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POSITIVE AND MONOTONE SYSTEMS IN A PARTIALLY ORDERED SPACE

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We investigate properties of positive and monotone differential systems with respect to a given cone in the phase space. We formulate criteria for the stability of linear positive systems in terms of monotonically invertible operators and develop methods for the comparison of systems in a partially ordered space.

1. Introduction

The positivity (monotonicity) of a dynamical system is equivalent to the positivity (monotonicity) of a certain operator (describing the motion of this system) with respect to a given cone in the phase space [1–3]. The Lyapunov and Riccati differential equations are examples of positive systems with respect to the cone of symmetric nonnegative-definite matrices. The properties of positive and monotone systems are used in various problems of analysis and synthesis. The investigation of the stability of a class of linear autonomous positive systems reduces to the solution of algebraic equations determined by the operator coefficients of the given systems [4–6].

In the present paper, we investigate conditions for the positivity and monotonicity of differential systems in a partially ordered Banach space. We propose methods for the analysis of the stability of linear positive systems based on the solution of algebraic equations with monotonically invertible operators. We also present analogs of comparison systems in a partially ordered space.

2. Operators in a Space with a Cone

A convex closed set $\mathcal{K}$ of a real normed space $\mathcal{E}$ is called a cone if $\mathcal{K} \cap -\mathcal{K} = \{0\}$ and $\alpha X + \beta Y \in \mathcal{K}$ for all $X, Y \in \mathcal{K}$ and $\alpha, \beta \geq 0$. A space with a cone is partially ordered: $X \leq Y$ $(X < Y)$ $\Leftrightarrow$ $Y - X \in \mathcal{K}$ $(Y - X \in \mathcal{K}_0)$, where $\mathcal{K}_0$ is the set of interior points of $\mathcal{K}$. A cone $\mathcal{K}$ is called normal if it follows from the inequality $0 \leq X \leq Y$ that $\|X\| \leq c\|Y\|$, where c is a universal constant. If $\mathcal{E} = \mathcal{K} - \mathcal{K}$, then the cone $\mathcal{K}$ is reproducing.

Let a cone $\mathcal{K}_1 \subset \mathcal{E}_1$ $(\mathcal{K}_2 \subset \mathcal{E}_2)$ be specified in a Banach space $\mathcal{E}_1$ $(\mathcal{E}_2)$. An operator $M: \mathcal{E}_1 \to \mathcal{E}_2$ is called monotone (monotone on the cone $\mathcal{K}$) if $MX \geq MY$ for $X \geq Y$ $(X \geq Y \geq 0)$. The monotonicity of a linear operator is equivalent to its positivity: $X \geq 0 \Rightarrow MX \geq 0$. The inequality between operators $M \leq L$ means that the operator $L - M$ is positive. If $M\mathcal{E}_1 \subset \mathcal{K}_2$, then the operator M is everywhere positive. A linear operator M is called monotonically invertible if, for every $Y \in \mathcal{K}_2$, the equation $MX = Y$ has a solution $X \in \mathcal{K}_1$. If $\mathcal{K}_2$ is a normal reproducing cone and $M_1 \leq M \leq M_2$, then the monotone invertibility of the operators M_1 and M_2 implies the monotone invertibility of the operator M; in this case, $M_2^{-1} \leq M^{-1} \leq M_1^{-1}$ [1].

We specify the class of linear operators [5]

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$$M = L - P, \quad P\mathcal{K}_1 \subset \mathcal{K}_2 \subset L\mathcal{K}_1,$$

where $\mathcal{K}_2$ is a normal reproducing cone. A criterion for the monotone invertibility of such operators is the inequality $\rho(T) < 1$, where $\rho(T)$ is the spectral radius of the pencil of operators $T(\lambda) = P - \lambda L$. If the cone $\mathcal{K}_2$ is solid, then this inequality is equivalent to the solvability of the equation $MX = Y$ in the form $X \geq 0$ for some $Y > 0$.

3. Positive and Monotone Systems

Let $\mathcal{E}$ be a Banach space partially ordered by a cone $\mathcal{K}$ and let $X(t) \in \mathcal{E}$ be a state of a dynamical system with continuous or discrete time $t \geq 0$. A system is called (t, t_0)-positive if the relation $X(t_0) = X_0 \in \mathcal{K}$ implies that $X(t) \in \mathcal{K}$. A system is positive if it is (t, t_0)-positive for all $t > t_0 \geq 0$. This property of a system is equivalent to the positivity of the operator of motion of the system $V(t, t_0): \mathcal{E} \to \mathcal{E}$ determining the transition from the state X_0 to the state $X(t) = V(t, t_0) X_0$ for $t > t_0 \geq 0$. A system is called monotone (monotone on the cone $\mathcal{K}$) if its operator of motion $V(t, t_0)$ is monotone (monotone on the cone $\mathcal{K}$) for $t > t_0 \geq 0$.

