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Robust Output Feedback Stabilization and Optimization of Control Systems *

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1 Introduction

The design of state and output feedback controllers for dynamic systems with the prescribed and desired properties is a key problem of control theory. At the same time, the properties of control systems such as asymptotic stability, robustness and optimality of the performance indexes are in the foreground. The main problem in H_∞-control theory is connected with suppression of external and initial perturbations (see, e.g., [1–5] as well as review papers [6, 7]).

It should be noted that the practical applications of many modern methods for synthesis of control systems are based on the construction and solution of *linear matrix inequalities* (LMI). For this purpose, sufficiently effective computational algorithms and appropriate tools are established in Matlab environment (see [8, 9]).

In this chapter, we consider classes of linear and nonlinear control systems for which closed loop systems can be represented in the *pseudolinear form*

$$\dot{x} = M(x,t)\,x, \quad x \in \mathbb{R}^n, \quad t \geq 0,$$

* Advances in Stability and Control Theory for Uncertain Dynamical Systems (Eds.: C. Cruz-Hernández, A.A. Martynyuk and A.G. Mazko). Stability, Oscillations and Optimization of Systems. Cambridge Scientific Publishers, Vol. 11, 2021, 37–62.

besides, a matrix function $M(x,t)$ can contain uncertain quantities belonging to certain sets. Matrix intervals, polytopes, affine sets of matrices and other objects may serve as the uncertainty sets. To define uncertainties and robust stability conditions for systems in semiordered spaces one can use cone inequalities and intervals [5, 10]. The applied control laws are of the form of static or dynamic output feedback. It should be noted that at the solution of many control problems the dynamic controllers have great potential as compared with the static controllers.

Our consideration includes the following types of problems:

- output feedback stabilization of control systems (Section 2);
- robust stabilization and optimization of control systems with polyhedral uncertainties (Section 3);
- robust stabilization and weighted suppression of perturbations in control systems (Section 4).

Throughout the paper, the following notations are used: I_n is the identity $n \times n$ matrix; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^\top > 0$ (≥ 0) is the symmetric *positive definite* (*semidefinite*) matrix X; $\mathrm{i}(X) = \{\mathrm{i}_+, \mathrm{i}_-, \mathrm{i}_0\}$ is the *inertia* of matrix $X = X^\top$ consisting of the numbers of positive, negative and zero eigenvalues; $\sigma(A)$ and $\rho(A)$ are the spectrum and the spectral radius of A, respectively; $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the maximum and the minimum eigenvalue of the Hermitian matrix X, respectively; A^+ is the pseudoinverse matrix; W_A is a matrix whose columns make up the bases of the kernel $\mathrm{Ker}\, A$; $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$; $\|w\|_P$ denotes the weighted L_2-norm of vector function $w(t)$; $\mathrm{Co}\{A_1, \ldots, A_\nu\}$ stands for a *polytope* in a matrix space described as a convex full of the set $\{A_1, \ldots, A_\nu\}$, i. e.

$$\mathrm{Co}\{A_1, \ldots, A_\nu\} = \Big\{ \sum_{i=1}^{\nu} \alpha_i A_i : \;\; \alpha_i \geq 0, \; i = \overline{1,\nu}, \; \sum_{i=1}^{\nu} \alpha_i = 1 \Big\}.$$

Note that matrix intervals and affine sets of matrices are described in terms of polytopes.

2 Output Feedback Stabilization of Nonlinear Systems

Consider the following affine nonlinear *time-invariant control system*

$$\dot{x} = A(x)x + B(x)u, \quad y = C(x)x + D(x)u, \tag{1}$$

where $x \in \mathbb{R}^n$ is state vector, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^l$ are input and output vectors, respectively, $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in some neighborhood $\mathcal{S}_0$ of the zero state $x = 0$. We will assume that $\operatorname{rank} B(x) \equiv m$ and $\operatorname{rank} C(x) \equiv l$ in $\mathcal{S}_0$.

Along with (1), consider the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{2}$$

where $A = A(0)$, $B = B(0)$, $C = C(0)$ and $D = D(0)$. Let $B^\perp$ and $C^\perp$ be the *orthogonal complements* of B and C, respectively, i.e.

$$B^\top B^\perp = 0,\ \det\left[B, B^\perp\right] \neq 0,\ C^\perp C^\top = 0,\ \det\left[C^\top, C^{\perp\top}\right] \neq 0.$$

2.1 Static controllers

Formulate stabilizability conditions of the zero state $x = 0$ for systems (1) and (2) through the *static output-feedback controller*

$$u = Ky, \quad K \in \mathcal{K}_D, \tag{3}$$

where $\mathcal{K}_D = \left\{K \in \mathbb{R}^{m\times l} : \det(I_m - KD) \neq 0\right\}$. *Closed loop system* (2), (3) has the form

$$\dot{x} = Mx, \quad M = A + B\mathbf{D}(K)C, \tag{4}$$

where $\mathbf{D}(K) = (I_m - KD)^{-1}K \equiv K(I_l - DK)^{-1}$ is a nonlinear operator with the following properties:

- if $K \in \mathcal{K}_D$, then $I_l + D\mathbf{D}(K) \equiv (I_l - DK)^{-1}$;
- if $K_1 \in \mathcal{K}_D$ and $K_2 \in \mathcal{K}_{D_1}$, then $K_1 + K_2 \in \mathcal{K}_D$ and

$$\mathbf{D}(K_1 + K_2) = \mathbf{D}(K_1) + (I_m - K_1D)^{-1}\mathbf{D}_1(K_2)\,(I_l - DK_1)^{-1}, \tag{5}$$

  where $\mathbf{D}_1(K_2) = (I_m - K_2D_1)^{-1}K_2$, $D_1 = (I_l - DK_1)^{-1}D$;
- if $-K_0 \in \mathcal{K}_D$, then $K = -\mathbf{D}(-K_0) \in \mathcal{K}_D$ and $\mathbf{D}(K) = K_0$.

To achieve the desired properties and, in particular, to stabilize system (4) it suffices to provide these properties for a system with the matrix $M_* = A + BKC$.

Definition 2.1 System (4) is *α-stable* if the spectrum $\sigma(M)$ lies in the open left half-plane $\mathbb{C}_\alpha^- = \{\lambda : \operatorname{Re}\lambda < -\alpha\}$, where $\alpha \geq 0$.

Theorem 2.1 *The following statements are equivalent:*
1) *There exists static controller* (3) *ensuring α-stability of system* (4).
2) *There exists matrix $X = X^\top > 0$ satisfying the relations*

$$B^{\perp\top}(AX + XA^\top + 2\alpha X)\, B^\perp < 0, \tag{6}$$

$$\mathrm{i}(\Delta) = \{l, n, 0\}, \quad \Delta = \begin{bmatrix} AX + XA^\top + 2\alpha X & XC^\top \\ CX & 0 \end{bmatrix}. \tag{7}$$

3) *There exist mutually inverse matrices $X = X^\top > 0$ and $Y = Y^\top > 0$ satisfying* (6) *and*

$$C^\perp(A^\top Y + YA + 2\alpha Y)\, C^{\perp\top} < 0. \tag{8}$$

When one of the statements 2 *or* 3 *is true, then the controller*

$$u = Ky, \quad K = -\mathbf{D}(-K_0) \in \mathcal{K}_D, \tag{9}$$

where K_0 is a solution of the LMI

$$AX + XA^\top + 2\alpha X + BK_0CX + XC^\top K_0^\top B^\top < 0, \tag{10}$$

ensures α-stability of closed loop system (4).

