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WEIGHTED DAMPING OF EXTERNAL AND INITIAL DISTURBANCES IN DESCRIPTOR CONTROL SYSTEMS

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We study the problem of generalized H_∞-control for a class of linear descriptor systems and suggest a criterion and sufficient conditions for the existence of the laws of control guaranteeing that the closed-loop system is regular, stable, and impulse-free and satisfies the desired estimate for the weighted level of damping of the external and initial disturbances. The main computational procedures for the synthesis of controllers are reduced to the solution of linear and quadratic matrix inequalities without rank constraints. An example of robust stabilization of a hydraulic system with three vessels is presented.

1. Introduction

Modern directions of investigations in the control theory are formed by the methods of robust stabilization and H_2/H_∞-optimization guaranteeing the robust stability of equilibrium states and minimizing the negative influence of external (exogenous) disturbances on the dynamics of controlled objects (see, e.g., [1–6]). To estimate and weaken the influence of bounded disturbances in control systems, it is possible to apply the methods aimed at the minimization of the characteristics used to describe the sizes of invariant sets of the vectors of state or output [6, 7]. A typical performance criterion in the problems of H_∞-optimization of continuous and discrete systems with zero initial state is the level of damping of external disturbances corresponding to the maximal value of the ratio of L_2-norms of the vectors of the controlled output of the object and disturbances. Thus, for a class of linear systems

$$E\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad x(0) = x_0, \tag{1.1}$$

this characteristic coincides with the H_∞-norm of the matrix transfer function

$$\|H\|_\infty = \sup_{\omega \in \mathbb{R}} \sqrt{\lambda_{\max}(H^\top(-i\omega)H(i\omega))}, \qquad H(\lambda) = C(\lambda E - A)^{-1}B + D,$$

where $x \in \mathbb{R}^n$, $z \in \mathbb{R}^k$, and $w \in \mathbb{R}^s$ are, respectively the vectors of state, controlled output, and input of system, E, A, B, C, and D are constant matrices of the corresponding orders, and $\lambda_{\max}(\cdot)$ is the maximal eigenvalue of the matrix.

In practice, it is reasonable to use weighted performance criteria for control systems of the form [8]

$$J = \sup_{(w,x_0) \in \mathcal{W}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \tag{1.2}$$

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where

$$\|z\|_Q^2 = \int_0^\infty z^\top Qz\,dt, \qquad \|w\|_P^2 = \int_0^\infty w^\top Pw\,dt,$$

$\mathcal{W}$ is the set of admissible pairs (w, x_0) of the system for which the inequality

$$0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty$$

is true and $P = P^\top > 0$, $Q = Q^\top > 0$, and $X_0 = X_0^\top \geq 0$ are given weight matrices (see also [9, 10]). The quantity J characterizes the weighed level of damping of the external disturbances and also of the initial disturbances caused by the nonzero initial vector. By using weight coefficients in these performance criteria, we can establish priorities between the components of controlled output and the unknown disturbances in the control system. Moreover, both the external disturbances acting upon the system and the errors of measured output can be components of the unknown disturbances.

System (1.1) is a differential-algebraic (descriptor) system if the matrix of coefficients of the derivatives E is degenerate. Descriptor systems are encountered in the design and investigation of the dynamics of controlled objects of mechanics, electrical engineering, economics, etc. (see, e.g., [11–18]). In constructing the equations of motion of these objects in terms of variables used to describe the actual physical processes, it is necessary to take into account not only differential but also algebraic relations and restrictions in the phase space. Thus, under general assumptions, the mechanical systems with constraints are described by the equations [13, 16]

$$A_2\ddot{q}(t) + A_1\dot{q}(t) + A_0 q(t) = Uu(t) + V\mu(t), \qquad G_1\dot{q}(t) + G_0 q(t) = 0, \tag{1.3}$$

where $q(t) \in \mathbb{R}^\nu$ is a position vector, $u(t) \in \mathbb{R}^s$ is the vector of acting external forces, $\mu(t) \in \mathbb{R}^r$ is the vector of Lagrange multipliers, A_2 is the matrix of inertial characteristics, A_1 is the matrix of damping (or of gyroscopic characteristics), A_0 is the stiffness matrix, and $V^\top$ is the Jacobian of the equation of constraints. The system of equations (1.3) takes the standard form (1.1) if

$$E = \begin{bmatrix} I_\nu & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I_\nu & 0 \\ -A_0 & -A_1 & V \\ G_0 & G_1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ U \\ 0 \end{bmatrix}, \qquad x = \begin{bmatrix} q \\ \dot{q} \\ \mu \end{bmatrix},$$

$$C = \begin{bmatrix} C_0 & C_1 & 0 \end{bmatrix}, \qquad D = 0, \qquad z = C_0 q + C_1\dot{q}.$$

The available methods of synthesis of the H_∞-control are based on the criteria of validity of the upper bounds for the corresponding performance criteria established in terms of matrix equations and linear matrix inequalities [1, 2, 19]. For the class of linear descriptor systems, similar statements were established in [20–23]. For the available methods of H_∞-optimization of these systems, see, e.g., [16, 20, 22, 24, 25].

In the present paper, we continue the investigations originated in [26, 27] and devoted to the problems of synthesis of generalized H_∞-control for linear descriptor systems. We propose new necessary and sufficient conditions for the existence of static and dynamic controllers guaranteeing the validity of the required estimate for the weighted level of influence of bounded disturbances on the quality of transient processes in descriptor systems with controlled and observed outputs. Practical applications of these conditions are reduced to solving linear and quadratic matrix inequalities for parametrized matrices without additional rank restrictions. As a specific feature of the obtained results as compared with the known data, we can mention the use of weighted performance criteria, which open new possibilities in the attainment of the required characteristics of descriptor control systems.

We use the following notation: I_n is the identity matrix of order n; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^\top > 0$ (≥ 0) is a positive-definite (nonnegative-definite) symmetric matrix X; $\sigma(A)$ is the spectrum of the matrix A; A^{-1} (A^+) is the inverse (pseudoinverse) matrix; $\operatorname{Ker} A$ is the kernel of the matrix A; W_A is the matrix whose columns form a basis of the kernel $\operatorname{Ker} A$; $\operatorname{Co}\{A_1, \dots, A_\nu\}$ is a convex polyhedron (polytope) with vertices $A_1, \dots, A_\nu$ in the space of matrices; $\|x\|$ is the Euclidean norm of a vector x, and $\|w\|_P$ is the weighted L_2-norm of a vector function $w(t)$.

