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ATTAINMENT OF WEIGHTED PERFORMANCE MEASURES AND THE LOCALIZATION OF SPECTRUM IN DESCRIPTOR CONTROL SYSTEMS

Dedicated to the 90th birthday of Academician Ivan Oleksandrovych Lukovsky

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We study the problems of estimation of the weighted level of damping of bounded disturbances and the localization of spectrum for linear descriptor systems with indefinite initial vector. In terms of linear matrix inequalities, we establish criteria and sufficient conditions for the validity of the desired estimation of the generalized performance measure and the localization of the system spectrum in a given region of the complex plane. New methods are proposed for synthesis of the static state and the output controllers guaranteeing the validity of the indicated properties of the analyzed closed-loop descriptor system. We also present a numerical example of descriptor control system.

1. Introduction

Methods of H_2/H_∞-optimization are extensively developed and widely used in the modern control theory. They guarantee the robust stability of equilibrium states and weakening of the negative influence of external disturbances on the dynamics of controlled objects (see, e.g., [1–3]). As the typical performance measure in these problems for systems with zero initial state, one can use the maximum value of the ratio of L_2-norms of the vectors of controlled output of the object and disturbances. For the class of linear autonomous systems, this characteristic coincides with the H_∞-norm of the matrix transfer function.

The methods of construction of the H_2/H_∞ controllers were developed in [4–10]. For a closed control system, they guarantee not only the desired level of damping of external disturbances but also the localization of its spectrum in a given domain of the complex plane. If the domain of desired localization of the spectrum belongs to the so-called LMI class, then the procedure of construction of the required controllers is reduced to solving systems of linear matrix inequalities (LMIs). The criteria for localization of the spectrum of a regular pencil of matrices in LMI domains make it possible to extend the analyzed approaches to the classes of linear descriptor (singular) control systems and also their discrete and stochastic analogs (see, e.g., [11–18]). Highly efficient software tools were created for the solution of LMIs, e.g., the LMI Toolbox of MATLAB [19]. The practical applications of synthesis methods with desired localization of the spectrum enable us to significantly improve the qualitative performance characteristics of the actual controlled objects.

In the present paper, we continue investigations originated in [20, 21] and dealing with the problems of synthesis of the generalized H_∞ control for a class of linear descriptor systems. These systems are used for the simulation of motion of various objects of mechanics, robotics, power engineering, electrical engineering, economics, etc. For the descriptor systems with controlled and observed outputs, we obtain new necessary and sufficient conditions for the existence of static state and output controllers guaranteeing the required estimate for the weighted damping level of bounded disturbances and the localization of finite spectrum in given LMI domains of the complex plane.

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The practical implementation of these conditions is reduced to the solution of strict LMI without additional rank restrictions. As a distinctive feature of the obtained results, we can mention the use of weighted performance measures, which extends the possibilities of attainment of the desired characteristics of descriptor control systems.

We use the following notation: I_n is the identity matrix of order n; $0_{n\times m}$ is the $n \times m$ null matrix, $X = X^\top > 0$ (≥ 0) is a positive-definite (negative-definite) matrix; $\sigma(A)$ is the spectrum of the matrix A; A^{-1} (A^+) is the inverse (pseudoinverse) matrix; $\operatorname{Ker} A$ is the kernel of the matrix A; W_A is a matrix whose columns form a basis of the kernel $\operatorname{Ker} A$; $\|x\|$ is the Euclidean norm of a vector x; $\|w\|_P$ is the weighted L_2-norm of a vector function $w(t)$, and $\mathbb{C}^-$ is the open half plane $\operatorname{Re}\lambda < 0$.

2. Definitions and Auxiliary Statements

Consider a linear descriptor system

$$E\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, and $w \in \mathbb{R}^s$ are, respectively, the vectors of state, controlled output, and external disturbances of the system, E, A, B, C, and D are constant matrices of the corresponding order, and moreover, $\operatorname{rank} E = r < n$. The vector of external disturbances in system (1) is unknown but must be bounded in the L_2-norm.

Definition 1. *System (1) is called admissible if the pair of matrices* $\{E, A\}$ *is regular, stable, and impulse-free [22], i.e.,* $\det F(\lambda) \not\equiv 0,\ \lambda \in \mathbb{C},\ \Sigma \subset \mathbb{C}^-$, *and* $\deg\{\det F(\lambda)\} = r = \rho$, *where* $\Sigma = \{\lambda_1, \dots, \lambda_\rho\}$ *is the finite spectrum of the pencil of matrices* $F(\lambda) = A - \lambda E$.

The pair of matrices $\{E, A\}$ is regular if and only if there exist nondegenerate matrices L and R reducing it to the canonical Weierstrass form [23], namely,

$$LER = \begin{bmatrix} I_\rho & 0 \\ 0 & N \end{bmatrix} \qquad \text{and} \qquad LAR = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-\rho} \end{bmatrix}. \tag{2}$$

The eigenvalues of the matrix A_1 in relations (2) form the finite spectrum of the pencil Σ and N is a nilpotent matrix with index ν $(N^\nu = 0)$. In this case, system (1) has ρ finite dynamic modes, $n - r$ nondynamic modes, and $r - \rho$ impulsive modes [24].

