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STABILITY OF POSITIVE AND MONOTONE SYSTEMS IN A PARTIALLY ORDERED SPACE

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We investigate properties of positive and monotone dynamical systems with respect to given cones in the phase space. Stability conditions for linear and nonlinear differential systems in a partially ordered space are formulated. Conditions for the positivity of dynamical systems with respect to the Minkowski cone are established. By using the comparison method, we solve the problem of the robust stability of a family of systems.

1. Introduction

Numerous real systems possess the properties of positivity and monotonicity. These properties are inherent in certain classes of systems describing the motion and interaction of objects of different nature. The positivity (monotonicity) of a dynamical system is equivalent to the positivity (monotonicity) of a certain operator that describes its motion with respect to given cones in the phase space. The Lyapunov and the Riccati differential equations are examples of positive systems with respect to a cone of symmetric nonnegative-definite matrices. Properties of positive systems are used in various problems of analysis and synthesis [1–4]. The investigation of the stability of a class of linear positive systems reduces to the solution of algebraic equations defined by the operator coefficients of given systems [3–8].

In the present work, we study properties of solutions of positive and monotone dynamical systems with respect to given cones in a partially ordered phase space. We present a generalized principle of comparison of systems and conditions for the robust stability of a family of nonlinear systems. We also consider multiply connected systems, which can be used for the description of physical objects and processes in an inhomogeneous medium.

2. Definitions and Auxiliary Facts

A convex closed set $\mathcal{K}$ of a real normed space $\mathcal{E}$ is called a cone if $\mathcal{K} \cap -\mathcal{K} = \{0\}$ and $\alpha\mathcal{K} + \beta\mathcal{K} \subset \mathcal{K}$ $\forall \alpha, \beta \geq 0$. The dual cone $\mathcal{K}^*$ consists of linear functionals $\varphi \in \mathcal{E}^*$ taking nonnegative values on the elements of $\mathcal{K}$. In this case, $\mathcal{K} = \{X \in \mathcal{E} : \varphi(X) \geq 0 \ \ \forall \varphi \in \mathcal{K}^*\}$.

The space with a cone is partially ordered: $X \leq Y \Leftrightarrow Y - X \in \mathcal{K}$. A cone $\mathcal{K}$ with the set of interior points $\mathcal{K}^0 = \{X : X > 0\} \neq \varnothing$ is solid. A cone $\mathcal{K}$ is called normal if the relation $0 \leq X \leq Y$ yields $\|X\| \leq c\,\|Y\|$, where c is a universal constant. If $\mathcal{E} = \mathcal{K} - \mathcal{K}$, then $\mathcal{K}$ is a reproducing cone.

Note that the property of the normality of a cone $\mathcal{K}$ is equivalent to the condition

$$U \leq X \leq V \ \Rightarrow \ \|X\| \leq \alpha\|U\| + \beta\|V\|, \tag{1}$$

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where $\alpha > 0$ and $\beta > 0$ are universal constants. Indeed, this condition coincides with the definition of the normality of a cone for $U = 0$. If the cone is normal, then the relation $0 \le X - U \le V - U$ yields

$$\|X\| - \|U\| \le \|X - U\| \le c\|V - U\| \le c\|V\| + c\|U\|,$$

and, in (1), in particular, we can set $\alpha = c + 1$ and $\beta = c$, where c is the normality constant of the cone $\mathcal{K}$.

Assume that a Banach space $\mathcal{E}_1(\mathcal{E}_2)$ contains a cone $\mathcal{K}_1(\mathcal{K}_2)$. An operator $M: \mathcal{E}_1 \to \mathcal{E}_2$ is called monotone if $MX \ge MY$ for $X \ge Y$. The monotonicity of a linear operator is equivalent to its positivity: $X \ge 0 \Rightarrow MX \ge 0$. The operator inequality $M \le L$ means that the operator $L - M$ is positive. If $M\mathcal{E}_1 \subset \mathcal{K}_2$, then the operator M is positive everywhere. A linear operator M is called monotonically invertible if, for any $Y \in \mathcal{K}_2$, the equation $MX = Y$ has a solution $X \in \mathcal{K}_1$. If $\mathcal{K}_2$ is a normal reproducing cone and $M_1 \le M \le M_2$, then the monotone invertibility of M_1 and M_2 yields the monotone invertibility of the operator M, and, furthermore, $M_2^{-1} \le M^{-1} \le M_1^{-1}$ [1].

Consider the class of linear operators $M = L - P$, $P\mathcal{K}_1 \subset \mathcal{K}_2 \subset L\mathcal{K}_1$, where $\mathcal{K}_2$ is a normal reproducing cone. For such operators, a criterion for monotone invertibility is the inequality $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the pencil of operators $T(\lambda) = P - \lambda L$. If the cone $\mathcal{K}_2$ is solid, then the inequality indicated is equivalent to the existence of elements $X \ge 0$ and $Y > 0$ satisfying the equation $MX = Y$.

Note that an arbitrary linear operator preserving the cone of Hermitian nonnegative-definite matrices is representable in the form [6]

$$MX = \sum_k A_k X A_k^* + \sum_s B_s X^T B_s^*, \quad A_k, B_s \in C^{n \times n}.$$

Let $M: \mathcal{E} \to \mathcal{E}$ and let cones $\mathcal{K}$, $\mathcal{K}_1 = S\mathcal{K}$, and $\mathcal{K}_2 = S^{-1}\mathcal{K}$ be given in the space $\mathcal{E}$ (here, S is an invertible operator). It is obvious that the relations $S\mathcal{K} \subset \mathcal{K}$, $S\mathcal{K}_1 \subset \mathcal{K}_1$, $S\mathcal{K}_2 \subset \mathcal{K}_2$, and $\mathcal{K}_1 \subset \mathcal{K} \subset \mathcal{K}_2$ are equivalent. Using properties of functions of operators, one can establish that

$$f(M)\mathcal{K} \subset \mathcal{K} \Leftrightarrow f(M_2)\mathcal{K}_1 \subset \mathcal{K}_1 \Leftrightarrow f(M_1)\mathcal{K}_2 \subset \mathcal{K}_2,$$

$$f(M)\mathcal{K}_1 \subset \mathcal{K}_1 \Leftrightarrow f(M_1)\mathcal{K} \subset \mathcal{K}, \quad f(M)\mathcal{K}_2 \subset \mathcal{K}_2 \Leftrightarrow f(M_2)\mathcal{K} \subset \mathcal{K},$$

where $M_1 = S^{-1}MS$ and $M_2 = SMS^{-1}$. The relations presented above can be useful in the investigation of stability conditions for a class of positive systems.