2. Definitions and Auxiliary Statements

We now consider the descriptor system (1.1), where $\operatorname{rank} E = \rho < n$, and the performance criterion (1.2). System (1.1) is called *admissible* if the pair of matrices (AE) is *regular*, *stable*, and *impulse-free* [15], i.e., respectively, $\det F(\lambda) \not\equiv 0,\ \lambda \in \mathbb{C},\ \operatorname{Re}\lambda_i < 0,\ i = \overline{1,\alpha}$, and $\alpha = \rho$, where $\Sigma = \{\lambda_1, \dots, \lambda_\alpha\}$ is the finite spectrum of the matrix pencil $F(\lambda) = A - \lambda E$. The pair of matrices (E, A) is regular if and only if there exist nondegenerate matrices L and R reducing it to the canonical Weierstrass form [28]

$$LAR = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-\alpha} \end{bmatrix}, \qquad LER = \begin{bmatrix} I_\alpha & 0 \\ 0 & N \end{bmatrix}, \tag{2.1}$$

where N is a nilpotent matrix of index ν and the spectrum of the matrix A_1 coincides with Σ. In particular, the equality $N = 0$ is equivalent to the rank condition [13]

$$\operatorname{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \rho$$

and means that the pair of matrices (A, E) is impulse-free.

In view of transformation (2.1), the admissible descriptor system (1.1) can be represented in the form

$$\dot{x}_1 = A_1 x_1 + B_1 w, \qquad z = C_1 x_1 + D_1 w, \qquad x_1(0) = x_{01}, \tag{2.2}$$

where $x_1 \in \mathbb{R}^\alpha$, A_1 is the Hurwitz matrix, $D_1 = D - C_2 B_2$,

$$x = R\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad x_0 = R\begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}, \qquad LB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad \text{and} \qquad CR = \begin{bmatrix} C_1, C_2 \end{bmatrix}.$$

Lemma 2.1 [22]. *System (1.1) is admissible if and only if the system of relations*

$$A^\top X + X^\top A < 0, \qquad E^\top X = X^\top E \geq 0$$

is consistent for X.

Let J be the performance criterion (1.2) of system (1.1) with the weight matrix $X_0 = E^\top H E$, where $H = H^\top > 0$ is a given matrix. For the admissible system,

$$x_0^\top X_0 x_0 = x_{01}^\top X_{01} x_{01},$$

where

$$X_{01} = L_1^\top H L_1 > 0, \quad L_1 = L^{-1} \begin{bmatrix} I_\alpha & 0 \end{bmatrix}^\top,$$

and the value J coincides with a similar performance criterion J_1 of type (1.2) obtained for subsystem (2.2).

Lemma 2.2 [23]. *Suppose that the system of equations*

$$\Psi(X) = \begin{bmatrix} A^\top X + X^\top A + C^\top QC & X^\top B + C^\top QD \\ B^\top X + D^\top QC & D^\top QD - \gamma^2 P \end{bmatrix} < 0, \tag{2.3}$$

$$0 \le E^\top X = X^\top E \le \gamma^2 X_0, \qquad \operatorname{rank}\left(E^\top X - \gamma^2 X_0\right) = \rho, \tag{2.4}$$

where $\gamma > 0,$ is consistent for X. Then system (1.1) is admissible and the estimate $J < \gamma$ holds. The converse assertion is true under the condition

$$\operatorname{rank} \begin{bmatrix} E^\top & C^\top QD \end{bmatrix} = \rho. \tag{2.5}$$

Under the conditions of Lemma 2.2, the null state of system (1.1) with structured indeterminacy of the vector

$$w = \frac{1}{\gamma}\Theta z$$

for $\Theta^\top P \Theta \le Q$ is robustly stable with a common Lyapunov function $v(x) = x^\top E^\top X x$. This statement is a corollary of transformation (2.1) and the theorem on robust stabilization of the linear system [8] (Theorem 3.3.1).

Let $E = E_l E_r^\top$ be the skeleton decomposition of the matrix E, where E_l and E_r are the matrices of full rank ρ with respect to columns, and let W_E and $W_{E^\top}$ be matrices whose columns form bases of the kernels of the matrices E and $E^\top$, respectively.

Lemma 2.3. *For the nondegenerate matrices X and Y connected by the equality*

$$XY = \gamma^2 I_n, \tag{2.6}$$

the following assertions are equivalent:

(i) the system of relations (2.4) is true;

(ii) there exist matrices $S = S^\top$ and G for which

$$X = SE + W_{E^\top} G, \quad 0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l; \tag{2.7}$$

(iii) there exist matrices $T = T^\top$ and F for which

$$Y = TE^\top + W_E F, \quad E_r^\top T E_r > \left(E_l^\top H E_l\right)^{-1}. \tag{2.8}$$

Proof. It is clear that each matrix X in (2.7) satisfies relation (2.4) because

$$E^\top X = E^\top SE \geq 0, \quad E^\top X - \gamma^2 X_0 = E_r \Big(E_l^\top SE_l - \gamma^2 E_l^\top HE_l \Big) E_r^\top.$$

Let L and R be nondegenerate matrices such that

$$E = L^{-1} \begin{bmatrix} I_\rho & 0 \\ 0 & 0 \end{bmatrix} R^{-1}, \qquad E_l = L^{-1} \begin{bmatrix} I_\rho \\ 0 \end{bmatrix}, \qquad E_r = R^{-1\top} \begin{bmatrix} I_\rho \\ 0 \end{bmatrix},$$

$$W_E = R \begin{bmatrix} 0 \\ I_{n-\rho} \end{bmatrix}, \qquad W_{E^\top} = L^\top \begin{bmatrix} 0 \\ I_{n-\rho} \end{bmatrix}.$$

Each matrix X in (2.4) has the following structure:

$$X = L^\top \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix} R^{-1}, \quad 0 < X_1 = X_1^\top < \gamma^2 E_l^\top HE_l. \tag{2.9}$$

Relations (2.9) take the form (2.7) if

$$S = L^\top \begin{bmatrix} S_1 & S_2^\top \\ S_2 & S_3 \end{bmatrix} L, \qquad G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} R^{-1},$$

where $S_1 = X_1,\ S_2 = X_2 - G_1,\ G_2 = X_3$, and G_1 and $S_3 = S_3^\top$ are arbitrary matrices of the corresponding dimensions. Thus, assertions (i) and (ii) are equivalent.

