direct sum of pairs $(\Phi_i, E) \oplus (\Phi_i^-, E)$, (Ψ_j, E), P_k, where $P_k = (J_n, E) \oplus (E, J_n)$ for $F_k = J_{2n}$, $P_k = (G_n, H_n) \oplus (G_n^T, H_n^T)$ for $F_k = J_{2n-1}$. By Kronecker's theorem two direct sums of the form (1) can be congruent only if one is gotten from the other by replacing some summands of the form Φ^+ by $(\Phi^-)^+$ or $\hat{\Phi} \oplus \hat{\Phi}$ and some summands of the form $\hat{\Psi} \oplus \hat{\Psi}$ by Ψ^+ ($\Phi = \Phi^-$, provided $\hat{\Phi}$ exists; cf. Lemma 1 (C)).

Replacing Φ^+ by $(\Phi^-)^+$ leads to a congruent direct sum, since $S\Phi^+S^T = (\Phi^-)^+$, where $S = (R, (\Phi^\vee R^{-1})^+$, $R\Phi^\vee R^{-1} = \Phi^-$. We show that if $\Phi_i \neq \Psi_j$ replacement of Φ_i^+ by $\hat{\Phi}_i \oplus \hat{\Phi}_i$ in the direct sum (1) leads to a noncongruent matrix.

By contradiction, let $SAS^T = B$, where $A = A_1 \oplus \ldots \oplus A_n$, $(n = p + q + r)$ is the matrix (1), $B = \hat{\Phi} \oplus C$, Φ being one of the matrices $\Phi_1, \ldots, \Phi_p$. We divide the matrices S, $S^\vee$ into n vertical and two horizontal strips corresponding to the partitions of A and B, and let $(S_1 | \ldots | S_n)$, $(R_1 | \ldots | R_n)$ be the upper horizontal strips of the matrices S, $R = S^\vee$. Then $S_1A_1S_1^T + \ldots + S_nA_nS_n^T = \hat{\Phi}$.

We get a contradiction with the nondegeneracy of $\hat{\Phi}$, if we prove that the last row of all the matrices $S_iA_iS_i^T$ is zero. Since $SA = BS^\vee$ and $SA^T = B^TS^\vee$ we get $S_iA_i = \hat{\Phi}R_i$ and $S_iA_i^T = \hat{\Phi}^TR_i$. From this, $S_iA_i = \hat{\Phi}\hat{\Phi}^\vee S_iA_i^T = \Phi S_iA_i^T$, $S_i\bar{A}_i = \Phi S_i$ (where $\bar{A}_i = A_iA_i^\vee$) for nondegenerate A_i.

If $i \le p$, then $A_i = \Phi_i^+$, $\bar{A}_i = \Phi_i \oplus \Phi_i^\vee$. From $S_i\bar{A}_i = \Phi S_i$ it follows that $P\Phi_i = \Phi P$, $Q\Phi_i^\vee = \Phi Q$, where $S_i = (P \mid Q)$. From this $\Phi QP^T\Phi^T = QP^T$ and by Lemma 1 (B), $QP^T = f(\Phi)\hat{\Phi}$. Hence

$$P\Phi_iQ^T = \Phi PQ^T = \hat{\Phi}f(\Phi^T),$$

the matrix

$$S_iA_iS_i^T = P\Phi_iQ^T + QP^T = (f(\Phi^{-1}) + f(\Phi))\hat{\Phi}$$

has last row zero.

If $i = p + j$ $(j \leqslant q)$, then $A_i = \hat{\Psi}_j$, $\bar{A}_i = \Psi_j$. Since $\Phi_i \neq \Psi_j$ for all i and j, one has $\Phi \neq \Psi_j$. By Lemma 1 (C) the eigenvalues of the blocks Φ, Ψ_j are equal to 1, so Φ and Ψ_j are of different sizes. Let the size of Φ be greater than the size of Ψ_j; then since $S_i\Psi_j = \Phi S_i$, the last row of the matrix S_i, and hence also $S_i\hat{\Psi}_jS_i^T$, is zero. Let the size of Φ be smaller than the size of Ψ_j; then the last row of the matrices S_i, $\hat{\Psi}_j$ and the first row of the matrix S_i^T have the forms, respectively, $(0 \ldots 0\ a)$, $(1\ 0 \ldots 0)$, $(0 \ldots 0)$, so the last row of the matrix $S_i\hat{\Psi}_jS_i^T$ is zero.

If $i = p + q + k$ $(k \leqslant r)$, then $A_i = F_k$. It follows from the relation $S_iF_k = \Phi S_iF_k^T$ that the columns with even indices of the matrix S_i are zero, so $S_iF_kS_i^T = 0$. The theorem is proved.

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