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Robust Output Feedback Stabilization and Optimization of Control Systems

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Abstract: The paper is devoted to the problems of output feedback stabilization, robust stabilization, quadratic optimization and generalized H_∞-control for some classes of linear and nonlinear dynamical systems. Sufficient stability conditions for the zero state are formulated with the joint quadratic Lyapunov function for a family of control systems with uncertain coefficient matrices. The solution of robust stabilization problem and evaluation of the quadratic performance criterion for a family of nonlinear control systems are proposed. Methods for construction of control laws providing a robust stability and specified evaluation of the weighted damping level of input signals and initial perturbations are proposed for a class of linear systems with controllable and observable outputs. The application of the main results reduces to solving the systems of linear matrix inequalities.

Keywords: *pseudolinear system; output feedback; robust stability; linear matrix inequality; quadratic Lyapunov function,* H_∞*-control.*

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1 Introduction

State and output feedback controllers design for dynamic systems with the prescribed and desired properties is a key problem of control theory. At the same time, the properties of control systems such as asymptotic stability, robustness and optimality of the performance indexes are in the foreground. The main problem in H_∞-control theory is connected with suppression of external and initial perturbations (see, e.g., [1–5] as well as review papers [6, 7]).

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It should be noted that the practical applications of many modern methods for synthesis of control systems are based on the construction and solution of linear matrix inequalities (LMI). For this purpose, sufficiently effective computational algorithms and appropriate tools are established in Matlab environment (see [8, 9]).

In this paper, we consider classes of linear and affine control systems for which closed loop systems can be represented in the pseudolinear form

$$\dot{x} = M(x,t)\,x, \quad x \in \mathbb{R}^n, \quad t \geq 0,$$

besides, a matrix function $M(x,t)$ can contain uncertain quantities belonging to certain sets. Intervals, polytopes, affine families of matrices and other objects may serve as the uncertainty sets. To define uncertainties and robust stability conditions for systems in semiordered spaces one can use cone inequalities and intervals [5, 10, 11]. The applied control laws are of the form of static or dynamic output feedback. It should be noted that at the solution of many control problems the dynamic controllers have great potential as compared with the static controllers.

Our consideration includes the following types of problems:

- output feedback stabilization of control systems (Section 2);
- robust stabilization and optimization of control systems with polyhedral uncertainties (Section 3);
- robust stabilization and weighted perturbation suppression in control systems (Section 4).

Throughout the paper, the following notations are used: I_n is the identity $n \times n$ matrix; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^T > 0$ (≥ 0) is the symmetric positive definite (semidefinite) matrix X; $\mathrm{i}(X) = \{\mathrm{i}_+, \mathrm{i}_-, \mathrm{i}_0\}$ is the inertia of Hermitian matrix $X = X^*$ consisting of the numbers of positive ($\mathrm{i}_+(X)$), negative ($\mathrm{i}_-(X)$) and zero ($\mathrm{i}_0(X)$) eigenvalues (taking into account the multiplicities); $\sigma(A)$ and $\rho(A)$ are the spectrum and the spectral radius of A, respectively; $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the maximum and the minimum eigenvalue of the Hermitian matrix X, respectively; A^+ is the pseudoinverse matrix; W_A is a matrix whose columns make up the bases of the kernel $\mathrm{Ker}\,A$; $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$; $\mathrm{Co}\{A_1, \ldots, A_\nu\}$ stands for a polytope in a matrix space described as a convex full of the set $\{A_1, \ldots, A_\nu\}$, i. e.

$$\mathrm{Co}\{A_1, \ldots, A_\nu\} = \Big\{ \sum_{i=1}^{\nu} \alpha_i A_i : \ \alpha_i \geq 0, \ i = \overline{1, \nu}, \ \sum_{i=1}^{\nu} \alpha_i = 1 \Big\}.$$

Note that matrix intervals and affine sets of matrices are described in terms of polytopes.

2 Output Feedback Stabilization of Nonlinear Systems

Consider the following affine nonlinear time-invariant control system

$$\dot{x} = A(x)x + B(x)u, \quad y = C(x)x + D(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is state vector, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ are input and output vectors, respectively, $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in some neighborhood $\mathcal{S}_0$ of the zero state $x = 0$. We will assume that $\mathrm{rank}\,B(x) \equiv m$ and $\mathrm{rank}\,C(x) \equiv l$ in $\mathcal{S}_0$.

Along with (1), consider the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{2}$$

where $A = A(0)$, $B = B(0)$, $C = C(0)$ and $D = D(0)$. Let $B^\perp$ and $C^\perp$ be the orthogonal complements of B and C, respectively, i.e.

$$B^T B^\perp = 0, \ \det\left[B, B^\perp\right] \neq 0, \ C^\perp C^T = 0, \ \det\left[C^T, C^{\perp T}\right] \neq 0.$$

2.1 Static controllers

Formulate stabilizability conditions of the zero state $x = 0$ for systems (1) and (2) through the static output-feedback controller

$$u = Ky, \quad K \in \mathcal{K}_D, \tag{3}$$

where $\mathcal{K}_D = \{K \in \mathbb{R}^{m\times l} : \det(I_m - KD) \neq 0\}$. Closed loop system (2), (3) has the form

$$\dot{x} = Mx, \quad M = A + B\mathbf{D}(K)C, \tag{4}$$

where $\mathbf{D}(K) = (I_m - KD)^{-1}K$ is a nonlinear operator with the following properties:
• if $K \in \mathcal{K}_D$, then

$$\mathbf{D}(K) \equiv K(I_l - DK)^{-1}, \quad I_l + D\mathbf{D}(K) \equiv (I_l - DK)^{-1}; \tag{5}$$

• if $K_1 \in \mathcal{K}_D$ and $K_2 \in \mathcal{K}_{D_1}$, then $K_1 + K_2 \in \mathcal{K}_D$ and

$$\mathbf{D}(K_1 + K_2) = \mathbf{D}(K_1) + (I_m - K_1 D)^{-1}\mathbf{D}_1(K_2)\,(I_l - DK_1)^{-1}, \tag{6}$$

where $\mathbf{D}_1(K_2) = (I_m - K_2 D_1)^{-1}K_2$, $D_1 = (I_l - DK_1)^{-1}D$;
• if $-K_0 \in \mathcal{K}_D$, then $K = -\mathbf{D}(-K_0) \in \mathcal{K}_D$ and

$$\mathbf{D}(K) = K_0. \tag{7}$$

According to (7), to achieve the desired properties and, in particular, to stabilize system (4) it suffices to provide a system with matrix $M_* = A + BKC$ with these properties.

Definition 2.1 System (4) is α-stable if the spectrum $\sigma(M)$ lies in the open left half-plane $\mathbb{C}_\alpha^- = \{\lambda : \operatorname{Re}\lambda + \alpha < 0\}$, where $\alpha \geq 0$.