3. Positive and Monotone Systems

Consider a dynamical system with continuous or discrete time $t \ge \theta$ whose states in the phase space $\mathcal{E}$ are defined by the relations

$$X(t) = \Omega(t, t_0)X_0, \quad \Omega(t_0, t_0) = E, \quad t \ge t_0 \ge \theta. \tag{2}$$

Here, $\Omega(t, t_0): \mathcal{E} \to \mathcal{E}$ is the operator that determines the transition from the initial state $X(t_0) = X_0$ to a state $X(t)$ for $t > t_0$, and E is the identity operator. If $\Omega(t, t_0)0 \equiv 0$, then $X(t) \equiv 0$ is the equilibrium state of the system.

Assume that the space $\mathcal{E}$ contains cones $\mathcal{K}$ and $\mathcal{K}_0$. In what follows, we use the order relations $\leq$ and $\geq$ ($\trianglelefteq$ and $\trianglerighteq$) generated by the cone $\mathcal{K}$ ($\mathcal{K}_0$). We define the following properties of system (2):

$$\Omega(t, t_0)\mathcal{K} \subset \mathcal{K} \quad \text{(positivity with respect to } \mathcal{K}),$$

$$\Omega(t, t_0)\mathcal{K}_0 \subset \mathcal{K} \quad \text{(positivity with respect to } \mathcal{K}_0 \text{ and } \mathcal{K}),$$

$$X_0 \leq Y_0 \;\Rightarrow\; X(t) \leq Y(t) \quad \text{(monotonicity with respect to } \mathcal{K}),$$

$$X_0 \trianglelefteq Y_0 \;\Rightarrow\; X(t) \leq Y(t) \quad \text{(monotonicity with respect to } \mathcal{K}_0 \text{ and } \mathcal{K}),$$

$$0 \leq X_0 \leq Y_0 \;\Rightarrow\; X(t) \leq Y(t) \quad \text{(monotonicity in } \mathcal{K}),$$

$$0 \trianglelefteq X_0 \trianglelefteq Y_0 \;\Rightarrow\; X(t) \leq Y(t) \quad \text{(monotonicity in } \mathcal{K}_0 \text{ with respect to } \mathcal{K}).$$

Here, $X(t)$ and $Y(t)$ are the states of the system for $t \geq t_0$, $X(t_0) = X_0$, and $Y(t_0) = Y_0$.

It is obvious that a continuous system possessing one of the properties indicated must satisfy the inclusion $\mathcal{K}_0 \subset \mathcal{K}$. Furthermore, the positivity (monotonicity) of a system with respect to $\mathcal{K}_0$ or $\mathcal{K}$ yields its positivity (monotonicity) with respect to $\mathcal{K}_0$ and $\mathcal{K}$.

Consider the differential system

$$\dot{X} + M(t)X = 0, \tag{3}$$

where $M(t)\colon \mathcal{E} \to \mathcal{E}$ is a linear operator. An arbitrary solution of system (3) has the form (2), where $\Omega(t, t_0) = W(t, t_0)$ is a linear evolution operator. The properties of positivity and monotonicity of system (3) with respect to $\mathcal{K}_0$ and $\mathcal{K}$ are equivalent to the positivity of the evolution operator. The positivity of the operator $W(t, t_0)$ for $t \geq t_0$ is equivalent to the positivity of the exponential operator $e^{-M(t)h}$ for $t \geq 0$ and $h \geq 0$. If two systems of the form (3) with operators $M_1(t)$ and $M_2(t)$ are positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$, then the system described by the operator $M_1(t) + M_2(t)$ possesses the same property [7, 9].

We generalize the differential system (3) as follows:

$$\dot{X} + M(t)X = G(X, t), \tag{4}$$

where $G(X, t)$ is a nonlinear operator that guarantees the existence and uniqueness of a solution $X(t) \in \mathcal{E}$ for $t \geq t_0$ and $X(t_0) = X_0$. Solutions of system (4) satisfy the integral equation

$$X(t) = W(t, t_0)X_0 + \int_{t_0}^{t} W(t, s)G(X(s), s)\,ds,$$

where $W(t, s)$ is the evolution operator of system (3). This implies that system (4) is positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$ if $W(t, t_0)\mathcal{K}_0 \subset \mathcal{K}$ and the operator $W(t, s)G(X, t)$ is positive on the cone $\mathcal{K}$ for any $t \geq s \geq t_0$.

Let us find conditions for the positivity and monotonicity of system (4) with respect to the cone $\mathcal{K}$ by using the dual cone of linear functionals $\mathcal{K}^*$. Let $\mathcal{F}_0$ and $\mathcal{F}$ denote the families of continuous operator functions $F(X, t)$ satisfying, for $t \geq \theta$, the corresponding conditions

$$X \in \mathcal{K},\ \ \varphi \in \mathcal{K}^*,\ \ \varphi(X) = 0 \ \Rightarrow\ \varphi(F(X, t)) \geq 0,$$

$$X - Y \in \mathcal{K},\ \ \varphi \in \mathcal{K}^*,\ \ \varphi(X - Y) = 0 \ \Rightarrow\ \varphi(F(X, t) - F(Y, t)) \geq 0.$$

Lemma 1. *If the cone $\mathcal{K}$ is solid, system (3) is positive with respect to $\mathcal{K}$, and $G \in \mathcal{F}_0$ $(G \in \mathcal{F})$, then system (4) is positive (monotone) with respect to $\mathcal{K}$. If system (4) is positive (monotone) with respect to $\mathcal{K}$, then $F \in \mathcal{F}_0$ $(F \in \mathcal{F})$, where $F(X, t) = G(X, t) - M(t)X$.*

Proof. Consider the auxiliary system

$$\dot{Z} = F(Z, t) + \varepsilon Q,$$

where $\varepsilon > 0$ and $Q > 0$ is an interior element of $\mathcal{K}$. Let $Z(t)$ be its solution satisfying the condition $Z(t_0) = Z_0 \geq 0$ and let $Z(t_1) = Z_1 \in \partial\mathcal{K}$ be a point of the boundary of the cone $\mathcal{K}$ for some $t_1 \geq t_0$. Then $\varphi(Z_1) = 0$ and $\varphi(Q) > 0$ for some $\varphi \in \mathcal{K}^*$, and $\varphi \neq 0$.