Consider the linear differential system

$$\dot{X}(t) + M(t)X(t) = G(t), \quad t \geq 0, \tag{1}$$

where $M(t)$ is a bounded operator acting in a partially ordered space $\mathcal{E}$ with cone $\mathcal{K}$. Assume that every initial condition $X(t_0) = X_0$ of system (1) determines the unique solution

$$X(t) = W(t, t_0)X_0 + \int_{t_0}^{t} W(t, s)G(s)ds, \tag{2}$$

where $W(t, s) = W(t, t_0)[W(s, t_0)]^{-1}$ is the evolution operator and $t \geq t_0 \geq 0$. The linear operator $W(t, t_0)$ is representable as the series

$$W(t, t_0) = I - \int_{t_0}^{t} M(t_1)dt_1 + \int_{t_0}^{t} M(t_2)\int_{t_0}^{t_2} M(t_1)dt_1\,dt_2 - \ldots$$

uniformly convergent with respect to the operator norm. According to (2), $X(t) \in \mathcal{K}$ for any initial condition $X_0 \in \mathcal{K}$ if

$$W(t, t_0) \geq 0, \quad \int_{t_0}^{t} W(t, s)G(s)ds \geq 0. \tag{3}$$

Here, the first inequality means the monotonicity of the operator with respect to the cone $\mathcal{K}$, and the second inequality means that the value of the function belongs to the given cone. The converse statement can easily be proved with regard for the fact that the cone $\mathcal{K}$ is closed. Consequently, system (1) is positive if and only if relations (3) hold for $t > t_0 \geq 0$.

Using formulas (2) and (3), one can establish the equivalence of the following statements:

(a) for any function $G(t) \geq 0$, system (1) is $(t, 0)$-positive for $t \geq 0$;

(b) system (1) is monotone;

(c) the operator $W(t, s)$ is monotone for $t > s \geq 0$;

(d) it follows from the relations $\dot{Z}(t) + M(t)Z(t) \geq 0$ and $Z(0) = Z_0 \geq 0$ that $Z(t) \geq 0$ for $t > 0$.

If $G(t) \geq 0$, then each of statements (a)–(d) is equivalent to the positivity of system (1).

Lemma 1. *The evolution operator $W(t, s)$ is monotone for $t \geq s \geq 0$ if and only if the exponential operator $e^{-M(t)h}$ is monotone for $t \geq 0$ and $h \geq 0$.*

Proof. We use the procedure of representation of the operator $W(t, s)$ in the form of a multiplicative integral [7]. Partitioning the segment $[s, t]$ by the points $t_{kn} = s + kh_n$, where $h_n = (t-s)/n$, $k = 0, \dots, n$, for large values of n we get

$$W(t, s) = W(t_{nn}, t_{n-1n})\, W(t_{n-1n}, t_{n-2n}) \dots W(t_{1n}, t_{0n}),$$

$$W(t_{kn}, t_{k-1n}) = e^{-M(\theta_{kn})h_n} + o(h_n), \qquad k = 1, \dots, n,$$

where $\theta_{kn} \in [t_{k-1n}, t_{kn}]$ are certain intermediate points. Therefore,

$$W(t, s) = \lim_{n\to\infty} \left[e^{-M(\theta_{nn})h_n} \dots e^{-M(\theta_{1n})h_n} \right].$$

If $e^{-M(t)h} \geq 0$ for all $t \geq 0$ and $h \geq 0$, then the operator $W(t, s)$ is the limit of a certain sequence of monotone operators and is monotone by virtue of the closedness of the cone of linear monotone operators.

The converse statement is proved by analogy on the basis of the relations

$$W(t, t-h/n) = e^{-M(\theta_n)h/n} + o(1/n),$$

$$e^{-M(t)h} = \lim_{n\to\infty} [W(t, t-h/n)]^n,$$

where $\theta_n \in [t-h/n, t]$, $n = 1, 2, \dots$.

The lemma is proved.