System (1) is impulse-free if and only if [25]

$$\operatorname{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + r. \tag{3}$$

Admissible system (1) is *internally stable*, i.e., its trivial state $x \equiv 0$ is asymptotically stable in the absence of disturbances $(w = 0)$. It is known [26] that system (1) is admissible if and only if there exists a matrix X satisfying the following LMI system:

$$A^\top X + X^\top A < 0, \qquad E^\top X = X^\top E \geq 0. \tag{4}$$

Moreover, under the rank condition $\operatorname{rank}(E^\top X) = r$, the second relation in (4) is equivalent to the representation of the matrix X in the form [27]

$$X = SE + W_{E^\top} G, \qquad E_l^\top S E_l > 0, \tag{5}$$

where $S = S^\top \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{(n-r)\times n}$. In particular, if $S > 0$, then

$$E_l^\top S E_l > 0,$$

where E_l is the factor of skeleton decomposition of the matrix

$$E = E_l E_r^\top, \qquad \operatorname{rank} E_l = \operatorname{rank} E_r = r, \qquad E_l \in \mathbb{R}^{n\times r}, \qquad E_r \in \mathbb{R}^{n\times r}.$$

Consider weighted performance measures of system (1) of the form [28]

$$J_0 = \sup_{0<\|w\|_P<\infty} \frac{\|z\|_Q}{\|w\|_P}, \qquad J = \sup_{\{w,x_0\}\in\mathcal{W}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}. \tag{6}$$

Here, $\mathcal{W}$ is the set of admissible pairs $\{w(t), x_0\}$ satisfying the inequality

$$0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty$$

and $\|z\|_Q$ and $\|w\|_P$ are weighted L_2-norms of the corresponding vector functions $z(t)$ and $w(t)$ of the form

$$\|z\|_Q = \sqrt{\int\limits_0^\infty z^\top Q z \, dt}, \qquad \|w\|_P = \sqrt{\int\limits_0^\infty w^\top P w \, dt},$$

where $P = P^\top > 0$, $Q = Q^\top > 0$, and $X_0 = E^\top H E$ $(H = H^\top > 0)$ are given weighted matrices (see also [27, 29]). Since $H > 0$, $x_0 \in \operatorname{Ker} E$ if and only if $x_0^\top X_0 x_0 = 0$. It is clear that $J_0 \leq J$. In the case of the identity matrices $P = I_s$ and $Q = I_k$, the characteristic J_0 coincides with the standard performance measure in the theory of H_∞-control, which is equal to the H_∞-norm of the matrix transfer function

$$\|\mathcal{H}\|_\infty = \sup_{\omega\in\mathbb{R}} \sqrt{\lambda_{\max}(\mathcal{H}^\top(-i\omega)\mathcal{H}(i\omega))}, \qquad \mathcal{H}(\lambda) = C(\lambda E - A)^{-1} B + D.$$

The quantity J characterizes the weighted level of damping of the external disturbances and also of the initial disturbances caused by the nonzero initial vector of the system x_0. By using weighting coefficients in these performance measures, it is possible to establish priorities between the components of controlled output and indefinite disturbances.

We say that the vector function of disturbance $w(t)$ and the initial vector of the system x_0 are the *worst possible* for the performance measure J if the supremum in (6) is attained for their values, i.e.,

$$\|z\|_Q^2 = J^2 \big(\|w\|_P^2 + x_0^\top X_0 x_0\big).$$

The vector function $w(t)$ is the *worst possible* for J_0 if $\|z\|_Q = J_0\|w\|_P$. Methods used to determine the worst possible pairs of vectors $\{w(t), x_0\}$ for linear systems based on the solution of quadratic Riccati-type equations were proposed in [29–31]. The analysis of the behavior of a closed system under the indicated worst conditions can be important for the construction and testing of actual controlled objects.

It is known [32] that system (1) is admissible and possesses the performance measure $J < \gamma$ if a certain matrix X satisfies the relations

$$\begin{bmatrix} A^\top X + X^\top A + C^\top QC & X^\top B + C^\top QD \\ B^\top X + D^\top QC & D^\top QD - \gamma^2 P \end{bmatrix} < 0, \tag{7}$$

$$0 \le E^\top X = X^\top E \le \gamma^2 X_0, \qquad \operatorname{rank}\left(E^\top X - \gamma^2 X_0\right) = r. \tag{8}$$

The converse statement holds under the additional rank condition

$$\operatorname{rank}\left[E^\top \; C^\top QD\right] = r. \tag{9}$$

We perform the nondegenerate transformation of system (1) by using the relations [20]

$$LER = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \qquad LAR = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \qquad x = R\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}, \quad \xi_1 \in \mathbb{R}^r, \quad \xi_2 \in \mathbb{R}^{n-r}, \tag{10}$$

where

$$L = \begin{bmatrix} E_l^+ \\ E_l^{\perp+} \end{bmatrix}, \qquad E_l^+ = (E_l^\top E_l)^{-1}E_l^\top, \qquad E_l^{\perp+} = (E_l^{\perp\top}E_l^\perp)^{-1}E_l^{\perp\top},$$

$$R = \left[E_r^{+\top} \; E_r^{\perp+\top}\right], \qquad E_r^+ = (E_r^\top E_r)^{-1}E_r^\top, \qquad E_r^{\perp+} = (E_r^{\perp\top}E_r^\perp)^{-1}E_r^{\perp\top},$$

$$A_1 = E_l^+ A E_r^{+\top}, \qquad A_2 = E_l^+ A E_r^{\perp+\top}, \qquad A_3 = E_l^{\perp+} A E_r^{+\top}, \qquad A_4 = E_l^{\perp+} A E_r^{\perp+\top}.$$

Here, $E_l^\perp \in \mathbb{R}^{n\times(n-r)}$ and $E_r^\perp \in \mathbb{R}^{n\times(n-r)}$ are the orthogonal complements of the corresponding matrices E_l and E_r satisfying the relations

$$E_l^\top E_l^\perp = 0, \qquad E_r^\top E_r^\perp = 0, \qquad \det\left[E_l \quad E_l^\perp\right] \ne 0, \quad \det\left[E_r \quad E_r^\perp\right] \ne 0.$$

In this case,

$$L^{-1} = \left[E_l \;\; E_l^\perp\right], \qquad R^{-1} = \begin{bmatrix} E_r^\top \\ E_r^{\perp\top} \end{bmatrix}, \qquad \xi_1 = E_r^\top x, \qquad \xi_2 = E_r^{\perp\top} x.$$

