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Weighted Performance Measure and Generalized H_∞ Control Problem for Linear Descriptor Systems

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Abstract: In this paper, the generalized problem of H_∞ control with transients is investigated for linear descriptor systems using a weighted performance measure that describes a mixed attenuation level of exogenous and initial disturbances. Based on a generalization of the bounded real lemma, involving special matrix variables, new necessary and sufficient conditions for the existence of static and dynamic output-feedback controller are proposed to ensure the admissibility of a closed-loop system with prescribed estimate of the weighted performance measure. The corresponding synthesis techniques are reduced to solving the linear and quadratic matrix inequalities with rank constraints. A numerical example is included to demonstrate the applicability of the present approaches.

Keywords: *descriptor system; robust stability; admissible system; weighted performance measure; H_∞ control.*

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1 Introduction

Descriptor (differential algebraic) equations arise naturally in many significant applications, for example, in constrained mechanical systems, power generation, chemical processing, network fluid flow, vehicle dynamics, robotics etc. (see, e. g., [5,6,11]). Problems of sensitivity reduction and exogenous disturbance attenuation in descriptor control systems are very important and, at the same time, insufficiently studied for practical applications. These problems are solved by the H_2/H_∞ control design for state-space systems that provide internal stability and minimize the negative influence of exogenous

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disturbances on the dynamics of controlled objects (see, e.g., [4, 8, 12]). As a typical performance measure in H_∞ control design, one can use the H_∞-norm of the transfer function matrix corresponding to the maximum ratio of the L_2-norms for the regulated output and bounded disturbances of the system.

Recently, attention has been paid to the problem of H_∞ control with transients for state-space systems when the initial states are uncertain and might be non-zero. In this regard, more general performance measures characterizing the damping level of external and initial disturbances caused by the nonzero initial vector were used in [1, 13, 18] for standard state-space systems. Known methods of H_∞ control design are based on using the upper bounds for applicable performance measures established via linear matrix inequalities (LMI) and Riccati-type equations (Bounded Real Lemma type statements), see, e.g., [2, 8, 25]. Necessary and sufficient conditions for H_∞ control with transients for state-space systems were proposed in terms of algebraic and differential Riccati equations [13], and in terms of LMIs [1].

The Bounded Real Lemma and H_∞ control theory have been extended for a class of descriptor systems (e.g., [3, 7, 10, 14–16, 22, 24]). A state-feedback controller design approach based on LMIs was proposed in [7] for solving the H_∞ control with transients problem for descriptor systems. Many important control issues including the H_∞ optimization problem for descriptor systems can be formulated as dissipativity with general quadratic supply functions (e.g., [6, 15, 22]).

This paper is concerned with a non-standard H_∞ control problem for linear time-invariant descriptor systems. The purpose of this paper is to extend the results obtained in [1, 7, 20, 21] via using the weighted performance measure taking into account the influence evaluation of both exogenous and initial disturbances in control systems. The application of weight coefficients in the generalized performance measures enables one to establish priorities between the regulated output components and bounded disturbances. Compared with [1, 7], the control system and performance measure studied in this paper are more general. In contrast to [20, 21], we use a special parametrization of the desired solutions of linear and quadratic matrix inequalities, which simplifies the proposed controller synthesis procedure. Furthermore, in some cases, resulting conditions for the existence of the weighted state- and output-feedback H_∞ controller contain only LMIs, which can be solved by existing numerical tools.

This paper is organised as follows. Section 2 contains some basic definitions and lemmas for linear descriptor systems. In Section 3, new necessary and sufficient conditions are proposed for the existence of stabilizing static and dynamic output-feedback controllers solving the weighted H_∞ control problem for descriptor systems. These conditions guarantee a prescribed upper bound for the weighted performance measure of a closed-loop system and, in general, have the LMIs form with additional rank constraints, as well as the form of the generalized algebraic Riccati inequalities (GARIs). In Section 4, the effectiveness of the proposed methods is illustrated by means of a numerical example. After that, a conclusion is given in Section 5. Finally, the solvability criteria for some matrix inequalities are stated in Appendix. In particular, new necessary and sufficient conditions for the solvability of quadratic matrix inequalities arising in the proposed methods for the weighted H_∞ control are presented.

Notations: I_n is the identity $n \times n$ matrix; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^\top > 0$ (≥ 0) is a positive (nonnegative) definite symmetric matrix X; $\sigma(A)$ is the spectrum of A; $\operatorname{Ker} A$ is the kernel of A; A^{-1} (A^+) is the inverse (pseudo-inverse) of A; W_A is the right null matrix of $A \in \mathbb{R}^{m\times n}$, that is, $AW_A = 0$, $W_A \in \mathbb{R}^{n\times(n-r)}$, $\operatorname{rank} W_A = n - r$, where $r = \operatorname{rank} A < n$ ($W_A = 0$ if $r = n$); $\operatorname{Co}\{A_1, \dots, A_\nu\}$ is the convex polyhedron (polytope) with vertices $A_1, \dots, A_\nu$ in a matrix space; $\|x\|$ is the Euclidean norm of x; $\|w\|_P$ is the weighted L_2-norm of a vector function $w(t)$.

2 Basic Definitions and Lemmas

Consider the following continuous-time descriptor system:

$$E\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^s$ is the exogenous input (disturbances) and $z \in \mathbb{R}^k$ is the output, E, A, B, C and D are constant matrices with compatible dimensions and $\operatorname{rank} E = \rho \le n$.

Definition 2.1 A matrix pair (E, A) is said to be *admissible*, if it is *regular*, *impulse-free* and *stable*, i.e., $\det F(\lambda) \not\equiv 0$, $\deg F(\lambda) = \rho$ and $\sigma(F) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$, respectively, where $F(\lambda) = A - \lambda E$ is a matrix pencil. Descriptor system (1) with the admissible pair (E, A) is *admissible.*

Lemma 2.1 (see [16]) *System* (1) *is admissible if and only if there exists a matrix* X *such that* $A^\top X + X^\top A < 0$ *and* $E^\top X = X^\top E \ge 0$.

A regular pair (E, A) can be transformed into the Weierstrass canonical form [9]

$$LER = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad LAR = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$

where L and R are nonsingular matrices, $\sigma(F) = \sigma(A_1)$, $r \le \rho$ and N is a nilpotent matrix. A pair (E, A) is impulse-free if and only if [5]

$$\operatorname{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \rho.$$

In this case, $N = 0$ and system (1) can be transformed into the following form:

$$\dot{x}_1 = A_1 x_1 + B_1 w, \quad x_2 = -B_2 w, \quad z = C_1 x_1 + D_1 w, \tag{2}$$

where

$$x_1 \in \mathbb{R}^r,\ x_2 \in \mathbb{R}^{n-r},\ x = R \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ LB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ CR = \begin{bmatrix} C_1, C_2 \end{bmatrix},\ D_1 = D - C_2 B_2.$$

Define the performance measure J for system (1) in the form

$$J = \sup_{(w,x_0)\in\mathcal{W}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \quad \|z\|_Q^2 = \int_0^\infty z^\top Q z\, dt, \quad \|w\|_P^2 = \int_0^\infty w^\top P w\, dt, \tag{3}$$

where $\mathcal{W}$ is a set of pairs (w, x_0) such that $0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty$ and system (1) has a solution, $P = P^\top > 0$, $Q = Q^\top > 0$ and $X_0 \ge 0$ are weight matrices. In the following, we consider $X_0 = E^\top H E$ with $H = H^\top > 0$.