We now establish the equivalence of assertions (i) and (iii). We rewrite relations (2.4) in terms of the matrix $Y = \gamma^2 X^{-1}$. To this end, we multiply the expressions from the left and from the right by $X^{-1\top}$ and X^{-1}, respectively:

$$0 \leq EY = Y^\top E^\top \leq Y^\top X_0 Y, \qquad \operatorname{rank} \Big(EY - Y^\top X_0 Y \Big) = \rho. \tag{2.10}$$

Each matrix Y in (2.8) satisfies relation (2.10) because

$$EY = ETE^\top \geq 0, \qquad EY - Y^\top X_0 Y = E_l T_1 \left(T_1^{-1} - E_l^\top HE_l \right) T_1 E_l^\top,$$

where $T_1 = E_r^\top TE_r$. Moreover, $T_1^{-1} < E_l^\top HE_l$ if and only if the matrix inequality in (2.8) holds.

Assume that relations (2.4) are true. By using (2.9) and computing the inverse matrix X^{-1}, we arrive at the relation

$$Y = \gamma^2 R \begin{bmatrix} X_1^{-1} & 0 \\ -X_3^{-1} X_2 X_1^{-1} & X_3^{-1} \end{bmatrix} L^{-1\top}.$$

Further, setting

$$T = R \begin{bmatrix} T_1 & T_2^\top \\ T_2 & T_3 \end{bmatrix} R^\top, \qquad F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} L^{-1\top},$$

where

$$T_1 = E_r^\top T E_r = \gamma^2 X_1^{-1}, \qquad T_2 = -F_1 - \gamma^2 X_3^{-1} X_2 X_1^{-1}, \qquad F_2 = \gamma^2 X_3^{-1},$$

and F_1 and $T_3 = T_3^\top$ are arbitrary matrices of the corresponding sizes, and taking into account the equivalence of the inequalities

$$X_1 < \gamma^2 E_l^\top H E_l \quad \text{and} \quad \gamma^2 X_1^{-1} > (E_l^\top H E_l)^{-1},$$

we arrive at relations (2.8). Thus, assertions (i) and (iii) are equivalent.

Lemma 2.3 is proved.

The vector of disturbance $w(t)$ and the initial vector x_0 are called the *worst* vectors in system (1.1) for the performance criterion J if the supremum is attained on their values in (1.2), i.e.,

$$\|z\|_Q^2 = J^2 \left(\|w\|_P^2 + x_0^\top X_0 x_0 \right).$$

The methods aimed at the determination of these vectors in special cases were proposed in [9, 29].

Lemma 2.4 [26]. *Suppose that system (1.1) is admissible and, for some matrix X, the following relations are true:*

$$A_1^\top X + X^\top A_1 + X^\top R_1 X + Q_1 = 0,$$

$$0 \le E^\top X = X^\top E \le \gamma^2 X_0,$$

where

$$A_1 = A + BR^{-1} D^\top Q C, \qquad R_1 = BR^{-1} B^\top, \qquad Q_1 = C^\top \left(Q + QDR^{-1} D^\top Q \right) C,$$

$$R = \gamma^2 P - D^\top Q D > 0, \qquad \text{and} \qquad \gamma = J.$$

Then the structured vector of external disturbances in the form of a linear inverse (with respect to the state) relationship

$$w = K_0 x, \quad K_0 = R^{-1} \left(B^\top X + D^\top Q C \right),$$

and an arbitrary initial vector $x_0 \in \operatorname{Ker} \left(E^\top X - J^2 X_0 \right)$ are the worst vectors for the performance criterion J of system (1.1).

We now reformulate the consistency criterion for the quadratic matrix inequalities of the form

$$A + B^\top X C + C^\top X^\top B + C^\top X^\top R X C < 0 \tag{2.11}$$

obtained in [26], where $A = A^\top \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{p \times n}$, $C \in \mathbb{R}^{q \times n}$, and $R = R^\top \in \mathbb{R}^{p \times p}$.

Lemma 2.5. *If* $\operatorname{rank} B < n,\ \operatorname{rank} C < n,$ *and* $R \geq 0,$ *then the matrix inequality (2.11) possesses a solution* $X \in \mathbb{R}^{p \times q}$ *if and only if the following conditions are satisfied:*

$$W_C^\top A W_C < 0, \qquad \Delta^\top \left(A - B^\top R^+ B\right) \Delta < 0, \qquad \Delta = \begin{cases} W_B, & r = 0, \\ I_n, & r = p, \\ W_{B_0}, & 1 \leq r < p, \end{cases}$$

where $B_0 = W_R^\top B,\ R^+$ *is a pseudoinverse matrix, and* $r = \operatorname{rank} R.$

3. Main Results

Consider a descriptor control system

$$\begin{aligned} E\dot{x} &= Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,\ w \in \mathbb{R}^s,\ z \in \mathbb{R}^k,$ and $y \in \mathbb{R}^l$ are the vectors of state, control, external disturbances, and controlled and observed outputs, respectively. All matrix coefficients in (3.1) are constant. Moreover, the pair (E, A) is regular and $\operatorname{rank} E = \rho \leq n.$ We are interested in the regularities of control guaranteeing the required estimate $J < \gamma$ for the performance criterion (1.2) of a closed system with respect to the controlled output $z.$ The static and dynamic controllers minimizing the performance criterion J are called J*-optimal*.