Lemma 2. *If $M(t) = M_1(t) + M_2(t)$ and the operators $W_{M_1}(t, s)$ and $W_{M_2}(t, s)$ are monotone for $t \geq s \geq 0$, then the operator $W_M(t, s)$ is also monotone for $t \geq s \geq 0$.*

The proof is based on Lemma 1 and the relations [5]

$$e^{-(M_1+M_2)\tau} = \lim_{k\to\infty} [E(\tau/k)]^k, \quad E(h) = \frac{1}{2}\left(e^{-M_1h}e^{-M_2h} + e^{-M_2h}e^{-M_1h}\right).$$

The positivity of a system can be used for the estimation of its solutions. If functions $X_1(t)$ and $X_2(t)$ satisfy the inequalities

$$\dot{X}_1(t) + M(t)X_1(t) \le G_1(t), \quad \dot{X}_2(t) + M(t)X_2(t) \ge G_2(t), \quad X_1(t_0) \le X_2(t_0),$$

then, under conditions (3), the following relations are true:

$$X_2(t) - X_1(t) \ge W(t,t_0)[X_2(t_0) - X_1(t_0)] + \int_{t_0}^{t} W(t,s)G(s)ds \ge 0,$$

where $G(t) = G_2(t) - G_1(t)$. This yields the following statement:

Lemma 3. *Let $X(t)$ be a solution of the positive system (1) and let functions $X_1(t)$ and $X_2(t)$ satisfy the inequalities*

$$\dot{X}_1(t) + M(t)X_1(t) \le \alpha G(t) \quad \text{and} \quad \dot{X}_2(t) + M(t)X_2(t) \ge \beta G(t),$$

where $\alpha \le 1$ and $\beta \ge 1$. Then the relation $X_1(t_0) \le X(t_0) \le X_2(t_0)$ implies that $X_1(t) \le X(t) \le X_2(t)$ for $t \ge t_0$.

If $\alpha = 0$, then the lower bound $X_1(t)$ of a solution of system (1) in Lemma 3 does not depend on the right-hand side $G(t)$ of system (1). In the case $\alpha = \beta = 1$, the statement of Lemma 3 is true if $W(t,s) \ge 0$, $t \ge s \ge t_0$.

We generalize the differential system (1) as follows:

$$\dot{X} + M(t)X = G(X,t), \quad t \ge 0, \tag{4}$$

where G is a nonlinear operator. The positivity (monotonicity) of system (4) is equivalent to the positivity (monotonicity) of the operator of shift along the trajectories $X(t) = V(t,t_0)X_0$. The solutions of system (4) satisfy the integral equation

$$X(t) = W(t,t_0)X_0 + \int_{t_0}^{t} W(t,s)G(X(s),s)ds, \tag{5}$$

where $W(t,s)$ is the evolution operator of the linear system (1). It follows from (5) that system (4) is positive if the operator $G(X,t)$ is everywhere positive and the operator $W(t,s)$ is monotone for $t \ge s \ge 0$.

Assume that the operator $W(t,s)$ is monotone with respect to a solid cone $\mathcal{K}$ and let $\mathcal{F}_0$ and $\mathcal{F}$ be the families of operators G defined by the conditions

$$X \geq 0, \quad \varphi \in \mathcal{K}^*, \quad \varphi(X) = 0 \quad \Rightarrow \quad \varphi(G(X,t)) \geq 0$$

and

$$X \geq Y, \quad \varphi \in \mathcal{K}^*, \quad \varphi(X-Y) = 0 \quad \Rightarrow \quad \varphi(G(X,t) - G(Y,t)) \geq 0,$$

respectively; here, $\mathcal{K}^*$ is the conjugate cone of linear functionals. Then the positivity (monotonicity) of system (4) can be established for $G \in \mathcal{F}_0$ ($G \in \mathcal{F}$).

Lemma 4. *Let $X(t)$ be a solution of system (4) with a monotone operator $W(t,s)$, and let functions $X_1(t)$ and $X_2(t)$ satisfy the relations*

$$\dot{X}_1 + M(t)X_1 = G_1(X_1, t), \qquad \dot{X}_2 + M(t)X_2 = G_2(X_2, t),$$

$$G_1(X.t) \leq G(X.t) \leq G_2(X.t), \qquad G_1,\, G_2 \in \mathcal{F}, \qquad t \geq t_0.$$

Then the relation $X_1(t_0) \leq X(t_0) \leq X_2(t_0)$ implies that $X_1(t) \leq X(t) \leq X_2(t)$ for $t \geq t_0$.

Examples. The nonlinear differential system

$$\dot{x} + A(t)x = g(x,t), \qquad x \in R^n, \quad t \geq 0,$$

where $A(t)$ is a matrix with nonpositive diagonal elements, is positive with respect to the cone of nonnegative vectors $\mathcal{K}$ if the vector function $g(x,t)$ satisfies the conditions [2]

$$x \geq 0, \qquad x_i = 0 \quad \Rightarrow \quad g_i(x,t) \geq 0, \quad i = 1, \dots, n,$$

and it is monotone with respect to the same cone if $g(x,t)$ is quasimonotone and nondecreasing with respect to x (the Ważewski condition), i.e.,

$$x \leq y, \qquad x_i = y_i \quad \Rightarrow \quad g_i(x,t) \leq g_i(y,t), \quad i = 1, \dots, n.$$

If both restrictions on $g(x,t)$ are satisfied for $0 \leq x \leq y$, then the given system is monotone on the cone $\mathcal{K}$.