For the equivalence of the statements 1 and 2 in Theorem 2.1, see [5]. Equivalence of the statements 2 and 3 follows from the relations (see [10, p. 147]) $\mathrm{i}_\pm(\Delta) = \mathrm{i}_\pm(\Delta_1) = \mathrm{i}_\pm(C^\perp L_1 C^{\perp\top}) + l$, where

$$\Delta_1 = R^\top \Delta R = \begin{bmatrix} C^\perp L_1 C^{\perp\top} & 0 \\ 0 & S \end{bmatrix}, \quad S = \begin{bmatrix} 0 & I_l \\ I_l & C^{+T} L_1 C^+ \end{bmatrix},$$

$$L_1 = A^\top Y + YA + 2\alpha Y,\ Y = X^{-1},\ R = \begin{bmatrix} YC^{\perp\top} & 0 & YC^+ \\ -C^{+T} L_1 C^{\perp\top} & I_l & 0 \end{bmatrix},$$

$\mathrm{i}(S) = \{l, l, 0\}$ and $\det R \neq 0$. For the equivalence of the statements 1 and 3, see also [4].

Theorem 2.2 [11] *Let one of the statements* 2 *or* 3 *of Theorem* 2.1 *holds for system* (2). *Then* (9) *and* (10) *determine static controller ensuring asymptotic stability of the state $x \equiv 0$ and quadratic Lyapunov function $v(x) = x^\top Yx$ of nonlinear closed loop system* (1), (9).

2.2 Dynamic controllers

The dynamic output feedback stabilization problem for system (1) consists in finding, if possible, a *dynamic control* law described by

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \tag{11}$$

where $\xi \in \mathbb{R}^r$, such that the zero state of closed loop system is asymptotically stable. Equations (1) and (11) may be represented by control system in the extended phase space $\mathbb{R}^{n+r}$ with static controller:

$$\dot{\widehat{x}} = \widehat{A}(\widehat{x})\widehat{x} + \widehat{B}(\widehat{x})\widehat{u}, \quad \widehat{y} = \widehat{C}(\widehat{x})\widehat{x} + \widehat{D}(\widehat{x})\widehat{u}, \quad \widehat{u} = \widehat{K}\widehat{y}, \tag{12}$$

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{y} = \begin{bmatrix} y \\ \xi \end{bmatrix}, \quad \widehat{u} = \begin{bmatrix} u \\ \dot{\xi} \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix},$$

$$\widehat{A}(\widehat{x}) = \begin{bmatrix} A(x) & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \quad \widehat{B}(\widehat{x}) = \begin{bmatrix} B(x) & 0_{n\times r} \\ 0_{r\times m} & I_r \end{bmatrix},$$

$$\widehat{C}(\widehat{x}) = \begin{bmatrix} C(x) & 0_{l\times r} \\ 0_{r\times n} & I_r \end{bmatrix}, \quad \widehat{D}(\widehat{x}) = \begin{bmatrix} D(x) & 0_{l\times r} \\ 0_{r\times m} & 0_{r\times r} \end{bmatrix}.$$

If $K \in \mathcal{K}_D$, then linear closed loop system (2), (11) has the form

$$\dot{\widehat{x}} = \widehat{M}\,\widehat{x}, \quad \widehat{M} = \widehat{A} + \widehat{B}\widehat{\mathbf{D}}(\widehat{K})\widehat{C}, \tag{13}$$

where $\widehat{A} = \widehat{A}(0)$, $\widehat{B} = \widehat{B}(0)$, $\widehat{C} = \widehat{C}(0)$, $\widehat{D} = \widehat{D}(0)$ and

$$\widehat{\mathbf{D}}(\widehat{K}) = \left[\begin{array}{c|c} \mathbf{D}(K) & (I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1} & Z + VD(I_m - KD)^{-1}U \end{array}\right],$$

$$\widehat{M} = \left[\begin{array}{c|c} M & B(I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1}C & Z + VD(I_m - KD)^{-1}U \end{array}\right].$$

Theorem 2.3 *The following statements are equivalent:*
1) *There exists dynamic controller* (11) *of order* $r \le n$ *ensuring* α-*stability of closed loop system* (13).
2) *There exist matrices* X *and* X_0 *satisfying* (6) *and*

$$\mathrm{i}(\Delta_0) = \bigl\{l, n, 0\bigr\}, \quad X \ge X_0 > 0, \quad \mathrm{rank}\,(X - X_0) \le r, \tag{14}$$

where

$$\Delta_0 = \begin{bmatrix} AX_0 + X_0 A^\top + 2\alpha X_0 & X_0 C^\top \\ C X_0 & 0 \end{bmatrix}.$$

3) *There exist matrices* X *and* Y *satisfying* (6), (8) *and*

$$W = \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq n + r. \tag{15}$$

Proof of Theorem 2.3 follows from the corresponding statements of Theorem 2.1 taking into account the structure of block matrices in (13) (see [11]). In [11], a computation algorithm of finding a stabilizing dynamic controller (11) for nonlinear systems (1) has been proposed on the basis of Theorem 2.3.

Remark 2.1 Note, that matrices X and X_0 in Theorem 2.3 satisfy statement 2 iff matrices X and $Y = X_0^{-1}$ satisfy statement 3. From (15) it follows that matrices X and Y are positive definite. The rank restriction in (15) always holds in case of full order $r = n$ dynamic regulator.

3 Robust Stabilization and Optimization of Nonlinear Systems

We formulate an auxiliary statement that will be used in the proofs of our main results. Consider the nonlinear operator

$$\mathbf{F}(K) = W + U^\top \mathbf{D}(K) V + V^\top \mathbf{D}^\top(K) U + V^\top \mathbf{D}^\top(K)\, R\, \mathbf{D}(K) V \tag{16}$$

with $\mathbf{D}(K) = (I_m - KD)^{-1} K$ and an ellipsoidal set of matrices

$$\mathcal{K} = \{K \in \mathbb{R}^{m \times l} : \, K^\top P K \leq Q\}, \tag{17}$$

where $P = P^\top > 0$, $Q = Q^\top > 0$, $R = R^\top \geq 0$, $W = W^\top$, U, V and D are matrices of suitable sizes. Matrix inequality in (17) is equivalent to the following $KQ^{-1}K^\top \leq P^{-1}$. Therefore, in case of $m = 1$ the ellipsoid $\mathcal{K}$ is described by a scalar inequality.

Lemma 3.1 [12] *Suppose that the matrix inequalities*

$$D^\top Q D + R < P, \quad \Omega = \begin{bmatrix} W & U^\top & V^\top \\ U & R-P & D^\top \\ V & D & -Q^{-1} \end{bmatrix} \leq 0 \ (< 0) \qquad (18)$$

hold. Then $\mathbf{F}(K) \leq 0$ (< 0) *for any matrix* $K \in \mathcal{K}$.

In Lemma 3.1 letting $D = 0$, $R = 0$, $P = \varepsilon I_m$ and $Q = \varepsilon I_l$, $\varepsilon > 0$, we get the sufficiency statement of *Petersen's lemma* on matrix uncertainty [13].