3.1. Static Controller. By using the static controller

$$u = Ky, \quad \det(I_m - K D_{22}) \neq 0, \tag{3.2}$$

we arrive at the closed system

$$E\dot{x} = A_* x + B_* w, \quad z = C_* x + D_* w, \tag{3.3}$$

where

$$A_* = A + B_2 K_* C_2, \qquad B_* = B_1 + B_2 K_* D_{21}, \qquad C_* = C_1 + D_{12} K_* C_2,$$

$$D_* = D_{11} + D_{12} K_* D_{21}, \qquad \text{and} \qquad K_* = (I_m - K D_{22})^{-1} K.$$

It is known [30] that there exists a matrix of controller K for which system (3.3) is admissible and has the performance criterion $J < \gamma$ if the system of relations (2.4), (2.6) is consistent for X and Y and

$$W_R^\top \begin{bmatrix} A^\top X + X^\top A + C_1^\top Q C_1 & X^\top B_1 + C_1^\top Q D_{11} \\ B_1^\top X + D_{11}^\top Q C_1 & D_{11}^\top Q D_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{3.4}$$

$$W_L^\top \begin{bmatrix} AY + Y^\top A^\top + B_1 P^{-1} B_1^\top & Y^\top C_1^\top + B_1 P^{-1} D_{11}^\top \\ C_1 Y + D_{11} P^{-1} B_1^\top & D_{11} P^{-1} D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{3.5}$$

where W_R and W_L are the matrices whose columns form bases for the kernels of the matrices $R = \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$ and $L = \begin{bmatrix} B_2^\top & D_{12}^\top \end{bmatrix}$, respectively. Moreover, the rank condition (2.5) for system (3.3) guarantees the validity of the converse assertion, i.e.,

$$\operatorname{rank} \begin{bmatrix} E^\top & C_*^\top Q D_* \end{bmatrix} = \rho, \tag{3.6}$$

and the required matrix K can be constructed in the form $K = K_*(I_l + D_{22}K_*)^{-1}$ by solving the linear matrix inequality for K_*

$$\begin{bmatrix} A_*^T X + X^\top A_* & X^\top B_* & C_*^\top \\ B_*^\top X & -\gamma^2 P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} = \widehat{L}^\top K_* \widehat{R} + \widehat{R}^\top K_*^\top \widehat{L} + \widehat{\Omega} < 0, \tag{3.7}$$

where

$$\widehat{R} = \begin{bmatrix} R & 0_{l\times k} \end{bmatrix}, \quad \widehat{L} = \begin{bmatrix} L & 0_{m\times s} \end{bmatrix} \widetilde{X},$$

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \qquad \widehat{\Omega} = \begin{bmatrix} A^\top X + X^\top A & X^\top B_1 & C_1^\top \\ B_1^\top X & -\gamma^2 P & D_{11}^\top \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix}.$$

By the Schur lemma, the matrix inequalities (2.3) and (3.7) for system (3.3) are equivalent. Condition (3.6) is independent of K and is satisfied, e.g., in the following cases:

$$D_{11} = 0, \quad D_{21} = 0, \tag{3.8}$$

$$D_{12} = 0, \quad \operatorname{rank} \begin{bmatrix} E^\top & C_1^\top Q D_{11} \end{bmatrix} = \rho. \tag{3.9}$$

The difficulties encountered in the application of this criterion may appear due to the presence of matrix inequalities in system (2.4), (2.6), (3.4), and (3.5) that should be solved. By using Lemma 2.3, we can remove these difficulties in the application of the static controller of state.

Theorem 3.1. *Suppose that the conditions*

$$C_2 = I_n, \quad D_{11}^\top Q D_{11} < \gamma^2 P, \quad D_{21} = 0, \quad D_{22} = 0 \tag{3.10}$$

are satisfied. Then, for system (3.1), there exists a static controller of state $u = Kx$ for which the closed system (3.3) is admissible and has the performance criterion $J < \gamma$ if the system of linear matrix inequalities (2.8) and (3.5) with nondegenerate matrix Y is consistent with $T = T^\top$ and F. The converse assertion is true if conditions (3.8) or (3.9) are satisfied together with (3.10).

Proof. Under conditions (3.10), we have $y \equiv x$, $W_R = \left[0_{s\times n}, I_s\right]^\top$, and the matrix inequality (3.4) is true and independent of X. In this case, in view of Lemma 2.2 and the equivalence of assertions (ii) and (iii) in Lemma 2.3, the role of sufficient conditions for the existence of the matrix of controller K is played by the consistency of relations (2.8) and (3.5) for $T = T^\top$ and F. Moreover, the matrix $K = K_*$ of this controller can be found as a solution of the linear matrix inequality (3.7), where $X = \gamma^2 Y^{-1}$.

Under conditions (3.10), we have $C_* = C_1 + D_{12}K$ and $D_* = D_{11}$ in (3.6). Hence, if one of the conditions (3.8) or (3.9) is satisfied together with (3.10), then the rank condition (3.6) is also true and, by Lemma 2.2, we obtain necessary and sufficient conditions for the existence of a static controller of state solely in terms of the linear matrix inequalities (2.8) and (3.5).

Theorem 3.1 is proved.

3.2. Dynamic Controller. Consider the control system (3.1) and the dynamic controller with zero initial vector

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \quad \xi(0) = 0, \tag{3.11}$$

where $\xi \in \mathbb{R}^p$, Z, V, U, and K are the required matrices of the corresponding sizes. We can rewrite this system in the extended phase space $\mathbb{R}^{n+p}$ as follows:

$$\begin{gathered} \widehat{E}\dot{\widehat{x}} = \widehat{A}\widehat{x} + \widehat{B}_1 w + \widehat{B}_2 \widehat{u}, \quad \widehat{x}(0) = \widehat{x}_0, \\ z = \widehat{C}_1 \widehat{x} + \widehat{D}_{11} w + \widehat{D}_{12}\widehat{u}, \\ \widehat{y} = \widehat{C}_2 \widehat{x} + \widehat{D}_{21} w \end{gathered} \tag{3.12}$$

by using a static controller of the controlled output

$$\widehat{u} = \widehat{K}_* \widehat{y}, \quad \det(I_m - KD_{22}) \neq 0, \tag{3.13}$$

