$$\leqslant r \sum_{m_1=1}^{r_1} \sum_{m=1}^{r_1} \frac{1}{m_1 m d^2} \sum_{j=1}^{k(m_1, m)} 1 = O(r \log^2 r/d\delta) = O(r/d).$$

The lemma is proved.

The proof of the estimate

$$\sum_{m_1=-2}^{-r_1} \sum_{m=1}^{r_1} S_{m_1 m} = O(r/d)$$

is almost a verbatim repetition of the proof of Lemma 4.

Combining the results of the last three lemmas, we obtain the estimate

$$\sum_{m_1=-r_1}^{-r_1} \sum_{m=0}^{r_1} S_{m_1 m} = O\left(\frac{r}{d} + \frac{r}{\delta^2}\right).$$

Consequently, if $u \in N_r$, then

$$|R_u| = O\left(\frac{r}{d} + \frac{r}{\delta^2} + \frac{n \log r}{d}\right).$$

It is now easy to see that under the conditions of the theorem we have

$$|R_u| = o(n^2/r).$$

At the same time, according to Lemma 1,

$$r - |N_r| = o(r).$$

The theorem is proved.

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SYMMETRIC REPRESENTATIONS OF ALGEBRAS WITH INVOLUTION

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Suppose K is a field of characteristic ≠2 with involution $k \to \bar{k}$ (possibly the identity mapping) and Λ is an algebra over K with involution, i.e., a mapping $\iota: \Lambda \to \Lambda$ such that $(\lambda + \mu)^\iota = \lambda^\iota + \mu^\iota$, $(\lambda\mu)^\iota = \mu^\iota\lambda^\iota$, $(k\lambda)^\iota = \bar{k}\lambda^\iota$, $\lambda^{\iota\iota} = \lambda$ for all $\lambda, \mu \in \Lambda$, $k \in K$.

By a representation of the algebra Λ by operators of a vector space V over K we mean a homeomorphism $\varphi: \Lambda \to \mathrm{End}(V)$. The representation is symmetric if to a conjugate element there is assigned the conjugate linear operator relative to a fixed scalar product in V: $\varphi(\lambda^\iota) = \varphi(\lambda)^\iota$. If we introduce in V the multiplication K, $F(v, w) = \varepsilon\overline{F(w, v)}$ we obtain an ε-Hermitian module defined as follows.

Definition. By an ε-Hermitian module (M, F), (M', F') we mean a pair (M, F), where M is a module over Λ that is finite-dimensional over K, $F(v, w) = \varepsilon\overline{F(w, v)}$ is a nondegenerate ε-Hermitian form on the vector space $_K M$ of the module M, and

$$F(\lambda v, w) = F(v, \lambda^\iota w), \quad \lambda \in \Lambda, \ v, w \in M. \tag{1}$$

Two ε-Hermitian modules (M, F), (M', F') are isomorphic if there exists a Λ-isomorphism φ: $M \simeq M'$, preserving the forms:

$$F(v, w) = F'(\varphi v, \varphi w), \quad v, w \in M. \tag{2}$$

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Examples. 1) $\Lambda = K[x]$, $x^\iota = x$. The module M over Λ is the vector space $_KM$ with fixed linear operator $v \to xv$. The problem of classifying ε-Hermitian modules is that of classifying self-conjugate linear operators in a finite-dimensional vector space with a nondegenerate ε-Hermitian form.

2) $\Lambda = K[x, x^{-1}]$, $x^\iota = x^{-1}$. The problem is to classify isometric operators in a space with a nondegenerate ε-Hermitian form.

3) Λ = KG is the group algebra of a group G with involution $(\sum k_g g)^\iota = \sum \bar{k}_g g^{-1}$. The problem is to classify representations of G by isometric operators in a space with a nondegenerate ε-Hermitian form.

4) $K = \mathbf{C}$ with a nonidentity involution. Then a 1-Hermitian module (M, F), where F is a positive definite Hermitian form, defines a symmetric representation of the algebra Λ by operators of the unitary space $(_{\mathbf{C}}M, F)$ (see [1, Chap. 2, Sec. 2.6]). In particular, if Λ = $\mathbf{C}G$ with involution $(\sum k_g g)^\iota = \sum \bar{k}_g g^{-1}$, then such a module defines a unitary representation of G (see [1, Chap. 2, Sec. 2.8]).