The value J describes the weighted damping level of the external and initial disturbances in system (1). For example, if the weight matrices P and Q are diagonal, then their diagonal elements are the *priority coefficients* for the corresponding components of input w and output z in system (1) with respect to J. System (1) is *nonexpansive* if $J \leq 1$. A pair (w, x_0) is the *worst* for system (1) with respect to J if in (3), a supremum is reached.

When $x_0 \in \operatorname{Ker} E$, we denote J as J_0. It is obvious that $J_0 \leq J$. If $P = I_s$ and $Q = I_l$, then J_0 coincides with the H_∞-norm of the transfer matrix function $\mathcal{H}_1(\lambda) = C_1(\lambda I_n - A_1)^{-1}B_1 + D_1$ of the dynamical subsystem in (2) (see, e.g., [5,8]). In this case, we have a standard performance index J_0 used in the H_∞ control theory. Note that the performance measure (3) was introduced in [13] when $E = I_n$, $P = I_s$ and $Q = I_l$.

Lemma 2.2 (see [19]) *Given a scalar $\gamma > 0$, the descriptor system* (1) *is admissible and satisfies $J < \gamma$ if there exists a matrix X such that*

$$0 \leq E^\top X = X^\top E \leq \gamma^2 X_0, \quad \operatorname{rank}(E^\top X - \gamma^2 X_0) = \rho, \tag{4}$$

$$\Psi(X) = \begin{bmatrix} A^\top X + X^\top A + C^\top QC & X^\top B + C^\top QD \\ B^\top X + D^\top QC & D^\top QD - \gamma^2 P \end{bmatrix} < 0. \tag{5}$$

The converse is true if

$$\operatorname{rank} \begin{bmatrix} E^\top & C^\top QD \end{bmatrix} = \rho. \tag{6}$$

Remark 2.1 Note that $E^\top X = X^\top E \geq 0$ if and only if the non-strict LMI

$$\begin{bmatrix} S_0 & S_0 - E^\top X \\ S_0 - X^\top E & 0 \end{bmatrix} \geq 0 \tag{7}$$

is feasible in the variables X and S_0, and moreover, $S_0 = E^\top X = X^\top E \geq 0$. It can be established that (5) is satisfied if and only if $D^\top QD < \gamma^2 P$ and $D_1^\top QD_1 < \gamma^2 P$, where $D_1 = D - CA^{-1}B$. Any matrix X satisfying (5) must be nonsingular.

Remark 2.2 If the LMIs (5) and (7) are feasible in the variables X and S_0, then system (1) is admissible with $J_0 < \gamma$. The converse is true under the additional condition (6). Moreover, if (5) and (7) hold, then system (1) with a structured uncertain input $w = \gamma^{-1}\Theta z$, where $\Theta^\top P \Theta \leq Q$, is robust stable and $v(x) = x^\top S_0 x$ is a common Lyapunov function of the system (see [18]).

Note that the conditions of Lemma 2.2 can be used in calculating the values of J_0 and J as the solutions of optimization problems. In particular, we have

$$J = \inf \left\{ \gamma : \ \Psi(X) < 0, \ \ 0 \leq E^\top X = X^\top E \leq \gamma^2 X_0 \right\}.$$

Lemma 2.3 (see [20]) *Let system* (1) *be admissible and there exist matrices X and S_0 satisfying* (7) *and the Riccati-type equation*

$$A_1^\top X + X^\top A_1 + X^\top R_1 X + Q_1 = 0,$$

where $A_1 = A + BR^{-1}D^\top QC$, $R_1 = BR^{-1}B^\top$, $Q_1 = C^\top\left(Q + QDR^{-1}D^\top Q\right)C$, $R = \gamma^2 P - D^\top QD > 0$ and $\gamma = J$. Then the structured input vector

$$w = K_0 x, \quad K_0 = R^{-1}(B^\top X + D^\top QC),$$

and any initial vector $x_0 \in \operatorname{Ker}(S_0 - J^2 X_0)$ form the worst pair for system (1) *with respect to J.*

3 Main Results

Consider the following descriptor system with constant coefficient matrices:

$$\begin{aligned} E\dot{x} &= Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{8}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^s$ is the exogenous input, $z \in \mathbb{R}^k$ is the regulated output and $y \in \mathbb{R}^l$ is the measured output. The rank of the matrix E is $\rho \le n$. We are interested in static and dynamic control laws that guarantee a desirable estimate for performance measure (3) with respect to the regulated output z of a resulting closed-loop system. Controllers that minimize the value of J are called *J-optimal.* If $P = I_s$ and $Q = I_k$, then the J_0-optimal controller is *H_∞-optimal.*

In studying a class of systems (8), their properties such as C-, R- and I-controllability, as well as adjoint C-, R- and I-observability, are important (see, e.g., [5,6]). Known H_∞ control methods for such systems use the stabilizability and I-controllability properties of the triple (E, A, B_2). It means that there exists a matrix K for which the pair $(E, A + B_2 K)$ is admissible. The criteria for I-controllability of the triple (E, A, B_2) and I-observability of the triple (E, A, C_2) are the corresponding rank conditions [5]

$$\operatorname{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B_2 \end{bmatrix} = n + \rho, \quad \operatorname{rank} \begin{bmatrix} E^\top & 0 & 0 \\ A^\top & E^\top & C_2^\top \end{bmatrix} = n + \rho.$$

3.1 Static output-feedback controller

When we apply the static output-feedback controller

$$u = Ky, \quad K \in \mathbb{R}^{m \times l}, \tag{9}$$

with the condition $\det(I_m - KD_{22}) \ne 0$, the closed-loop system is given by

$$E\dot{x} = A_* x + B_* w, \quad z = C_* x + D_* w, \quad x(0) = x_0, \tag{10}$$

where $A_* = A + B_2 K_* C_2$, $B_* = B_1 + B_2 K_* D_{21}$, $C_* = C_1 + D_{12} K_* C_2$, $D_* = D_{11} + D_{12} K_* D_{21}$, $K_* = (I_m - KD_{22})^{-1} K$. Let, for simplicity, $D_{22} = 0$, then $K_* = K$.

Applying Lemma 2.2 for system (10), we will use the special structure of a matrix X in (4), (5) and the skeletal decomposition $E = E_l E_r^\top$, where E_l and E_r are full column rank ρ matrices.

Lemma 3.1 *Given a scalar $\gamma > 0$ and matrices X and Y satisfying $XY = \gamma^2 I_n$, the following statements are equivalent:*

(i) *the conditions* (4) *of Lemma* 2.2 *hold;*

(ii) *there are matrices $S = S^\top$ and G such that*

$$X = SE + W_{E^\top} G, \quad 0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l; \tag{11}$$

(iii) *there are matrices $T = T^\top$ and F such that*

$$Y = TE^\top + W_E F, \quad E_r^\top T E_r > (E_l^\top H E_l)^{-1}. \tag{12}$$

Proof. (ii) $\Rightarrow$ (i) Considering (11), we have

$$E^\top X = E^\top SE \geq 0, \quad E^\top X - \gamma^2 X_0 = E_r\big(E_l^\top SE_l - \gamma^2 E_l^\top HE_l\big)E_r^\top.$$

Hence, (4) hold true.