Consider a nonlinear control system in the vector-matrix form

$$E(x)\dot{x} = A(x,t)\,x + B(x,t)\,u, \quad y = C(x,t)\,x + D(x,t)\,u, \qquad (19)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ and $t \geq 0$. We construct a set of the static controllers

$$u = K(x,t)\,y, \quad K(x,t) = K_*(x,t) + \widetilde{K}(x,t), \quad \widetilde{K}(x,t) \in \mathcal{K}, \qquad (20)$$

where $\mathcal{K}$ is an ellipsoidal set of matrices of the form (17). We assume that the matrices E, A, B, C, D, K and K_* continuously depend on x and t and the equilibrium state $x \equiv 0$ is isolated, i.e., the neighborhood $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \leq h\}$ does not contain other equilibrium states of this system. If $K \in \mathcal{K}_D$, then the closed loop system (19), (20) can be represented as

$$E(x)\dot{x} = M(x,t)x, \quad M(x,t) = A + B\mathbf{D}(K)C. \qquad (21)$$

Let the zero state of this system for $K \equiv K_*$ be asymptotically stable. When looking for the stabilizing matrix K_* in the class of autonomous systems (1), one can use Theorem 2.1 and its special cases. The problem is to construct conditions under which the zero state of system (21) is Lyapunov asymptotically stable for every matrix $\widetilde{K}(x,t) \in \mathcal{K}$. We find a solution for our problem in terms of a quadratic Lyapunov function (see [5, 12]).

Theorem 3.1 *Let for some matrix functions* $X(t) = X^\top(t)$ *and* $K_*(x,t)$ *at* $x = 0$ *and* $t \geq 0$ *the relations*

$$\varepsilon_1 I_n \leq X(t) \leq \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \leq \varepsilon_2, \qquad (22)$$

$$\begin{bmatrix} E^\top \dot{X} E + M_*^\top X E + E^\top X M_* + \varepsilon_0 I_n & E^\top X B_* & C_*^\top \\ B_*^\top X E & -P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{23}$$

hold with $\varepsilon_0 > 0$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_* D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$, $D_* = D(I_m - K_* D)^{-1}$. *Then any control* (20) *ensures asymptotic stability of the zero state* $x \equiv 0$ *for system* (21) *and a common Lyapunov function* $v(x,t) = x^\top E_0^\top X(t) E_0 x$, *where* $E_0 = E(0)$.

Consider control system (19) with quadratic quality functional

$$J(u, x_0) = \int_0^\infty \varphi(x, u, t)\, dt, \tag{24}$$

where

$$x_0 = x(0), \quad \varphi(x,u,t) = \left[x^\top, u^\top\right] \Phi(t) \begin{bmatrix} x \\ u \end{bmatrix}, \quad \Phi(t) = \begin{bmatrix} S & N \\ N^\top & R \end{bmatrix},$$

$R > 0$, $S \geq NR^{-1}N^\top + \eta I_n$, $\eta > 0$ and $t \geq 0$.

Theorem 3.2 *Let for some matrix functions* $X(t) = X^\top(t)$ *and* $K_*(x,t)$ *at* $x = 0$ *and* $t \geq 0$ *the relations* (22) *and*

$$\begin{bmatrix} \mathbf{W}(X) & \mathbf{U}^\top(X) & C_*^\top \\ \mathbf{U}(X) & R_* - P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{25}$$

hold with $\varepsilon_0 > 0$, $\mathbf{W}(X) = E^\top \dot{X} E + M_*^\top X E + E^\top X M_* + \Phi_* + \varepsilon_0 I_n$, $\mathbf{U}(X) = B_*^\top X E + N_*^\top + R_* K_* C$, $\Phi_* = L_*^\top \Phi L_*$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_* D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$, $D_* = D(I_m - K_* D)^{-1}$, $R_* = (I_m - K_* D)^{-1\top} R\, (I_m - K_* D)^{-1}$, $N_* = N(I_m - K_* D)^{-1}$, $L_*^\top = \left[I_n, C^\top \mathbf{D}^\top(K_*)\right]$. *Then any control* (20) *ensures asymptotic stability of the zero state* $x \equiv 0$ *for system* (21), *a common Lyapunov function* $v(x,t) = x^\top E_0^\top X(t) E_0 x$, *where* $E_0 = E(0)$, *and evaluation* $J(u, x_0) \leq v(x_0, 0)$.

Corollary 3.1 *Let for some matrices $X = X^\top > 0$ and K_* the matrix inequalities*

$$\begin{bmatrix} \mathbf{W}_{ijk}(X) & \mathbf{U}_{jks}^\top(X) & C_{*k}^\top \\ \mathbf{U}_{jks}(X) & R_* - P & D_*^\top \\ C_{*k} & D_* & -Q^{-1} \end{bmatrix} < 0, \tag{26}$$

holds with $\mathbf{W}_{ijk}(X) = M_{*ijk}^\top X E_s + E_s^\top X M_{*ijk} + L_{*k}^\top \Phi L_{*k}$, $\mathbf{U}_{jks}(X) = B_{*j}^\top X E_s + N_*^\top + R_* K_* C_k$, $M_{*ijk} = A_i + B_j \mathbf{D}(K_*) C_k$, $B_{*j} = B_j (I_m - K_* D)^{-1}$, $C_{*k} = (I_l - D K_*)^{-1} C_k$, $D_* = D(I_m - K_* D)^{-1}$, $R_* = (I_m - K_* D)^{-1\top} R \, (I_m - K_* D)^{-1}$, $N_* = N(I_m - K_* D)^{-1}$, $L_{*k}^\top = \left[I_n, C_k^\top \mathbf{D}^\top(K_*) \right]$, $i = \overline{1,\alpha}$, $j = \overline{1,\beta}$, $k = \overline{1,\gamma}$, $s = \overline{1,\delta}$. *Then any control* (20) *ensures asymptotic stability of the zero state* $x \equiv 0$ *for system* (21) *with uncertainties (robust stability)*

$$\begin{aligned} A(0,t) &\in \mathrm{Co}\{A_1, \ldots, A_\alpha\}, \quad B(0,t) \in \mathrm{Co}\{B_1, \ldots, B_\beta\}, \\ C(0,t) &\in \mathrm{Co}\{C_1, \ldots, C_\gamma\}, \quad E(0) \in \mathrm{Co}\{E_1, \ldots, E_\delta\}, \end{aligned} \tag{27}$$

and evaluation $J(u, x_0) \le \omega = \max\limits_{1 \le s \le \delta} x_0^\top E_s^\top X E_s x_0$.

Note that the proof of Theorems 3.1 and 3.2 follows directly from Lemma 3.1 and Lyapunov theorem on asymptotic stability taking into account representation of derivative of Lyapunov function $v(x,t)$ with respect to system (21) in the form of a quadratic function with matrix of the form (16) and application of formula (5) (see [5, 12]).

4 Generalized H_∞-Control

4.1 Weighted level of perturbation suppression

Consider a dynamical system with external perturbations

$$\dot{x} = f(x, w, t), \quad y = g(x, w, t), \quad x(0) = x_0, \quad t \ge 0, \tag{28}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^s$ and $y \in \mathbb{R}^l$ are the state, the norm-limited external perturbations and the output vector, respectively.

Definition 4.1 The system (28) is called *nonexpansive*, if

$$\int_0^\tau y(t)^\top Qy(t)dt \le \int_0^\tau w(t)^\top Pw(t)dt + x_0^\top X_0 x_0$$

for all square-integrable functions $w(t)$ and $\tau > 0$, where Q, P and X_0 are weight symmetric positive definite matrices.