where

$$\begin{gathered} \widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad \widehat{y} = \begin{bmatrix} y - D_{22}u \\ \xi \end{bmatrix}, \quad \widehat{u} = \begin{bmatrix} u \\ \dot{\xi} \end{bmatrix}, \quad \widehat{E} = \begin{bmatrix} E & 0_{n\times p} \\ 0_{p\times n} & I_p \end{bmatrix}, \\ \widehat{A} = \begin{bmatrix} A & 0_{n\times p} \\ 0_{p\times n} & 0_{p\times p} \end{bmatrix}, \quad \widehat{B}_1 = \begin{bmatrix} B_1 \\ 0_{p\times s} \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} B_2 & 0_{n\times p} \\ 0_{p\times m} & I_p \end{bmatrix}, \\ \widehat{C}_1 = \begin{bmatrix} C_1 & 0_{k\times p} \end{bmatrix}, \quad \widehat{D}_{11} = D_{11}, \quad \widehat{D}_{12} = \begin{bmatrix} D_{12} & 0_{k\times p} \end{bmatrix}, \\ \widehat{C}_2 = \begin{bmatrix} C_2 & 0_{l\times p} \\ 0_{p\times n} & I_p \end{bmatrix}, \quad \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{p\times s} \end{bmatrix}, \quad \widehat{K}_* = \begin{bmatrix} K_* & U_* \\ V_* & Z_* \end{bmatrix} = (I_{m+p} - \widehat{K}\widehat{D}_{22})^{-1}\widehat{K}, \\ \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix} = (I_{m+p} + \widehat{K}_*\widehat{D}_{22})^{-1}\widehat{K}_*, \quad \widehat{D}_{22} = \begin{bmatrix} D_{22} & 0_{l\times p} \\ 0_{p\times m} & 0_{p\times p} \end{bmatrix}. \end{gathered} \tag{3.14}$$

The closed system takes the form

$$\widehat{E}\dot{\widehat{x}} = \widehat{A}_*\widehat{x} + \widehat{B}_*w, \quad z = \widehat{C}_*\widehat{x} + \widehat{D}_*w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{3.15}$$

where

$$\widehat{A}_* = \widehat{A} + \widehat{B}_2\widehat{K}_*\widehat{C}_2, \qquad \widehat{B}_* = \widehat{B}_1 + \widehat{B}_2\widehat{K}_*\widehat{D}_{21},$$

$$\widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_*\widehat{C}_2, \qquad \text{and} \qquad \widehat{D}_* = \widehat{D}_{11} + \widehat{D}_{12}\widehat{K}_*\widehat{D}_{21}.$$

Let $\widehat{J}$ be a performance criterion of the form (1.2) for system (3.15) with weight matrices

$$P = P^\top > 0, \quad Q = Q^\top > 0, \quad \widehat{X}_0 = \widehat{E}^\top \widehat{H}\widehat{E}, \quad \widehat{H} = \begin{bmatrix} H & H_1^\top \\ H_1 & H_2 \end{bmatrix} > 0.$$

Since $\xi_0 = 0$, the criterion $\widehat{J}$ is independent of H_1 and H_2 and its value coincides with J.

Theorem 3.2. *Suppose that the conditions*

$$R_0 = D_{12}^\top Q D_{12} > 0, \quad R_1 = \gamma^2 P - D_{11}^\top Q_1 D_{11} > 0 \tag{3.16}$$

are satisfied. If the system of equations (3.4) is consistent for $\Theta = \Theta^\top \geq 0,\ S = S^\top$, *and* G *and, in addition,*

$$E_l^\top \Theta E_l < E_l^\top S E_l < \gamma^2 E_l^\top H E_l, \tag{3.17}$$

$$A_2^\top \widetilde{X} + \widetilde{X}^\top A_2 + \widetilde{X}^\top R_2 \widetilde{X} + Q_2 < 0, \tag{3.18}$$

where

$$X = SE + W_{E^\top}G, \quad \widetilde{X} = (S - \Theta)E + W_{E^\top}G,$$

$$A_2 = A_1 + B_{11}R_1^{-1}D_{11}^\top Q_1 C_1, \quad A_1 = A - B_2 R_0^{-1} D_{12}^\top Q C_1,$$

$$R_2 = B_{11}R_1^{-1}B_{11}^\top - B_2 R_0^{-1} B_2^\top, \quad B_{11} = B_1 - B_2 R_0^{-1} D_{12}^\top Q D_{11},$$

$$Q_1 = Q - Q D_{12} R_0^{-1} D_{12}^\top Q, \quad Q_2 = C_1^\top \left(Q_1 + Q_1 D_{11} R_1^{-1} D_{11}^\top Q_1\right) C_1,$$

then, for system (3.1), there exists the dynamic controller (3.11) of order $p = \operatorname{rank}\Theta$ *for which the closed system (3.15) is admissible and possesses the performance criterion* $J < \gamma$.

Proof. By using the representation of system (3.12) and Lemmas 2.2 and 2.3, we now present the relations guaranteeing the existence of the static controller (3.13) for which the closed system (3.15) is admissible and its performance criterion $\widehat{J} = J < \gamma$. We rewrite the matrix inequality (2.3) for system (3.15) in the form of a quadratic matrix inequality for $\widehat{K}_*$:

$$\widehat{A}_0 + \widehat{B}_0^\top \widehat{K}_* \widehat{C}_0 + \widehat{C}_0^\top \widehat{K}_*^\top \widehat{B}_0 + \widehat{C}_0^\top \widehat{K}_*^\top \widehat{R}_0 \widehat{K}_* \widehat{C}_0 < 0, \tag{3.19}$$

where

$$\widehat{A}_0 = \begin{bmatrix} \widehat{A}^\top \widehat{X} + \widehat{X}^\top \widehat{A} + \widehat{C}_1^\top Q \widehat{C}_1 & \widehat{X}^\top \widehat{B}_1 + \widehat{C}_1^\top Q D_{11} \\ \widehat{B}_1^\top \widehat{X} + D_{11}^\top Q \widehat{C}_1 & D_{11}^\top Q D_{11} - \gamma^2 P \end{bmatrix}, \quad \widehat{X} = \begin{bmatrix} X & X_3 \\ X_1 & X_2 \end{bmatrix},$$

$$\widehat{B}_0 = \begin{bmatrix} \widehat{B}_2^\top \widehat{X} + \widehat{D}_{12}^\top Q \widehat{C}_1 & \widehat{D}_{12}^\top Q D_{11} \end{bmatrix}, \quad \widehat{C}_0 = \begin{bmatrix} \widehat{C}_2 & \widehat{D}_{21} \end{bmatrix}, \quad \widehat{R}_0 = \widehat{D}_{12}^\top Q \widehat{D}_{12}.$$