(i) $\Rightarrow$ (ii) Let L and R be nonsingular matrices such that

$$E = L^{-1}\begin{bmatrix} I_\rho & 0 \\ 0 & 0 \end{bmatrix} R^{-1}, \quad E_l = L^{-1}\begin{bmatrix} I_\rho \\ 0 \end{bmatrix}, \quad E_r = R^{-1\top}\begin{bmatrix} I_\rho \\ 0 \end{bmatrix},$$

$$W_E = R\begin{bmatrix} 0 \\ I_{n-\rho} \end{bmatrix}, \quad W_{E^\top} = L^\top \begin{bmatrix} 0 \\ I_{n-\rho} \end{bmatrix}.$$

Then any matrix X in (4) can be expressed as

$$X = L^\top \begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix} R^{-1}, \quad 0 < X_1 = X_1^\top < \gamma^2 E_l^\top HE_l. \tag{13}$$

Assuming $S_1 = X_1$, $S_2 = X_2 - G_1$, $G_2 = X_3$, $S_3 = S_3^\top$, $G_1 \in \mathbb{R}^{(n-\rho)\times\rho}$ and

$$S = L^\top \begin{bmatrix} S_1 & S_2^\top \\ S_2 & S_3 \end{bmatrix} L, \quad G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} R^{-1},$$

we get (11).

(iii) $\Rightarrow$ (i) Note that conditions (4) hold if and only if

$$0 \leq EY = Y^\top E^\top \leq Y^\top X_0 Y, \quad \operatorname{rank}\big(EY - Y^\top X_0 Y\big) = \rho, \tag{14}$$

where $Y = \gamma^2 X^{-1}$. Considering (12), we have

$$EY = ETE^\top \geq 0, \quad EY - Y^\top X_0 Y = E_l T_1 \big(T_1^{-1} - E_l^\top HE_l\big) T_1 E_l^\top,$$

where $T_1 = E_r^\top TE_r$. Besides, $T_1^{-1} < E_l^\top HE_l$ if and only if $T_1 > (E_l^\top HE_l)^{-1}$. Hence, (14) and (4) hold true.

(i) $\Rightarrow$ (iii) Suppose that (4) hold true. Using (13), we have

$$Y = \gamma^2 X^{-1} = \gamma^2 R \begin{bmatrix} X_1^{-1} & 0 \\ -X_3^{-1}X_2X_1^{-1} & X_3^{-1} \end{bmatrix} L^{-1\top}.$$

Let $T_1 = \gamma^2 X_1^{-1}$, $T_2 = -F_1 - \gamma^2 X_3^{-1}X_2X_1^{-1}$, $F_2 = \gamma^2 X_3^{-1}$, $T_3 = T_3^\top$, $F_1 \in \mathbb{R}^{(n-\rho)\times\rho}$ and

$$T = R\begin{bmatrix} T_1 & T_2^\top \\ T_2 & T_3 \end{bmatrix} R^\top, \quad F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} L^{-1\top}.$$

Since $T_1 = E_r^\top TE_r$, we obtain (12) using the equivalence of the matrix inequalities $X_1 < \gamma^2 E_l^\top HE_l$ and $\gamma^2 X_1^{-1} > (E_l^\top HE_l)^{-1}$. This completes the proof.

Theorem 3.1 *Let there exist matrices X and Y such that* (11) *and the following conditions hold:*

$$W_R^\top \begin{bmatrix} A^\top X + X^\top A + C_1^\top QC_1 & X^\top B_1 + C_1^\top QD_{11} \\ B_1^\top X + D_{11}^\top QC_1 & D_{11}^T QD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{15}$$

$$W_L^\top \begin{bmatrix} AY+Y^\top A^\top + B_1P^{-1}B_1^\top & Y^\top C_1^\top + B_1P^{-1}D_{11}^\top \\ C_1Y + D_{11}P^{-1}B_1^\top & D_{11}P^{-1}D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{16}$$

$$\operatorname{rank} \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} = n, \tag{17}$$

where $R = \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$, $L = \begin{bmatrix} B_2^\top & D_{12}^\top \end{bmatrix}$. *Then there exists a static output-feedback controller* (9) *such that closed-loop system* (10) *is admissible and its performance measure* $J < \gamma$. *Conversely, if system* (10) *is admissible with* $J < \gamma$ *and*

$$\operatorname{rank} \begin{bmatrix} E^\top & C_*^\top Q D_* \end{bmatrix} = \rho \tag{18}$$

for some controller (9), *then conditions* (11) *and* (15) – (17) *are feasible in* X *and* Y.

Proof. Taking into account the Schur complement, we rewrite matrix inequality (5) in Lemma 2.2 for a closed-loop system (10) as the LMI with respect to K_*:

$$\begin{bmatrix} A_*^T X + X^\top A_* & X^\top B_* & C_*^\top \\ B_*^\top X & -\gamma^2 P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} = \widehat{L}^\top K_* \widehat{R} + \widehat{R}^\top K_*^\top \widehat{L} + \Omega < 0, \tag{19}$$

where $\widehat{R} = \begin{bmatrix} R & 0_{l\times k} \end{bmatrix}$, $\widehat{L} = \begin{bmatrix} L & 0_{m\times s} \end{bmatrix} \widetilde{X}$, and

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^\top X + X^\top A & X^\top B_1 & C_1^\top \\ B_1^\top X & -\gamma^2 P & D_{11}^\top \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix}.$$

There exists K_* satisfying (19) if and only if (see the condition (d) in Lemma A.1)

$$W_{\widehat{R}}^\top \Omega W_{\widehat{R}} < 0, \quad W_{\widehat{L}}^\top \Omega W_{\widehat{L}} < 0. \tag{20}$$

Since

$$W_{\widehat{R}} = \begin{bmatrix} W_R & 0 \\ 0 & I_k \end{bmatrix}, \quad W_{\widehat{L}} = \widetilde{X}^{-1} \begin{bmatrix} W_L & 0 \\ 0 & I_s \end{bmatrix},$$

the conditions (20) are reduced to (15) and (16) with $Y = \gamma^2 X^{-1}$, respectively. The last equality is equivalent to the rank condition (17).

Thus, if (11) and (15) – (17) hold for some matrices $S = S^\top$, $G \in \mathbb{R}^{(n-\rho)\times n}$ and Y, then taking into account the equivalence of statements (i) and (ii) in Lemma 3.1, we can construct a controller (9) provided the admissibility of system (10) with $J < \gamma$. The gain matrix K of the controller can be defined as any solution $K = K_*$ of LMI (19).

Conversely, if system (10) is admissible with $J < \gamma$ and (18) holds for some controller (9), then (11) and (15) – (17) are feasible in X and Y (see Lemma 2.2).