We will show (see the theorem) that the classification of ε-Hermitian modules reduces to that of ordinary modules over Λ and Hermitian forms over a skew field. This follows from [2, Chap. 7, Theorem 10.9], but we will use [3, 4] in order to obtain the reduction in a more explicit form. We will apply the reduction to symmetric representations of algebras with involution in pseudo-unitary and pseudo-Euclidean spaces (see Corollary 1) and in unitary, Euclidean, and complex Euclidean spaces (see Corollary 2).

By the orthogonal sum of ε-Hermitian modules we mean the ε-Hermitian module $(M, F) \perp (M', F') = (M \oplus M', F \oplus F')$.

Suppose M is a module over Λ. We define the dual module M* over Λ as the module whose vector space is the space of semilinear forms f: $_KM \to K$, with multiplication by elements $\lambda \in \Lambda$ defined by $\lambda f = f\lambda^\iota$. We also define the ε-Hermitian module $M^{(\varepsilon)} = (M \oplus M^*, F)$, where

$$F(v \oplus f,\ w \oplus g) = g(v) + \overline{\varepsilon f(w)} \quad (3)$$

(all sesquilinear forms are regarded as semilinear in the first argument and linear in the second).

Let ind(Λ) be a fixed complete system of nonisomorphic modules over Λ that are indecomposable into a direct sum and finite-dimensional over K. Let $\mathrm{ind}_0^\varepsilon(\Lambda)$ denote the set of all $N \in \mathrm{ind}(\Lambda)$, for which there exists an ε-Hermitian module (N, F), and fix one such module (N, F_N) [in this case $N \rightrightarrows N^*, v \mapsto F_N(?, v)$]. In the set $\mathrm{ind}_1^\varepsilon(\Lambda)$ we include all $M \in \mathrm{ind}(\Lambda)$, $M^* \simeq M \notin \mathrm{ind}_0^\varepsilon(\Lambda)$, and one module from each pair $\{M, N\} \subset \mathrm{ind}(\Lambda)$, $M \not\simeq M^* \simeq N$.

Suppose $N \in \mathrm{ind}_0^\varepsilon(\Lambda)$. In the algebra End(N) of endomorphisms we define an involution $\varphi \mapsto \varphi^\iota$, where φ^ι is the conjugate endomorphism relative to F_N:

$$F_N(\varphi v, w) = F_N(v, \varphi^\iota w), \qquad v, w \in N.$$

The algebra of endomorphisms of an indecomposable module is local, hence the quotient algebra by the radical, T(N) = End(N)/R, is a skew field with involution $(\varphi + R)^\iota = \varphi^\iota + R$. For each $0 \neq t = t^\iota \in T(N)$ we fix $\varphi_t = \varphi_t^\iota \in t$ [we can take $\varphi_t = 1/2 \cdot (\varphi + \varphi^\iota)$, where $\varphi \in t$) and define an ε-Hermitian form $F_N^t(v, w) = F_N(v, \varphi_t w)$. For each Hermitian form $\varphi(x) = x_1^\iota t_1 x_1 + \ldots + x_r^\iota t_r x_r$ over the skew field $T(N)$ $(0 \neq t_i = t_i^\iota \in T(N))$ we put

$$N^{\varphi(x)} = (N, F_N^{t_1}) \perp \ldots \perp (N, F_N^{t_r}).$$

THEOREM. Each ε-Hermitian module over Λ is isomorphic to an orthogonal sum

$$M_1^{(\varepsilon)} \perp \ldots \perp M_m^{(\varepsilon)} \perp N_1^{\varphi_1(x)} \perp \ldots \perp N_n^{\varphi_n(x)},$$

where $M_i \in \mathrm{ind}_1^\varepsilon(\Lambda)$, $N_j \in \mathrm{ind}_0^\varepsilon(\Lambda)$, $N_j \neq N_{j'}$ for j ≠ j'. This orthogonal sum is uniquely determined to within a rearrangement of the summands and the replacement of $N_j^{\varphi_j(x)}$ by $N_j^{\psi_j(x)}$, where $\varphi_j(x), \psi_j(x)$ are equivalent Hermitian forms over the skew field $T(N_j)$.