Examples of differential equations positive with respect to the cone of symmetric nonnegative-definite matrices are the Lyapunov and Riccati differential equations and the more general equation

$$\dot{X} - A(t)X - XA(t)^T - \sum_k B_k(t)XB_k(t)^T = XC(t)X + D(t),$$

where $A(t)$, $B_k(t)$, $C(t) = C(t)^T \geq 0$, and $D(t) = D(t)^T \geq 0$ are given matrices. In the case $C(t) \equiv 0$, the given system also possesses the property of monotonicity. In the example considered, the operator $M(t)$ has the following structure:

$$M(t) = L(t) - P(t), \quad L(t)X = -A(t)X - XA(t)^T, \quad P(t)X = \sum_k B_k(t)XB_k(t)^T.$$

In this case, the monotonicity of the evolution operator $W_M(t,s)$ follows from Lemma 2 and the relations

$$W_L(t,s)X = \Omega(t,s)X\Omega(t,s)^T \geq 0, \qquad W_{-P}(t,s)X \geq 0 \quad \forall X = X^T \geq 0,$$

where $\Omega(t,s)$ is the matrizant of the system $\dot{x} = A(t)x$, $t \geq s \geq 0$.

A linear matrix differential equation of the form

$$\dot{X} - A(t)X - XA(t)^T - \sum_k B_k(t)XB_k(t)^T = 0$$

is known as the equation of second moments for the Itô stochastic system

$$dx(t) - A(t)x(t)dt = \sum_k B_k(t)x(t)dw_k(t),$$

where w_k are components of the standard Wiener process. This equation possesses the properties of positivity and monotonicity and is used in the theory of stability of stochastic systems.

4. Stability of Linear Positive Systems

Let a normal reproducing cone $\mathcal{K}$ be specified in the phase space of the linear autonomous system

$$\dot{Z}(t) + MZ(t) = 0, \qquad t \geq 0, \tag{6}$$

where M is a bounded operator. The positivity of this system is equivalent to the monotonicity of the operator e^{-Mt} with respect to the cone $\mathcal{K}$ for $t \geq 0$. Therefore, one can use the properties of one-parameter positive semigroups [3] for the investigation of conditions of stability of the positive system (6).

We define the limit of growth of the operator exponential and the spectral bound as follows:

$$\gamma_M = \lim_{t\to\infty} \frac{1}{t} \ln \left\| e^{-Mt} \right\| < \infty, \qquad \alpha_M = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(M)\}.$$

It follows from the theorem on the mapping of the spectrum of a bounded operator that $\gamma_M = -\alpha_M$. The spectral radius of a monotone linear operator is a point of its spectrum (see the Krein–Bonsall–Karlin theorems [1]). Therefore, for the positive system (6), we have $\alpha_M \in \sigma(M)$.

Lemma 5. *If system (6) is positive, then the relations* $(M+\gamma I)^{-1} \geq 0$ *and* $\gamma > \gamma_M$ *are equivalent. If* $(M+\gamma I)^{-1} \geq 0$ *for every* $\gamma \geq \gamma_0$, *then system (6) is positive and* $\gamma_0 > \gamma_M$.

Proof. If system (6) is positive, then, for every $\gamma > \gamma_M$, the following relation is true:

$$(M+\gamma I)^{-1} = \int_0^\infty e^{-\gamma t} e^{-Mt}\, dt \ \geq\ 0.$$

Conversely, if the operator $M+\gamma I$ is monotonically invertible for every $\gamma \geq \gamma_0$, where γ_0 is a real number, then

$$e^{-Mt} = \lim_{k\to\infty} \left[t_k (M+t_k I)^{-1}\right]^k \geq 0, \qquad t_k = \frac{k}{t}, \quad t \geq 0.$$

Let us show that, for the positive system (6), the operator $M + \gamma I$ is not monotonically invertible for $\gamma \leq \gamma_M$. Assume that the operators $M_1 = M + \gamma_1 I$ and $M_2 = M + \gamma_2 I$ are monotonically invertible for some numbers γ_1 and γ_2 such that $\gamma_1 < \gamma_M < \gamma_2$. Then it follows from the relation $M_1 \leq M + \gamma_M I \leq M_2$ and the theorem on a two-sided estimate of monotonically invertible operators that the operator $M + \gamma_M I$ is also monotonically invertible [1]. However, this contradicts the condition $\alpha_M = -\gamma_M \in \sigma(M)$.