Theorem 2.1 *The following statements are equivalent:*
1) *There exists static controller* (3) *ensuring α-stability of system* (4).
2) *There exists matrix $X = X^T > 0$ satisfying the relations*

$$B^{\perp T}(AX + XA^T + 2\alpha X)\,B^\perp < 0, \tag{8}$$

$$\mathrm{i}(\Delta) = \{l, n, 0\}, \quad \Delta = \begin{bmatrix} AX + XA^T + 2\alpha X & XC^T \\ CX & 0 \end{bmatrix}. \tag{9}$$

3) *There exist mutually inverse matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying the relations* (8) *and*

$$C^\perp(A^T Y + YA + 2\alpha Y)\,C^{\perp T} < 0. \tag{10}$$

When one of the statements 2) *or* 3) *is true, then the controller*

$$u = Ky, \quad K = -\mathbf{D}(-K_0) \in \mathcal{K}_D, \tag{11}$$

where K_0 *is a solution of the LMI*

$$AX + XA^T + 2\alpha X + BK_0CX + XC^TK_0^TB^T < 0, \tag{12}$$

ensures α*-stability of closed loop system* (4)*.*

For the equivalence of the statements 1) and 2) in Theorem 2.1, see [5]. Equivalence of the statements 2) and 3) follows from the correlations (see [10, p. 147])

$$\mathrm{i}_\pm(\Delta) = \mathrm{i}_\pm(\Delta_1) = \mathrm{i}_\pm(C^\perp L_1 C^{\perp T}) + l,$$

where

$$\Delta_1 = R^T \Delta R = \left[\begin{array}{c|cc} C^\perp L_1 C^{\perp T} & 0 & C^\perp L_1 C^+ \\ \hline 0 & 0 & I_l \\ C^{+T} L_1 C^{\perp T} & I_l & C^{+T} L_1 C^+ \end{array}\right],$$

$$L_1 = A^TY + YA + 2\alpha Y, \quad Y = X^{-1}, \quad R = \begin{bmatrix} YC^{\perp T} & 0 & YC^+ \\ 0 & I_l & 0 \end{bmatrix}, \quad \det R \neq 0.$$

For the equivalence of the statements 1) and 3), see also [4].

Theorem 2.2 [12] *Let one of the statements* 2) *or* 3) *of Theorem* 2.1 *hold for system* (2)*. Then relations* (11) *and* (12) *determine static controller ensuring asymptotic stability of the state* $x \equiv 0$ *and quadratic Lyapunov function* $v(x) = x^TYx$ *of nonlinear closed loop system* (1)*,* (11)*.*

2.2 Dynamic controllers

The dynamic output feedback stabilization problem for system (1) consists in finding, if possible, a dynamic control law described by

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \tag{13}$$

where $\xi \in \mathbb{R}^r$ and $r \leq n$, such that the zero state of closed loop system is asymptotically stable. Equations (1) and (13) may be represented by control system in the extended phase space $\mathbb{R}^{n+r}$ with static controller

$$\dot{\widehat{x}} = \widehat{A}(\widehat{x})\widehat{x} + \widehat{B}(\widehat{x})\widehat{u}, \quad \widehat{y} = \widehat{C}(\widehat{x})\widehat{x} + \widehat{D}(\widehat{x})\widehat{u}, \quad \widehat{u} = \widehat{K}\widehat{y}, \tag{14}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{y} = \begin{bmatrix} y \\ \xi \end{bmatrix}, \quad \widehat{u} = \begin{bmatrix} u \\ \dot{\xi} \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix},$$

$$\widehat{A}(\widehat{x}) = \begin{bmatrix} A(x) & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \quad \widehat{B}(\widehat{x}) = \begin{bmatrix} B(x) & 0_{n\times r} \\ 0_{r\times m} & I_r \end{bmatrix},$$

$$\widehat{C}(\widehat{x}) = \begin{bmatrix} C(x) & 0_{l\times r} \\ 0_{r\times n} & I_r \end{bmatrix}, \quad \widehat{D}(\widehat{x}) = \begin{bmatrix} D(x) & 0_{l\times r} \\ 0_{r\times m} & 0_{r\times r} \end{bmatrix}.$$

If $K \in \mathcal{K}_D$, then linear closed loop system (2), (13) has the form

$$\dot{\widehat{x}} = \widehat{M}\,\widehat{x}, \quad \widehat{M} = \widehat{A} + \widehat{B}\widehat{\mathbf{D}}(\widehat{K})\widehat{C}, \tag{15}$$

where $\widehat{A} = \widehat{A}(0)$, $\widehat{B} = \widehat{B}(0)$, $\widehat{C} = \widehat{C}(0)$, $\widehat{D} = \widehat{D}(0)$, $\widehat{\mathbf{D}}(\widehat{K}) = (I_{m+r} - \widehat{K}\widehat{D})^{-1}\widehat{K}$, and

$$\widehat{\mathbf{D}}(\widehat{K}) = \left[\begin{array}{c|c} \mathbf{D}(K) & (I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1} & Z + VD(I_m - KD)^{-1}U \end{array}\right],$$

$$\widehat{M} = \left[\begin{array}{c|c} M & B(I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1}C & Z + VD(I_m - KD)^{-1}U \end{array}\right].$$

Theorem 2.3 *The following statements are equivalent:*

1) *There exists dynamic controller* (13) *of order $r \le n$ ensuring α-stability of closed loop system* (15)*.*

2) *There exist matrices X and X_0 satisfying the relations* (8) *and*

$$\mathrm{i}(\Delta_0) = \{l, n, 0\}, \quad X \ge X_0 > 0, \quad \mathrm{rank}\,(X - X_0) \le r, \tag{16}$$

where

$$\Delta_0 = \begin{bmatrix} AX_0 + X_0A^T + 2\alpha X_0 & X_0C^T \\ CX_0 & 0 \end{bmatrix}.$$

3) *There exist matrices X and Y satisfying the relations* (8), (10) *and*

$$W = \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \ge 0, \quad \mathrm{rank}\, W \le n + r. \tag{17}$$

Proof of Theorem 2.3 follows from the corresponding statements of Theorem 2.1 taking into account the structure of block matrices in (15) (see [12]). In [12], a computation algorithm of finding a stabilizing dynamic controller (13) for nonlinear systems (1) has been proposed on the basis of Theorem 2.3.

Remark 2.1 Note, that matrices X and X_0 satisfy statement 2) iff matrices X and $Y = X_0^{-1}$ satisfy statement 3). From (17) it follows that matrices X and Y are positive definite. The rank restriction in (17) always holds in case of full order $r = n$ dynamic regulator.