We introduce the performance criterion of system (28) with respect to observable output y:

$$J = \sup_{0<\|w\|_P^2 + x_0^\top X_0 x_0 < \infty} \varphi(w, x_0), \quad \varphi(w, x_0) = \frac{\|y\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \tag{29}$$

where $\|y\|_Q$ and $\|w\|_P$ are *weighted* L_2*-norms* of y and w, respectively,

$$\|y\|_Q^2 = \int_0^\infty y^\top Qy dt, \quad \|w\|_P^2 = \int_0^\infty w^\top Pw dt.$$

In case of $x_0 = 0$, we denote J by J_0. It is obvious, that $J_0 \le J$ and $J \le 1$ for a nonexpansive system. The value J describes the weighted damping level of external and initial perturbation in system (28). A pair (w, x_0) is the *worst* for system (28) with respect to the performance criterion J, if in (29) a supremum is reached. If $P = I_s$, $Q = I_l$ and $X_0 = \rho I_n$, then J coincides with known performance criterion of systems [14]. In this case, the value J_0 for a class of linear systems

$$\dot{x} = Ax + Bw, \quad y = Cx + Dw, \tag{30}$$

with zero initial vector coincides with H_∞-norm of the transfer matrix function $H(\lambda) = C(\lambda I_n - A)^{-1}B + D$ (see, e.g., [3]).

Lemma 4.1 *Let A be a Hurwitz matrix. Then $J_0 < \gamma$ for system* (30) *iff the LMI*

$$\Phi = \begin{bmatrix} A^\top X + XA + C^\top QC & XB + C^\top QD \\ B^\top X + D^\top QC & D^\top QD - \gamma^2 P \end{bmatrix} < 0 \tag{31}$$

has a solution $X = X^\top > 0$. Moreover, $J < \gamma$ iff the LMI (31) *has a solution X such that*

$$0 < X < \gamma^2 X_0. \tag{32}$$

Proof. *Sufficiency.* Construct the quadratic Lyapunov function $v(x) = x^\top X x$ for system (30) and evaluate the expression

$$\dot v(x) + y^\top Q y - \gamma^2 w^\top P w = [x^\top, w^\top]\Phi \begin{bmatrix} x \\ w \end{bmatrix},$$

where $\dot v(x)$ is the derivative of $v(x)$ with respect to the system. Integrating given expression and in view of (29) and (31), we have

$$\|y\|_Q^2 \le \gamma^2(\|w\|_P^2 + x_0^\top X_0 x_0), \quad \varphi(w, x_0) \le \gamma.$$

The strict matrix inequalities (31) and (32) hold if we replace γ by $\gamma - \varepsilon$ for some small $\varepsilon > 0$. Therefore, $\varphi(w, x_0) \le \gamma - \varepsilon$ and $J < \gamma$. In particular, in case of $x_0 = 0$ the inequality $J_0 < \gamma$ holds.

Necessity. Use the Cholesky decompositions $Q = \widetilde Q^\top \widetilde Q$, $P = \widetilde P^\top \widetilde P$, $X_0 = \widetilde X_0^\top \widetilde X_0$ and transform system (30):

$$\dot{\widetilde x} = \widetilde A \widetilde x + \widetilde B \widetilde w, \quad \widetilde y = \widetilde C \widetilde x + \widetilde D \widetilde w, \quad \widetilde x(0) = \widetilde x_0,$$

where $\widetilde x = \widetilde X_0 x$, $\widetilde y = \widetilde Q y$, $\widetilde w = \widetilde P w$, $\widetilde A = \widetilde X_0 A \widetilde X_0^{-1}$, $\widetilde B = \widetilde X_0 B \widetilde P^{-1}$, $\widetilde C = \widetilde Q C \widetilde X_0^{-1}$ and $\widetilde D = \widetilde Q D \widetilde P^{-1}$. Then

$$J = \widetilde J = \sup_{0<\|\widetilde w\|_{I_s}^2 + \widetilde x_0^\top \widetilde x_0 < \infty} \frac{\|\widetilde y\|_{I_l}}{\sqrt{\|\widetilde w\|_{I_s}^2 + \widetilde x_0^\top \widetilde x_0}}.$$

If $\widetilde J < \gamma$, then for some matrix $\widetilde X = \widetilde X^\top$ (see [14, Theorem 1])

$$0 < \widetilde X < \gamma^2 I_n, \quad \widetilde\Omega = \begin{bmatrix} \widetilde A^\top \widetilde X + \widetilde X \widetilde A & \widetilde X \widetilde B & \widetilde C^\top \\ \widetilde B^\top \widetilde X & -\gamma^2 I_s & \widetilde D^\top \\ \widetilde C & \widetilde D & -I_l \end{bmatrix} < 0,$$

$$0 < X < \gamma^2 X_0, \quad \Omega = S^\top \widetilde\Omega S = \begin{bmatrix} A^\top X + XA & XB & C^\top \\ B^\top X & -\gamma^2 P & D^\top \\ C & D & -Q^{-1} \end{bmatrix} < 0,$$

where $X = \widetilde X_0^\top \widetilde X \widetilde X_0$, $S = \operatorname{diag}\left\{\widetilde X_0, \widetilde P, \widetilde Q^{-1\top}\right\}$. By Schur complement, the last matrix inequality reduces to the form (31) $\square$.

It follows from Lemma 4.1 that the values J and J_0 for system (30) can be computed from the corresponding optimization problems:

$$J_0 = \inf\left\{\gamma : \Phi < 0,\ X > 0\right\},\ J = \inf\left\{\gamma : \Phi < 0,\ 0 < X < \gamma^2 X_0\right\}.$$

Remark 4.1 If $\Phi < 0$, then system (30) with a structurally uncertain input

$$w = \frac{1}{\gamma}\Theta y, \quad \Theta^\top P \Theta \le Q, \tag{33}$$

is robust stable with a common Lyapunov function $v(x) = x^\top X x$ (see [12, Theorem 1]). Note that (33) implies $\varphi(w, 0) \ge \gamma$ and $\varphi(w, 0) = \gamma$, if $\Theta^\top P \Theta = Q$.

Remark 4.2 By Schur complement, $\Phi < 0$, if and only if

$$A_1^\top X + X A_1 + X R_1 X + Q_1 < 0, \tag{34}$$

where $A_1 = A + BR^{-1}D^\top QC$, $Q_1 = C^\top\big(Q + QDR^{-1}D^\top Q\big)C$, $R_1 = BR^{-1}B^\top$ and $R = \gamma^2 P - D^\top QD > 0$. If the pair (A, B) is controllable, the pair (A, C) is observable and $J_0 < \gamma$, then the *Riccati equation*

$$A_1^\top X + X A_1 + X R_1 X + Q_1 = 0 \tag{35}$$

has solutions $X_\pm$ such that $\sigma(A_1 + R_1 X_\pm) \subset \mathbb{C}^\pm$, $X_- < X_+$ and $X_- < X < X_+$ for any solution X of (34) [1,2]. Moreover, if $J < \gamma$ and X satisfies (35), then $X < \gamma^2 X_0$. Indeed, setting $v(x) = x^\top X x$ and

$$w = K_0 x, \quad K_0 = R^{-1}(B^\top X + D^\top QC), \tag{36}$$

we have

$$\dot v(x) + y^\top Q y - \gamma^2 w^\top P w = 0, \quad \|y\|_Q^2 - \gamma^2\|w\|_P^2 = x_0^\top X x_0 < \gamma^2 x_0^\top X_0 x_0,$$

for $x_0 \ne 0$. If $J = \gamma$, then considering (35) and (36) we have $x_0^\top X x_0 = \gamma^2 x_0^\top X_0 x_0$ and $(X - \gamma^2 X_0)x_0 = 0$ for some $x_0 \ne 0$. Moreover, $\|y\|_Q^2 = J^2(\|w\|_P^2 + x_0^\top X_0 x_0)$, i.e. in (29) a supremum is reached. Therefore, the expression (36) and any vector $x_0 \in \operatorname{Ker}(X - \gamma^2 X_0)$ corresponding to the stabilizing solution of Riccati equation (35) represent the worst external and initial perturbations in system (30).