Eliminating the variable $\xi_2 = -A_4^{-1}\left(A_3\xi_1 + B_2 w\right)$, under the condition $\det A_4 \ne 0$, i.e.,

$$\det(E_l^{\perp\top} A E_r^\perp) \ne 0, \tag{11}$$

we can represent the dynamic part of the impulse-free system (1) in the form

$$\dot{\xi}_1 = \bar{A}\xi_1 + \bar{B}w, \qquad z = \bar{C}\xi_1 + \bar{D}w, \qquad \xi_1(0) = \xi_{10}, \tag{12}$$

where

$$\bar{A} = A_1 - A_2A_4^{-1}A_3, \qquad \bar{B} = B_1 - A_2A_4^{-1}B_2,$$

$$\bar{C} = C_1 - C_2A_4^{-1}A_3, \qquad \bar{D} = D - C_2A_4^{-1}B_2,$$

$$LB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad CR = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

Lemma 1. *The following assertions are equivalent:*

1. *System (1) is admissible and the following estimate holds:* $J < \gamma$.
2. *There exist matrices* $S = S^\top \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{(n-r)\times n}$, *and* $U \in \mathbb{R}^{(n-r)\times s}$ *satisfying the relations*

$$\begin{bmatrix} A^\top X + X^\top A & X^\top B + A^\top W_{E^\top}U & C^\top \\ B^\top X + U^\top W_{E^\top}^\top A & B^\top W_{E^\top}U + U^\top W_{E^\top}^\top B - \gamma^2 P & D^\top \\ C & D & -Q^{-1} \end{bmatrix} < 0, \tag{13}$$

$$X = SE + W_{E^\top}G, \qquad 0 < E_l^\top SE_l < \gamma^2 E_l^\top HE_l. \tag{14}$$

3. *System (12) is internally stable with the performance measure*

$$\bar{J} = \sup_{\{w,\xi_{10}\}\in\bar{\mathcal{W}}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + \xi_{10}^\top \bar{H}\xi_{10}}} < \gamma,$$

where $\bar{H} = E_l^\top HE_l$, $\bar{\mathcal{W}}$ *is the set of admissible pairs of vectors* $\{w(t), \xi_{10}\}$ *for which*

$$0 < \|w\|_P^2 + \xi_{10}^\top \bar{H}\xi_{10} < \infty.$$

4. *Condition (11) is satisfied and there exists a matrix* $X = X^\top \in \mathbb{R}^{r\times r}$ *satisfying the LMI system*

$$0 < X < \gamma^2 \bar{H}, \tag{15}$$

$$\begin{bmatrix} \bar{A}^\top X + X\bar{A} + \bar{C}^\top Q\bar{C} & X\bar{B} + \bar{C}^\top Q\bar{D} \\ \bar{B}^\top X + \bar{D}^\top Q\bar{C} & \bar{D}^\top Q\bar{D} - \gamma^2 P \end{bmatrix} < 0. \tag{16}$$

Proof. The equivalence of assertions 1 and 2 was established in [33]. System (12) is constructed on the basis of relations (10) and (11). Moreover, inequality (11) holds if and only if the original system (1) is impulse-free. Indeed, conditions (3) and (11) are equivalent because

$$\mathcal{L}\begin{bmatrix} E & 0 \\ A & E \end{bmatrix}\mathcal{R} = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & A_4 & 0 & 0 \end{bmatrix}, \qquad \det \mathcal{L} \neq 0, \qquad \det \mathcal{R} \neq 0,$$

where

$$\mathcal{L} = \left[\begin{array}{cc|cc} I_r & 0 & 0 & 0 \\ 0 & I_{n-r} & 0 & 0 \\ \hline -A_1 & 0 & I_r & 0 \\ -A_3 & 0 & 0 & I_{n-r} \end{array}\right]\begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}, \qquad \mathcal{R} = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix}\left[\begin{array}{cc|cc} I_r & 0 & 0 & 0 \\ 0 & I_{n-r} & 0 & 0 \\ \hline 0 & -A_2 & I_r & 0 \\ 0 & 0 & 0 & I_{n-r} \end{array}\right].$$

The spectrum of the matrix $\bar{A}$ in system (12) determines the finite spectrum of the matrix pencil $A - \lambda E$ because

$$\begin{bmatrix} I_\rho & -A_2 A_4^{-1} \\ 0 & I_{n-r} \end{bmatrix} LF(\lambda)R \begin{bmatrix} I_\rho & 0 \\ -A_4^{-1}A_3 & I_{n-r} \end{bmatrix} = \begin{bmatrix} \bar{A} - \lambda I_r & 0 \\ 0 & A_4 \end{bmatrix}.$$

The performance measure $\bar{J}$ of this system is independent of ξ_2 and coincides with J. This fact follows from the relations $L^{-1} = \begin{bmatrix} E_l & E_l^{\perp} \end{bmatrix}$ and

$$x_0^\top X_0 x_0 = \begin{bmatrix} \xi_{10}^\top & \xi_{20}^\top \end{bmatrix} R^\top E^\top L^\top L^{-1\top} H L^{-1} L E R \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = \xi_{10}^\top \bar{H} \xi_{10}.$$

Thus, assertions 1 and 3 are equivalent. For the equivalence of assertions 3 and 4, see [28, Lemma 5.1].

The lemma is proved.

Remark 1. In Lemma 1, under condition (9), we can use expression (13) with $U = 0$, i.e.,

$$\begin{bmatrix} A^\top X + X^\top A & X^\top B & C^\top \\ B^\top X & -\gamma^2 P & D^\top \\ C & D & -Q^{-1} \end{bmatrix} < 0. \tag{17}$$

By the Schur lemma [1], the matrix inequalities (7) and (17) are equivalent.

Remark 2. Assertions 2 and 4 of Lemma 1, without using the relations

$$E_l^\top S E_l < \gamma^2 \bar{H} \quad \text{and} \quad X < \gamma^2 \bar{H}$$

in (14) and (15), respectively, specify necessary and sufficient conditions under which system (1) is admissible and the estimate $J_0 < \gamma$ holds.