3 Robust Stabilization and Optimization of Nonlinear Systems

We formulate an auxiliary statement that will be used in the proofs of our main results. Consider a nonlinear operator

$$\mathbf{F}(K) = W + U^T\mathbf{D}(K)V + V^T\mathbf{D}^T(K)U + V^T\mathbf{D}^T(K)\,R\,\mathbf{D}(K)V \tag{18}$$

with $\mathbf{D}(K) = (I_m - KD)^{-1}K$ and an ellipsoidal set of matrices

$$\mathcal{K} = \{K \in \mathbb{R}^{m\times l} :\ K^TPK \le Q\}, \tag{19}$$

where $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T \ge 0$, $W = W^T \le 0$, U, V and D are matrices of suitable sizes. Matrix inequality in (19) is equivalent to the following $KQ^{-1}K^T \le P^{-1}$. Therefore, in case of $m = 1$ the ellipsoid $\mathcal{K}$ is described by a scalar inequality.

Lemma 3.1 [13] *Suppose that the following matrix inequalities hold:*

$$D^TQD + R < P, \quad \Omega = \begin{bmatrix} W & U^T & V^T \\ U & R-P & D^T \\ V & D & -Q^{-1} \end{bmatrix} \le 0 \ (< 0). \tag{20}$$

Then $\mathbf{F}(K) \le 0$ (< 0) *for every matrix* $K \in \mathcal{K}$.

Note that Lemma 3.1 is a generalization of the sufficiency statement for a criterion known as Petersen's lemma on matrix uncertainty [14] (see also [15]). In Lemma 3.1 letting $D = 0$, $R = 0$, $P = \varepsilon I_m$ and $Q = \varepsilon I_l$, where $\varepsilon > 0$, we get the sufficiency statement of Petersen's lemma.

Consider a nonlinear control system in the vector-matrix form

$$E(x)\dot{x} = A(x,t)\,x + B(x,t)\,u, \quad y = C(x,t)\,x + D(x,t)\,u, \quad t \ge 0, \tag{21}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$. We construct a set of the static controllers

$$u = K(x,t)\,y, \quad K(x,t) = K_*(x,t) + \widetilde{K}(x,t), \quad \widetilde{K}(x,t) \in \mathcal{K}, \tag{22}$$

where $\mathcal{K}$ is an ellipsoidal set of matrices of the form (19). We assume that the matrices E, A, B, C, D, K and K_* depend on x and t continuously and the equilibrium state $x \equiv 0$ is isolated, i.e., the neighborhood $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \le h\}$ does not contain other equilibrium states of this system. If $K \in \mathcal{K}_D$, then the closed loop system (21), (22) can be represented as

$$E(x)\dot{x} = M(x,t)x, \quad M(x,t) = A + B\mathbf{D}(K)C. \tag{23}$$

Let the zero state of this system for $K \equiv K_*$ be asymptotically stable. When looking for the stabilizing matrix K_* in the class of autonomous systems (1), one can use Theorem 2.1 and its special cases. The problem is to construct conditions under which the zero state of system (23) is Lyapunov asymptotically stable for every matrix $\widetilde{K}(x,t) \in \mathcal{K}$. We find a solution for our problem in terms of a quadratic Lyapunov function (see [5,13]).

Theorem 3.1 *Let for some matrix functions* $X(t) = X^T(t)$ *and* $K_*(x,t)$ *at* $x = 0$ *and* $t \ge 0$ *the correlations*

$$\varepsilon_1 I_n \le X(t) \le \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \le \varepsilon_2, \tag{24}$$

$$\begin{bmatrix} E^T\dot{X}E + M_*^TXE + E^TXM_* + \varepsilon_0 I_n & E^TXB_* & C_*^T \\ B_*^TXE & -P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{25}$$

hold with $\varepsilon_0 > 0$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_*D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$, $D_* = D(I_m - K_*D)^{-1}$. *Then any control* (22) *ensures asymptotic stability of the zero state* $x \equiv 0$ *for system* (23) *and a common Lyapunov function* $v(x,t) = x^TE_0^TX(t)E_0x$, *where* $E_0 = E(0)$.

Consider control system (21) with quadratic quality functional

$$J(u,x_0) = \int_0^\infty \varphi(x,u,t)\,dt, \tag{26}$$

where

$$x_0 = x(0), \quad \varphi(x,u,t) = \left[x^T, u^T\right] \Phi(t) \begin{bmatrix} x \\ u \end{bmatrix},$$

$$\Phi(t) = \begin{bmatrix} S & N \\ N^T & R \end{bmatrix}, \quad R > 0, \quad S \geq NR^{-1}N^T + \eta\, I_n, \quad \eta > 0, \quad t \geq 0.$$

Theorem 3.2 *Let for some matrix functions $X(t) = X^T(t)$ and $K_*(x,t)$ at $x = 0$ and $t \geq 0$ the correlations* (24) *and*

$$\begin{bmatrix} E^T\dot{X}E + M_*^T XE + E^T XM_* + \Phi_* + \varepsilon_0 I_n & E^T XB_* + N_* + C^T K_*^T R_* & C_*^T \\ B_*^T XE + N_*^T + R_* K_* C & R_* - P & D_*^T \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{27}$$

hold with $\varepsilon_0 > 0$, $\Phi_* = L_*^T \Phi L_*$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_*D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$, $D_* = D(I_m - K_*D)^{-1}$, $R_* = (I_m - K_*D)^{-1T} R\,(I_m - K_*D)^{-1}$, $N_* = N(I_m - K_*D)^{-1}$, $L_*^T = \left[I_n, C^T\mathbf{D}^T(K_*)\right]$. *Then any control* (22) *ensures asymptotic stability of the zero state $x \equiv 0$ for system* (23)*, a common Lyapunov function $v(x,t) = x^T E_0^T X(t) E_0 x$, where $E_0 = E(0)$, and a bound on the functional $J(u,x_0) \leq v(x_0,0)$.*

Corollary 3.1 *Let for some matrix $X = X^T > 0$ and K_* the system of LMI*

$$\begin{bmatrix} M_{*ijk}^T XE_s + E_s^T XM_{*ijk} + L_{*k}^T \Phi L_{*k} & E_s^T XB_{*j} + N_* + C_k^T K_*^T R_* & C_{*k}^T \\ B_{*j}^T XE_s + N_*^T + R_* K_* C_k & R_* - P & D_*^T \\ C_{*k} & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{28}$$

$$i = \overline{1,\alpha}, \quad j = \overline{1,\beta}, \quad k = \overline{1,\gamma}, \quad s = \overline{1,\delta},$$

hold with $M_{*ijk} = A_i + B_j\mathbf{D}(K_*)C_k$, $B_{*j} = B_j(I_m - K_*D)^{-1}$, $C_{*k} = (I_l - DK_*)^{-1}C_k$, $D_* = D(I_m - K_*D)^{-1}$, $R_* = (I_m - K_*D)^{-1T} R\,(I_m - K_*D)^{-1}$, $N_* = N(I_m - K_*D)^{-1}$, $L_{*k}^T = \left[I_n, C_k^T \mathbf{D}^T(K_*)\right]$. *Then any control* (22) *ensures asymptotic stability of the zero state $x \equiv 0$ for system* (23) *with uncertainties*

$$\begin{aligned} A(0,t) &\in \mathrm{Co}\{A_1, \ldots, A_\alpha\}, \quad B(0,t) \in \mathrm{Co}\{B_1, \ldots, B_\beta\}, \\ C(0,t) &\in \mathrm{Co}\{C_1, \ldots, C_\gamma\}, \quad E(0) \in \mathrm{Co}\{E_1, \ldots, E_\delta\}, \end{aligned} \tag{29}$$

and a bound on the functional $J(u,x_0) \leq \omega = \max\limits_{1 \leq s \leq \delta} x_0^T E_s^T X E_s x_0$.