It follows from the monotonicity of the exponential operator $e^{-M(t)h}$ and the relation

$$M(t)Z = \lim_{h \to 0+} \frac{1}{h}\left(Z - e^{-M(t)h}Z\right)$$

that $\varphi(M(t_1)Z_1) \leq 0$. If, in addition, $G \in \mathcal{F}_0$, then $F \in \mathcal{F}_0$, and, under the continuity conditions, for a certain $\delta > 0$ we get

$$\varphi(\dot{Z}(t_1)) = \varphi(F(Z_1, t_1)) + \varepsilon\varphi(Q) > 0,$$

$$\int_{t_1}^{t_1+\delta} \varphi(\dot{Z}(t))\,dt = \varphi(Z(t_1 + \delta)) > 0.$$

Hence, the trajectory $Z(t)$ does not leave the cone $\mathcal{K}$ for $t > t_1$, i.e., $Z(t) \geq 0$ for $t_1 \leq t \leq t_1 + \delta$. Otherwise, the inequality $\varphi(Z(t_1 + \delta)) < 0$ must be satisfied for some $\varphi \in \mathcal{K}^*$ and $\delta > 0$. By virtue of the closedness of the cone, we get $Z(t) \to X(t) \geq 0$ as $\varepsilon \to 0$ for any $Z_0 = X_0 \geq 0$ and $t \geq t_0$, i.e., system (4) is positive with respect to $\mathcal{K}$.

The fact that the condition $F \in \mathcal{F}_0$ is necessary for system (4) positive with respect to $\mathcal{K}$ follows, for sufficiently small $\delta > 0$, from the relations

$$\varphi(X(t_1 + \delta)) = \delta\varphi(F(X(\tau), \tau)), \qquad \varphi(X_1) = 0,$$

where $X(t_1) = X_1 \in \partial\mathcal{K}$, $\varphi \in \mathcal{K}^*$, and $t_1 \leq \tau \leq t_1 + \delta$.

By analogy, we establish the required necessary and sufficient conditions for the monotonicity of system (4) with respect to $\mathcal{K}$.

Lemma 1 is proved.

In the case of the solid cone $\mathcal{K}$, the positivity (monotonicity) of the differential systems described by the operators $F_1(X, t)$ and $F_2(X, t)$ with respect to $\mathcal{K}$ yields the positivity (monotonicity) of the differential system described by the operator $F(X, t) = F_1(X, t) + F_2(X, t)$ with respect to $\mathcal{K}$.

Example 1. The nonlinear differential system

$$\dot{x} + A(t)x = g(x, t), \qquad x \in R^n,$$

where $A(t)$ is a matrix with nonpositive off-diagonal elements, is positive with respect to the cone of nonnegative vectors $\mathcal{K}$ if the vector function $g(x, t)$ satisfies the conditions [10]

$$x \geq 0, \;\; x_i = 0 \;\;\Rightarrow\;\; g_i(x, t) \geq 0, \;\; t \geq \theta, \;\; i = \overline{1, n},$$

and is monotone with respect to $\mathcal{K}$ if $g(x, t)$ is quasimonotone and nondecreasing in x (the Wazewski condition), i.e.,

$$x \leq y, \;\; x_i = y_i \;\;\Rightarrow\;\; g_i(x, t) \leq g_i(y, t), \;\; t \geq \theta, \;\; i = \overline{1, n}$$

If both restrictions are satisfied for $0 \leq x \leq y$, then the system considered is monotone in $\mathcal{K}$.

Example 2. Consider a nonlinear control system with dynamical feedback

$$\dot{x} = f(x, u, t), \qquad \dot{u} = g(x, u, t), \qquad x \in R^n, \quad u \in R^1.$$

In the phase space R^{n+1}, we consider the circular Minkowski cone [11]

$$\mathcal{K} = \left\{ z \in R^{n+1} : z^T = \left[x^T, u \right], \; \| x \| \leq u \right\}.$$

This cone is normal, solid, and self-dual. The last property means that $l^T z \geq 0 \;\; \forall x \in \mathcal{K} \Leftrightarrow l \in \mathcal{K}$. Using Lemma 1, we can represent the criterion for the positivity of the system with respect to $\mathcal{K}$ in the form

$$\| y \| = 1, \;\; u \geq 0 \;\;\Rightarrow\;\; y^T f(u y, u, t) \leq g(u y, u, t) \;\;\; \forall t \geq \theta.$$

In the case of a linear system, we set $f(x, u, t) = A(t)x + b(t)u$ and $g(x, u, t) = c(t)^T x + d(t)u$. In this case, each of the conditions

$$\frac{\lambda_{\max}\left(A(t) + A(t)^T\right)}{2} + \| b(t) - c(t) \| \leq d(t),$$

$$\begin{bmatrix} -A(t)-A(t)^T & c(t)-b(t) \\ c(t)^T-b(t)^T & 2d(t) \end{bmatrix} \geq 0,$$

where $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of the symmetric matrix, guarantees the positivity and monotonicity of the system with respect to $\mathcal{K}$.

Example 3. The Lyapunov and the Riccati differential equations and the more general equation

$$\dot{X} - A(t)X - XA(t)^T - \sum_k B_k(t)XB_k(t)^T = XC(t)X + D(t),$$

where $A(t)$, $B_k(t)$, $C(t)=C(t)^T \geq 0$, and $D(t)=D(t)^T \geq 0$ are given matrices and $t \geq \theta$, are positive with respect to the cone of symmetric nonnegative-definite matrices $\mathcal{K}$. In the case where $C(t) \in \mathcal{K}$, this equation is also monotone with respect to $\mathcal{K}$. In the example under consideration, we have $M(t)=L(t)-P(t)$, where

$$L(t)X = -A(t)X - XA(t)^T, \qquad P(t)X = \sum_k B_k(t)XB_k(t)^T.$$

The evolution operator $W_M(t, t_0)$ of the system is positive because

$$W_L(t, t_0)X = \Xi(t, t_0)X\Xi(t, t_0)^T \geq 0, \qquad W_{-P}(t, t_0)X \geq 0 \quad \forall X = X^T \geq 0,$$

where $\Xi(t, t_0)$ is the matrizant of the system $\dot{x} = A(t)x$, $t \geq t_0 \geq \theta$.

The matrix differential equation

$$\dot{X} = A(t)X + XA(t)^T + \sum_k B_k(t)XB_k(t)^T$$

is known as the second-moment equation for the Itô stochastic system

$$dx(t) - A(t)x(t)dt = \sum_k B_k(t)x(t)dw_k(t),$$

where w_k are the components of a standard Wiener process. This equation possesses the properties of positivity and monotonicity with respect to $\mathcal{K}$ and is used in the theory of stability of stochastic systems.

Note that if system (3) with a monotone operator M is positive with respect to the cone of nonnegative-definite matrices, then this operator is representable in the form $MX = \alpha X$, where $\alpha \geq 0$. To prove the last statement, it suffices to consider the case where $MX = AXA^T$ and establish that all eigenvalues of the matrix A are real and identical.

4. Stability of Systems in a Partially Ordered Space

In the phase space $\mathcal{E}$, we consider a dynamical system whose states are described by continuous differentiable functions $X(t)$ of the form (2). Let $\Omega(t, t_0)0 \equiv 0$ and $\mathcal{K}_0 \subset \mathcal{K}$, where $\mathcal{K}$ is a normal cone and $\mathcal{K}_0$ is a reproducing cone, which generate the corresponding order relations in $\mathcal{E}$.