3. Estimation of the Performance Measures and Localization of the Spectrum of Admissible Systems

We study conditions under which the descriptor system (1) is admissible, its performance measure (6) is smaller than a given value, and the finite spectrum Σ is located in a specified domain of the half plane $\mathbb{C}^-$. Since $\Sigma = \sigma(\bar{A})$, by using condition (11) instead of (1), we consider the dynamical system (12), which must be internally stable.

Definition 2. *A pair of matrices $\{E, A\}$ is called Λ-stable if a finite spectrum of the matrix pencil $F(\lambda) = A - \lambda E$ belongs to Λ. System (1) is called Λ-admissible if the pair of matrices $\{E, A\}$ is regular, Λ-stable, and impulse-free.*

We first consider an algebraic domain described by the Hermitian function f:

$$\Lambda_f^+ = \left\{ \lambda \in \mathbb{C} : \, f(\lambda, \overline{\lambda}) \triangleq \sum_{i,j=0}^{q} \gamma_{ij} \lambda^i \overline{\lambda}^j > 0 \right\} \neq \varnothing, \tag{18}$$

where γ_{ij} are elements of the given matrix $\Gamma = \Gamma^\top$. It is known that the spectrum of the matrix $\bar{A}$ belongs to Λ_f^+ if the generalized matrix Lyapunov inequality

$$\mathbf{L}_f X \triangleq \sum_{i,j=0}^{q} \gamma_{ij} \bar{A}^i X \bar{A}^{j\top} > 0 \tag{19}$$

possesses the solution $X = X^\top > 0$. The converse statement is true if the matrix Γ is Lorentzian, i.e., $\mathrm{i}_+(\Gamma) = 1$, where $\mathrm{i}_+(\Gamma)$ is the number of positive eigenvalues of Γ with regard for multiplicities. The maximum class of algebraic and transcendental domains of type (18) for which the criterion of localization of the spectrum of a matrix holds in the form of the generalized Lyapunov theorem was described in [34] (see also Theorem 1.10 in [28]).

Hence, according to Lemma 1, if for a certain matrix $X = X^\top > 0$, the LMI system (15), (16), and (19) is true, then system (12) is internally stable with the performance measure $\bar{J} < \gamma$ and its spectrum belongs to the domain $\Lambda = \Lambda_f^+ \cap \mathbb{C}^-$. In this case, system (1) is Λ-admissible and $J < \gamma$. In particular, if $\mathrm{i}_+(\Gamma) = 1$, then system (1) is Λ-admissible for the domain Λ and has the performance measure $J < \gamma$ if and only if condition (11) is satisfied and the LMI system (15) and (16) has a solution of the form

$$X = \mathbf{L}_f \widetilde{X},$$

where $\widetilde{X} = \widetilde{X}^\top > 0$.

It is worth noting that, in the general case, the application of the presented methods for localization of the spectrum in control problems may lead to serious computational difficulties caused by the nonlinear dependence of an operator of the type $\mathbf{L}_f$ on the matrices of the system and the required controllers.

In the half plane $\mathbb{C}^-$, we consider a domain Λ, which can be represented in the form

$$\Lambda = \big\{ \lambda \in \mathbb{C}^- : \, \Delta(\lambda, \overline{\lambda}) = \Theta + \lambda \Gamma + \overline{\lambda} \Gamma^\top < 0 \big\}, \tag{20}$$

where $\Theta = \Theta^\top \in \mathbb{R}^{\delta \times \delta}$ and $\Gamma \in \mathbb{R}^{\delta \times \delta}$ are given matrices. The sets of complex plane of the form (20) are called LMI domains [4]. The class of these domains includes, in particular, half planes, strips, conic sections, disks, ellipses, and all possible intersections of domains of these types. Note that the class of domains

$$\Lambda = \left\{\lambda \in \mathbb{C}^- :\ R_0 + \lambda R_1 + \overline{\lambda} R_1^\top + \lambda\overline{\lambda} R_2 R_2^\top < 0\right\}$$

used in [35–37] can be represented in the form (20) with the following matrices:

$$\Theta = \begin{bmatrix} R_0 & 0 \\ 0 & -I \end{bmatrix} \qquad \text{and} \qquad \Gamma = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}.$$

By applying Theorem 2.2 in [4] to the matrix $\bar{A}$ of the transformed system (12) whose spectrum coincides with Σ, we obtain the following statement:

Lemma 2. *System (1) is Λ-admissible for domain (20) if and only if condition (11) is satisfied and the LMI system*

$$\Theta \otimes X + \Gamma \otimes (X\bar{A}) + \Gamma^\top \otimes (X\bar{A})^\top < 0, \quad X = X^\top > 0, \tag{21}$$

where $\otimes$ is the operation of Kronecker product, is consistent for $X \in \mathbb{R}^{r\times r}$.

Corollary 1. *If condition (11) is satisfied and the LMI system (15), (16), and (21) is consistent for X, then system (1) is Λ-admissible for domain (20) and has the performance measure $J < \gamma$.*

We now formulate conditions directly in terms of the pair of matrices $\{E, A\}$. Under these conditions, system (1) is Λ-admissible for domain (20).

Lemma 3. *System (1) is Λ-admissible for domain (20) if and only if, for some matrices $X \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{(n-r)\times n}$, the following relations are true:*

$$E^\top X = X^\top E \geq 0, \tag{22}$$

$$\mathbf{M}(X, V) < 0, \tag{23}$$

where

$$\mathbf{M}(X, V) = \Theta \otimes (E^\top X) + \Gamma \otimes (X^\top A) + \Gamma^\top \otimes (A^\top X) + I_\delta \otimes (V^\top W_{E^\top}^\top A + A^\top W_{E^\top} V).$$