Remarks. 1) Suppose M is a module over Λ and A_λ $(\lambda \in \Lambda)$ is the matrix of the linear operator $v \mapsto \lambda v$ $(v \in M)$ in the basis $e_1, \ldots, e_n$ of the space $_KM$. Then in the dual basis $e_1^*, \ldots, e_n^*$ of the space of the module M* the operator $f \mapsto \lambda f$ $(f \in M^*)$ is defined by the matrix

$A^*_{\lambda^\iota}$ [for each matrix $A = (a_{ij})$ we define the matrix $A^* = (\bar{a}_{ji})$).. In the basis $e_1, \ldots, e_n$, $e_1^*, \ldots, e_n^*$ of the space of the module $M^{(\varepsilon)} = (M \oplus M^*, F)$ the linear operator $w \mapsto \lambda w$ $(w \in M \oplus M^*)$ and the ε-Hermitian form F are defined by the matrices $\begin{pmatrix} A_\lambda & 0 \\ 0 & A^*_{\lambda^\iota} \end{pmatrix}$ and $\begin{pmatrix} 0 & E \\ \varepsilon E & 0 \end{pmatrix}$.

2) (See [2, Chap. 7, Theorem 4.5].) For each $N^* \simeq N \in \operatorname{ind}(\Lambda)$ there exists a 1-Hermitian or (−1)-Hermitian module (N, F). Indeed, suppose $\varphi\colon N \rightleftarrows N^*$, $N \in \operatorname{ind}(\Lambda)$.. Consider the dual isomorphism $\varphi^*\colon N = N^{**} \rightleftarrows N^*$, $v = v^{**} \mapsto v^{**}\varphi$. Since the algebra End(Λ) of endomorphisms is local, the invertibility of $2\varphi = (\varphi + \varphi^*) + (\varphi - \varphi^*)$ implies the invertibility of $\varphi + \varphi^*$ or $\varphi - \varphi^*$. Consequently, there exists an isomorphism $\psi = \varepsilon\psi^*\colon N \rightleftarrows N^*$, $\varepsilon \in \{1, -1\}$, hence the module (N, F), F(v, w) = ψ(w)(v) is ε-Hermitian.

3) If K is a field with a nonidentity involution, then $\operatorname{ind}_0(\Lambda)$ consists of all $N \in \operatorname{ind}(\Lambda)$, $N \simeq N^*$. It suffices to use the preceding remark and the fact that over the field K each ε-Hermitian form can be made Hermitian by multiplying it by $1 + \bar{\varepsilon}$ if $\varepsilon \neq -1$, or by $k - \bar{k} \neq 0$ $(k \in K)$ if $\varepsilon = -1$.

Proof of the Theorem. It is only in proving the theorem that we will assume as known the definitions and notation of [4].

We represent Λ as a quotient algebra of a free algebra with generators $x_1, x_2, \ldots$:

$$\Lambda = K\langle x_1, x_2, \ldots \rangle / K\langle f_1, f_2, \ldots \rangle,$$

where the $f_i(x_1, x_2, \ldots)$ are certain noncommutative polynomials. Then the $\lambda_j = x_j + K\langle f_1, f_2, \ldots \rangle$ are generators of Λ. The involution in Λ is defined by certain relations

$$\lambda_j^\iota = g_j(\lambda_1, \lambda_2, \ldots). \tag{4}$$

Suppose (M, F) is an ε-Hermitian module over Λ. Fix a basis of the vector space $_K M$. Let A_j be the matrix of the linear operator $v \mapsto \lambda_j v$ $(v \in M)$, and $B = \varepsilon B^*$ the matrix of the ε-Hermitian form F. The set of matrices A_j must satisfy the relations satisfied by the elements λ_j of Λ, hence

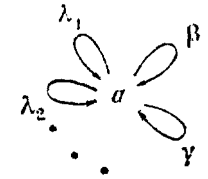
$$f_i(A_1, A_2, \ldots) = 0. \tag{5}$$

It follows from these relations [1, 4] that

$$A_j^* B = B g_j(A_1, A_2, \ldots). \tag{6}$$

Conversely, any set consisting of a nondegenerate ε-Hermitian matrix $B = \varepsilon B^*$ and square matrices A_j of the same size satisfying relations (5) and (6) defines some ε-Hermitian module (M, F).