The state $X \equiv 0$ of system (2) is called stable from $\mathcal{K}_0$ into $\mathcal{K}$ if, for any $\varepsilon > 0$ and $t_0 \geq \theta$, one can indicate $\delta > 0$ such that the relations $\|X_0\| \leq \delta$ and $X_0 \trianglerighteq 0$ yield $\|X(t)\| \leq \varepsilon$ and $X(t) \geq 0$ for $t > t_0$. If, in addition, for some $\delta_0 > 0$ the relation $\|X_0\| \leq \delta_0$ implies that $\|X(t)\| \to 0$ as $t \to \infty$, then the solution $X \equiv 0$ is asymptotically stable from $\mathcal{K}_0$ into $\mathcal{K}$. If the solution $X \equiv 0$ of system (2) positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$ is Lyapunov stable (asymptotically Lyapunov stable), then it is stable (asymptotically stable) from $\mathcal{K}_0$ into $\mathcal{K}_0$.

Lemma 2. *Suppose that the states of system (2) satisfy the conditions*

$$X_0 \trianglerighteq 0 \;\Rightarrow\; X(t) \geq 0, \quad \dot{X}(t) \leq 0, \tag{5}$$

$$X_0 = X_+ - X_- \;\Rightarrow\; -X_-(t) \leq X(t) \leq X_+(t), \tag{6}$$

where $X_\pm \trianglerighteq 0$, $X_\pm(t) = \Omega(t, t_0)X_\pm$, *and* $t > t_0$. *Then the state* $X \equiv 0$ *of this system is Lyapunov stable.*

Proof. A. $X_0 \in \mathcal{K}_0$. According to the Lagrange theorem, we have

$$X(t) - X(t_0) = \dot{X}(\xi)(t - t_0), \qquad \xi \in (t, t_0), \qquad t > t_0.$$

In view of (5), this yields $0 \leq X(t) \leq X_0$, whence $\|X(t)\| \leq c\|X_0\|$, where c is the normality constant of the cone $\mathcal{K}$. Therefore, for any $\varepsilon > 0$, the relation $\|X_0\| \leq \delta = \varepsilon/c$ yields $\|X(t)\| \leq \varepsilon$.

B. $X_0 \in \mathcal{E}$. The reproducing cone $\mathcal{K}_0$ is unflattened [1], i.e., $X_0 = X_+ - X_-$ and $\|X_\pm\| \leq \gamma\|X_0\|$, where $\gamma > 0$ is a universal constant.

Let $\varepsilon > 0$. We choose $\delta_\pm$ according to step A so that the relation $\|X_\pm\| \leq \delta_\pm$ yields $\|X_+(t)\| \leq \varepsilon/(2\beta)$ and $\|X_-(t)\| \leq \varepsilon/(2\alpha)$. To this end, we set $\delta_+ = \varepsilon/(2\beta c)$ and $\delta_- = \varepsilon/(2\alpha c)$. If $\|X_0\| \leq \delta$, where $\delta = \min\{\delta_+, \delta_-\}/\gamma$, then, using (1) and (6), we get

$$\|X(t)\| \leq \alpha\|X_-(t)\| + \beta\|X_+(t)\| \leq \varepsilon.$$

This means that the zero state of system (2) is stable.

Lemma 2 is proved.

Condition (5) guarantees the stability of the zero state of system (2) from $\mathcal{K}_0$ into $\mathcal{K}$. Condition (6) is always satisfied, e.g., for the class of systems positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$ with the linear operator $\Omega(t, t_0)$. Condition (6) is also satisfied if the operator $\Omega(t, t_0)$ is monotone and the operator $\hat{\Omega}(t, t_0)X = \Omega(t, t_0)X + \Omega(t, t_0)(-X)$ is positive with respect to the cones $\mathcal{K}_0$ and $\mathcal{K}$ for $t \geq t_0$.

Lemma 3. *The state* $X \equiv 0$ *of system (2) monotone with respect to* $\mathcal{K}_0$ *and* $\mathcal{K}$ *is Lyapunov stable if* $\dot{X}(t) \in \mp\mathcal{K}$ *for any* $X_0 \in \pm\mathcal{K}_0$ *and* $t > t_0$.

Proof. It follows from the monotonicity of the operator $\Omega(t, t_0)$ and the condition $\Omega(t, t_0)0 \equiv 0$ that $X(t) \in \pm\mathcal{K}$ for any $X_0 \in \pm\mathcal{K}_0$. If $X_0 \in \mathcal{K}_0$, then $0 \le X(t) \le X_0$ and, hence, $\|X(t)\| \le c\|X_0\|$ (see the proof of Lemma 2). This estimate is also satisfied in the case where $X_0 \in -\mathcal{K}_0$ because $0 \le -X(t) \le -X_0$. In the general case, $X_0 = X_+ - X_- \in \mathcal{E}$, and, by virtue of the monotonicity of the system with respect to $\mathcal{K}_0$ and $\mathcal{K}$, we get $\Omega(t, t_0)(-X_-) \le X(t) \le \Omega(t, t_0)X_+$, where $X_\pm \in \mathcal{K}_0$. Hence, with regard for relation (1) and the fact that the reproducing cone $\mathcal{K}_0$ is unflattened, we obtain the estimate $\|X(t)\| \le c\gamma(\alpha + \beta)\|X_0\|$, which yields the stability of the state $X \equiv 0$ of system (2).

Lemma 3 is proved.

Under the conditions of Lemma 3, the state $X \equiv 0$ of system (2) monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}$ is stable from $\mathcal{K}_0$ into $\mathcal{K}$ and from $-\mathcal{K}_0$ into $-\mathcal{K}$. This guarantees the Lyapunov stability of the state $X \equiv 0$ of the system considered. For linear systems, the stability of the zero state from $\mathcal{K}_0$ into $\mathcal{K}$ is equivalent to its stability from $-\mathcal{K}_0$ into $-\mathcal{K}$.

Lemma 3 can be used for the construction of stability conditions for the classes of monotone differential systems (3) and (4) in terms of the operators $M(t)$ and $G(X, t)$. For system (4), the conditions of Lemma 3 mean that the inequalities $G(X, t) \le M(t)X$ and $G(X, t) \ge M(t)X$ are satisfied for its solutions with initial values from $\mathcal{K}_0$ and $-\mathcal{K}_0$, respectively.