Further, instead of conditions (2.4), for the block matrix $\widehat{X}$, we consider the relations

$$\widehat{X} = \widehat{S}\widehat{E} + W_{\widehat{E}^\top} \widehat{G}, \quad 0 < \widehat{E}_l^\top \widehat{S} \widehat{E}_l < \gamma^2 \widehat{E}_l^\top \widehat{H} \widehat{E}_l, \tag{3.20}$$

where

$$\widehat{S} = \begin{bmatrix} S & S_1^\top \\ S_1 & S_2 \end{bmatrix}, \quad \widehat{G} = \begin{bmatrix} G & G_1 \end{bmatrix}, \quad W_{\widehat{E}^\top} = \begin{bmatrix} W_{E^\top} \\ 0 \end{bmatrix}, \quad \widehat{E}_l = \begin{bmatrix} E_l & 0 \\ 0 & I_p \end{bmatrix}.$$

It can be shown that, under these conditions, the matrix $\widehat{X}$ and its diagonal blocks X and $X_2 = S_2$ are nondegenerate. By Lemma 2.5, we get the following consistency criterion for the quadratic inequality (3.19):

$$W_{\widehat{C}_0}^\top \widehat{A}_0 W_{\widehat{C}_0} < 0, \quad W_{\widetilde{B}_0}^\top \left(\widehat{A}_0 - \widehat{B}_0^\top \widehat{R}_0^+ \widehat{B}_0 \right) W_{\widetilde{B}_0} < 0, \tag{3.21}$$

where $\widetilde{B}_0 = W_{\widehat{R}_0}^\top \widehat{B}_0$. The first inequality in (3.21) is reduced to (3.4) because

$$\widehat{C}_0 = \begin{bmatrix} C_2 & 0_{l\times p} & D_{21} \\ 0_{p\times n} & I_p & 0_{p\times s} \end{bmatrix}, \quad W_{\widehat{C}_0} = \begin{bmatrix} I_n & 0_{n\times s} \\ 0_{p\times n} & 0_{p\times s} \\ 0_{s\times n} & I_s \end{bmatrix} W_R.$$

By using the relations

$$W_{\widehat{R}_0} = \begin{bmatrix} 0_{m\times p} \\ I_p \end{bmatrix}, \quad W_{\widetilde{B}_0} = \begin{bmatrix} I_n & 0_{n\times s} \\ -X_2^{-1} X_1 & 0_{p\times s} \\ 0_{s\times n} & I_s \end{bmatrix}, \quad \widetilde{B}_0 = \begin{bmatrix} X_1 & X_2 & 0_{p\times s} \end{bmatrix},$$

$$\widehat{R}_0^+ = \begin{bmatrix} R_0^{-1} & 0_{k\times p} \\ 0_{p\times k} & 0_{p\times p} \end{bmatrix}, \quad \widehat{B}_0 W_{\widetilde{B}_0} = \begin{bmatrix} B_2^\top \widetilde{X} + D_{12}^\top Q C_1 & D_{12}^\top Q D_{11} \\ 0_{p\times n} & 0_{p\times s} \end{bmatrix},$$

$$W_{\widetilde{B}_0}^\top \widehat{A}_0 W_{\widetilde{B}_0} = \begin{bmatrix} A^\top \widetilde{X} + \widetilde{X}^\top A + C_1^\top Q_1 C_1 & \widetilde{X}^\top B_1 + C_1^\top Q D_{11} \\ B_1^\top \widetilde{X} + D_{11}^\top Q C_1 & D_{11}^\top Q D_{11} - \gamma^2 P \end{bmatrix},$$

where $\widetilde{X} = X - X_3 X_2^{-1} X_1$ and $X_1 = X_3^\top E = S_1 E$, we can transform the second inequality in (3.21).

Let $X_2 = S_2 = I_p$ and let $X_3 \in \mathbb{R}^{n\times p}$ be a factor in the decomposition of the nonnegative-definite matrix $\Theta = X_3 X_3^\top \geq 0$. Then $\widetilde{X} = X - \Theta E$ and, by virtue of (3.16), the second inequality in (3.21) takes the form

$$\begin{bmatrix} A_1^\top \widetilde{X} + \widetilde{X}^\top A_1 + C_1^\top Q_1 C_1 - \widetilde{X}^\top B_2 R_0^{-1} B_2^\top \widetilde{X} & \widetilde{X}^\top B_{11} + C_1^\top Q_1 D_{11} \\ B_{11}^\top \widetilde{X} + D_{11}^\top Q_1 C_1 & -R_1 \end{bmatrix} < 0.$$

By the Schur lemma, this inequality is equivalent to (3.18).

In order that relations (3.20) be true under conditions (3.17), it is sufficient to set

$$S_1 = X_3^\top - G_1^\top W_{E^\top}^\top \quad \text{and} \quad H_1 = \gamma^{-2} S_1$$

and to choose arbitrary matrices G_1 and $H_2 > \gamma^{-2} I_p$. In this case,

$$E_l^\top S E_l > E_l^\top S_1^\top S_1 E_l = E_l^\top \Theta E_l.$$

Theorem 3.2 is proved.

Remark 3.1. The assertion converse to Theorem 3.2 on consistency conditions for the system of equations (3.4), (3.17), and (3.18) can be established by applying Lemma 2.2 to the closed system (3.15) and using the additional assumption (3.6) or, in particular, conditions (3.8) or (3.9).

On the basis of Theorem 3.2, we propose an algorithm for the construction of the required dynamic controller (3.11):

(i) to find the matrices $\Theta = \Theta^\top \geq 0,\ S = S^\top$, and G satisfying the system of equations (3.4), (3.17), and (3.18);

(ii) to construct the spectral decomposition of the nonnegative-definite matrix $\Theta = T\Lambda T^\top$, where $T \in \mathbb{R}^{n\times r}$, $\Lambda = \operatorname{diag}\{\theta_1, \ldots, \theta_r\} > 0$, and $r = \operatorname{rank}\Theta$;

(iii) to form the complementary blocks $X_1 = T^\top E,\ X_2 = \Lambda^{-1}$, and $X_3 = T$ of the matrix $\widehat{X}$ for $p = r$;

(iv) to solve the matrix inequality (3.19) for $\widehat{K}_*$;

(v) to compute the matrix coefficients of controller (3.11) by using relations (3.14).

Remark 3.2. For $\Theta = 0$, the conditions of Theorem 3.2 guarantee the existence of the static controller (3.2) for which the closed system (3.3) is admissible and the estimate $J < \gamma$ holds (see [27], Theorem 3.1). Thus, if the system of equations (3.4), (3.17), and (3.18) is solved for $S = S^\top$ and G with $\Theta = 0$, then the matrix of the required static controller can be found in the form $K = K_*(I_l + D_{22}K_*)^{-1}$, where K_* is the solution of the linear matrix inequalities (3.7) for $X = SE + W_{E^\top} G$.