Proof. *Sufficiency.* Relations (22) and (23) guarantee the regularity and Λ-stability of the pair $\{E, A\}$. Indeed, if the pair $\{E, A\}$ is not regular, then, for any $\lambda \in \mathbb{C}$, there exists a vector $v_\lambda \neq 0$ for which $(A - \lambda E)\, v_\lambda = 0$. In particular, v_λ can be the eigenvector corresponding to the eigenvalue $\lambda \in \Sigma$. We multiply inequality (23) from the left and right, respectively, by $I_\delta \otimes v_\lambda^*$ and $I_\delta \otimes v_\lambda$. By using the properties of the Kronecker product [38], we obtain

$$(I_\delta \otimes v_\lambda^*)\, \mathbf{M}(X, V)\, (I_\delta \otimes v_\lambda) = v_\lambda^* E^\top X v_\lambda\, \Delta(\lambda, \overline{\lambda}) < 0, \qquad v_\lambda^* E^\top X v_\lambda > 0, \quad \Delta(\lambda, \overline{\lambda}) < 0.$$

Thus, $\lambda \in \Lambda$ for any $\lambda \in \mathbb{C}$, i.e., $\Lambda = \mathbb{C}$, which contradicts the assumption that $\Lambda \subset \mathbb{C}^-$.

Any matrix X in relation (22) has the following structure:

$$X = L^\top \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix} R^{-1}, \quad X_1 = X_1^\top \geq 0, \tag{24}$$

where L and R are the transformation matrices from (10). Moreover, $X_1 > 0$ if $\operatorname{rank}(E^\top X) = r$. Since

$$W_{E^\top}^\top = \begin{bmatrix} 0 & I_{n-r} \end{bmatrix} L,$$

we have

$$\mathbf{M}(X, V) = (I_\delta \otimes R^{-1\top})\, M\, (I_\delta \otimes R^{-1}),$$

where

$$M = \Theta \otimes \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} + \Gamma \otimes \begin{bmatrix} U_1 & U_3 \\ U_2 & U_4 \end{bmatrix} + \Gamma^\top \otimes \begin{bmatrix} U_1^\top & U_2^\top \\ U_3^\top & U_4^\top \end{bmatrix} + I_\delta \otimes \begin{bmatrix} W_1 & W_2^\top \\ W_2 & W_3 \end{bmatrix} < 0, \tag{25}$$

$$U_1 = X_1 A_1 + X_2^\top A_3, \qquad U_2 = X_3^\top A_3, \qquad U_3 = X_1 A_2 + X_2^\top A_4, \qquad U_4 = X_3^\top A_4,$$

$$W_1 = A_3^\top V_1 + V_1^\top A_3, \qquad W_2 = A_4^\top V_1 + V_2^\top A_3, \qquad W_3 = A_4^\top V_2 + V_2^\top A_4, \qquad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix} R^{-1}.$$

This implies that $\det A_4 \neq 0$, i.e., system (1) is impulse-free. Otherwise, the equality $\hat{c}^\top M \hat{c} = 0$, where $\hat{c} = I_\delta \otimes \begin{bmatrix} 0 & c^\top \end{bmatrix}^\top$, holds for a certain nonzero vector $c \in \operatorname{Ker} A_4$. Thus, under conditions (22) and (23), system (1) is Λ-admissible for domain (20).

Necessity. Assume that system (1) is Λ-admissible for domain (20). It is necessary to show that the system of relations (22) and (23) is consistent for X and V. This means that relations (24) and (25) must be true for some matrices $X_1,\ X_2,\ X_3,\ V_1$, and V_2 .

Since the pair of matrices $\{E, A\}$ is regular, it is possible to assume that the matrices L and R in relations (10), (24), and (25) transform this pair into the canonical Weierstrass form and, moreover, $A_2 = 0,\ A_3 = 0$, and $A_4 = I_{n-r}$. We now set

$$X_2 = 0, \qquad X_3 = 0, \qquad V_1 = 0, \qquad \text{and} \qquad V_2 = \alpha I_{n-r}.$$

Then

$$M = \Theta \otimes \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} + \Gamma \otimes \begin{bmatrix} U_1 & 0 \\ 0 & 0 \end{bmatrix} + \Gamma^\top \otimes \begin{bmatrix} U_1^\top & 0 \\ 0 & 0 \end{bmatrix} + I_\delta \otimes \begin{bmatrix} 0 & 0 \\ 0 & W_3 \end{bmatrix}$$

$$= \begin{bmatrix} \theta_{ij} X_1 + \gamma_{ij} X_1 A_1 + \gamma_{ji} A_1^\top X_1 & 0 \\ 0 & 2\delta_{ij} \alpha I_{n-r} \end{bmatrix}_{1 \leq i,j \leq \delta}$$

$$= \mathcal{P} \begin{bmatrix} M_1 & 0 \\ 0 & 2\alpha I_{(n-r)\delta} \end{bmatrix} \mathcal{P}^\top, \qquad \mathcal{P} = \mathcal{P}_{2,3} \mathcal{P}_{3,5} \ldots \mathcal{P}_{\delta, 2\delta-1},$$

where

$$M_1 = \Theta \otimes X_1 + \Gamma \otimes (X_1 A_1) + \Gamma^\top \otimes (X_1 A_1)^\top, \quad \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

θ_{ij} and γ_{ij} are elements of the corresponding matrices Θ and Γ, $\mathcal{P}$ is a nondegenerate matrix, and $\mathcal{P}_{j,2j-1}$ $\left(\mathcal{P}_{j,2j-1}^\top\right)$ is the matrix with rearrangement of the block columns (rows) of the matrix M with the numbers j and $2j-1$, $j = \overline{2, \delta}$.

Since the spectrum of the matrix A_1 coincides with the finite spectrum of the matrix pencil $A - \lambda E$, by Lemma 2, there exists a matrix $X_1 = X_1^\top > 0$ for which $M_1 < 0$. Moreover, $M < 0$ for $\alpha < 0$. Hence, inequalities (22) and (23) are true for the indicated matrices X and V.

The lemma is proved.