In the investigation of the stability of the state $X \equiv 0$ of system (2), one can use various estimates for $X(t)$ or $\Omega(t, t_0)$ with respect to the cones $\mathcal{K}_0$ and $\mathcal{K}$. For example, if, for any $X_0 = X_+ - X_- \in \mathcal{E}$, one has

$$-\Delta_-(t, t_0)|X_0| \le X(t) \le \Delta_+(t, t_0)|X_0|, \qquad t \ge t_0, \tag{7}$$

where $|X_0| = X_+ + X_-$, $X_\pm \in \mathcal{K}_0$, and $\Delta_\pm(t, t_0)$ are uniformly bounded linear operators, then the stability of the state $X \equiv 0$ follows from the estimate

$$\|X(t)\| \le 2\gamma(\alpha\nu_- + \beta\nu_+)\|X_0\|, \qquad \nu_\pm = \sup\|\Delta_\pm(t, t_0)\| < \infty,$$

which is established with the use of relations (1) and (7) and the assumptions concerning the cones $\mathcal{K}_0$ and $\mathcal{K}$. An analogous statement is valid under the condition

$$-\Delta_-(t, t_0)X_+ - \Delta_+(t, t_0)X_- \le X(t) \le \Delta_+(t, t_0)X_+ + \Delta_-(t, t_0)X_-, \tag{8}$$

which, in the case of a linear system, is equivalent to the two-sided estimate

$$-\Delta_-(t, t_0) \le \Omega(t, t_0) \le \Delta_+(t, t_0), \qquad t \ge t_0.$$

Note that, under the condition of the positivity of $\Delta_\pm(t, t_0)\mathcal{K}_0 \subset \mathcal{K}$, the operators $\Delta_\pm(t, t_0)$ in (7) and (8) should be bounded in norm [2].

Below, we formulate a corollary of Lemmas 2 and 3 for system (3) in terms of the evolution operator $W(t, t_0)$.

Theorem 1. *If the evolution operator* $W(t, t_0)$ *of the differential system (3) satisfies the conditions*

$$W(t, t_0)\mathcal{K}_0 \subset \mathcal{K}, \quad M(t)W(t, t_0)\mathcal{K}_0 \subset \mathcal{K}, \quad t > t_0, \tag{9}$$

then this system is stable.

Consider the class of linear stationary systems

$$\dot{X} + MX = 0. \tag{10}$$

In this case, we have $W(t, t_0) = e^{-M(t-t_0)}$, and the positivity of system (10) with respect to $\mathcal{K}_0$ and $\mathcal{K}$ is equivalent to the condition $e^{-Mt}\mathcal{K}_0 \subset \mathcal{K}$, $t \geq 0$. The following statement is true [5, 7]:

Theorem 2. *If system (10) is positive with respect to* $\mathcal{K}$, *then it is exponentially stable iff the operator* M *is monotonically invertible, i.e.,* $\mathcal{K} \subset M\mathcal{K}$. *If the operator* $M + \gamma I$ *is monotonically invertible for any* $\gamma \geq 0$, *then system (10) is positive with respect to* $\mathcal{K}$ *and exponentially stable.*

Note that if conditions (9) are satisfied, $M(t) \equiv M$, and $\mathcal{K} \subset M\mathcal{K}_0$, then the following systems of inclusions are true:

(a) $\mathcal{K}_0 \subset M\mathcal{K}_0$, $e^{-Mt}\mathcal{K}_0 \subset \mathcal{K}_0$, $t > 0$,

(b) $\mathcal{K} \subset M\mathcal{K}$, $e^{-Mt}\mathcal{K} \subset \mathcal{K}$, $t > 0$.

Each of these systems guarantees the exponential stability of system (10). If $M\mathcal{K}_0 \subset \mathcal{K}$, then conditions (9) follow from inclusions (a) or (b).

In [7, 9], analogous conditions for the exponential stability of certain classes of nonstationary systems (3) were established.

5. Positivity and Stability of Discrete Systems

Consider the discrete system

$$X_{k+1} = M_k X_k + G(X_k, k), \quad k = 0, 1, \ldots, \tag{11}$$

where $M_k : \mathcal{E} \to \mathcal{E}$ is a linear operator, $G(X, k)$ is a nonlinear operator function, and $\mathcal{E}$ is a Banach space with cones $\mathcal{K}_0$ and $\mathcal{K}$. If the inclusion $X_0 \in \mathcal{K}_0$ yields $X_k \in \mathcal{K}$ for any $k = 0, 1, \ldots$, then system (11) is positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$.

Every solution of system (11) satisfies the relation

$$X_{k+1} = W_{k0}X_0 + \sum_{s=0}^{k} W_{ks+1}G(X_s, s),$$

where $W_{k\,k+1} = E$ and $W_{ks} = M_k \ldots W_s$, $k \ge s$. Therefore, system (11) is positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$ if $W_{k0}\mathcal{K}_0 \subset \mathcal{K}$ and the operator functions $W_{k\,s+1}G(X, s)$ are positive on $\mathcal{K}$ for $k \ge s \ge 0$. In the general case, these conditions are not necessary for positivity with respect to $\mathcal{K}_0$ and $\mathcal{K}$. If $G(X, k) \equiv 0$, then the positivity of system (11) with respect to $\mathcal{K}_0$ and $\mathcal{K}$ is equivalent to the positivity of all operators W_{k0}, $k \ge 0$.

Example 4. Consider the discrete control system with dynamical feedback

$$x_{k+1} = Ax_k + bu_k, \quad u_{k+1} = c^T x_k + du_k, \quad k = 0, 1, \ldots, \tag{12}$$

where x_k is the state vector and u_k is a control. We rewrite this system in the form

$$z_{k+1} = Mz_k, \quad M = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}, \quad z_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}$$

and consider a Minkowski cone $\mathcal{K}$ (see Example 2) in the phase space. The positivity of system (12) with respect to $\mathcal{K}$ is equivalent to the inclusion $M\mathcal{K} \subset \mathcal{K}$, and, in view of the fact that the cone is self-dual, it reduces to the inequality $l^T Mz \ge 0$, which must be satisfied for any $l, z \in \mathcal{K}$. Using the Cauchy inequality, we obtain a sufficient condition for the positivity of system (12) with respect to $\mathcal{K}$:

$$\sqrt{\lambda_{\max}(A^T A)} + \|b\| + \|c\| \le d.$$

The fact that a vector z belongs to the cone $\mathcal{K}$ can be described in terms of nonnegative-definite matrices as follows:

$$z = \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{K} \Leftrightarrow u \ge 0, \quad u^2 I \ge xx^T \Leftrightarrow S_z = \begin{bmatrix} uI & x \\ x^T & u \end{bmatrix} \ge 0.$$

Therefore, the positivity of system (12) with respect to $\mathcal{K}$ is equivalent to the condition

$$S_z \ge 0 \Rightarrow S_{Mz} = \begin{bmatrix} (c^T x + du)I & Ax + bu \\ x^T A^T + ub^T & c^T x + du \end{bmatrix} \ge 0.$$