Note that the proof of Theorems 3.1 and 3.2 follows directly from Lemma 3.1 and Lyapunov theorem on asymptotic stability taking into account representation of derivative of Lyapunov function $v(x,t)$ with respect to system (23) in the form of a quadratic function with matrix of the form (18) and application of formula (6) (see [5,13]).

4 Generalized H_∞-Control

4.1 Weighted level of perturbation suppression

Consider a dynamical system with external perturbations

$$\dot{x} = f(x,w,t), \quad y = g(x,w,t), \quad x(0) = x_0, \quad t \geq 0, \tag{30}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^s$ and $y \in \mathbb{R}^l$ are the state, the norm-limited external perturbations and the output vector, respectively.

Definition 4.1 The dynamical system (30) is called *nonexpansive*, if

$$\int_0^T y(t)^T Q y(t) dt \le \int_0^T w(t)^T P w(t) dt + x_0^T X_0 x_0$$

for all square-integrable functions $w(t)$ and $T > 0$, where Q, P and X_0 are weight symmetric positive definite matrices.

We introduce the performance criterion of system (30) with respect to output y:

$$J = \sup_{0<\|w\|_P^2 + x_0^T X_0 x_0 < \infty} \varphi(w, x_0), \tag{31}$$

where

$$\varphi(w, x_0) = \frac{\|y\|_Q}{\sqrt{\|w\|_P^2 + x_0^T X_0 x_0}}, \quad \|y\|_Q^2 = \int_0^\infty y^T Q y \, dt, \quad \|w\|_P^2 = \int_0^\infty w^T P w \, dt.$$

In case of $x_0 = 0$, we denote J by J_0. It is obvious, that $J_0 \le J$ and $J \le 1$ for a nonexpansive system. The value J describes the weighted level of external and initial perturbation suppression in system (30). If $P = I_s$, $Q = I_l$ and $X_0 = \rho I_n$, then J and J_0 coincide with known performance criteria of dynamical systems [16]. For the class of linear systems

$$\dot{x} = Ax + Bw, \quad y = Cx + Dw, \quad x(0) = x_0, \tag{32}$$

the value J_0 is equal to H_∞-norm of the transfer function $H(\lambda) = C(\lambda I_n - A)^{-1} B + D$ at $x_0 = 0$ (see, e.g., [3]).

Lemma 4.1 *Let A be a Hurwitz matrix. Then an evaluation $J_0 < \gamma$ for system* (32) *holds iff the LMI*

$$\Phi_\gamma = \begin{bmatrix} A^T X + XA + C^T Q C & XB + C^T Q D \\ B^T X + D^T Q C & D^T Q D - \gamma^2 P \end{bmatrix} < 0 \tag{33}$$

has a solution $X = X^T > 0$. To perform the evaluation $J < \gamma$ it is necessary and sufficient that LMI (33) *has a solution X such that*

$$0 < X < \gamma^2 X_0. \tag{34}$$

Proof. *Sufficiency.* Construct the quadratic Lyapunov function $v(x) = x^T X x$ for system (32) and evaluate the expression

$$\dot{v}(x) + y^T Q y - \gamma^2 w^T P w = [x^T, w^T] \Phi_\gamma \begin{bmatrix} x \\ w \end{bmatrix},$$

where $\dot{v}(x)$ is the derivative of $v(x)$ with respect to system. Integrating given expression and in view of (31) and (33), we have

$$\|y\|_Q^2 \le \gamma^2 (\|w\|_P^2 + x_0^T X_0 x_0), \quad \varphi(w, x_0) \le \gamma.$$

The strict matrix inequalities (33) and (34) hold if we replace γ by $\gamma - \varepsilon$ for some small $\varepsilon > 0$. Therefore, $\varphi(w, x_0) \le \gamma - \varepsilon$ and $J < \gamma$. In particular, in case of $x_0 = 0$ the inequality $J_0 < \gamma$ holds.

Necessity. Use the expansions $Q = \widetilde{Q}^T\widetilde{Q}$, $P = \widetilde{P}^T\widetilde{P}$, $X_0 = \widetilde{X}_0^T\widetilde{X}_0$ and transform system (32):

$$\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}\widetilde{w}, \quad \widetilde{y} = \widetilde{C}\widetilde{x} + \widetilde{D}\widetilde{w}, \quad \widetilde{x}(0) = \widetilde{x}_0,$$

where $\widetilde{x} = \widetilde{X}_0 x$, $\widetilde{y} = \widetilde{Q}y$, $\widetilde{w} = \widetilde{P}w$, $\widetilde{A} = \widetilde{X}_0 A \widetilde{X}_0^{-1}$, $\widetilde{B} = \widetilde{X}_0 B \widetilde{P}^{-1}$, $\widetilde{C} = \widetilde{Q}C\widetilde{X}_0^{-1}$ and $\widetilde{D} = \widetilde{Q}D\widetilde{P}^{-1}$. Then the performance criterion (31) has the form

$$\widetilde{J} = \sup_{0<\|\widetilde{w}\|_{I_m}^2 + \widetilde{x}_0^T\widetilde{x}_0<\infty} \frac{\|\widetilde{y}\|_{I_l}}{\sqrt{\|\widetilde{w}\|_{I_m}^2 + \widetilde{x}_0^T\widetilde{x}_0}}.$$

If $\widetilde{J} < \gamma$, then for some matrix $\widetilde{X} = \widetilde{X}^T$ (see [16, Theorem 1])

$$0 < \widetilde{X} < \gamma^2 I_n, \quad \widetilde{\Omega} = \begin{bmatrix} \widetilde{A}^T\widetilde{X} + \widetilde{X}\widetilde{A} & \widetilde{X}\widetilde{B} & \widetilde{C}^T \\ \widetilde{B}^T\widetilde{X} & -\gamma^2 I_m & \widetilde{D}^T \\ \widetilde{C} & \widetilde{D} & -I_l \end{bmatrix} < 0$$

or

$$0 < X < \gamma^2 X_0, \quad \Omega = S^T\widetilde{\Omega}S = \begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -\gamma^2 P & D^T \\ C & D & -Q^{-1} \end{bmatrix} < 0,$$

where $X = \widetilde{X}_0^T\widetilde{X}\widetilde{X}_0$, $S = \operatorname{diag}\{\widetilde{X}_0, \widetilde{P}, \widetilde{Q}^{-1T}\}$. By Schur's lemma, the last matrix inequality reduces to the form (33) $\square$.