Consequently, under the condition of the positivity of system (6), the operator $M+\gamma I$ is monotonically invertible only for $\gamma > \gamma_M$.

The lemma is proved.

Lemma 6. *If* $(M-\alpha I)^{-1} \geq 0$ *for every* $\alpha \leq \alpha_0$, *then the spectrum of the operator* M *lies in the half-plane* $\operatorname{Re}\lambda > \alpha_0$.

Proof. It follows from the invertibility of the operator $M - \alpha I$ that the operator M does not have real points of its spectrum on the interval $(-\infty, \alpha_0]$. The spectral radius of the monotone operator $(M-\alpha I)^{-1}$ is equal to $1/(\alpha_* - \alpha)$, where α_* is a real point of the spectrum $\sigma(M)$ such that $|\lambda - \alpha| \geq \alpha_* - \alpha > 0$ $\forall\, \lambda \in \sigma(M)$. In this case, $\alpha_* > \alpha_0 \geq \alpha$ and α_* does not depend on α. If $\operatorname{Re}\lambda \leq \alpha_0$, then one can choose α so that the opposite inequality is satisfied: $|\lambda - \alpha| < \alpha_* - \alpha$. Consequently, $\operatorname{Re}\lambda > \alpha_0$ for $\lambda \in \sigma(M)$. In this case, α_* coincides with α_M.

The lemma is proved.

If $\alpha_M > 0$, then system (6) is exponentially stable, i.e.,

$$\|Z(t)\| \leq \beta e^{-\gamma(t-t_0)} \|Z_0\|, \qquad t \geq t_0,$$

where γ and β are positive constants independent of the choice of a solution and $\gamma < \alpha_M$. For the positive system (6) having the partial solution $Z(t) = e^{-\alpha_M (t-t_0)} V$, where $V \neq 0$, the converse statement is also true. Taking into account Lemmas 5 and 6 for $\gamma_0 = \alpha_0 = 0$, we formulate the following statement:

Theorem 1. *If the operator* $M+\gamma I$ *is monotonically invertible for every* $\gamma \geq 0$, *then system (6) is positive and exponentially stable. If system (6) is positive, then it is exponentially stable if and only if the operator* M *is monotonically invertible.*

Note that the exponential stability of system (6) follows from the monotone invertibility of the operators M and $M+\gamma_0 I$, where $\gamma_0 > 0$ is a sufficiently large number. Indeed, for $\gamma \in [0, \gamma_0]$, every operator $M + \gamma I$

must be monotonically invertible; in this case, $|\lambda + \gamma_0| \geq \alpha_M + \gamma_0 > 0 \ \ \forall\, \lambda \in \sigma(M)$ (see the proofs of Lemmas 5 and 6). For sufficiently large γ_0, this yields the inequality $\alpha_M > 0$, under which system (6) is exponentially stable.

The fact that the monotone invertibility of the operator M is a necessary condition for the exponential stability of the positive system (6) can also be established by using the results of [6]. If the operator M is monotonically invertible with respect to a solid cone $\mathcal{K}$, then there exist elements $X > 0$ and $Y > 0$ satisfying the equation $MX = Y$, and the positive system (6) must be exponentially stable [6, p. 38]. The known criteria for the asymptotic mean-square stability of the Itô stochastic systems (see Sec. 3) are corollaries of Theorem 1.

Now consider nonautonomous systems of the form

$$\dot{X}(t) + M(t)X(t) = 0, \qquad t \geq 0. \tag{7}$$

System (7) is called positive reducible if there exists a Lyapunov transformation $X(t) = Q(t)Z(t)$ that reduces it to the positive autonomous system (6). In this definition, $Q(t)$ is a uniformly bounded differential operator having the uniformly bounded inverse $Q^{-1}(t)$ and satisfying the operator differential equation

$$\dot{Q}(t) + M(t)Q(t) - Q(t)M = 0, \qquad t \geq 0.$$

In this case, conditions for the stability of systems (6) and (7) are equivalent [7]. Consequently, the positive-reducible system (7) is exponentially stable if and only if the operator M is monotonically invertible.

Examples of reducible systems are the ω-periodic systems (7) for which

$$M(t+\omega) = M(t), \qquad W(t+\omega) = W(t)W(\omega),$$

$$Q(t) = W(t)e^{Mt}, \qquad M = -\omega^{-1}\ln W(\omega),$$

and the spectrum of the monodromy operator $W(\omega) = W(\omega, 0)$ does not surround zero.