Consider the *affine system* with bounded external perturbations

$$\dot x = A(x)x + B(x)w, \quad y = C(x)x + D(x)w, \quad x(0) = x_0, \tag{37}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in some neighbourhood $\mathcal{S}_0$ of the point $x = 0$. We can formulate the following lemma for local characteristics J_0 and J of system (37) (see the proof of sufficiency statement of Lemma 4.1).

Lemma 4.2 *Suppose that there exists a matrix $X = X^\top > 0$ satisfying the matrix inequality*

$$\begin{bmatrix} A^\top(x)X + XA(x) + C^\top(x)QC(x) & XB(x) + C^\top(x)QD(x) \\ B^\top(x)X + D^\top(x)QC(x) & D^\top(x)QD(x) - \gamma^2 P \end{bmatrix} < 0$$

for all $x \in \mathcal{S}_0$. Then $J_0 \leq \gamma$ and the zero state $x \equiv 0$ of system (37) *with uncertainty* (33) *is robust stable with a common Lyapunov function $v(x) = x^\top Xx$. In addition, if $0 < X \leq \gamma^2 X_0$, then $J \leq \gamma$.*

4.2 Static controllers with perturbations

Consider control systems (1), (2) and the performance criteria J and J_0 of the form (29). We are interested in control laws that ensure nonexpansivity property of closed loop system and minimize J and J_0. A control law is said to be J-*optimal*, if corresponding closed loop system has minimum performance criteria J. An J_0-optimal control law is H_∞-*optimal* in case of the identity weight matrices P and Q.

Primarily, we consider the static output-feedback controller

$$u = Ky + w, \tag{38}$$

where $w \in \mathbb{R}^m$ is a vector of bounded perturbations and $K \in \mathcal{K}_D$ is an unknown matrix. Assuming that $\det\big[I_m - KD(x)\big] \neq 0$, $x \in \mathcal{S}_0$, we rewrite the corresponding closed loop systems in the form

$$\dot{x} = M(x)x + N(x)w, \quad z = F(x)x + G(x)w, \quad x(0) = x_0, \tag{39}$$

$$\dot{x} = Mx + Nw, \quad y = Fx + Gw, \quad x(0) = x_0, \tag{40}$$

where $M(x) = A(x) + B(x)\big[I_m - KD(x)\big]^{-1}KC(x)$, $N(x) = B(x)\big[I_m - KD(x)\big]^{-1}$, $F(x) = \big[I_l - D(x)K\big]^{-1}C(x)$, $G(x) = \big[I_l - D(x)K\big]^{-1}D(x)$, $M = M(0)$, $N = N(0)$, $F = F(0)$, $G = G(0)$.

Theorem 4.1 [16] *For linear system* (2), *there exists an output-feedback controller* (38) *such that* $J < \gamma$ *iff the following relations are feasible:*

$$W_R^\top \begin{bmatrix} A^\top X + XA + C^\top QC & XB + C^\top QD \\ B^\top X + D^\top QC & D^\top QD - \gamma^2 P \end{bmatrix} W_R < 0, \qquad (41)$$

$$W_L^\top \begin{bmatrix} AY + YA^\top + BP^{-1}B^\top & YC^\top + BP^{-1}D^\top \\ CY + DP^{-1}B^\top & DP^{-1}D^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \quad (42)$$

$$0 < X < \gamma^2 X_0, \quad XY = \gamma^2 I_n, \qquad (43)$$

where $R = [C, D]$, $L = [B^\top, D^\top]$. *The gain matrix* K *of the controller may be constructed in the form* $K = K_0(I_l + DK_0)^{-1}$, *where* K_0 *is a solution of the LMI*

$$L_0^\top K_0 R_0 + R_0^\top K_0^\top L_0 + \Omega < 0 \qquad (44)$$

with $R_0 = \left[R, 0_{l\times l}\right]$, $L_0 = \left[L, 0_{m\times m}\right] \widetilde{X}$ *and*

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_m & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^\top X + XA & XB & C^\top \\ B^\top X & -P & D^\top \\ C & D & -Q^{-1} \end{bmatrix}.$$

Lemma 4.3 (Projection Lemma [15]) *LMI* (44) *has a solution* K_0 *if and only if*

$$W_{L_0}^\top \Omega W_{L_0} < 0, \quad W_{R_0}^\top \Omega W_{R_0} < 0, \qquad (45)$$

where W_{L_0} (W_{R_0}) *is a matrix whose columns make up the bases of the kernel* $\operatorname{Ker} L_0$ $(\operatorname{Ker} R_0)$.

4.3 Dynamic controllers with perturbations

Consider control systems (1) and (2) with the dynamic output-feedback controller

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky + w, \quad \xi(0) = 0, \qquad (46)$$

where $w \in \mathbb{R}^m$ is a vector of bounded perturbations, Z, V, U and K are unknown coefficient matrices. If $K \in \mathcal{K}_D$, then linear closed loop system (2), (46) reduces to the form

$$\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad y = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{47}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \widehat{M} = \begin{bmatrix} A+BK_0C & BU_0 \\ V_0C & Z_0 \end{bmatrix}, \widehat{N} = \begin{bmatrix} B+BK_0D \\ V_0D \end{bmatrix},$$

$$\widehat{F} = \begin{bmatrix} C + DK_0C, DU_0 \end{bmatrix}, \quad \widehat{G} = D + DK_0D, \ \ K_0 = \mathbf{D}(K),$$

$$U_0 = (I_m - KD)^{-1}U, V_0 = V(I_l - DK)^{-1}, Z_0 = Z + VD(I_m - KD)^{-1}U.$$

We give the following auxiliary statement (see [14] in case of $\gamma = 1$).

Lemma 4.4 *Gain matrices $X > 0$, $Y > 0$ and a scalar $\gamma > 0$, there are matrices $X_1 \in \mathbb{R}^{r\times n}$, $X_2 \in \mathbb{R}^{r\times r}$, $Y_1 \in \mathbb{R}^{r\times n}$ and $Y_2 \in \mathbb{R}^{r\times r}$ such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^\top \\ X_1 & X_2 \end{bmatrix} > 0, \ \ \widehat{Y} = \begin{bmatrix} Y & Y_1^\top \\ Y_1 & Y_2 \end{bmatrix} > 0, \ \ \widehat{X}\widehat{Y} = \gamma^2 I_{n+r}, \tag{48}$$

if and only if

$$W = \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq n + r. \tag{49}$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (47), we get the following result.

Theorem 4.2 [16] *There exists a dynamic controller* (46) *such that the evaluation $J < \gamma$ holds for linear closed loop system* (47)*, iff the LMI system* (32)*,* (41)*,* (42) *and* (49) *is solvable with respect to $X = X^\top > 0$ and $Y = Y^\top > 0$.*

The coefficient matrices of the controller (46) in Theorem 4.2 may be constructed in the form

$$\begin{gathered} K = (I_m + K_0D)^{-1}K_0, \quad U = (I_m + K_0D)^{-1}U_0, \\ V = V_0(I_l + DK_0)^{-1}, \quad Z = Z_0 - V_0D(I_m + K_0D)^{-1}U_0, \end{gathered} \tag{50}$$

by solving the LMI

$$\widehat{L}^\top \widehat{K}_0 \widehat{R} + \widehat{R}^\top \widehat{K}_0^\top \widehat{L} + \widehat{\Omega} < 0, \tag{51}$$

where

$$\widehat{\Omega} = \begin{bmatrix} A^\top X + XA & A^\top X_1^\top & XB & C^\top \\ X_1 A & 0 & X_1 B & 0 \\ B^\top X & B^\top X_1^\top & -P & D^\top \\ C & 0 & D & -Q^{-1} \end{bmatrix}, \widehat{L}^\top = \begin{bmatrix} XB & X_1^\top \\ X_1 B & X_2 \\ 0 & 0 \\ D & 0 \end{bmatrix},$$

$$\widehat{R} = \begin{bmatrix} C & 0 & D & 0 \\ 0 & I_r & 0 & 0 \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}.$$

Here X, X_1 and X_2 are blocks of matrix $\widehat{X}$ in (48).