Remark 3. In view of the representation of the required matrix X in the form (5), the application of Lemma 3 is reduced to solving the system of strict LMIs for $S = S^\top$, G, and V. In particular, system (1) is Λ-admissible for domain (20) if relations (5) and

$$\Theta \otimes (E^\top X) + \Gamma \otimes (X^\top A) + \Gamma^\top \otimes (A^\top X) < 0 \tag{26}$$

are consistent with respect to $S = S^\top$ and G. It is possible to show that, under the additional conditions

$$\Gamma + \Gamma^\top > 0 \quad \text{or} \quad \Gamma + \Gamma^\top < 0, \tag{27}$$

relations (5) and (26) are not only sufficient but also necessary for system (1) to be Λ-admissible for domain (20).

By using Lemmas 1 and 2, we can formulate the following statement:

Lemma 4. *System (1) is Λ-admissible for domain (20) and has the performance measure $J_0 < \gamma$ if relations (5), (13), and (23) are consistent for $S = S^\top$, G, U, and V. System (1) is Λ-admissible for domain (20) and has the performance measure $J < \gamma$ if relations (13), (14), and (23) are consistent for $S = S^\top$, G, U, and V.*

4. Synthesis of Static Controllers for Descriptor Systems

Consider the descriptor control system

$$\begin{aligned} E\dot{x} &= Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{28}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^s$, $z \in \mathbb{R}^k$, and $y \in \mathbb{R}^l$ are the vectors of state, control, external disturbances, and controlled and observed outputs, respectively. All matrix coefficients in (28) are constant. Moreover, the pair (E, A) is regular and $\operatorname{rank} E = r \leq n$. The role of components of the indefinite disturbances can be played either by external disturbances acting upon the system and/or the errors of the observed output.

We are interested in stabilizing control laws guaranteeing that a closed system has weighted performance criteria $J_0 < \gamma$ or $J < \gamma$ of the form (6) and is Λ-admissible with respect to a given LMI domain (20). Static and dynamic controllers minimizing the performance measure J are called *J-optimal*. The J_0-optimal control in the case of unit weight matrices P and Q is *H_∞-optimal*.

Assume that the control in system (28) is carried out by a static output-feedback controller

$$u = Ky, \quad K \in \mathbb{R}^{m\times l}. \tag{29}$$

Then, under the condition $\det(I_m - KD_{22}) \neq 0$, the closed system takes the form

$$E\dot{x} = A_* x + B_* w, \quad z = C_* x + D_* w, \tag{30}$$

where

$$A_* = A + B_2 K_* C_2, \qquad B_* = B_1 + B_2 K_* D_{21}, \qquad C_* = C_1 + D_{12} K_* C_2,$$

$$D_* = D_{11} + D_{12} K_* D_{21}, \qquad \text{and} \qquad K_* = (I_m - KD_{22})^{-1} K.$$

Consider a special case

$$C_2 = I_n, \qquad D_{21} = 0, \qquad \text{and} \qquad D_{22} = 0. \tag{31}$$

Under conditions (31), $y = x$, $K_* = K$, and (29) is the static state controller of the system.

Lemma 5 [39]**.** *For nondegenerate matrices X and Y satisfying the equality*

$$XY = \gamma^2 I_n, \tag{32}$$

the following assertions are equivalent:

(i) the system of relations (8) holds;

(ii) the matrix X admits representation (14);

(iii) the matrix Y admits the following representation:

$$Y = TE^\top + W_E F, \qquad E_r^\top T E_r > (E_l^\top H E_l)^{-1}, \tag{33}$$

where $T = T^\top \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{(n-r)\times n}$.

Theorem 1. *Suppose that conditions (31) are satisfied and that relations (33) and*

$$\begin{bmatrix} \gamma^2(AY + Y^\top A^\top + B_2 Z + Z^\top B_2^\top) & \gamma^2 B_1 & Y^\top C_1^\top + Z^\top D_{12}^\top \\ \gamma^2 B_1^\top & -\gamma^2 P & D_{11}^\top \\ C_1 Y + D_{12} Z & D_{11} & -Q^{-1} \end{bmatrix} < 0, \tag{34}$$

$$\Theta \otimes (Y^\top E^\top) + \Gamma \otimes (AY + B_2 Z) + \Gamma^\top \otimes (AY + B_2 Z)^\top < 0, \tag{35}$$

where Y is a nondegenerate matrix, are true for certain matrices $T = T^\top$, F, and Z. Then the closed system (30) *corresponding to the static state controller*

$$u = Kx, \quad K = ZY^{-1}, \tag{36}$$

is Λ*-admissible and has the performance measure* $J < \gamma$*.*

Proof. Assume that the nondegenerate matrices X and Y are related by equality (32). Hence, by Lemma 5, their corresponding representations (14) and (33) are equivalent. Multiplying the first block row and the first block column of expression (17) formed for the closed system with controller (36) by $Y^\top$ and Y, respectively, we arrive at the matrix inequality (34). Similarly, the matrix inequality (35) is obtained as a result of multiplication of expression (26) for this system by $I_\delta \otimes Y^\top$ and $I_\delta \otimes Y$ from the left and from the right, respectively. Thus, the assertion of Theorem 1 follows from Lemmas 4 and 5 and relations (34)–(36) (see also Remark 3).

The theorem is proved.