In this case, for any vectors $z \in \mathcal{K}$ and $v \in R^{n+1}$, the following inequality must be satisfied:

$$v^T S_{Mz} v = l_v^T z \ge 0,$$

where

$$l_v = \begin{bmatrix} v^T S_{g_1} v \\ \vdots \\ v^T S_{g_n} v \\ v^T S_g v \end{bmatrix}, \quad S_{g_i} = \begin{bmatrix} c_i I & a_i \\ a_i^T & c_i \end{bmatrix}, \quad S_g = \begin{bmatrix} dI & b \\ b^T & d \end{bmatrix},$$

$$g_i = \begin{bmatrix} a_i \\ c_i \end{bmatrix}, \qquad g = \begin{bmatrix} b \\ d \end{bmatrix},$$

a_i are the columns of the matrix A, and c_i are the elements of the vector c, $i = \overline{1, n}$. The condition $l_v \in \mathcal{K}$, which is equivalent to the inequality $S_{l_v} \geq 0$, is satisfied if

$$S = \begin{bmatrix} S_g & \cdots & 0 & S_{g_1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & S_g & S_{g_n} \\ S_{g_1} & \cdots & S_{g_n} & S_g \end{bmatrix} \geq 0. \tag{13}$$

If $g > 0$ is an interior point of the cone $\mathcal{K}$, then conditions for the positivity of system (12) with respect to $\mathcal{K}$ have the form

$$S_g > 0, \qquad S_g \geq \sum_i S_{g_i} S_g^{-1} S_{g_i}. \tag{14}$$

The condition $g > 0$ means that $d > \sqrt{b^T b}$. If $d = \sqrt{b^T b} > 0$, then it is necessary that $A = d^{-1} b c^T$ for the positivity of system (12) with respect to $\mathcal{K}$.

The asymptotic stability of system (12) is equivalent to each of the following conditions:

(a) $|\lambda| < 1$, $\lambda \in \sigma(M)$;

(b) $M^k \to 0$, $k \to \infty$.

If inequality (13) or (14) is satisfied, then the following conditions may also serve as criteria for the asymptotic stability of system (12):

(c) the matrix $I - M$ is asymptotically invertible;

(d) for some $w > 0$, the equation $z - Mz = w$ has a solution $z \geq 0$.

These conditions can be used for the determination of the parameters c and d of a dynamical compensator that stabilizes system (12).

6. Comparison Systems

Comparison methods based on the mapping of the space of states of a system under study into the spaces of states of auxiliary systems are used in various applied and theoretical investigations. In the analysis of stability problems, it is reasonable to use classes of positive and monotone systems with respect to appropriate cones and nonlinear systems satisfying the conditions of Chaplygin-type and Wazewski-type theorems [12, 13] as comparison systems.

Consider the differential system

$$\dot{x} = f(x, t), \quad x \in \mathcal{X}, \quad t \geq t_0, \tag{15}$$

where f is an operator that guarantees the existence of a unique solution $x(t)$ with values in a Banach phase space $\mathcal{X}$. Let $\mathcal{E}$ be a Banach space partially ordered by a normal cone $\mathcal{K} \subset \mathcal{E}$. In the space $\mathcal{E}$, we construct classes of differential systems

$$\dot{X} = F(X, t), \quad X \in \mathcal{E}, \quad t \geq t_0, \tag{16}$$

which are used as comparison systems for system (15). In $\mathcal{E}$, inequalities for the values of functions at the initial time t_0 will be defined with respect to a certain reproducing cone $\mathcal{K}_0 \subset \mathcal{K}$.

Let Σ_+ denote the class of systems (16) whose solutions can be associated with the solutions of the corresponding differential inequalities

$$\dot{Z} \leq F(Z, t), \quad Z \in \mathcal{E}, \quad t \geq t_0, \tag{17}$$

so that the relation $Z(t_0) \trianglelefteq X(t_0)$ yields $Z(t) \leq X(t)$ for $t > t_0$. It is obvious that every system of the class Σ_+ is monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}$. If $F(0, t) \geq 0$, then system (16) of the class Σ_+ is positive and monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}$.

Let $V(x, t)$ be an operator continuously mapping a certain neighborhood of the point $0 \in \mathcal{X}$ for $t \geq t_0$ into the space $\mathcal{E}$. If $V(x, t)$ and its generalized derivative along solutions of system (15) satisfy the relation

$$D_t V(x, t)\big|_{(15)} \leq F\big(V(x, t), t\big), \tag{18}$$

then system (16) of the class Σ_+ is an upper comparison system, i.e.,

$$V\big(x(t_0), t_0\big) \trianglelefteq X(t_0) \;\Rightarrow\; V\big(x(t), t\big) \leq X(t), \quad t > t_0. \tag{19}$$

In (18), the derivative along solutions of system (15) can be defined as follows:

$$D_t V(x, t)\big|_{(15)} = \limsup_{h \to 0+} \frac{1}{h}[\, V(x + h f(x, t), t + h) - V(x, t)\,].$$

By analogy, we introduce the class of systems Σ_- and the lower comparison systems (16) for system (15). In this case, all inequality signs in (17)–(19) defined by the cones $\mathcal{K}_0$ and $\mathcal{K}$ in the space $\mathcal{E}$ are replaced by the opposite ones.

Let $\mathcal{F}_\pm$ denote the families of operators $F(X, t)$ describing the corresponding classes of systems $\Sigma_\pm$ of the form (16). If $F \in \mathcal{F}_+$ or $F \in \mathcal{F}_-$, then system (16) is monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}$.

If the equality

$$D_t V(x, t)\big|_{(15)} = F\big(V(x, t), t\big) \tag{20}$$

is satisfied instead of (18), then it follows from the definition of the monotonicity of system (16) with respect to $\mathcal{K}_0$ and $\mathcal{K}$ that

$$X_1(t_0) \trianglelefteq V\big(x(t_0), t_0\big) \trianglelefteq X_2(t_0) \Rightarrow X_1(t) \le V\big(x(t), t\big) \le X_2(t) \qquad \forall t \ge t_0, \tag{21}$$

where $X_1(t)$ and $X_2(t)$ are certain solutions of system (16). Therefore, relation (20) defines the class of systems (16) monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}$ that are simultaneously lower and upper comparison systems for system (15).

Estimates (19) and (20) can be used for the comparison of dynamical properties of systems (15) and (16) and for the construction of the attraction region in the phase space of system (15). For example, if the operator V is chosen so that the inequality $V(x, t) \le 0$ is possible only for $x = 0$, then we get $x(t) \to 0$, $t \to \infty$, provided that conditions (19) are satisfied and $X(t) \to 0$.