If $K \in \mathcal{K}_D$, then $\det\left[I_m - KD(x)\right] \neq 0$ for all $x \in \mathcal{S}_0$, and nonlinear closed loop system (1), (46) reduces to the form

$$\dot{\widehat{x}} = \widehat{M}(\widehat{x})\widehat{x} + \widehat{N}(\widehat{x})w, \quad y = \widehat{F}(\widehat{x})\widehat{x} + \widehat{G}(\widehat{x})w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{52}$$

where all coefficient matrices are continuous in $\mathcal{S}_0$. Therefore, the dynamic controller (46) with (50) ensures robust stability of the zero state $\widehat{x} \equiv 0$ of system (52) with structured uncertainty (33) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^\top \widehat{X}\widehat{x}$. To evaluate local characteristics J_0 and J of system (52), we can apply Lemma 4.2.

4.4 Control systems with controlled and observed outputs

Consider the control system

$$\begin{aligned} &\dot{x} = Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ &z = C_1 x + D_{11} w + D_{12} u, \\ &y = C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{53}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^s$, $z \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$ are the state, the control, the norm-limited external perturbations, the controlled and observed outputs, respectively. We are interested in static and dynamic control laws that ensure nonexpansivity property of closed loop system and minimize the performance criteria J and J_0 of the form (29) with respect to controlled output z.

4.4.1 Static controllers

If we use the static output feedback controller

$$u = Ky, \quad \det\big(I_m - KD_{22}\big) \neq 0, \tag{54}$$

then closed loop system (53), (54) has the form

$$\dot{x} = Mx + Nw, \quad z = Fx + Gw, \quad x(0) = x_0, \tag{55}$$

where $M = A + B_2K_0C_2$, $N = B_1 + B_2K_0D_{21}$, $F = C_1 + D_{12}K_0C_2$, $G = D_{11} + D_{12}K_0D_{21}$, $K_0 = (I_m - KD_{22})^{-1}K$. To formulate an analog of Theorem 4.1 we construct the following LMI

$$W_R^\top \begin{bmatrix} A^\top X + XA + C_1^\top QC_1 & XB_1 + C_1^\top QD_{11} \\ B_1^\top X + D_{11}^\top QC_1 & D_{11}^\top QD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{56}$$

$$W_L^\top \begin{bmatrix} AY + YA^\top + B_1P^{-1}B_1^\top & YC_1^\top + B_1P^{-1}D_{11}^\top \\ C_1Y + D_{11}P^{-1}B_1^\top & D_{11}P^{-1}D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{57}$$

where $R = \big[\, C_2, D_{21} \,\big]$, $L = \big[\, B_2^\top, D_{12}^\top \,\big]$.

Theorem 4.3 *For linear system* (53)*, there exists a static output feedback controller* (54) *such that* $J < \gamma$ *iff the system of relations* (43)*,* (56) *and* (57) *is feasible.*

If we use a static state feedback $u = Kx$, then $C_2 = I_n$, $D_{21} = 0$ and $D_{22} = 0$. In this case the relations (43) and (56) can be written as

$$\begin{bmatrix} X_0 & I_n \\ I_n & Y \end{bmatrix} > 0, \quad D_{11}^\top QD_{11} - \gamma^2 P < 0. \tag{58}$$

Corollary 4.1 *For linear system* (53)*, there exists a state feedback controller* $u = Kx$ *such that* $J < \gamma$ *iff the LMI system* (57) *and* (58) *is solvable for some matrix* $Y = Y^\top > 0$.

Remark 4.3 The gain matrix K in Theorem 4.3 and Corollary 4.1 may be constructed as $K = K_0(I_l + D_{22}K_0)^{-1}$, where K_0 is an arbitrary solution of LMI:

$$\widehat{L}^\top K_0 \widehat{R} + \widehat{R}^\top K_0^\top \widehat{L} + \Omega < 0,$$

$\widehat{R} = \left[R, 0_{l\times k}\right]$, $R = \left[\, C_2, D_{21}\,\right]$, $\widehat{L} = \left[L, 0_{m\times s}\right] \widetilde{X}$, $L = \left[\, B_2^\top, D_{12}^\top\,\right]$,

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^\top X + XA & XB_1 & C_1^\top \\ B_1^\top X & -\gamma^2 P & D_{11}^\top \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix}.$$

4.4.2 Dynamic controllers

If we use the dynamic output feedback

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \quad \xi(0) = 0, \tag{59}$$

and $\det\left(I_m - KD_{22}\right) \neq 0$, then closed loop system (53), (59) has the form

$$\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad z = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{60}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \widehat{M} = \begin{bmatrix} A + B_2K_0C_2 & B_2U_0 \\ V_0C_2 & Z_0 \end{bmatrix} = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2,$$

$$\widehat{N} = \begin{bmatrix} B_1 + B_2K_0D_{21} \\ V_0D_{21} \end{bmatrix} = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{F} = \left[C_1 + D_{12}K_0C_2, D_{12}U_0\right] = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2,$$

$$\widehat{G} = D_{11} + D_{12}K_0D_{21} = D_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21},$$

$$\widehat{A} = \begin{bmatrix} A & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \ \widehat{B}_2 = \begin{bmatrix} B_2 & 0_{n\times r} \\ 0_{r\times m} & I_r \end{bmatrix}, \ \widehat{C}_2 = \begin{bmatrix} C_2 & 0_{l\times r} \\ 0_{r\times n} & I_r \end{bmatrix},$$

$$\widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}, \ \widehat{B}_1 = \begin{bmatrix} B_1 \\ 0_{r\times s} \end{bmatrix}, \ \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{r\times s} \end{bmatrix},$$

$$\widehat{C}_1 = \left[C_1, 0_{k\times r}\right], \quad \widehat{D}_{12} = \left[D_{12}, 0_{k\times r}\right],$$

$$K_0 = (I_m - KD_{22})^{-1}K, \quad U_0 = (I_m - KD_{22})^{-1}U,$$

$$V_0 = V(I_l - D_{22}K)^{-1}, \quad Z_0 = Z + VD_{22}(I_m - KD_{22})^{-1}U.$$

Applying Lemmas 4.3, 4.4 and Theorem 4.1 to system (60), we get the following result.