Remark 4.1 If $J_0 < \gamma$, then system (32) with an uncertainty

$$w = \frac{1}{\gamma}\Theta y, \quad \Theta^T P \Theta \le Q, \tag{35}$$

is robust stable and has a common Lyapunov function $v(x) = x^TXx$. This fact follows from Lemma 4.1 and [13, Theorem 1]. The functional $\varphi(w, x_0)$ on the set of functions (35) accepts the minimum value, if $\Theta^T P\Theta = Q$. In particular, if $k \le s$ and $x_0 = 0$, then we have $\varphi(w, 0) = \gamma$ for

$$\Theta = (\sqrt{P})^{-1}E\sqrt{Q}, \quad E = \begin{cases} I_k, & k = s, \\ \left[I_k, 0_{k\times s-k}\right]^T, & k < s. \end{cases}$$

It follows from Lemma 4.1 that the performance criteria J and J_0 of system (32) may be computed as the solutions of the corresponding optimization problems:

$$J_0 = \inf\{\gamma : \Phi_\gamma < 0,\ X > 0\}, \quad J = \inf\{\gamma : \Phi_\gamma < 0,\ 0 < X < \gamma^2 X_0\}. \tag{36}$$

Consider the affine system with norm-limited external perturbations

$$\dot{x} = A(x)x + B(x)w, \quad y = C(x)x + D(x)w, \quad x(0) = x_0, \tag{37}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in $\mathcal{S}_0$. We can formulate the following lemma (see the proof of sufficiency statement of Lemma 4.1).

Lemma 4.2 *Suppose that there exists a matrix $X = X^T > 0$ satisfying the matrix inequality*

$$\Phi_\gamma(x) = \begin{bmatrix} A^T(x)X + XA(x) + C^T(x)QC(x) & XB(x) + C^T(x)QD(x) \\ B^T(x)X + D^T(x)QC(x) & D^T(x)QD(x) - \gamma^2 P \end{bmatrix} < 0 \qquad (38)$$

for all $x \in \mathcal{S}_0$. Then $J_0 \le \gamma$ and the zero state $x \equiv 0$ of system (37) *with uncertainty* (35) *is robust stable with a common Lyapunov function $v(x) = x^T X x$. In addition, if the restriction $0 < X \le \gamma^2 X_0$ holds, then $J \le \gamma$.*

4.2 Static controllers with perturbations

Consider control systems (1), (2) and the performance criteria J and J_0 of the form (31). We are interested in control laws that ensure nonexpansivity property of close loop system and minimize J and J_0. A control law is said to be *J-optimal*, if corresponding close loop system has minimum performance criteria J. A J_0-optimal control law is *H_∞-optimal* in case of the identity weight matrices P and Q.

Primarily, we consider the static output-feedback controller

$$u = K_* y + w, \qquad (39)$$

where $w \in \mathbb{R}^m$ is a vector of bounded perturbations and $K_* \in \mathcal{K}_D$ is an unknown matrix. Assuming that $\det \big[I_m - K_* D(x)\big] \neq 0$, $x \in \mathcal{S}_0$, we rewrite the corresponding close loop systems in the form

$$\dot{x} = A_*(x)x + B_*(x)w, \quad y = C_*(x)x + D_*(x)w, \quad x(0) = x_0, \qquad (40)$$

$$\dot{x} = A_* x + B_* w, \quad y = C_* x + D_* w, \quad x(0) = x_0, \qquad (41)$$

where $A_*(x) = A(x) + B(x)\big[I_m - K_* D(x)\big]^{-1} K_* C(x)$, $B_*(x) = B(x)\big[I_m - K_* D(x)\big]^{-1}$, $C_*(x) = \big[I_l - D(x)K_*\big]^{-1} C(x)$, $D_*(x) = \big[I_l - D(x)K_*\big]^{-1} D(x)$, $A_* = A_*(0)$, $B_* = B_*(0)$, $C_* = C_*(0)$, $D_* = D_*(0)$.

Theorem 4.1 [18] *For linear system* (2), *there exists an output-feedback controller* (39) *such that $J < \gamma$ iff the following correlations are feasible:*

$$W_R^T \begin{bmatrix} A^T X + XA + C^T QC & XB + C^T QD \\ B^T X + D^T QC & D^T QD - \gamma^2 P \end{bmatrix} W_R < 0, \qquad (42)$$

$$W_L^T \begin{bmatrix} AY + YA^T + BP^{-1}B^T & YC^T + BP^{-1}D^T \\ CY + DP^{-1}B^T & DP^{-1}D^T - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \qquad (43)$$

$$0 < X < \gamma^2 X_0, \quad XY = \gamma^2 I_n, \qquad (44)$$

where $R = [C, D]$, $L = [B^T, D^T]$. Herewith, the zero states $x \equiv 0$ of systems (40) *and* (41) *with uncertainty* (35) *are robust stable with common Lyapunov function $v(x) = x^T X x$.*

Remark 4.2 The gain matrix K_* in Theorem 4.1 may be constructed in the form

$$K_* = K_0(I_l + DK_0)^{-1}, \quad -K_0 \in \mathcal{K}_D, \qquad (45)$$

Here K_0 is an arbitrary solution of the LMI

$$L_0^T K_0 R_0 + R_0^T K_0^T L_0 + \Omega < 0, \qquad (46)$$

where $R_0 = \left[R, 0_{l\times l}\right]$, $L_0 = \left[L, 0_{m\times m}\right]\widetilde{X}$,

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_m & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -P & D^T \\ C & D & -Q^{-1} \end{bmatrix}.$$

Lemma 4.3 [17] *LMI* (46) *has a solution* K_0 *if and only if*

$$W_{L_0}^T \Omega W_{L_0} < 0, \quad W_{R_0}^T \Omega W_{R_0} < 0, \tag{47}$$

where W_{L_0} (W_{R_0}) *is a matrix whose columns make up the bases of the kernel* $\operatorname{Ker} L_0$ $(\operatorname{Ker} R_0)$.

4.3 Dynamic controllers with perturbations

Consider control systems (1) and (2) with the dynamic output-feedback controller

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky + w, \quad \xi(0) = 0, \tag{48}$$

where $w \in \mathbb{R}^m$ is a vector of bounded perturbations, Z, V, U and K are unknown coefficient matrices. If $K \in \mathcal{K}_D$, then linear close loop system (2), (48) reduces to the form

$$\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad y = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{49}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad \widehat{M} = \begin{bmatrix} A + BK_0C & BU_0 \\ V_0C & Z_0 \end{bmatrix}, \quad \widehat{N} = \begin{bmatrix} B + BK_0D \\ V_0D \end{bmatrix},$$

$$\widehat{F} = \left[C + DK_0C, DU_0\right], \quad \widehat{G} = D + DK_0D, \ K_0 = \mathbf{D}(K),$$

$$U_0 = (I_m - KD)^{-1}U, \ V_0 = V(I_l - DK)^{-1}, \ Z_0 = Z + VD(I_m - KD)^{-1}U.$$

We give the following auxiliary statement (see also [16] in case of $\gamma = 1$).