In the space $\mathcal{E}$, we consider two systems

$$\dot{X}_1 = F_1(X_1, t), \quad X_1 \in \mathcal{E}, \quad t \ge t_0, \tag{22}$$

$$\dot{X}_2 = F_2(X_2, t), \quad X_2 \in \mathcal{E}, \quad t \ge t_0, \tag{23}$$

of the classes Σ_- and Σ_+, respectively. Assume that

$$F_1\big(V(x, t), t\big) \le D_t V(x, t)\big|_{(15)} \le F_2\big(V(x, t), t\big), \quad t \ge t_0. \tag{24}$$

Then a solution of the original system (15) satisfies estimate (21), where $X_1(t)$ and $X_2(t)$ are solutions of the corresponding systems (22) and (23).

Assume that the original system (15) and the comparison systems (22) and (23) have isolated equilibrium states, i.e., $f(0, t) \equiv 0$, $F_1(0, t) \equiv 0$, and $F_2(0, t) \equiv 0$. Also assume that the operator V possesses the following properties:

$$V(0, t) \equiv 0, \quad V(x, t) \ne 0, \quad x \ne 0, \quad t \ge t_0. \tag{25}$$

Theorem 3. *Suppose that $F_1 \in \mathcal{F}_-$, $F_2 \in \mathcal{F}_+$, and the operator V satisfies relations (24) and (25). The zero solution of system (15) is Lyapunov stable (asymptotically Lyapunov stable) if the zero solutions of systems (22) and (23) are stable (asymptotically stable) from $-\mathcal{K}_0$ to $-\mathcal{K}$ and from $\mathcal{K}_0$ to $\mathcal{K}$, respectively.*

Proof. Since the cone $\mathcal{K}_0$ is reproducing and unflattened, we get

$$-X_-^0 \trianglelefteq V(x_0, t_0) = X_+^0 - X_-^0 \trianglelefteq X_+^0, \qquad \|X_\pm^0\| \le \gamma \|V(x_0, t_0)\|,$$

where $X_\pm^0 \in \mathcal{K}_0$ and $\gamma > 0$ is a universal constant.

Let $X_1(t)$ and $X_2(t)$ be solutions of systems (22) and (23) with the initial conditions $X_1(t_0) = -X_-^0$ and $X_2(t_0) = X_+^0$. Since $F_1 \in \mathcal{F}_-$ and $F_2 \in \mathcal{F}_+$, we get $X_1(t) \le 0$ and $X_2(t) \ge 0$ for $t \ge t_0$. Taking into account relation (21) and the normality of the cone $\mathcal{K}$, we get

$$\| V(x(t), t) \| \le \alpha \| X_1(t) \| + \beta \| X_2(t) \|, \quad t > t_0,$$

where $\alpha > 0$ and $\beta > 0$ are universal constants.

It follows from the continuity of the function $V(x, t)$ and conditions (25) that, for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that $\| x(t) \| \le \varepsilon$ whenever $\| V(x(t), t) \| \le \delta_0$. We now use the property of stability of the zero solutions of systems (22) and (23) from $-\mathcal{K}_0$ to $-\mathcal{K}$ and from $\mathcal{K}_0$ to $\mathcal{K}$, respectively. We choose $\delta_1 > 0$ and $\delta_2 > 0$ so that the inequalities $\| X_-^0 \| \le \delta_1$ and $\| X_+^0 \| \le \delta_2$ yield

$$\| X_1(t) \| \le \frac{\delta_0}{2\alpha}, \quad \| X_2(t) \| \le \frac{\delta_0}{2\beta}, \quad t > t_0.$$

Finally, we choose $\delta > 0$ so that $\| x_0 \| \le \delta$ yields $\| V(x_0, t_0) \| \le \min\{ \delta_1, \delta_2 \} / \gamma$. Then, with regard for the arguments presented above, we get $\|x(t)\| \le \varepsilon$ for $t > t_0$, i.e., the zero solution of system (15) is Lyapunov stable. Moreover, $\| x(t) \| \to 0$ if $\| X_1(t) \| \to 0$ and $\| X_2(t) \| \to 0$ as $t \to \infty$.

Theorem 3 is proved.

The analysis of the stability of the zero solution of system (15) can be carried out on the basis of the construction of only upper comparison systems under certain additional restrictions on the operator V.

Theorem 4. *Suppose that $F \in \mathcal{F}_+$, the operator V satisfies relations (18) and (25), and $V(x, t) \ge 0$ for $x \in \mathcal{X}$ and $t \ge t_0$. Then the zero solution of system (15) is Lyapunov stable (asymptotically Lyapunov stable) if the zero solution of system (16) is stable (asymptotically stable) from $\mathcal{K}_0$ to $\mathcal{K}$.*

The proofs of Theorems 3 and 4 are analogous.

Remark 1. Under the conditions of Theorem 4, the comparison system (16) must be positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$. In the construction of the upper comparison systems (16) positive or monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}$, the operator V can be chosen from the class of positive operators. Theorem 4 remains true if the condition $V(x, t) \ge 0$ is replaced by the weaker requirement that $\varphi_0(V(x, t)) > 0$, $x \ne 0$, $t \ge t_0$, in a certain neighborhood of the point $x = 0$ for some $\varphi_0 \in \mathcal{K}^*$.

As an example, for the linear system $\dot{x} = A(t)x$, $x \in R^n$, we present the upper matrix comparison system

$$\dot{X} = A(t)X + XA(t)^T + P(t)X + Y(t), \quad X \in R^{n \times n}, \tag{26}$$

which is constructed on the basis of (18) with the operator $V(x) = xx^T$. Here, $P(t)$ is a linear operator monotone with respect to the cone of symmetric nonnegative-definite matrices $\mathcal{K}$ and $Y(t) = Y(t)^T \ge 0$. Equation (26) is a system of the class Σ_+ positive with respect to $\mathcal{K}$. The asymptotic stability of Eq. (26) yields the asymptotic stability of the original system.

Note that the upper and lower comparison systems for system (15) can be constructed in different partially ordered spaces $\mathcal{E}_1$ and $\mathcal{E}_2$. In this case, the properties of the corresponding operators $V_1(x, t)$ and $V_2(x, t)$ and the order relations defined by the cones $\mathcal{K}_1 \subset \mathcal{E}_1$ and $\mathcal{K}_2 \subset \mathcal{E}_2$ in the relations

$$V_1(x(t), t) \ge X_1(t), \quad V_2(x(t), t) \le X_2(t), \quad t \ge t_0,$$

must be coordinated for the investigation of certain characteristics of the original system (15). For example, one can require that the system of inequalities $V_1(x, t) \ge 0$ and $V_2(x, t) \le 0$ be satisfied only for $x = 0$. In this case, one may expect that $x(t) \to 0$ as $t \to \infty$ if $X_1(t) \to 0$ and $X_2(t) \to 0$, where $X_1(t)$ $(X_2(t))$ is a solution of the lower (upper) comparison system.