Consider the subclass of systems (7) described by a functionally commutative operator $M(t)$, i.e.,

$$M(t)M(\tau) \equiv M(\tau)M(t), \qquad t \geq 0, \quad \tau \geq 0. \tag{8}$$

In this case, the evolution operator is defined by the relations [8]

$$W(t, s) = e^{-N(t,s)}, \qquad N(t, s) = \int_s^t M(\tau)\,d\tau, \quad t \geq s. \tag{9}$$

Assume that there exists the limiting bounded operator

$$M = \lim_{t\to\infty} \frac{1}{\varphi(t)} \int_{t_0}^{t} M(\tau)\,d\tau, \tag{10}$$

where $\varphi(t) > 0$ is a function such that $\varphi(t) \to \infty$ as $t \to \infty$.

Theorem 2. *Suppose that conditions (8) are satisfied and system (6) with operator (10) is positive. Then the monotone invertibility of operator (10) yields the asymptotic stability of system (7).*

Proof. Relations (8)–(10) yield

$$M(t)N(t,\tau) = N(t,\tau)M(t), \qquad MN(t,t_0) = N(t,t_0)M,$$

$$M\Delta(t,t_0) = \Delta(t,t_0)M, \qquad \Delta(t,t_0) = \frac{1}{\varphi(t)}N(t,t_0) - M,$$

where $\Delta(t,t_0) \to 0$ as $t \to \infty$. Therefore, in view of (9), an arbitrary solution of system (7) can be represented in the form

$$X(t) = e^{-\varphi(t)[M+\Delta(t,t_0)]}X_0 = e^{-\varphi(t)M}e^{-\varphi(t)\Delta(t,t_0)}X_0.$$

It follows from the monotone invertibility of the operator M of the positive system (6) that $\alpha_M > 0$ (see the proof of Lemma 5). For every $\varepsilon > 0$, there exists $t_1 \geq t_0$ such that $\|\Delta(t,t_0)\| < \varepsilon$ for $t > t_1$. In this case, the following estimate is true:

$$\|X(t)\| \leq \beta e^{-\varphi(t)(\alpha_M-\varepsilon)}e^{\varphi(t)\varepsilon}\|X_0\| = \beta e^{-\varphi(t)(\alpha_M-2\varepsilon)}\|X_0\|,$$

where $\beta > 0$ is a certain constant. Setting $\varepsilon < \alpha_M/2$ and taking into account that $\varphi(t) \to \infty$, we establish that $\|X(t)\| \to 0$ as $t \to \infty$. Consequently, system (7) is asymptotically stable.

The theorem is proved.

Example. Consider the matrix system (7) with

$$M(t) = \begin{bmatrix} a(t) & -b(t) \\ -b(t) & a(t) \end{bmatrix},$$

where $a(t)$ and $b(t)$ are given functions. The matrix $M(t)$ obviously satisfies the condition of functional commutativity (8). Assume that

$$\varphi(t) = \int_{t_0}^{t} b(s)ds \to \infty, \qquad \frac{1}{\varphi(t)}\int_{t_0}^{t} a(s)ds \to c, \qquad t \to \infty.$$

Then the limiting matrix (10) has the form

$$M = \begin{bmatrix} c & -1 \\ -1 & c \end{bmatrix}.$$

The autonomous system (6) with matrix M is positive with respect to the cone of nonnegative vectors (see Sec. 2). The condition of the monotone invertibility of the matrix M reduces to the inequality $c > 1$. In this case, according to Theorem 2, the original system (7) is asymptotically stable.

5. Robust Stability

One of important applied problems is the problem of the stability of a given family of systems with indefinite parameters. Consider a family of differential systems of the form

$$\dot{X} + M(t)X = G(X,t), \qquad \underline{M}(t) \le M(t) \le \overline{M}(t), \quad t \ge 0, \tag{11}$$

$$G_1(t) - M_1(t)X \le G(X,t) \le G_2(t) - M_2(t)X. \tag{12}$$

In this family, we select the following two systems:

$$\dot{X}_1 + \left[\overline{M}(t) + M_1(t)\right]X_1 = G_1(t), \qquad t \ge 0, \tag{13}$$

$$\dot{X}_2 + \left[\underline{M}(t) + M_2(t)\right]X_2 = G_2(t), \qquad t \ge 0. \tag{14}$$

Lemma 7. *Suppose that the evolution operator of system (13) is monotone and inequalities (12) are satisfied for* $X \in \mathcal{K}$. *Then solutions* $X(t) \ge 0$ *of every system (11), (12) are bounded by solutions of systems (13) and (14), i.e.,*

$$X_1(t_0) \le X(t_0) \le X_2(t_0) \quad \Rightarrow \quad X_1(t) \le X(t) \le X_2(t), \qquad t \ge t_0.$$