Lemma 4.4 *Gain matrices* $X > 0$, $Y > 0$ *and number* $\gamma > 0$, *there are matrices* $X_1 \in \mathbb{R}^{r\times n}$, $X_2 \in \mathbb{R}^{r\times r}$, $Y_1 \in \mathbb{R}^{r\times n}$ *and* $Y_2 \in \mathbb{R}^{r\times r}$ *such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad \widehat{Y} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_2 \end{bmatrix} > 0, \quad \widehat{X}\widehat{Y} = \gamma^2 I_{n+r}, \tag{50}$$

if and only if

$$W = \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq n + r. \tag{51}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (49), we get the following result.

Theorem 4.2 [18] *There exists a dynamic controller* (48) *such that the evaluation* $J < \gamma$ *holds for linear system* (49), *iff the LMI system* (34), (42), (43) *and* (51) *is solvable with respect to* $X = X^T > 0$ *and* $Y = Y^T > 0$. *Herewith, a close loop system* (49) *with uncertainty* (35) *is robust stable.*

Remark 4.3 The coefficient matrices of dynamic controller (48) in Theorem 4.2 may be constructed in the form

$$\begin{aligned} K &= (I_m + K_0 D)^{-1} K_0, \quad U = (I_m + K_0 D)^{-1} U_0, \\ V &= V_0 (I_l + D K_0)^{-1}, \quad Z = Z_0 - V_0 D (I_m + K_0 D)^{-1} U_0, \end{aligned} \tag{52}$$

by solving LMI

$$\widehat{L}^T \widehat{K}_0 \widehat{R} + \widehat{R}^T \widehat{K}_0^T \widehat{L} + \widehat{\Omega} < 0, \tag{53}$$

where

$$\widehat{\Omega} = \begin{bmatrix} A^T X + XA & A^T X_1^T & XB & C^T \\ X_1 A & 0 & X_1 B & 0 \\ B^T X & B^T X_1^T & -P & D^T \\ C & 0 & D & -Q^{-1} \end{bmatrix}, \quad \widehat{L}^T = \begin{bmatrix} XB & X_1^T \\ X_1 B & X_2 \\ 0 & 0 \\ D & 0 \end{bmatrix},$$

$$\widehat{R} = \begin{bmatrix} C & 0 & D & 0 \\ 0 & I_r & 0 & 0 \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix},$$

Here X, X_1 and X_2 are blocks of matrix $\widehat{X}$ in (50).

If $K \in \mathcal{K}_D$, then $\det \left[I_m - K D(x) \right] \neq 0$ for all $x \in \mathcal{S}_0$, where $\mathcal{S}_0$ is some neighbourhood of the point $x = 0$, and nonlinear close loop system (1), (48) reduces to the form

$$\dot{\widehat{x}} = \widehat{M}(\widehat{x}) \widehat{x} + \widehat{N}(\widehat{x}) w, \quad y = \widehat{F}(\widehat{x}) \widehat{x} + \widehat{G}(\widehat{x}) w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{54}$$

where all coefficient matrices are continuous in $\mathcal{S}_0$. Therefore, the dynamic controller (48), (52) ensures robust stability of the zero state $\widehat{x} \equiv 0$ of system (54) with uncertainty (35) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$. To evaluate characteristics J_0 and J of system (54), we can apply Lemma 4.2.

4.4 Control systems with controlled and observed outputs

Consider the control system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{55}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^s$, $z \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$ are the state, the control, the norm-limited external perturbations, the controlled and observed outputs, respectively. We are interested in static and dynamic control laws that ensure nonexpansivity property of close loop system and minimize the performance criteria J and J_0 with respect to controlled output z of the form (31), where

$$\varphi(w, x_0) = \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^T X_0 x_0}}, \quad \|z\|_Q^2 = \int_0^\infty z^T Q z \, dt, \quad \|w\|_P^2 = \int_0^\infty w^T P w \, dt.$$

4.4.1 Static controllers

If we use the static output feedback controller

$$u = Ky, \quad K \in \mathcal{K}_{D_{22}}, \tag{56}$$

then closed loop system (55), (56) has the form

$$\dot{x} = Mx + Nw, \quad z = Fx + Gw, \quad x(0) = x_0, \tag{57}$$

where $M = A+B_2K_0C_2$, $N = B_1+B_2K_0D_{21}$, $F = C_1+D_{12}K_0C_2$, $G = D_{11}+D_{12}K_0D_{21}$, $K_0 = (I_m - KD_{22})^{-1}K$. To formulate an analog of Theorem 4.1 we construct the following LMI

$$W_R^T \begin{bmatrix} A^TX + XA + C_1^TQC_1 & XB_1 + C_1^TQD_{11} \\ B_1^TX + D_{11}^TQC_1 & D_{11}^TQD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{58}$$

$$W_L^T \begin{bmatrix} AY + YA^T + B_1P^{-1}B_1^T & YC_1^T + B_1P^{-1}D_{11}^T \\ C_1Y + D_{11}P^{-1}B_1^T & D_{11}P^{-1}D_{11}^T - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{59}$$

where $R = \left[C_2, D_{21} \right]$, $L = \left[B_2^T, D_{12}^T \right]$.

Theorem 4.3 *For linear system* (55)*, there exists an output feedback controller* (56) *such that* $J < \gamma$ *iff the system of correlations* (44)*,* (58) *and* (59) *is feasible. Herewith, system* (57) *with uncertainty*

$$w = \frac{1}{\gamma}\Theta z, \quad \Theta^T P \Theta \le Q, \tag{60}$$

is robust stable with common Lyapunov function $v(x) = x^TXx$.

If we use a static state feedback $u = Kx$, then $C_2 = I_n$, $D_{21} = 0$ and $D_{22} = 0$. In this case the correlations (44) and (58) can be written as

$$\begin{bmatrix} X_0 & I_n \\ I_n & Y \end{bmatrix} > 0, \quad D_{11}^TQD_{11} - \gamma^2 P < 0. \tag{61}$$

Corollary 4.1 *For linear system* (55)*, there exists a state feedback controller* $u = Kx$ *such that* $J < \gamma$ *iff the LMI system* (59) *and* (61) *is solvable for some matrix* $Y = Y^T > 0$*. Herewith, system* (57) *with uncertainty* (60) *is robust stable with common Lyapunov function* $v(x) = \gamma^2 x^TY^{-1}x$.