7. Robust Stability of a Family of Systems

In applied studies, one encounters the problem of the stability of a given family of systems described by differential or difference equations with uncertain parameters (the problem of robust stability). Below, we describe a method for the analysis of the robust stability of the family of systems

$$\dot{X} = F(X, t), \quad F(0, t) \equiv 0, \tag{27}$$

$$\underline{F}(X, t) \le F(X, t) \le \overline{F}(X, t), \quad X \subset \mathcal{E}, \quad t \ge t_0, \tag{28}$$

where the inequalities are defined by a normal cone $\mathcal{K} \subset \mathcal{E}$. The inequalities for the values of the functions at initial time t_0 are defined, as above, with respect to the reproducing cone $\mathcal{K}_0 \subset \mathcal{K}$.

We select the following two systems in family (27), (28):

$$\dot{\underline{X}} = \underline{F}(\underline{X}, t), \quad \underline{F}(0, t) \equiv 0, \tag{29}$$

$$\dot{\overline{X}} = \overline{F}(\overline{X}, t), \quad \overline{F}(0, t) \equiv 0. \tag{30}$$

If $\underline{F} \in \mathcal{F}_-$ and $\overline{F} \in \mathcal{F}_+$, then the solutions of every system of the given family are bounded by the corresponding solutions of systems (29) and (30), i.e.,

$$\underline{X}(t_0) \trianglelefteq X(t_0) \trianglelefteq \overline{X}(t_0) \Rightarrow \underline{X}(t) \le X(t) \le \overline{X}(t) \quad \forall\, t \ge t_0. \tag{31}$$

Therefore, systems (29) and (30) can be regarded as, respectively, the lower and the upper comparison systems for system (27). By setting $V(X, t) \equiv X$ in Theorem 3, we obtain the following conditions for the robust stability of the family of systems (27), (28):

Theorem 5. *If $\underline{F} \in \mathcal{F}_-$, $\overline{F} \in \mathcal{F}_+$, and the zero solutions of systems (29) and (30) are stable (asymptotically stable) from $-\mathcal{K}_0$ to $-\mathcal{K}$ and from $\mathcal{K}_0$ to $\mathcal{K}$, respectively, then the zero solution of every system of family (27), (28) is Lyapunov stable (asymptotically Lyapunov stable).*

In using Theorems 3–5, it is necessary to establish that operators belong to the classes $\mathcal{F}_\pm$. If $\mathcal{E} = R^n$, then the classes $\mathcal{F}_\pm$ defined with the use of the cone of nonnegative vectors contain functions satisfying the Wazewski conditions. The generalized property of quasimonotonicity with respect to a cone $\mathcal{K} \subset \mathcal{E}$ is possessed by operator functions $F \in \mathcal{F}$ (see Sec. 3).

Lemma 4. *If a cone* $\mathcal{K}$ *is solid, then* $\mathcal{F} \subset \mathcal{F}_+ \cap \mathcal{F}_-$.

Proof. We proceed by analogy with the proof of Lemma 1. Assume that $F \in \mathcal{F}$, the functions $Y(t)$ and $Z(t)$ satisfy the relations

$$\dot{Y} = F(Y, t) + \varepsilon Q, \quad \dot{Z} \le F(Z, t), \quad t \ge t_0,$$

where $\varepsilon > 0$ and $Q > 0$, and, furthermore, for some $\varphi \in \mathcal{K}^*$ and $\tau \ge t_0$, we have

$$Z(\tau) \le Y(\tau), \quad \varphi(Z(\tau)) = \varphi(Y(\tau)),$$

$$\varphi(Z(t)) > \varphi(Y(t)), \quad \tau < t \le \tau + \delta.$$

Taking into account the assumptions made, we get

$$\dot{Y}(\tau) - \dot{Z}(\tau) \ge F(Y(\tau), \tau) - F(Z(\tau), \tau) + \varepsilon Q,$$

$$\varphi\big(\dot{Y}(\tau) - \dot{Z}(\tau)\big) \ge \varepsilon\varphi(Q) > 0.$$

Therefore, for some $\delta > 0$, we obtain

$$\int_{\tau}^{\tau+\delta} \varphi\big(\dot{Y}(t) - \dot{Z}(t)\big) dt = \varphi(Y(\tau+\delta)) - \varphi(Z(\tau+\delta)) \ge 0,$$

which contradicts the assumption.

Consequently, $Z(t) \le Y(t)$. As $\varepsilon \to 0$, we get $Z(t) \le X(t)$, where $X(t)$ is a solution of system (27), i.e., $F \subset \mathcal{F}_+$. In this case, the inequality $Z(t_0) \trianglelefteq X(t_0)$ can be considered with respect to an arbitrary cone $\mathcal{K}_0 \subset \mathcal{K}$. Similarly, $F \subset \mathcal{F}_-$.

Lemma 4 is proved.

8. Multiply Connected Systems

The operation of interrelated objects combined in a large-scale system can be described as follows:

$$\dot{X}_i + A_i(t)X_i = G_i(X, t), \quad t \ge t_0, \quad i = \overline{1, s}, \tag{32}$$

where $X_i \in \mathcal{E}_i$ are the states of subsystems forming a phase vector $X \in \mathcal{E}$, $A_i(t)$ are given operators, and G_i are coupling functions. In the investigation of stability conditions for solutions of such systems, it is necessary to take into account the structure of the phase space $\mathcal{E}$, which can be inhomogeneous by virtue of physical properties of the components of subsystems (see, e.g., [14, 15].

Assume that $\mathcal{E}_i = R^{n_i}$ and a dominant subsystem with vector $X_s = U$ is selected in system (32) in the sense that, for $t > t_0$, the inclusion $X(t_0) \in \mathcal{K}_0$ yields

$$X(t) \in \mathcal{K}_\alpha = \left\{ X \in R^n : \max_{1 \le i \le s-1} \| X_i \| \le (1+\alpha) \min_{1 \le j \le n_s} u_j \right\}, \tag{33}$$

where $\alpha \ge 0$, $n = n_1 + \ldots + n_s$, and u_j are the components of the vector U. As U, one can take, e.g., the control vector. One can show that the set $\mathcal{K}_\alpha$ is a normal solid cone in $\mathcal{E}$. Therefore, condition (33) expresses the property of the positivity of a system with respect to $\mathcal{K}_0$ and $\mathcal{K}_\alpha$.

In using Lemma 2, it is necessary that, along with (33), the following condition be satisfied:

$$\max_{1 \le i \le s-1} \| A_i(t)X_i - G_i(X,t) \| \le \min_{1 \le j \le n_s} \left(\sum_k a_{jk}^{(s)}(t) u_k - g_{sj}(X,t) \right),$$

where $a_{jk}^{(s)}(t)$ are the elements of the matrix $A_s(t)$. If system (33) is monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}_\alpha$, then the stability of its solutions can be established by using Lemma 3.

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