Theorem 4.4 *For linear system* (53)*, there exists a dynamic controller* (59) *such that* $J < \gamma$ *iff the matrix system* (32), (49), (56) *and* (57) *is feasible.*

Remark 4.4 The coefficient matrices of dynamic controller (59) in Theorem 4.4 may be constructed in the form

$$\begin{aligned} K = (I_m+K_0D_{22})^{-1}K_0, \;\; U = (I_m+K_0D_{22})^{-1}U_0, \\ V = V_0(I_l+D_{22}K_0)^{-1}, \;\; Z = Z_0-V_0D_{22}(I_m+K_0D_{22})^{-1}U_0, \end{aligned} \tag{61}$$

by solving the LMI

$$\widehat{L}^\top \widehat{K}_0 \widehat{R} + \widehat{R}^\top \widehat{K}_0^\top \widehat{L} + \widehat{\Omega} < 0, \tag{62}$$

where $\widehat{R} = \big[\widehat{C}_2, \widehat{D}_{21}, 0_{l+r\times k}\big]$, $\widehat{L} = \big[\widehat{B}_2^\top \widetilde{X}, 0_{m+r\times s}, \widehat{D}_{12}^\top\big]$,

$$\widetilde{X} = \begin{bmatrix} \widehat{X} & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \; \widehat{X} = \begin{bmatrix} X & X_1^\top \\ X_1 & X_2 \end{bmatrix}, \; \widehat{\Omega} = \begin{bmatrix} \widehat{A}^\top \widehat{X} + \widehat{X}\widehat{A} & \widehat{X}\widehat{B}_1 & \widehat{C}_1^\top \\ \widehat{B}_1^\top \widehat{X} & -\gamma^2 P & D_{11}^\top \\ \widehat{C}_1 & D_{11} & -Q^{-1} \end{bmatrix}.$$

We give the following algorithm for constructing stabilizing dynamic controller (59) satisfying Theorem 4.4.

Algorithm 4.1 1) calculate the matrices W_R and W_L, where $R = \big[\, C_2, D_{21} \,\big]$ and $L = \big[\, B_2^\top, D_{12}^\top \,\big]$;

2) find the matrices $X = X^\top > 0$ and $Y = Y^\top > 0$ satisfying (32), (49), (56) and (57);

3) construct decomposition $Z = Y - \gamma^2 X^{-1} = V^\top V$, $V \in \mathbb{R}^{r\times n}$, $\ker V = \ker Z$ and form the block matrix

$$\widehat{X} = \begin{bmatrix} X & X_1^\top \\ X_1 & X_2 \end{bmatrix} > 0, \quad X_1 = \frac{1}{\gamma} V X, \quad X_2 = \frac{1}{\gamma^2} V X V^\top + I_r;$$

4) solve the LMI (62) under restriction $\det(I_m + K_0 D_{22}) \neq 0$;

5) calculate the coefficient matrices of dynamic controller (59) by formula (61).

Remark 4.5 Note, that we have necessary and sufficient conditions for an evaluation $J_0 < \gamma$ represented by the corresponding statements of Theorems 4.1 – 4.4 without usage of additional restriction

$X < \gamma^2 X_0$. With the use of static state feedback or full order $r = n$ dynamic controllers the problems under consideration are reduced to solving LMI systems. We can formulate analogs of Theorems 4.1 – 4.4 for the corresponding control systems with a polyhedral uncertainties of the matrices A, B_1, C_1 and D_{11}. In addition, sufficient statements of these theorems can be generalized for the corresponding affine control systems with continuous coefficient matrices (see Lemma 4.2).

Example 4.1 Consider a controlled linear damped oscillator described by system (53) with

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -\delta \end{bmatrix}, \; B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$D_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, D_{21} = D_{22} = 0, x = \begin{bmatrix} \varphi \\ \dot{\varphi} \end{bmatrix}, z = \begin{bmatrix} \varphi \\ u \end{bmatrix}, y = \varphi.$$

For system without control, we get $J_0 = 1,00124$ and $J = 1,29005$ assuming that $\delta = 0,1$, $\omega_0 = 1$, $P = 1$, $Q = \mathrm{diag}\{q_1, q_2\}$ and $X_0 = \mathrm{diag}\{\rho_1, \rho_2\}$, where $q_1 = 0,01$, $q_2 = 0,1$, $\rho_1 = \rho_2 = 0,04$. Figure 1 shows the dependence J of δ and ω_0. The damping level of input signals and initial perturbations of oscillator decreases with the increase of its natural frequency ω_0 and does not change with the increase of the damping factor δ.

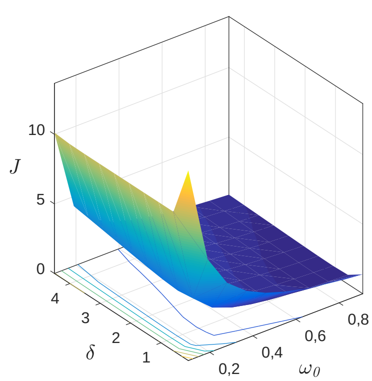

Figure 1: The dependence $J(\delta, \omega_0)$.

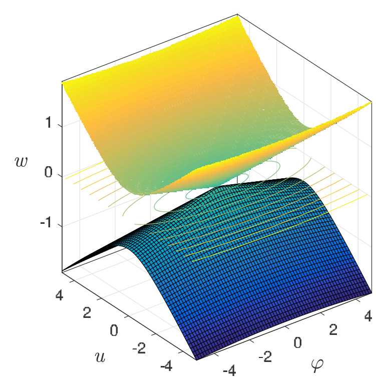

Figure 2: Uncertainty region (63).

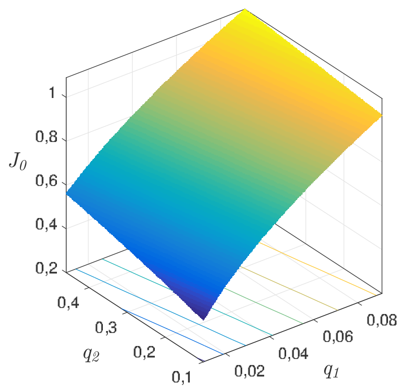

Figure 3: The dependence $J_0(q_1, q_2)$ (closed loop system).

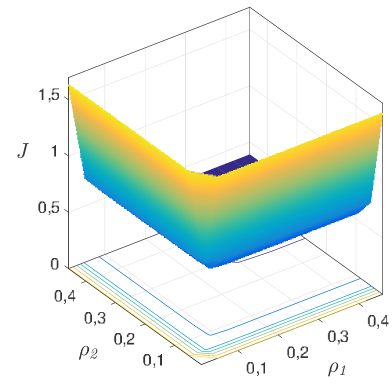

Figure 4: The dependence $J(\rho_1, \rho_2)$ (closed loop system).

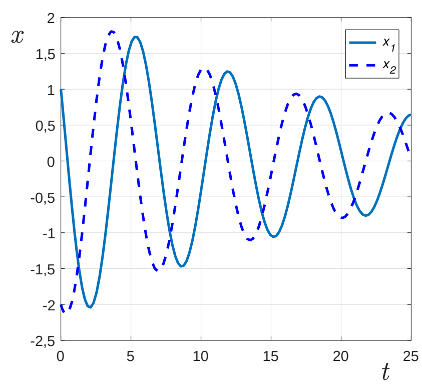

Figure 5: System behavior without control.

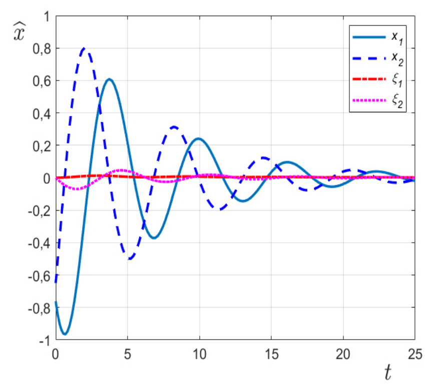

Figure 6: Closed loop system behavior.