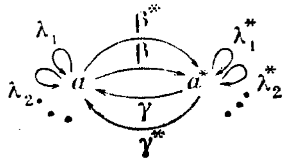
Consequently, an ε-Hermitian module (M, F) defines a representation of a digraph with relations (cf. [4, digraph (9)])

S: 

$$f_i(\lambda_1, \lambda_2, \ldots) = 0, \quad \lambda_j^* \beta = \beta g_j(\lambda_1, \lambda_2, \ldots), \quad \beta = \varepsilon\beta^*,\ \gamma\beta = 1_a,\ \beta\gamma = 1_{a^*}.$$

and each such representation defines an ε-Hermitian module.

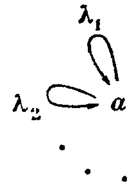
The quiver with involution of the digraph S is

$\bar{S}$: 

$$f_i(\lambda_1, \lambda_2, \ldots) = 0, \quad \lambda_j^* \beta = \beta g_j(\lambda_1, \lambda_2, \ldots), \quad \beta = \varepsilon\beta^*,\ \gamma\beta = 1_a,\ \beta\gamma = 1_{a^*}. \tag{7}$$

We do not include in (7) the conjugate relations, but they follow from the relations (7) since the involution $\lambda \mapsto \lambda^\iota$ in Λ is compatible with addition and multiplication.

Consider the quiver

Q:  $f_i(\lambda_1, \lambda_2, \ldots) = 0$

defining modules over Λ. We extend each representation $A \in \mathrm{ind}(Q)$ to a representation of the quiver $\bar{S}$ by putting $A_\beta = A_{\beta^*} = A_\gamma = A_{\gamma^*} = 1$, $A_{\lambda_j^*} = g_j(A_{\lambda_1}, A_{\lambda_2}, \ldots)$, the resulting representations form a set $\mathrm{ind}(\bar{S})$. We can therefore identify ind(Q) and $\mathrm{ind}(\bar{S})$. Furthermore, the dual module M^* can be identified with the conjugate representation A^0, the module $M^{(\varepsilon)}$ with the representation A^+, and the set $\mathrm{ind}_i^\varepsilon(\Lambda)$ with the set $\mathrm{ind}_i(\bar{S})$, i = 0, 1. To prove the theorem we need only use [4, Theorem 1].

COROLLARY 1. Suppose K is one of the following fields of characteristic ≠2:

a) an algebraically closed field with the identity involution;

b) an algebraically closed field with a nonidentity involution;

c) a maximal ordered field [i.e., $1 < (K_{alg}:K) < \infty$, where K_{alg} is an algebraic closure of K, e.g., $K = \mathbf{R}$);

d) a finite field.

Then each ε-Hermitian module is isomorphic to a uniquely defined (to within a rearrangement of the summands) orthogonal sum of ε-Hermitian modules of the form ($M \in \mathrm{ind}_1^\varepsilon(\Lambda)$, $N \in \mathrm{ind}_0^\varepsilon(\Lambda)$)

a) $M^{(\varepsilon)}$, (N, F_N);

b) $M^{(\varepsilon)}$, (N, F_N), $(N, -F_N)$;

c) $M^{(\varepsilon)}$, (N, tF_N), where t = 1 if T(N) is an algebraically closed field with the identity involution or the skew field of quaternions with involution different from a + bi + cj + dk → a − bi − cj − dk, and $t \in \{-1, 1\}$ otherwise;

d) $M^{(\varepsilon)}$, (N, tF_N), where t = 1 for a nonidentity involution on the field T(N), t is equal to 1 or a fixed nonsquare in T(N) for the identity involution, and for each N the orthogonal sum contains at most one summand (N, tF_N) with t ≠ 1.

The proof follows from the theorem and [2, Theorem 2].

COROLLARY 2. Suppose (M, F), (M', F') are 1-Hermitian modules in which $({}_KM, F)$, $({}_KM', F')$ are Euclidean, or unitary, or complex Euclidean spaces ($K = \mathbf{R}$, or $K = \mathbf{C}$ with a nonidentity involution, or $K = \mathbf{C}$ with the identity involution, respectively).

1) (M, F) ≃ (M', F') if and only if M ≃ M'.

2) (M, F) is uniquely (to within isomorphism of summands) decomposable into an orthogonal sum of orthogonally indecomposable 1-Hermitian modules.

3) If (M, F) is indecomposable into an orthogonal sum, then either M is indecomposable into a direct sum, or (only in the case of a complex Euclidean space) $M \simeq N \oplus N^*$, where N is indecomposable into a direct sum.

The proof follows easily from the law of inertia for Hermitian forms and Corollary 1.

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