Note that the rank constraint (18) does not depend on K if one of the following conditions is satisfied:

$$D_{11} = 0, \quad D_{21} = 0; \tag{21}$$

$$D_{12} = 0, \quad \operatorname{rank} \begin{bmatrix} E^\top & C_1^\top Q D_{11} \end{bmatrix} = \rho. \tag{22}$$

It can also be established that (18) follows from

$$\operatorname{rank} \begin{bmatrix} E^\top & C_1^\top Q D_{11} & C_1^\top Q D_{12} & C_2^\top \end{bmatrix} = \rho.$$

Corollary 3.1 *Assume that*

$$C_2 = I_n, \quad D_{11}^\top Q D_{11} < \gamma^2 P, \quad D_{21} = 0, \quad D_{22} = 0. \tag{23}$$

Then there is a static state-feedback controller $u = Kx$ such that a closed-loop system (10) *is admissible and its performance measure $J < \gamma$ if two LMIs* (12) *and* (16) *with nonsingular Y are feasible in the variables $T = T^\top$ and F. The converse statement is true if* (23) *and either* (21) *or* (22) *hold.*

Proof. Considering (23), we have $y \equiv x$, $W_R = \left[0_{s\times n}, I_s\right]^T$. In this case, matrix inequality (15) holds in Theorem 3.1 and does not depend on X. Taking into account the equivalence of the statements (i) and (iii) in Lemma 3.1, a sufficient condition for the existence of a static state-feedback controller in Theorem 3.1 is the solvability of (12) and (16) with respect to $T = T^\top$ and F. The gain matrix K of the controller can be defined as a solution $K = K_*$ of the LMI (19) with $X = \gamma^2 Y^{-1}$.

Given (23), we also have $C_* = C_1 + D_{12}K$ and $D_* = D_{11}$. Therefore, in the converse statement of Theorem 3.1, the rank condition (18) is true and does not depend on K if either (21) or (22) holds.

Remark 3.1 Note that Y in (12) is nonsingular if such is $FW_{E^\top}$. In particular, we can search for F in the form $F = \widetilde{F}E^\top + CW_{E^\top}^\top$, where $\widetilde{F}$ is a new required matrix and C is nonsingular. Then Y in Corollary 3.1 is nonsingular (see the proof of Lemma 3.1).

Theorem 3.2 *Assume that*

$$R_0 = D_{12}^\top Q D_{12} > 0, \quad R_1 = \gamma^2 P - D_{11}^\top Q_1 D_{11} > 0, \quad Q_1 = Q - QD_{12}R_0^{-1}D_{12}^\top Q \tag{24}$$

and there exist matrices $S = S^\top$ and G such that (11), (15) *and the GARI*

$$A_2^\top X + X^\top A_2 + X^\top R_2 X + Q_2 < 0 \tag{25}$$

hold with $A_2 = A - B_2 R_0^{-1} D_{12}^\top Q C_1 + B_{11} R_1^{-1} D_{11}^\top Q_1 C_1$, $R_2 = B_{11} R_1^{-1} B_{11}^\top - B_2 R_0^{-1} B_2^\top$, $B_{11} = B_1 - B_2 R_0^{-1} D_{12}^\top Q D_{11}$, $Q_2 = C_1^\top (Q_1 + Q_1 D_{11} R_1^{-1} D_{11}^\top Q_1) C_1$. Then there exists a static output-feedback controller (9) *such that closed-loop system* (10) *is admissible and its performance measure $J < \gamma$.*

Proof. To apply Lemmas 2.2 and 3.1, we rewrite the expression $\Psi(X) < 0$ for system (10) in the form of a quadratic matrix inequality with respect to K_*:

$$A_0 + B_0^\top K_* C_0 + C_0^\top K_*^\top B_0 + C_0^\top K_*^\top R_0 K_* C_0 < 0, \tag{26}$$

where

$$A_0 = \begin{bmatrix} A^\top X + X^\top A + C_1^\top Q C_1 & X^\top B_1 + C_1^\top Q D_{11} \\ B_1^\top X + D_{11}^\top Q C_1 & D_{11}^\top Q D_{11} - \gamma^2 P \end{bmatrix},$$

$$B_0 = \begin{bmatrix} B_2^\top X + D_{12}^\top Q C_1 & D_{12}^\top Q D_{11} \end{bmatrix}, \quad C_0 = \begin{bmatrix} C_2 & D_{21} \end{bmatrix}.$$

Since $R_0 > 0$, the solvability conditions for (26) are of the form $W_{C_0}^\top A_0 W_{C_0} < 0$ and $A_0 < B_0^\top R_0^{-1} B_0$ (see the conditions (a) and (b) in Lemma A.2). The first inequality coincides with (15), and the second inequality takes the form (25) via the Schur complement.

Note that on the basis of Lemmas 2.2 and 3.1, as well as the generalized uncertainty lemma for inequality (26) (see [17]), we can construct an ellipsoidal set of gain matrices $\mathcal{K} = \left\{K : (K - K_*)^\top P_0 (K - K_*) \le Q_0\right\}$, where $P_0 = P_0^\top > 0$ and $Q_0 = Q_0^\top > 0$, for which closed-loop system (10) is admissible and its performance measure $J < \gamma$.

3.2 Dynamic output-feedback controller

Consider system (8) with the dynamic output-feedback controller

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \quad \xi(0) = 0, \tag{27}$$

where $\xi \in \mathbb{R}^p$, Z, V, U and K denote constant matrices with appropriate dimensions to be determined. The combined system in an extended state space $\mathbb{R}^{n+p}$ is represented by

$$\begin{aligned} \widehat{E}\dot{\widehat{x}} &= \widehat{A}\widehat{x} + \widehat{B}_1 w + \widehat{B}_2 \widehat{u}, \quad \widehat{x}(0) = \widehat{x}_0, \\ z &= \widehat{C}_1 \widehat{x} + D_{11} w + \widehat{D}_{12} \widehat{u}, \\ \widehat{y} &= \widehat{C}_2 \widehat{x} + \widehat{D}_{21} w, \end{aligned} \tag{28}$$

using the static output-feedback controller

$$\widehat{u} = \widehat{K}_* \widehat{y}, \quad \widehat{K}_* \in \mathbb{R}^{(m+p)\times(l+p)}, \tag{29}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \ \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \ \widehat{y} = \begin{bmatrix} y - D_{22}u \\ \xi \end{bmatrix}, \ \widehat{u} = \begin{bmatrix} u \\ \dot{\xi} \end{bmatrix},$$

$$\widehat{E} = \begin{bmatrix} E & 0 \\ 0 & I_p \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & 0_{n\times p} \\ 0_{p\times n} & 0_{p\times p} \end{bmatrix}, \quad \widehat{B}_1 = \begin{bmatrix} B_1 \\ 0_{p\times s} \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} B_2 & 0_{n\times p} \\ 0_{p\times m} & I_p \end{bmatrix},$$

$$\widehat{C}_1 = \begin{bmatrix} C_1 & 0_{k\times p} \end{bmatrix}, \ \widehat{D}_{12} = \begin{bmatrix} D_{12} & 0_{k\times p} \end{bmatrix}, \ \widehat{C}_2 = \begin{bmatrix} C_2 & 0_{l\times p} \\ 0_{p\times n} & I_p \end{bmatrix}, \ \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{p\times s} \end{bmatrix},$$

$$\widehat{K}_* = \begin{bmatrix} K_* & U_* \\ V_* & Z_* \end{bmatrix} = (I_{m+p} - \widehat{K}\widehat{D}_{22})^{-1}\widehat{K}, \quad \widehat{D}_{22} = \begin{bmatrix} D_{22} & 0_{l\times p} \\ 0_{p\times m} & 0_{p\times p} \end{bmatrix},$$

$$\widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix} = (I_{m+p} + \widehat{K}_* \widehat{D}_{22})^{-1} \widehat{K}_*. \tag{30}$$

Here $\det(I_m - KD_{22}) \neq 0$. Let, for simplicity, $D_{22} = 0$, then $\widehat{K}_* = \widehat{K}$.