Remark 3.3. Theorems 3.1 and 3.2 can be extended to the class of systems (3.1) under the conditions of polyhedral indeterminacy of the matrix coefficients

$$A \in \operatorname{Co}\{A_1, \ldots, A_{\nu_1}\}, \quad B_1 \in \operatorname{Co}\left\{B_1^1, \ldots, B_1^{\nu_2}\right\},$$

$$C_1 \in \operatorname{Co}\left\{C_1^1, \ldots, C_1^{\nu_3}\right\}, \quad D_{11} \in \operatorname{Co}\left\{D_{11}^1, \ldots, D_{11}^{\nu_4}\right\}.$$

To this end, instead of (3.4), (3.5), and (3.18), it is necessary to apply the corresponding systems of matrix inequalities formed for all possible collections of vertices of the given polytopes. Note that the matrix intervals and affine sets can be described in the form of polytopes. Thus, the matrix interval

$$\mathcal{A} = \left\{ A \in \mathbb{R}^{n\times m} : \underline{A} \le A \le \overline{A} \right\},$$

where

$$\underline{A} = \|\underline{a}_{ij}\|_{i,j=1}^{n,m}, \qquad \overline{A} = \|\overline{a}_{ij}\|_{i,j=1}^{n,m},$$

and $\le$ is the sign of inequality for a cone of nonnegative matrices, describes a polytope with $\nu = 2^{nm}$ vertices:

$$\operatorname{Co}\{A_1,\ldots,A_\nu\} = \left\{ \sum_{k=1}^{\nu} a_k A_k : \, a_k \ge 0, \; k = \overline{1,\nu}, \; \sum_{k=1}^{\nu} a_k = 1 \right\},$$

$$A_k = \left\| a_{ij}^k \right\|_{i,j=1}^{n,m}, \quad a_{ij}^k \in \left\{ \underline{a}_{ij}, \overline{a}_{ij} \right\}, \quad i = \overline{1,n}, \quad j = \overline{1,m}, \quad k = \overline{1,\nu}.$$

4. Example. Robust Stabilization of a Hydraulic System

Consider a linearized model of hydraulic system with three vessels connected in series. This model is described by the descriptor control system (3.1) with the following matrices [31]:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -k_1 & 0 & 0 \\ k_1 & -k_2 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \quad D_{11} = 0_{1\times 2}, \quad D_{12} = 1,$$

$$C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{22} = 0_{2\times 1}.$$

The components of the vector of state $x = \left[x_1, x_2, x_3\right]^\top$ determine the levels of liquid in the corresponding vessels, the vector $w = \left[w_1, w_2\right]^\top$ is formed by the uncontrolled disturbances w_1 and the error w_2 of measurements $y = \left[x_1, x_2 + w_2\right]^\top$, the controlled output $z = x_3 + u$, and the role of control u regulating the level of liquid in the first two vessels is played by the debit (flow) of liquid through the pump from the third vessel into the first vessel (Fig. 1).

In this example, $n = 3,\ m = 1,\ k = 1,\ s = 2,\ l = 2$, the couple of matrices (E, A) is admissible, and system (3.1) is impulsively controlled and impulsively observed. The system without control has the following performance criterion: $J = 7.66027$.

We choose the weight matrices of the performance criterion (1.2) as follows: $P = \operatorname{diag}\{2, 1\},\ Q = 1,$ $X_0 = \operatorname{diag}\{1, 1, 0\}$, and $H = I_3$. The admissible values of the parameters are chosen as

$$\underline{k}_1 = 0.01 \le k_1 \le 0.1 = \overline{k}_1, \quad \underline{k}_2 = 0.1 \le k_2 \le 1.2 = \overline{k}_2. \tag{4.1}$$

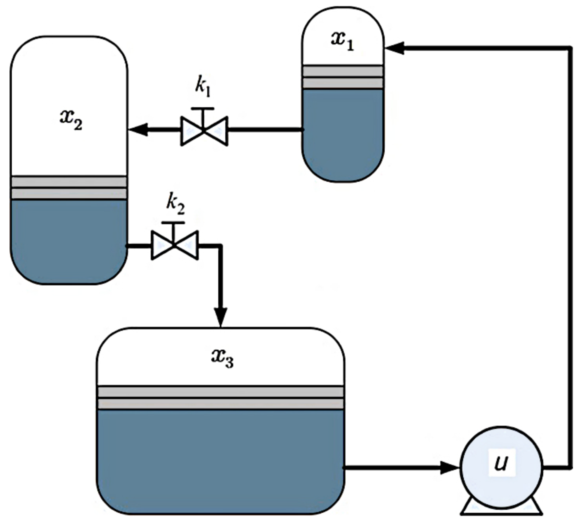

Fig. 1. Hydraulic system with three vessels.

By using the Mathcad Prime software, for $\gamma = 2.2$ and $\Theta = 0,$ we obtain the matrices

$$S = \begin{bmatrix} 2.75851 & 1.86824 & 0.02321 \\ 1.86824 & 3.15258 & 0.19783 \\ 0.02321 & 0.19783 & 0.23992 \end{bmatrix}, \quad G = \begin{bmatrix} 0.80714 & 0.64032 & -0.85066 \end{bmatrix},$$

satisfying relation (3.17) and the system of eight matrix inequalities formed according to (3.4) and (3.18) for the following values of the pair (k_1, k_2): $(\underline{k}_1, \underline{k}_2)$, $(\underline{k}_1, \overline{k}_2)$, $(\overline{k}_1, \underline{k}_2)$, $(\overline{k}_1, \overline{k}_2)$. Further, we determine the matrix of the static controller (3.2):

$$K = \begin{bmatrix} -1.12507 & -0.76271 \end{bmatrix}$$

as a solution of the system of linear matrix inequalities (3.7) for the indicated values of the pair (k_1, k_2) and the matrix

$$X = SE + W_{E^\top} G = \begin{bmatrix} 2.75851 & 1.86824 & 0 \\ 1.86824 & 3.15258 & 0 \\ 0.83034 & 0.83815 & -0.85066 \end{bmatrix}.$$

In this case, the closed system (3.3) is admissible and its performance criterion $J = 1.67775 < \gamma$ for all values of parameters (4.1) (see Remarks 3.2 and 3.3).