If inequalities (12) are satisfied for $X \in \mathcal{E}$, *then the positivity of system (13) yields the positivity of every system (11), (12) and, furthermore,*

$$0 \le X_1(t_0) \le X(t_0) \le X_2(t_0) \;\Rightarrow\; 0 \le X_1(t) \le X(t) \le X_2(t), \qquad t \ge t_0.$$

Proof. Subtracting relations (13) and (11) from (11) and (14), respectively, and using (12), we obtain the differential inequalities

$$\dot{H}_1(t) + \left[\overline{M}(t) + M_1(t)\right]H_1(t) \ge \left[\overline{M}(t) - M(t)\right]X(t),$$

$$\dot{H}_1(t) + \left[M(t) + M_1(t)\right]H_1(t) \ge \left[\overline{M}(t) - M(t)\right]X_1(t),$$

$$\dot{H}_2(t) + \left[M(t) + M_2(t)\right]H_2(t) \ge \left[M(t) - \underline{M}(t)\right]X_2(t),$$

$$\dot{H}_2(t) + \left[\underline{M}(t) + M_2(t)\right]H_2(t) \ge \left[M(t) - \underline{M}(t)\right]X(t),$$

where $H_1(t) = X(t) - X_1(t)$ and $H_2(t) = X_2(t) - X(t)$. In this case, the following relations are true:

$$\overline{M}(t) + M_1(t) \ge M(t) + M_1(t) \ge M(t) + M_2(t) \ge \underline{M}(t) + M_2(t).$$

If system (13) is positive, then, according to (3), its evolution operator $W_{\overline{M}+M_1}(t,s)$ must be monotone. The monotonicity of the operator $W_{\overline{M}+M_1}(t,s)$ yields the monotonicity of the operators $W_{M+M_1}(t,s)$, $W_{M+M_2}(t,s)$, and $W_{\underline{M}+M_2}(t,s)$ (see Sec. 2).

If $X(t) \geq 0$ or $X_1(t) \geq 0$, then the relation $H_1(t_0) \geq 0$ implies that $H_1(t) \geq 0$, i.e., $X_1(t) \leq X(t)$ for $t \geq t_0$. Similarly, if $X(t) \geq 0$ or $X_2(t) \geq 0$, then the relation $H_2(t_0) \geq 0$ yields $H_2(t) \geq 0$, i.e., $X(t) \leq X_2(t)$ for $t \geq 0$. Therefore, the positivity of system (13) yields the positivity of every system of family (11), (12). In the case $X(t) \geq 0$, inequalities (12) are used only for $X \in \mathcal{K}$.

The lemma is proved.

Theorem 3. *If system (14) is asymptotically stable and system (13) is positive, then every linear system from family (11), (12) is asymptotically stable and positive.*

The proof is carried out using Lemma 7 and the properties of a normal reproducing cone.

Note that, for stationary systems, the monotone invertibility of the operators $\underline{M}$ and $\overline{M}$ yields the monotone invertibility of every operator M from the segment $\underline{M} \leq M \leq \overline{M}$. In particular, it follows from Theorems 1 and 3 that the operator M is monotonically invertible if the relations $e^{-\overline{M}t} \geq 0$ and $\operatorname{Re}\lambda > 0$ hold for every $t \geq 0$ and $\lambda \in \sigma(\underline{M})$.

6. Comparison Systems

In stability theory, one uses methods of comparison based on the mapping of the space of states of the original system to the spaces of states of well-studied auxiliary systems. As comparison systems, one can use the classes of positive and monotone systems, in particular, systems satisfying the conditions of theorems of the Chaplygin and Ważewski type [9–11]. In this case, Theorems 1–3 and Lemmas 3, 4, and 7 may turn out to be helpful.

Consider the differential system

$$\dot{x} = f(x, t), \quad x \in \mathcal{X}, \quad t \geq 0, \tag{15}$$

where f is an operator that guarantees the existence and uniqueness of solutions $x(t)$. In the space $\mathcal{E}$ partially ordered by a normal reproducing cone $\mathcal{K}$, we construct systems of the form

$$\dot{X} = F(X, t), \quad X \in \mathcal{E}, \quad t \geq 0, \tag{16}$$

which are used as comparison systems for system (15).