The closed-loop system has the form

$$\widehat{E}\dot{\widehat{x}} = \widehat{A}_* \widehat{x} + \widehat{B}_* w, \quad z = \widehat{C}_* \widehat{x} + \widehat{D}_* w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{31}$$

where $\widehat{A}_* = \widehat{A} + \widehat{B}_2 \widehat{K}_* \widehat{C}_2$, $\widehat{B}_* = \widehat{B}_1 + \widehat{B}_2 \widehat{K}_* \widehat{D}_{21}$, $\widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12} \widehat{K}_* \widehat{C}_2$, $\widehat{D}_* = D_{11} + \widehat{D}_{12} \widehat{K}_* \widehat{D}_{21}$. Since $\xi_0 = 0$, the performance measure $\widehat{J}$ of the form (3) for system (31) with the weight matrices P, Q, and

$$\widehat{X}_0 = \widehat{E}^\top \widehat{H} \widehat{E}, \quad \widehat{H} = \begin{bmatrix} H & H_1^\top \\ H_1 & H_2 \end{bmatrix} > 0,$$

does not depend on H_1 and H_2, and its value coincides with J.

Lemma 3.2 *Given a scalar $\gamma > 0$ and matrices*

$$X = SE + W_{E^\top} G, \quad Y = TE^\top + W_E F, \tag{32}$$

where $S = S^\top$, $T = T^\top$ and $G, F \in \mathbb{R}^{(n-\rho)\times n}$, the following statements are equivalent:

(i) *X and Y are nonsingular, $\Delta = \gamma^2 I_n - XY \neq 0$ and*

$$0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l, \quad \Gamma = \begin{bmatrix} E_l^\top S E_l & \gamma I_\rho \\ \gamma I_\rho & E_r^\top T E_r \end{bmatrix} \geq 0, \tag{33}$$

$$\operatorname{rank}\Gamma = \rho + \delta, \quad \delta = \operatorname{rank}\Delta; \tag{34}$$

(ii) *there are matrices $S_1 \in \mathbb{R}^{\delta\times n}$, $S_2 = S_2^\top \in \mathbb{R}^{\delta\times\delta}$, $G_1 \in \mathbb{R}^{(n-\rho)\times\delta}$, $T_1 \in \mathbb{R}^{\delta\times n}$, $T_2 = T_2^\top \in \mathbb{R}^{\delta\times\delta}$, $F_1 \in \mathbb{R}^{(n-\rho)\times\delta}$, $H_1 \in \mathbb{R}^{\delta\times n}$ and $H_2 = H_2^\top \in \mathbb{R}^{\delta\times\delta}$ such that*

$$0 < \widehat{E}_l^\top \widehat{S} \widehat{E}_l < \gamma^2 \widehat{E}_l^\top \widehat{H} \widehat{E}_l, \quad \widehat{X}\widehat{Y} = \gamma^2 I_{n+\delta}, \tag{35}$$

where

$$\widehat{X} = \begin{bmatrix} X & X_3 \\ X_1 & X_2 \end{bmatrix} = \widehat{S}\widehat{E} + W_{\widehat{E}^\top}\widehat{G}, \quad \widehat{S} = \begin{bmatrix} S & S_1^\top \\ S_1 & S_2 \end{bmatrix}, \quad \widehat{G} = \begin{bmatrix} G & G_1 \end{bmatrix}, \tag{36}$$

$$\widehat{Y} = \begin{bmatrix} Y & Y_3 \\ Y_1 & Y_2 \end{bmatrix} = \widehat{T}\widehat{E}^\top + W_{\widehat{E}}\widehat{F}, \quad \widehat{T} = \begin{bmatrix} T & T_1^\top \\ T_1 & T_2 \end{bmatrix}, \quad \widehat{F} = \begin{bmatrix} F & F_1 \end{bmatrix}, \tag{37}$$

$$\widehat{E} = \begin{bmatrix} E & 0 \\ 0 & I_\delta \end{bmatrix} = \widehat{E}_l \widehat{E}_r^\top, \quad W_{\widehat{E}} = \begin{bmatrix} W_E \\ 0 \end{bmatrix}, \quad W_{\widehat{E}^\top} = \begin{bmatrix} W_{E^\top} \\ 0 \end{bmatrix},$$

$$\widehat{E}_l = \begin{bmatrix} E_l & 0 \\ 0 & I_\delta \end{bmatrix}, \quad \widehat{E}_r = \begin{bmatrix} E_r & 0 \\ 0 & I_\delta \end{bmatrix}, \quad \widehat{H} = \begin{bmatrix} H & H_1^\top \\ H_1 & H_2 \end{bmatrix} > 0.$$

Proof. (ii) $\Rightarrow$ (i) We rewrite (35) as

$$0 < \begin{bmatrix} E_l^\top S E_l & E_l^\top S_1^\top \\ S_1 E_l & S_2 \end{bmatrix} < \gamma^2 \begin{bmatrix} E_l^\top H E_l & E_l^\top H_1^\top \\ H_1 E_l & H_2 \end{bmatrix}, \tag{38}$$

$$XY + X_3 Y_1 = \gamma^2 I_n, \quad XY_3 + X_3 Y_2 = 0, \quad X_1 Y + X_2 Y_1 = 0, \quad X_1 Y_3 + X_2 Y_2 = \gamma^2 I_\delta. \tag{39}$$

Obviously, (38) implies $0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l$ and $X_2 = S_2 > 0$. In addition, $Y_2 = T_2 > 0$ because $\widehat{E}_r^\top \widehat{T} \widehat{E}_r > (\widehat{E}_l^\top \widehat{H} \widehat{E}_l)^{-1}$ (see Lemma 3.1).

From (39) it follows that $X(Y - Y_3 Y_2^{-1} Y_1) = (X - X_3 X_2^{-1} X_1)\, Y = \gamma^2 I_n$. Therefore, X and Y must be nonsingular. Besides, $E^\top X = X^\top E \geq 0$, $EY = Y^\top E^\top \geq 0$, $X_1 = X_3 E$ and $Y_1 = Y_3^\top E^\top$. Next, we use the following transformation of Γ:

$$\Phi^\top \Gamma \Phi = \begin{bmatrix} E^\top S E & 0 \\ 0 & \Xi \end{bmatrix} \geq 0, \quad \Phi = \begin{bmatrix} E_r^\top & -\gamma E_r^\top X^{-1} \\ 0 & E_l^\top \end{bmatrix}, \tag{40}$$

where $E^\top S E \geq 0$ and $\Xi = E(Y - \gamma^2 X^{-1}) = Y_1^\top Y_2^{-1} Y_1 \geq 0$. Since Φ is the full row rank matrix, it yields $\Gamma \geq 0$. Moreover, the rank conditions (34) hold because $\operatorname{rank}(E^\top S E) = \rho$, $\Xi = -EX^{-1}\Delta$, $\Delta = X_3 Y_1$, $\operatorname{rank}\Delta \leq \operatorname{rank} Y_1 = \operatorname{rank}\Xi \leq \operatorname{rank}\Delta$ and, hence, $\operatorname{rank}\Xi = \delta$ and $\operatorname{rank}\Gamma = \rho + \delta$.