Further, for the closed system with $k_1 = 0.1$ and $k_2 = 1.2,$ we construct the worst disturbance

$$w = K_0 x, \quad K_0 = \begin{bmatrix} 0.41169 & 0.17803 & 0 \\ -0.40743 & -0.08191 & -0.34139 \end{bmatrix} \tag{4.2}$$

and the worst initial vector

$$x_0 = \begin{bmatrix} 0.53574 & 0.26573 & -0.80148 \end{bmatrix}^\top$$

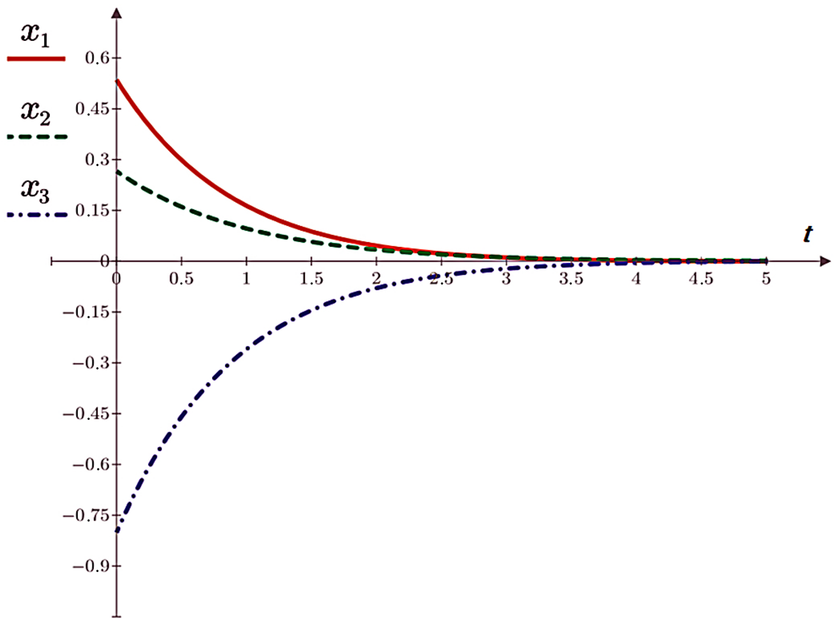

Fig. 2. Behavior of the closed system.

for the performance criterion J (see Lemma 2.4). The behavior of the solution of the closed system for the worst disturbance

$$E\dot{x} = A_0 x, \quad A_0 = A + B_2 K C_2 + B_1 K_0, \quad x(0) = x_0, \tag{4.3}$$

is shown in Fig. 2. For this disturbance, function (4.2) is depicted in Fig. 3. System (4.3) is admissible and has the finite spectrum

$$\Sigma = \{-0.98151 \pm 0.17470\, i\}.$$

Its solution is constructed in the form $x(t) = T\widetilde{x}(t)$, where T is the matrix of complete rank for which the following relations are true:

$$A_0 T = ET\Lambda \qquad \text{and} \qquad \operatorname{rank} T = \operatorname{rank}(ET) = \operatorname{rank} E,$$

and $\widetilde{x}(t)$ is the solution of the ordinary system $\dot{\widetilde{x}} = \Lambda\widetilde{x}$. Moreover,

$$\sigma(\Lambda) = \Sigma \qquad \text{and} \qquad x_0 = T\widetilde{x}_0.$$

Similar calculations were also performed for this system in order to find the dynamic controller (3.11). By using the algorithm presented above, we construct the dynamic controller of the first order with the following matrices:

$$K = \begin{bmatrix} -0.95250 & -1.09756 \end{bmatrix}, \quad U = -0.31040,$$

$$V = \begin{bmatrix} -0.73469 & -1.28101 \end{bmatrix}, \quad Z = -0.72267.$$

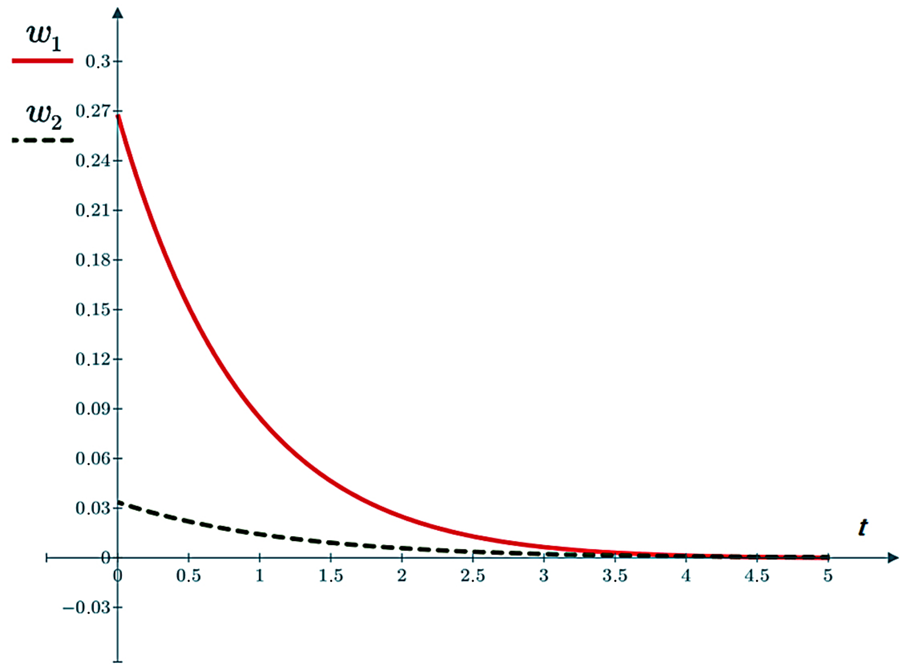

Fig. 3. The worst disturbance for the performance criterion J.

For this controller, the closed system (3.15) is admissible and the estimate $J < \gamma = 2.2$ is true for all values of parameters (4.1). In particular, for $k_1 = 0.1$ and $k_2 = 1.2$, system (3.15) possesses the finite spectrum

$$\Sigma = \{-0.37981; -1.29769 \pm 0.32064\, i\}$$

and $J = 1.75205$.

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