We now apply the equivalent transformation (10) to system (28). Eliminating the variable

$$\xi_2 = -A_4^{-1}\big(A_3\xi_1 + B_{12}w + B_{22}u\big),$$

under condition (11), we obtain the following ordinary system:

$$\begin{aligned} \dot{\xi}_1 &= \bar{A}\xi_1 + \bar{B}_1 w + \bar{B}_2 u, \quad \xi_1(0) = \xi_{10}, \\ z &= \bar{C}_1\xi_1 + \bar{D}_{11}w + \bar{D}_{12}u, \\ y &= \bar{C}_2\xi_1 + \bar{D}_{21}w + \bar{D}_{22}u, \end{aligned} \tag{37}$$

where

$$\bar{A} = A_1 - A_2A_4^{-1}A_3, \qquad \bar{B}_1 = B_{11} - A_2A_4^{-1}B_{12}, \qquad \bar{B}_2 = B_{21} - A_2A_4^{-1}B_{22},$$

$$\bar{C}_1 = C_{11} - C_{12}A_4^{-1}A_3, \qquad \bar{D}_{11} = D_{11} - C_{12}A_4^{-1}B_{12}, \qquad \bar{D}_{12} = D_{12} - C_{12}A_4^{-1}B_{22},$$

$$\bar{C}_2 = C_{21} - C_{22}A_4^{-1}A_3, \qquad \bar{D}_{21} = D_{21} - C_{22}A_4^{-1}B_{12}, \qquad \bar{D}_{22} = D_{22} - C_{22}A_4^{-1}B_{22},$$

$$LB_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \qquad LB_2 = \begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix}, \qquad C_1R = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \qquad C_2R = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}.$$

The performance measure $\bar{J}$ of this system coincides with J because $x_0^\top X_0 x_0 = \xi_{10}^\top \bar{H}\xi_{10}$, where $\bar{H} = E_l^\top H E_l$ (see Sec. 2).

For system (37), we construct the static controller (29) and apply it to the original system (28). Under the condition $\det(I_m - K\bar{D}_{22}) \neq 0$, the closed system takes the form

$$\dot{\xi}_1 = A_*\xi_1 + B_*w, \qquad z = C_*\xi_1 + D_*w, \tag{38}$$

where $A_* = \bar{A} + \bar{B}_2K_0\bar{C}_2$, $B_* = \bar{B}_1 + \bar{B}_2K_0\bar{D}_{21}$, $C_* = \bar{C}_1 + \bar{D}_{12}K_0\bar{C}_2$, $D_* = \bar{D}_{11} + \bar{D}_{12}K_0\bar{D}_{21}$, and $K_0 = (I_m - K\bar{D}_{22})^{-1}K$.

Let

$$\operatorname{rank} \bar{C}_2 = r \le l, \qquad \bar{D}_{21} = 0, \qquad \text{and} \qquad \bar{D}_{22} = 0, \tag{39}$$

Conditions (39) are satisfied if, e.g.,

$$\operatorname{rank}(C_2 E_r) = r, \qquad C_2 E_r^{\perp} = 0, \qquad D_{21} = 0, \qquad \text{and} \qquad D_{22} = 0.$$

Based on Lemmas 1 and 2, we get the following statement for system (38):

Theorem 2. *Let conditions (11) and (39) be satisfied and let the LMI system*

$$Y = Y^{\top} > \bar{H}^{-1}, \tag{40}$$

$$\begin{bmatrix} \gamma^2(\bar{A}Y + Y\bar{A}^{\top} + \bar{B}_2 Z + Z^{\top}\bar{B}_2^{\top}) & \gamma^2 \bar{B}_1 & Y\bar{C}_1^{\top} + Z^{\top}\bar{D}_{12}^{\top} \\ \gamma^2 \bar{B}_1^{\top} & -\gamma^2 P & \bar{D}_{11}^{\top} \\ \bar{C}_1 Y + \bar{D}_{12} Z & \bar{D}_{11} & -Q^{-1} \end{bmatrix} < 0, \tag{41}$$

$$\Theta \otimes Y + \Gamma \otimes (\bar{A}Y + \bar{B}_2 Z) + \Gamma^{\top} \otimes (\bar{A}Y + \bar{B}_2 Z)^{\top} < 0 \tag{42}$$

be consistent for $Y \in \mathbb{R}^{r\times r}$ and $Z \in \mathbb{R}^{m\times r}$. Then, for system (28), there exists the static controller (29) for which the closed system is Λ-admissible and its performance measure is such that $J < \gamma$. The matrix of this controller can be defined as the solution of the linear equation $K\bar{C}_2 Y = Z$, i.e.,

$$K = \begin{cases} Z(\bar{C}_2 Y)^{-1}, & r = l, \\ ZY^{-1}\big(\bar{C}_2^{+} + C\bar{C}_2^{\perp\top}\big), & r < l, \end{cases}$$

where $C \in \mathbb{R}^{r\times(l-r)}$ is an arbitrary matrix.

Remark 4. If the pair of matrices $\{E, A\}$ in system (28) is impulsive but there exists a matrix $K_1 \in \mathbb{R}^{m\times l}$ such that

$$\det(I_m - K_1 D_{22}) \ne 0 \qquad \text{and} \qquad \det\big[E_1^{\perp\top}(A + B_2 K_{10} C_2)E_2^{\perp}\big] \ne 0, \tag{43}$$

where $K_{10} = K_{11}K_1$ and $K_{11} = (I_m - K_1 D_{22})^{-1}$, then we can seek control for system (28) in the form $u = K_1 y + v$. Here, K_1 is the matrix of additional static output controller and v is a new control constructed according to Theorem 2 for the system

$$E\dot{x} = \widetilde{A}x + \widetilde{B}_1 w + \widetilde{B}_2 v, \qquad z = \widetilde{C}_1 x + \widetilde{D}_{11} w + \widetilde{D}_{12} v, \qquad y = \widetilde{C}_2 x + \widetilde{D}_{21} w + \widetilde{D}_{22} v, \tag{44}$$

where

$$\widetilde{A} = A + B_2 K_{10} C_2, \qquad \widetilde{B}_1 = B_1 + B_2 K_{10} D_{21}, \qquad \widetilde{B}_2 = B_2 K_{11},$$

$$\widetilde{C}_1 = C_1 + D_{12}K_{10}C_2, \qquad \widetilde{D}_{11} = D_{11} + D_{12}K_{10}D_{21}, \qquad \widetilde{D}_{12} = D_{12}K_{11},$$

$$\widetilde{C}_2 = C_2 + D_{22}K_{10}C_2, \qquad \widetilde{D}_{21} = D_{21} + D_{22}K_{10}D_{21}, \qquad \widetilde{D}_{22} = D_{22}K_{11}.$$

Under condition (43), system (44) is impulse-free and, in addition, impulsively controlled and impulsively observable (see, e.g., [40]).