(i) $\Rightarrow$ (ii) Assume that (33) and (34) hold with nonsingular X and Y. Given (33) and (40), we have the decomposition $\Xi = E(Y - \gamma^2 X^{-1}) = \Lambda^\top \Lambda \geq 0$, where $\Lambda \in \mathbb{R}^{\delta\times n}$ is a certain full row rank matrix. Then there exists $\Upsilon \in \mathbb{R}^{n\times\delta}$ such that $\Upsilon\Lambda = \Delta$. Indeed,

$$\begin{aligned} \operatorname{rank}\Lambda &\leq \operatorname{rank}\begin{bmatrix} \Lambda^\top & \Delta^\top \end{bmatrix} = \operatorname{rank}(\Lambda^\top\Lambda + \Delta^\top\Delta) = \operatorname{rank}\left[(\Delta^\top - EX^{-1})\Delta\right] \\ &\leq \operatorname{rank}\Delta = \operatorname{rank}\Xi = \operatorname{rank}\Lambda \end{aligned}$$

and hence $\operatorname{rank}\begin{bmatrix} \Lambda^\top & \Delta^\top \end{bmatrix} = \operatorname{rank}\Lambda$. Moreover, $\Xi = -EX^{-1}\Upsilon\Lambda = \Lambda^\top\Lambda$ implies $\Lambda^\top = -EX^{-1}\Upsilon$.

Setting in (36) and (37) $S_1 = \Upsilon^\top - G_1^\top W_{E^\top}^\top$, $S_2 = \gamma^2 I_\delta - \Lambda\Upsilon$, $T_1 = -\Upsilon^\top X^{-1\top} - F_1^\top W_E^\top$ and $T_2 = I_\delta$, where $F_1 \in \mathbb{R}^{(n-\rho)\times\delta}$ and $G_1 \in \mathbb{R}^{(n-\rho)\times\delta}$ are arbitrary matrices, we have $X_1 = \Upsilon^\top E$, $X_2 = \gamma^2 I_\delta - \Lambda\Upsilon$, $X_3 = \Upsilon$, $Y_1 = \Lambda$, $Y_2 = I_\delta$ and $Y_3 = -X^{-1}\Upsilon$.

Considering $\Lambda = -\Upsilon^\top X^{-1\top} E^\top$ and $EX^{-1} = X^{-1\top}E^\top$, it is easy to verify (39). The first matrix inequality in (38) follows from the Schur complement. Indeed, $E_l^\top S E_l > 0$ and

$$
\begin{aligned}
&S_2 - S_1 E_l (E_l^\top S E_l)^{-1} E_l^\top S_1^\top \\
&= \gamma^2 I_\delta - \Lambda\Upsilon - \Upsilon^\top E_l (E_l^\top S E_l)^{-1} E_l^\top \Upsilon \\
&= \gamma^2 I_\delta + \Upsilon^\top X^{-1\top}\big[E^\top - X^\top E_l S E_l (E_l^\top S E_l)^{-1} E_l^\top\big]\Upsilon \\
&= \gamma^2 I_\delta + \Upsilon^\top X^{-1\top} E_r \big[I_\rho - E_l^\top S E_l (E_l^\top S E_l)^{-1}\big] E_l^\top \Upsilon = \gamma^2 I_\delta > 0.
\end{aligned}
$$

Here, it is also taken into account that $W_{E^\top} = W_{E_l^\top}$ and $X = SE + W_{E^\top} G$.

The second matrix inequality in (38) holds, if, for instance, $H_1 = \gamma^{-2} S_1$ and $H_2 > \gamma^{-2} S_2$. This completes the proof.

Theorem 3.3 *Let the LMIs* (15) *and* (16) *with* (32) *as well as* (33) *and the rank conditions* (34) *are feasible in the variables $S = S^\top$, $T = T^\top$, G and F. Then there exists a dynamic controller* (27) *of the order $p = \delta$ such that closed-loop system* (31) *is admissible and its performance measure $J < \gamma$. Conversely, if system* (31) *is admissible with $J < \gamma$ and satisfies* (18) *for some controller* (27)*, then* (15), (16) *and* (32)–(34) *are feasible.*

Proof. According to Theorem 3.1, we can find a static controller (29) for extending system (28) such that closed-loop system (31) is admissible and its performance measure $J < \gamma$ if there exist matrices $\widehat{S} = \widehat{S}^\top$, $\widehat{T} = \widehat{T}^\top$, $\widehat{G}$ and $\widehat{F}$ satisfying (35) – (37) and

$$
W_{\widehat{R}}^\top \begin{bmatrix} \widehat{A}^\top \widehat{X} + \widehat{X}^\top \widehat{A} + \widehat{C}_1^\top Q \widehat{C}_1 & \widehat{X}^\top \widehat{B}_1 + \widehat{C}_1^\top Q D_{11} \\ \widehat{B}_1^\top \widehat{X} + D_{11}^\top Q \widehat{C}_1 & D_{11}^\top Q D_{11} - \gamma^2 P \end{bmatrix} W_{\widehat{R}} < 0, \tag{41}
$$

$$
W_{\widehat{L}}^\top \begin{bmatrix} \widehat{A}\widehat{Y} + \widehat{Y}^\top \widehat{A}^\top + \widehat{B}_1 P^{-1} \widehat{B}_1^\top & \widehat{Y}^\top \widehat{C}_1^\top + \widehat{B}_1 P^{-1} D_{11}^\top \\ \widehat{C}_1 \widehat{Y} + D_{11} P^{-1} \widehat{B}_1^\top & D_{11} P^{-1} D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_{\widehat{L}} < 0, \tag{42}
$$

where $\widehat{R} = \begin{bmatrix} \widehat{C}_2 & \widehat{D}_{21} \end{bmatrix}$, $\widehat{L} = \begin{bmatrix} \widehat{B}_2^\top & \widehat{D}_{12}^\top \end{bmatrix}$. Moreover, all diagonal blocks of the matrices $\widehat{X}$ and $\widehat{Y}$ are nonsingular. Using the block structure of coefficient matrices of the system and the following matrix representations:

$$
W_{\widehat{R}} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_s \end{bmatrix} W_R, \quad W_{\widehat{L}} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_k \end{bmatrix} W_L,
$$

one can establish the equivalence of matrix inequalities (15) and (41), as well as (16) and (42). Considering the equivalence of statements (i) and (ii) in Lemma 3.2, there exists a dynamic controller (27) such that closed-loop system (31) is admissible and its performance measure $J < \gamma$ if (15), (16) and (32) – (34) hold for some $S = S^\top$, $T = T^\top$, G and F. An additional constraint in the converse statement of Lemma 2.2 for system (31) has the form $\operatorname{rank}\begin{bmatrix} \widehat{E}^\top & \widehat{C}_*^\top Q \widehat{D}_* \end{bmatrix} = \rho + p$. This equality is equivalent to (18).