Let Σ_+ denote the class of systems (16) for which one can establish a correspondence between their solutions and solutions of the differential inequalities

$$\dot{Z} \leq F(Z, t), \quad Z \in \mathcal{E}, \quad t \geq 0,$$

such that the relation $X(t_0) \geq Z(t_0)$ implies that $X(t) \geq Z(t)$ for $t > t_0 \geq 0$. Let $E(x, t)$ be an operator that continuously maps a certain neighborhood of the point $x = 0 \in \mathcal{X}$ for $t \geq t_0$ to the space $\mathcal{E}$. If the expression $E(x, t)$ and its generalized derivative along the solutions of system (15) satisfy the relation

$$D_t E(x, t)\big|_{(15)} \leq F(E(x, t), t), \tag{17}$$

then system (16) from the class Σ_+ is an upper comparison system for system (15), i.e.,

$$E(x(t_0), t_0) \le X(t_0) \quad \Rightarrow \quad E(x(t), t) \le X(t), \quad t \ge t_0. \tag{18}$$

In (17), the derivative along the solutions of system (15) can be defined as follows:

$$D_t E(x,t)\big|_{(15)} = \limsup_{h\to 0+} \frac{1}{h}\big[E\big(x + hf(x,t),\ t+h\big) - E(x,t)\big].$$

By analogy, one can define the class of systems Σ_- and lower comparison systems; in this case, all inequality signs used in the space $\mathscr{E}$ are replaced by the opposite ones. It is obvious that, under the condition $F(0,t) \ge 0$, every system of the class Σ_+ must be positive, and every system of $\Sigma_+ \bigcup \Sigma_-$ must be monotone.

If the equality

$$D_t E(x,t)\big|_{(15)} = F(E(x,t), t) \tag{19}$$

holds in (17), then, using the definition of the property of monotonicity of system (16), we obtain

$$X_1(t_0) \le E(x(t_0), t_0) \le X_2(t_0) \Rightarrow X_1(t) \le E(x(t), t) \le X_2(t), \quad t \ge t_0, \tag{20}$$

where $X_1(t)$ and $X_2(t)$ are certain solutions of the given system. This means that, under condition (19), the monotone system (16) is simultaneously a lower comparison system and an upper comparison system for system (15).

Estimates (18) and (20) can be used for the comparison of the dynamical properties of systems (15) and (16). For example, if the operator E is chosen so that $E(x,t) \le 0$ only for $x = 0$, then relation (18) and the fact that $X(t) \to 0$ imply that $x(t) \to 0$ as $t \to \infty$. In constructing comparison systems positive or monotone on the cone, the operator E can be chosen to be everywhere positive. For example, for a linear system $\dot{x} = A(t)x$, setting $E(x,t) = xx^T \ge 0$ and using (17), we obtain the upper matrix comparison system

$$\dot{X} - A(t)X - XA^T(t) = G(X,t) \ge 0, \qquad X \in R^{n\times n}. \tag{21}$$

This system is positive with respect to the cone of nonnegative-definite matrices. The asymptotic stability of the matrix equation (21) yields the asymptotic stability of the original system.

Lower and upper comparison systems of the form (16) can be constructed in different partially ordered spaces $\mathscr{E}_1$ and $\mathscr{E}_2$ simultaneously. In this case, the properties of the corresponding operators $E_1(x,t)$ and $E_2(x,t)$ and the order relations defined by the cones $\mathcal{K}_1 \subset \mathscr{E}_1$ and $\mathcal{K}_2 \subset \mathscr{E}_2$ in the relations

$$E_1(x(t), t) \ge X_1(t) \qquad \text{and} \qquad E_2(x(t), t) \le X_2(t)$$

must be coordinated with the purpose of studying certain characteristics of the original system (15). For example, one can require that the system of inequalities $E_1(x,t) \ge 0$ and $E_2(x,t) \le 0$ be satisfied only for $x = 0$. In this case, one should expect that $x(t) \to 0$ as $t \to \infty$ if $X_1(t) \to 0$ and $X_2(t) \to 0$. If the operators E_1 and E_2 coincide, then this property follows from the lemma on two policemen in a partially ordered space [1].

The problem of comparison of systems (15) and (16) becomes more complicated without the requirement of the uniqueness of solutions. In [9, 10], comparison systems of the form (16) were considered in a partially ordered space R^n using the notion of maximum and minimum solutions together with the generalized Dini derivatives of the vector function $E(x,t)$ on the solutions of the original system (15). In this case, the function

$E(x, t)$ must be locally Lipschitzian with respect to x, and the vector function $F(X, t)$ must be quasimonotonically nondecreasing in X with respect to the cone $\mathcal{K} \subset R^n$, i.e., $F \in \mathcal{F}$. This property guarantees that system (16) belongs to the classes Σ_+ and Σ_-; in the case of the cone of nonnegative vectors, it reduces to the Ważewski conditions (see Sec. 3).

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