5. Example

Consider a linearized model of hydraulic system with three tanks connected in series [41]. This model is described by the descriptor control system (28), where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} -0.0104 & 0 & 0 \\ 0.0033 & -0.0042 & 0 \\ 1 & 2.3328 & 1 \end{bmatrix}, \qquad \text{and} \qquad B_1 = B_2 = \begin{bmatrix} 0.019 \\ 0 \\ 0 \end{bmatrix}.$$

The components of the state vector $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^\top$ determine the levels of liquid in the corresponding tanks and the role of control u characterized by the presence of an indefinite disturbance w and regulating the levels of liquid in the first two tanks is played by the debit (flow) of liquid in the pump running from the third tank into the first one.

Let $z = x_3 + u$ be the controlled output of the system determined from (28) for $C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $D_{11} = 0_{1\times 2}$, and $D_{12} = 1$. We choose the following weighting coefficients of the performance measures (6): $P = 1,\ Q = 1,$ and $H = 10I_3$; moreover, the domain of desirable location of eigenvalues of the system is selected in the form

$$\Lambda = \Lambda_1 \cap \Lambda_2, \qquad \Lambda_1 = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda < -\alpha\}, \qquad \Lambda_2 = \{\lambda \in \mathbb{C}\colon \operatorname{Re}\lambda \tan\varphi + |\operatorname{Im}\lambda| < 0\},$$

where $\alpha > 0,\ 0 < \varphi < \pi/2,$ and Λ_1 and Λ_2 are, respectively, the half plane and the conic sector located in $\mathbb{C}^-$. The embedding $\Sigma \subset \Lambda,$ where Σ is the finite spectrum of the system, enables one to estimate its characteristics, namely, the stability level μ and oscillation ω:

$$\mu = \min_{\lambda\in\Sigma} |\operatorname{Re}\lambda| > \alpha \qquad \text{and} \qquad \omega = \max_{\lambda\in\Sigma} \left|\frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda}\right| < \tan\,\varphi.$$

The domain Λ can be represented in the form (20) with the following matrices:

$$\Theta = \begin{bmatrix} 2\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \Gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\varphi & \cos\varphi \\ 0 & -\cos\varphi & \sin\varphi \end{bmatrix}.$$

We can easily show that the matrix Γ satisfies the additional condition (27). The system without control has the following characteristics:

$$\Sigma = \{-0.0042; -0.0104\}, \qquad J_0 = 5.17551, \qquad J = 9.28537, \qquad \mu = 0.0042, \qquad \omega = 0.$$

By using the LMI Toolbox tool of the MATLAB software system, on the basis of Theorem 1, we obtain the following matrices:

$$Y = \begin{bmatrix} 5.07914 & -2.29281 & 0 \\ -2.29281 & 1.15603 & 0 \\ 0.13457 & -0.33339 & -2.54847 \end{bmatrix}, \qquad Z = \begin{bmatrix} -0.15064 & -0.03166 & -2.90130 \end{bmatrix},$$

satisfying relations (33)–(35) for $\gamma = 7.4,\ \alpha = 0.01,$ and $\varphi = \pi/5,$ and also the static state-feedback controller

$$u = Kx, \qquad K = \begin{bmatrix} 0.72621 & 1.74127 & 1.13845 \end{bmatrix},$$

for which the closed system is Λ-admissible and has the following characteristics:

$$\Sigma = \{-0.01122 \pm 0.00285i\}, \quad J_0 = 2.36195, \quad J = 7.34228, \quad \mu = 0.01122, \quad \text{and} \quad \omega = 0.25393.$$

Let $y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top$ be the observed output of system (28), let

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad D_{21} = 0_{2\times 1}, \qquad D_{22} = 0_{2\times 1},$$

and let conditions (39) be satisfied. By using Theorem 2, we obtain the following matrices:

$$Y = \begin{bmatrix} 1473.78795 & -668.14755 \\ -668.14755 & 303.03001 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} -90.66637 & 40.87726 \end{bmatrix},$$

satisfying the LMI system (40)–(42) for given values of the parameters and the static output-feedback controller

$$u = Ky, \qquad K = \begin{bmatrix} -0.89913 & -1.84760 \end{bmatrix},$$

for which the closed system is Λ-admissible and has the following characteristics:

$$\Sigma = \{-0.01140; -0.02028\}, \quad J_0 = 1.78861, \quad J = 7.29300, \quad \mu = 0.01140, \quad \text{and} \quad \omega = 0.$$

6. Conclusions

In the present paper, we establish new necessary and sufficient conditions under which the descriptor system with external disturbances is regular, impulse-free, and Λ-stable with weighted performance measures $J_0 < \gamma$ or $J < \gamma$ for any LMI domain Λ of localization of the spectrum. We develop constructive methods aimed at the estimation and attainment of the desirable damping level of external and initial disturbances in descriptor control systems with the help of static state- and output-feedback controllers under the conditions of localization of the spectrum in a given LMI domain. The practical implementation of the developed methods is carried out on the basis of the solution of LMI systems constructed in terms of matrix coefficients either of the original system

(Lemmas 3 and 4 and Theorem 1) or of the additional system of lower order obtained as a result of special equivalent transformation (Lemma 2, Corollary 1, Theorem 2).

As the main specific features and advantages of the accumulated results as compared to the known data (see, e.g., [4, 8, 10]), we can mention the reduction of the problems of generalized H_∞ optimization and localization of the spectrum of a descriptor system with degenerate matrix E at the derivatives to the analogous problems for the ordinary systems of order $r = \operatorname{rank} E$ and the application of weighted performance measures of type J.

In the course of future investigations, we plan to obtain similar results for the other classes of descriptor systems with external disturbances (discrete, pseudolinear, etc.). For these systems, it is also advisable to develop the methods of generalized H_∞ optimization with the help of dynamic controllers.

Conflict of Interest. The author declares that he has no potential conflict of interest in connection with the study in the present paper.

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