Remark 3.2 Note that the required matrices of dynamic controller (27) in Theorem 3.3 can be determined according to (30), where $\widehat{K}_*$ is the gain matrix of a static controller (29) found for extending system (28) via the LMI technique (see the proof of Theorem 3.1). In the case $\delta = 0$, Theorem 3.3 yields sufficient and necessary conditions for the existence of a static controller in Theorem 3.1.

Corollary 3.2 *Let the LMIs* (15) *and* (16) *with*

$$X = \big(S + W_{E^\top}\widetilde{G}\big)E + \gamma W_{E^\top} C W_E^+, \quad Y = \big(T + W_E\widetilde{F}\big)E^\top + \gamma W_E C^{-1} W_{E^\top}^+, \tag{43}$$

where C denotes any nonsingular matrix, and

$$0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l, \quad \Gamma = \begin{bmatrix} E_l^\top S E_l & \gamma I_\rho \\ \gamma I_\rho & E_r^\top T E_r \end{bmatrix} > 0 \tag{44}$$

hold for some $S = S^\top$, $T = T^\top$, $\widetilde{G}$ and $\widetilde{F}$. Then there exists a dynamic controller (27) *of the order $p = \rho$ such that closed-loop system* (31) *is admissible with $J < \gamma$.*

Proof. Denote $\mathcal{L} = \begin{bmatrix} E & W_{E^\top} \end{bmatrix}$, $\mathcal{R} = \begin{bmatrix} E^\top & W_E \end{bmatrix}$. Since $\mathcal{L}$ and $\mathcal{R}$ are full row rank matrices, $\operatorname{rank}(\mathcal{L}^\top X \mathcal{R}) = n$ and $\operatorname{rank}(\mathcal{R}^\top Y \mathcal{L}) = n$, where

$$\mathcal{L}^\top X \mathcal{R} = \begin{bmatrix} E^\top S E E^\top & 0 \\ W_{E^\top}^\top X E^\top & \gamma W_{E^\top}^\top W_{E^\top} C \end{bmatrix}, \quad \mathcal{R}^\top Y \mathcal{L} = \begin{bmatrix} E T E^\top E & 0 \\ W_E^\top Y E & \gamma W_E^\top W_E C^{-1} \end{bmatrix},$$

$$E^\top S E E^\top = E_r E_l^\top S E_l E_r^\top E_r E_l^\top, \quad E T E^\top E = E_l E_r^\top T E_r E_l^\top E_l E_r^\top,$$

$$E_l^\top S E_l > 0,\ E_r^\top T E_r > 0,\ W_{E^\top}^\top W_{E^\top} > 0,\ W_E^\top W_E > 0,$$

X and Y in (43) are nonsingular.

Next, we use the following transformation of matrix $\Delta = \gamma^2 I_n - XY$:

$$\mathcal{L}^\top \Delta \mathcal{L} = \begin{bmatrix} E_r & 0 \\ 0 & I_{n-\rho} \end{bmatrix} \begin{bmatrix} D E_l^\top E_l & 0 \\ -W_{E^\top}^\top X Y E_l & 0 \end{bmatrix} \begin{bmatrix} E_r^\top & 0 \\ 0 & I_{n-\rho} \end{bmatrix},$$

where $D = \gamma^2 I_\rho - E_l^\top S E_l E_r^\top T E_r$. Then, due to (44), $\det D \neq 0$ and $\operatorname{rank}\Delta = \rho$. Hence, the rank conditions in (34) hold with $\operatorname{rank}\Gamma = 2\rho$ and $\delta = \rho$. The statement of Corollary 3.2 follows from Theorem 3.3.

Note that Theorems 3.1 and 3.3 can be extended to a class of descriptor systems (8) with the following polyhedral uncertainties:

$$A \in \operatorname{Co}\big\{A_1, \ldots, A_\alpha\big\}, \quad B_1 \in \operatorname{Co}\big\{B_{11}, \ldots, B_{1\beta}\big\},$$

$$C_1 \in \operatorname{Co}\big\{C_{11}, \ldots, C_{1\mu}\big\}, \quad D_{11} \in \operatorname{Co}\big\{D_{111}, \ldots, D_{11\nu}\big\},$$

where $\operatorname{Co}\{A_1, \ldots, A_\alpha\} = \Big\{ \sum\limits_{i=1}^{\alpha} a_i A_i : \ a_i \geq 0,\ i = \overline{1, \alpha},\ \sum\limits_{i=1}^{\alpha} a_i = 1 \Big\}$. For this, instead of (15) and (16), we can use the corresponding LMIs systems

$$W_R^\top \begin{bmatrix} A_i^\top X + X^\top A_i + C_{1p}^\top Q C_{1p} & X^\top B_{1j} + C_{1p}^\top Q D_{11q} \\ B_{1j}^\top X + D_{11q}^\top Q C_{1p} & D_{11q}^\top Q D_{11q} - \gamma^2 P \end{bmatrix} W_R < 0,$$

$$W_L^\top \begin{bmatrix} A_i Y + Y^\top A_i^\top + B_{1j} P^{-1} B_{1j}^\top & Y^\top C_{1p}^\top + B_{1j} P^{-1} D_{11q}^\top \\ C_{1p} Y + D_{11q} P^{-1} B_{1j}^\top & D_{11q} P^{-1} D_{11q}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0$$

for $i = \overline{1,\alpha}$, $j = \overline{1,\beta}$, $p = \overline{1,\mu}$, $q = \overline{1,\nu}$.

Considering formulas (28) and (29) with constraints (24) and the conditions (a) and (d) in Lemma A.2, we can formulate an analog of Theorem 3.2 that yields conditions for the existence of a dynamic controller (27) in terms of LMIs and GARIs such that closed-loop system (31) is admissible and its performance measure $J < \gamma$.

4 Numerical Example: Controlled Electrical Circuit

Consider the electrical circuit given in Fig. 1. The dynamics of this system is described

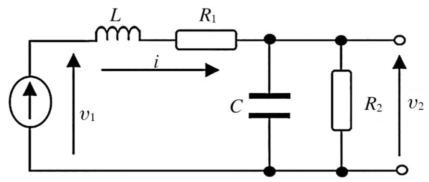

Figure 1: The electrical circuit.

by descriptor form (8) with the following data [23]:

$$E = \begin{bmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & -1 & 1 \\ 0 & -1/R_2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; D_{11} = D_{21} = D_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $x = \begin{bmatrix} i & v_2 & v_1 \end{bmatrix}^\top$, $i(t)$ denotes the current flow, v_1 and v_2 are the voltages, $i(t) = u(t) + w(t)$, $u(t)$ is the control input, $w(t)$ is the bounded disturbance, L and C are the inductance and the capacitance, respectively, R_1 and R_2 are the resistances, α is a constant parameter. In this example, $n = 3$, $m = s = 1$ and $k = l = 2$. Regulated and measured outputs of the system have the form $z = \begin{bmatrix} v_2 & \alpha v_1 + u \end{bmatrix}^\top$ and $y = \begin{bmatrix} v_2 & v_1 \end{bmatrix}^\top$, respectively.