Remark 4.4 The gain matrix K in Theorem 4.3 and Corollary 4.1 may be constructed as

$$K = K_0(I_l + D_{22}K_0)^{-1}, \quad -K_0 \in \mathcal{K}_{D_{22}}, \tag{62}$$

where K_0 is an arbitrary solution of LMI:

$$\widehat{L}^TK_0\widehat{R} + \widehat{R}^TK_0^T\widehat{L} + \Omega < 0,$$

where $\widehat{R} = \left[R, 0_{l\times k}\right]$, $R = \left[C_2, D_{21} \right]$, $\widehat{L} = \left[L, 0_{m\times s}\right]\widetilde{X}$, $L = \left[B_2^T, D_{12}^T \right]$,

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^TX + XA & XB_1 & C_1^T \\ B_1^TX & -\gamma^2 P & D_{11}^T \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix}.$$

4.4.2 Dynamic controllers

If we use the dynamic output feedback

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \quad \xi(0) = 0, \tag{63}$$

with $K \in \mathcal{K}_{D_{22}}$, then closed loop system (55), (63) has the form

$$\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad z = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{64}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \quad \widehat{M} = \begin{bmatrix} A + B_2K_0C_2 & B_2U_0 \\ V_0C_2 & Z_0 \end{bmatrix} = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2,$$

$$\widehat{N} = \begin{bmatrix} B_1 + B_2K_0D_{21} \\ V_0D_{21} \end{bmatrix} = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{F} = \big[C_1 + D_{12}K_0C_2, D_{12}U_0\big] = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2,$$

$$\widehat{G} = D_{11} + D_{12}K_0D_{21} = D_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{A} = \begin{bmatrix} A & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \widehat{B}_2 = \begin{bmatrix} B_2 & 0_{n\times r} \\ 0_{r\times m} & I_r \end{bmatrix}, \widehat{C}_2 = \begin{bmatrix} C_2 & 0_{l\times r} \\ 0_{r\times n} & I_r \end{bmatrix},$$

$$\widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}, \widehat{B}_1 = \begin{bmatrix} B_1 \\ 0_{r\times s} \end{bmatrix}, \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{r\times s} \end{bmatrix},$$

$$\widehat{C}_1 = \big[C_1, 0_{k\times r}\big], \quad \widehat{D}_{12} = \big[D_{12}, 0_{k\times r}\big].$$

Here the blocks of matrix $\widehat{K}_0$

$$K_0 = (I_m - KD_{22})^{-1}K, \quad U_0 = (I_m - KD_{22})^{-1}U,$$
$$V_0 = V(I_l - D_{22}K)^{-1}, \quad Z_0 = Z + VD_{22}(I_m - KD_{22})^{-1}U,$$

are unknown, and

$$\begin{aligned} K = (I_m + K_0D_{22})^{-1}K_0, \quad U = (I_m + K_0D_{22})^{-1}U_0, \\ V = V_0(I_l + D_{22}K_0)^{-1}, \quad Z = Z_0 - V_0D_{22}(I_m + K_0D_{22})^{-1}U_0. \end{aligned} \tag{65}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (64), we get the following result.

Theorem 4.4 *For linear system* (55)*, there exists a dynamic controller* (63) *such that* $J < \gamma$ *iff the system of correlations* (34)*,* (51)*,* (58) *and* (59) *is feasible. Herewith, system* (64) *with uncertainty* (60) *is robust stable.*

Remark 4.5 The coefficient matrices of dynamic controller (63) in Theorem 4.4 may be constructed in the form (65) by solving the LMI

$$\widehat{L}^T\widehat{K}_0\widehat{R} + \widehat{R}^T\widehat{K}_0^T\widehat{L} + \widehat{\Omega} < 0, \tag{66}$$

where

$$\widehat{R} = \big[\widehat{R}_1, 0_{l+r\times k}\big], \ \widehat{R}_1 = \big[\widehat{C}_2, \widehat{D}_{21}\big], \ \widehat{L} = \big[\widehat{L}_1, 0_{m+r\times s}\big]\widetilde{X}, \ \widehat{L}_1 = \big[\widehat{B}_2^T, \widehat{D}_{12}^T\big],$$

$$\widetilde{X} = \begin{bmatrix} \widehat{X} & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \quad \widehat{\Omega} = \begin{bmatrix} \widehat{A}^T\widehat{X} + \widehat{X}\widehat{A} & \widehat{X}\widehat{B}_1 & \widehat{C}_1^T \\ \widehat{B}_1^T\widehat{X} & -\gamma^2 P & D_{11}^T \\ \widehat{C}_1 & D_{11} & -Q^{-1} \end{bmatrix}, \quad \widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix}.$$

Herewith, system (64) with uncertainty (60) has common Lyapunov function $v(\widehat{x}) = \widehat{x}^T\widehat{X}\widehat{x}$.

We give the following algorithm for constructing stabilizing dynamic controller (63) satisfying Theorem 4.4.

Algorithm 4.1 1) calculate the matrices W_R and W_L, where $R = \big[\, C_2, D_{21} \,\big]$ and $L = \big[\, B_2^T, D_{12}^T \,\big]$;

2) find the matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying (34), (51), (58) and (59);

3) construct the expansion $Z = Y - \gamma^2 X^{-1} = V^T V$, $V \in \mathbb{R}^{r\times n}$, $\ker V = \ker Z$ and form the block matrix

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad X_1 = \frac{1}{\gamma} V X, \quad X_2 = \frac{1}{\gamma^2} V X V^T + I_r;$$

4) solve the LMI (66) under restriction $\det(I_m + K_0 D_{22}) \neq 0$;

5) calculate the coefficient matrices of dynamic controller (63) by formula (65).

Static and dynamic output-feedback controllers (56) and (63) with $K \in \mathcal{K}_{D_{22}}$ may be applied to a class of affine systems

$$\begin{aligned} \dot{x} &= A(x)x + B_1(x)w + B_2(x)u, \quad x(0) = x_0, \\ z &= C_1(x)x + D_{11}(x)w + D_{12}(x)u, \\ y &= C_2(x)x + D_{21}(x)w + D_{22}(x)u. \end{aligned} \tag{67}$$

So, close loop system (63), (67) reduces to the form

$$\dot{\widehat{x}} = \widehat{M}(\widehat{x})\widehat{x} + \widehat{N}(\widehat{x})w, \quad z = \widehat{F}(\widehat{x})\widehat{x} + \widehat{G}(\widehat{x})w, \quad \widehat{x}(0) = \widehat{x}_0. \tag{68}$$

As a result, the dynamic controller (63), (65) ensures robust stability of the zero state $\widehat{x} \equiv 0$ of system (68) with uncertainty (60) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^T\widehat{X}\widehat{x}$. To evaluate characteristics J_0 and J of system (68), we can apply Lemma 4.2.

Remark 4.6 Note, that we have necessary and sufficient conditions for an evaluation $J_0 < \gamma$ represented by the corresponding statements of Theorems 4.1 – 4.4 without usage of additional restriction $X < \gamma^2 X_0$. With the use of static state feedback or full order dynamic controllers the problems under consideration are reduced to the solution of LMI systems. We can formulate analogs of Theorems 4.1 – 4.4 for the corresponding control systems with the uncertainties

$$A \in \mathrm{Co}\big\{A^1, \ldots, A^{\nu_1}\big\}, \ B_1 \in \mathrm{Co}\big\{B_1^1, \ldots, B_1^{\nu_2}\big\},$$

$$C_1 \in \mathrm{Co}\big\{C_1^1, \ldots, C_1^{\nu_3}\big\}, \ D_{11} \in \mathrm{Co}\big\{D_{11}^1, \ldots, D_{11}^{\nu_4}\big\}.$$

In addition, sufficient statements of these theorems may be generalized for the corresponding affine control systems with continuous coefficient matrices (see Lemma 4.2).

Example 4.1 Consider a controlled linear damped oscillator described by system (55) with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\delta \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1, 0 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = D_{22} = 0, \quad x = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}, \quad z = \begin{bmatrix} \varphi \\ u \end{bmatrix}, \quad y = \varphi.$$

Taking into account (36) in the absence of control, we get $J_0 = 1,001$ and $J = 1,289$ assuming that

$$\delta = 0,1, \quad \omega_0 = 1, \quad P = 1, \quad Q = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}, \quad X_0 = \begin{bmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{bmatrix},$$

where $q_1 = 0,01$, $q_2 = 0,1$, $\rho_1 = \rho_2 = 0,04$. Figure 1 shows the dependence J of δ and ω_0. The damping level of input signals and initial perturbations of oscillator decreases with the increase of its natural frequency ω_0 and does not change with the increase of the damping factor δ.

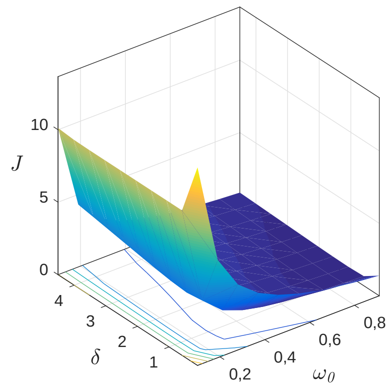

Figure 1: The dependence $J(\delta, \omega_0)$.

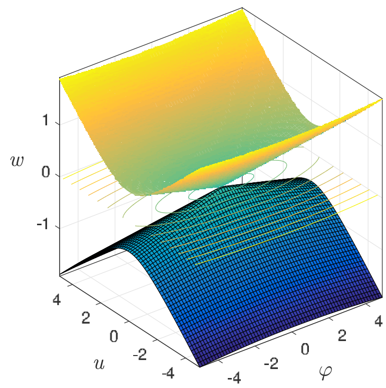

Figure 2: Uncertainty region.

Next, using Algorithm 4.1, we performed minimization of the parameter γ satisfying Theorem 4.4. As a result for $\gamma = 0,865$, we constructed an approximate J-optimal dynamic controller (63) with the coefficient matrices

$$Z = \begin{bmatrix} -0,06612 & -0,09307 \\ 0,23117 & -1,05843 \end{bmatrix}, \quad V = \begin{bmatrix} -0,00037 \\ 0,11011 \end{bmatrix},$$

$$U = \begin{bmatrix} -0,31404 & 3,90247 \end{bmatrix}, \quad K = -0,23776,$$

that provides a robust stability and nonexpansiveness of close loop system. This regulator significantly reduced the damping level of input signals and initial perturbations of oscillator. For example, for the indicated values of parameters we have $J_0 = 0,39062$

and $J = 0,86181 < 1$. The oscillator with constructed regulator preserves asymptotic stability for any perturbation function (see Figure 2)

$$w(t) = \frac{1}{\gamma}\Theta z(t),\ \Theta = \left[\theta_1, \theta_2\right], \quad \frac{\theta_1^2}{q_1} + \frac{\theta_2^2}{q_2} \leq 1,\ |w| \leq \frac{1}{\gamma}\sqrt{q_1\varphi^2 + q_2 u^2}. \tag{69}$$

The dependences $J_0(q_1, q_2)$ and $J(\rho_1, \rho_2)$ for close loop system are shown in Figures 3 and 4, respectively.

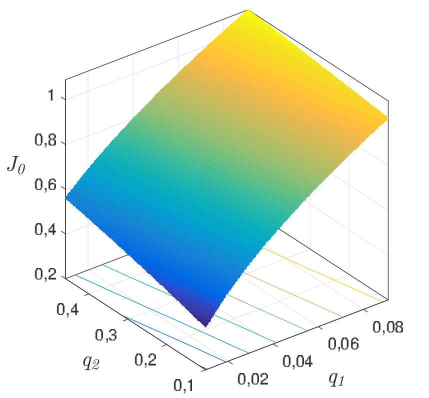

Figure 3: The dependence $J_0(q_1, q_2)$ (close loop system).

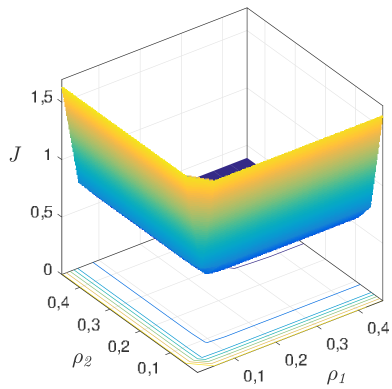

Figure 4: The dependence $J(\rho_1, \rho_2)$ (close loop system).

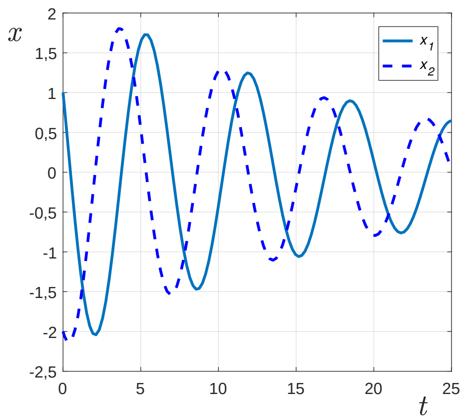

Figure 5: System behavior without control.

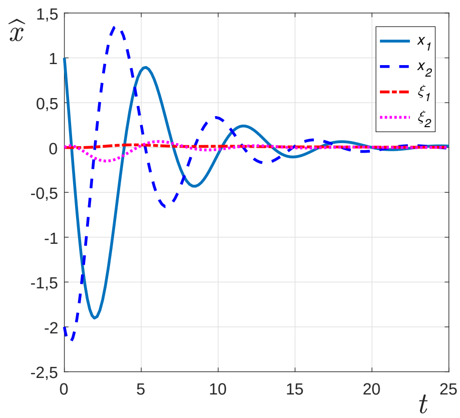

Figure 6: Close loop system behavior.

Figure 5 shows system behavior without control for the initial vector $x_0 = [1, -2]^T$ and Figure 6 shows close loop system behavior for the initial vector $\widehat{x}_0 = [1, -2, 0, 0]^T$, wherein the perturbation function w has the form (69) with $\Theta = \sqrt{P}^{-1} E\sqrt{Q} = \left[\sqrt{q_1/2}, \sqrt{q_2/2}\right]$, where $E = \left[1/\sqrt{2}, 1/\sqrt{2}\right]$, $E^T E \leq I_2$.

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