Let $L = 3$, $C = 2$, $R_1 = 2$, $R_2 = 1$ and $\alpha = 1$. Then the pair (E, A) is impulsive, the triple (E, A, B_2) is I-controllable and the triple (E, A, C_2) is I-observable.

Assume that the standard performance index J_0 and performance measure J of the form (3) are defined by the weight matrices $P = 1$, $Q = I_2$ and $X_0 = E^\top E$. Let $\gamma = 0.5$, then conditions (24) of Theorem 3.2 are satisfied. Using the Mathcad Prime 6.0 system, we found the matrices

$$S = \begin{bmatrix} 0.06402 & 0.03218 & -0.02942 \\ 0.03218 & 0.23409 & 0.00678 \\ -0.02942 & 0.00678 & 0.16182 \end{bmatrix}, \quad X = \begin{bmatrix} 0.19206 & 0.06436 & 0 \\ 0.09654 & 0.46818 & 0 \\ 0.07355 & 0.46119 & -0.33001 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.16181 & 0.44763 & -0.33001 \end{bmatrix},$$

satisfying (11), (15) and (25). Further, we determine the gain matrix $K = \begin{bmatrix} 0.01512 & -1.81255 \end{bmatrix}$ of controller (9) for which $J_0 = 0.44823$, $J = 0.48232 < \gamma$ and a closed-loop system (10) with the finite spectrum $\{-0.67528 \pm 0.36681i\}$ is admissible. Also, by using Lemma 2.3, the structured worst disturbance

$$w = K_0 x, \quad K_0 = \begin{bmatrix} -0.72013 & 0.28805 & 1.14308 \end{bmatrix}, \tag{45}$$

and the worst initial vector $x_0 = \begin{bmatrix} 0.08668 & 0.88792 & 0.17938 \end{bmatrix}^\top$ are determined.

The finite spectrum of system (10) with the worst pair (w, x_0)

$$E\dot{x} = (A_* + B_* K_0)x, \quad x(0) = x_0, \tag{46}$$

is computed as $\{-0.59864, -1.42448\}$. Figs. 2 and 3 show the behavior of system (46) and function $w(t)$ in (45), respectively.

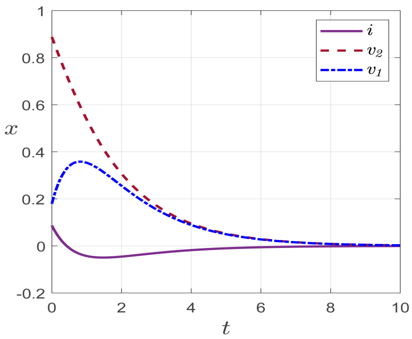

Figure 2: Closed-loop system behavior with the worst pair (w, x_0).

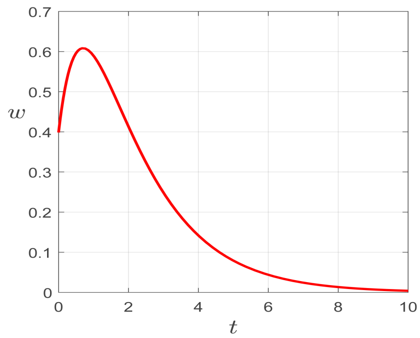

Figure 3: The worst disturbance.

Computational experiments have shown that the decrease of parameter α in the interval $[0, 1]$ leads to the increase of the minimum possible characteristics J_0 and J for a closed-loop system using static controllers of the form (9).

Also, on the basis of Theorem 3.3 (see Remark 3.2), a dynamic controller (27) with

$$\begin{bmatrix} K & U \\ V & Z \end{bmatrix} = \left[\begin{array}{cc|cc} -0.08957 & -0.96093 & 0.57234 & 2.61512 \\ \hline 0.03600 & -0.07555 & -0.38338 & -0.00721 \\ -0.00232 & -0.00068 & 0.08507 & -0.41471 \end{array}\right]$$

is determined, for which system (31) is admissible with $J_0 = 0.25070$ and $J = 0.47138$.

5 Conclusion

This paper presents new approaches to the generalized problem of H_∞ control with transients for continuous-time descriptor systems. The weighted performance measure used takes into account the influence of both exogenous disturbances and initial states. Necessary and sufficient conditions for the solvability of this problem via static and dynamic controllers have been proposed in terms of LMIs and GARIs with special matrix variables.

New auxiliary Lemmas 3.1 and 3.2 are obtained here, and used in the synthesis of static and dynamic controllers, respectively. These lemmas make it possible to search for solutions of the arising linear and quadratic matrix inequalities in parametric form (11) and (12) with considering the skeletal decomposition of system matrix E. This makes the main results (Theorems 3.1, 3.2 and 3.3) more constructive in comparison with [20,21]. Moreover, with this approach, it is possible to formulate necessary and sufficient conditions for the existence of generalized H_∞ controllers exclusively in terms of LMIs (see Corollaries 3.1 and 3.2). Projection Lemma (Lemma A.1) and its new generalization (Lemma A.2) gives criteria for the solvability of linear and quadratic matrix inequalities, respectively. The presented synthesis approaches have been illustrated by a numerical example of the controlled electrical circuit.

A Solvability of Some Matrix Inequalities

Lemma A.1 (see [8]) *Given matrices* $A = A^\top \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{p\times n}$ *and* $C \in \mathbb{R}^{q\times n}$, *the LMI*

$$A + B^\top XC + C^\top X^\top B < 0 \tag{47}$$

is solvable for $X \in \mathbb{R}^{p\times q}$ *if and only if one of the following conditions holds:*
(a) $\operatorname{rank} B = n$, $\operatorname{rank} C = n$; (b) $\operatorname{rank} B < n$, $\operatorname{rank} C = n$, $W_B^\top A W_B < 0$;
(c) $\operatorname{rank} B = n$, $\operatorname{rank} C < n$, $W_C^\top A W_C < 0$;
(d) $\operatorname{rank} B < n$, $\operatorname{rank} C < n$, $W_B^\top A W_B < 0$, $W_C^\top A W_C < 0$.

Consider the following quadratic matrix inequality:

$$A + B^\top XC + C^\top X^\top B + C^\top X^\top RXC < 0, \tag{48}$$

where $A = A^\top \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{p\times p}$. Suppose that matrices C and R are nonzero and $R = R^\top \geq 0$.

Lemma A.2 (see [21]) *There exist a matrix* $X \in \mathbb{R}^{p\times q}$ *satisfying* (48) *if and only if*
(a) *either* $\operatorname{rank} C = n$ *or* $\operatorname{rank} C < n$ *and* $W_C^\top A W_C < 0$;
and one of the following conditions hold:
(b) $R > 0$, $A < B^\top R^{-1} B$; (c) $\operatorname{rank} R < p$, $\operatorname{rank} B_0 = n$;
(d) $\operatorname{rank} R < p$, $\operatorname{rank} B_0 < n$, $W_{B_0}^\top \big(A - B^\top R^+ B\big) W_{B_0} < 0$;
where $B_0 = W_R^\top B$ *and* R^+ *is a pseudo-inverse of* R.

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