Next, using Algorithm 4.1, we performed minimization of the parameter γ satisfying Theorem 4.4. As a result for $\gamma = 0,865$, we constructed an approximate J-optimal dynamic controller (59) with the coefficient matrices $K = -0,23768$, $U = \left[\, -0,34024 \;\; 3,90359 \,\right]$,

$$V = \begin{bmatrix} -0,00081 \\ 0,11005 \end{bmatrix}, \quad Z = \begin{bmatrix} -0,02029 & -0,08965 \\ 0,24404 & -1,05858 \end{bmatrix},$$

that provides a robust stability and nonexpansiveness of closed loop system

(60). This regulator significantly reduced the damping level of input signals and initial perturbations of oscillator. For example, for the indicated values of parameters we have $J_0 = 0,39131$ and $J = 0,86275 < 1$. The oscillator with constructed regulator preserves asymptotic stability for any perturbation function (see Figure 2)

$$w(t) = \frac{1}{\gamma}(\theta_1\varphi + \theta_2 u), \quad \frac{\theta_1^2}{q_1} + \frac{\theta_2^2}{q_2} \le 1, \ |w| \le \frac{1}{\gamma}\sqrt{q_1\varphi^2 + q_2 u^2}. \tag{63}$$

For closed loop system, the worst perturbation w and the worst initial vector $\widehat{x}_0$ with respect to J were also found (see Remark 4.2):

$$w = \widehat{\Theta}_0\widehat{x}, \quad \widehat{\Theta}_0 = \left[\begin{array}{cccc} 0,00298 & 0,03650 & -0,00263 & 0,07191 \end{array}\right],$$

$$\widehat{x}_0 = \left[\begin{array}{cccc} -0,76067 & -0,64914 & 0 & 0 \end{array}\right]^\top.$$

The dependences $J_0(q_1, q_2)$ and $J(\rho_1, \rho_2)$ for closed loop system are shown in Figures 3 and 4, respectively. Figure 5 shows system behavior without control and Figure 6 shows closed loop system behavior for the worst perturbation w and the worst initial vector $\widehat{x}_0$:

$$\dot{\widehat{x}} = \widehat{M}_0\widehat{x}, \quad \widehat{M}_0 = \widehat{M} + \widehat{N}\widehat{\Theta}_0, \quad \widehat{x}(0) = \widehat{x}_0,$$

$$\sigma(\widehat{M}_0) = \big\{-0,05019, \ -0,79024, \ -0,15097 \pm 1.01506\, i\big\}.$$

4.5 H_∞-Control problem for descriptor systems

We can formulate analogs of Theorems 4.1 – 4.4 for a class of descriptor control systems. Consider a linear continuous-time *descriptor system* with bounded perturbations

$$E\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad x(0) = x_0, \tag{64}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^s$, $z \in \mathbb{R}^k$ and $\operatorname{rank} E = \rho \le n$.

Definition 4.2 A matrix pair (E, A) is said to be *admissible* if it is *regular*, *impulse-free* and *stable*, i.e. $\det F(\lambda) \not\equiv 0$, $\deg F(\lambda) = \rho$ and $\sigma(F) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$, respectively, where $F(\lambda) = A - \lambda E$. Descriptor system (64) with admissible pair (E, A) is *admissible*.

Lemma 4.5 [17] *System* (64) *is admissible if and only if there exists matrix* X *such that* $A^\top X + X^\top A < 0$ *and* $E^\top X = X^\top E \ge 0$.

We introduce an analog of the performance (29) for system (64):

$$J = \sup_{(w,x_0)\in\mathcal{W}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \tag{65}$$

where $P > 0$, $Q > 0$ and $X_0 \geq 0$ are weight matrices, $\mathcal{W}$ is a set of pairs (w, x_0) such that system (64) has a solution and $0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty$. To formulate the following analog of the Bounded Real Lemma for system (64), we suppose that $X_0 = E^\top H E \geq 0$, where $H = H^\top > 0$.

Lemma 4.6 [18] *Given $\gamma > 0$, system (64) is admissible and satisfies $J < \gamma$ if there exist matrices X and $S = S^\top \geq 0$ such that*

$$0 \leq E^\top X = X^\top E = S \leq \gamma^2 X_0, \quad \operatorname{rank}(S - \gamma^2 X_0) = \rho, \tag{66}$$

$$\begin{bmatrix} A^\top X + X^\top A + C^\top Q C & X^\top B + C^\top Q D \\ B^\top X + D^\top Q C & D^\top Q D - \gamma^2 P \end{bmatrix} < 0. \tag{67}$$

Conversely, if system (64) is admissible with $J < \gamma$ and $\operatorname{rank}\begin{bmatrix} E^\top & C^\top Q D \end{bmatrix} = \rho$, then relations (66) and (67) are feasible.

Remark 4.6 If system (1) is admissible and there exist matrices X and S such that (66) and

$$A_1^\top X + X^\top A_1 + X^\top R_1 X + Q_1 = 0 \tag{68}$$

hold with $\gamma = J$ (see Remark 4.2), then (36) and $x_0 \in \operatorname{Ker}(S - \gamma^2 X_0)$ represent the worst external and initial perturbations relatively J for system (64).

Consider the descriptor control system

$$\begin{array}{l} E\dot{x} = Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z = C_1 x + D_{11} w + D_{12} u, \\ y = C_2 x + D_{21} w + D_{22} u, \end{array} \tag{69}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathbb{R}^s$, $z \in \mathbb{R}^k$ and $y \in \mathbb{R}^l$. Using the dynamic controller (59) a closed loop system has the form

$$\widehat{E}\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad z = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{70}$$

where $\widehat{E} = \operatorname{diag}\{E, I_r\}$ (see (60)).

Theorem 4.5 [19] *If there exist matrices X, Y, $S = S^\top \geq 0$ and $\Theta = \Theta^\top \geq 0$ such that* (66) *and*

$$EY = Y^\top E^\top \geq 0, \quad \operatorname{rank} \begin{bmatrix} X - \Theta E & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} = n, \quad \operatorname{rank} \Theta = r, \tag{71}$$

$$W_R^\top \begin{bmatrix} A^\top X + X^\top A + C_1^\top Q C_1 & X^\top B_1 + C_1^\top Q D_{11} \\ B_1^\top X + D_{11}^\top Q C_1 & D_{11}^\top Q D_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{72}$$

$$W_L^\top \begin{bmatrix} AY + Y^\top A^\top + B_1 P^{-1} B_1^\top & Y^\top C_1^\top + B_1 P^{-1} D_{11}^\top \\ C_1 Y + D_{11} P^{-1} B_1^\top & D_{11} P^{-1} D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0 \tag{73}$$

hold with $L = \left[B_2^\top, D_{12}^\top\right]$ and $R = \left[C_2, D_{21}\right]$, then there exists an r-order dynamic controller (59) *provided the admissibility and evaluation $J < \gamma$ for a closed loop system* (70).

Theorem 4.6 [19] *Let*

$$R_0 = D_{12}^\top Q D_{12} > 0, \quad R_1 = \gamma^2 P - D_{11}^\top Q_1 D_{11} > 0.$$

If there exist matrices X, G, $S = S^\top \geq 0$ and $\Theta = \Theta^\top \geq 0$ such that (66), (72) *and*

$$X - \Theta E = G, \quad \Theta = \Theta^\top \geq 0, \quad \operatorname{rank} \Theta = r, \tag{74}$$

$$A_2^\top G + G^\top A_2 + G^\top R_2 G + Q_2 < 0 \tag{75}$$

hold with $A_2 = A_1 + B_{11} R_1^{-1} D_{11}^\top Q_1 C_1$, $A_1 = A - B_2 R_0^{-1} D_{12}^\top Q C_1$, $R_2 = B_{11} R_1^{-1} B_{11}^\top - B_2 R_0^{-1} B_2^\top$, $B_{11} = B_1 - B_2 R_0^{-1} D_{12}^\top Q D_{11}$, $Q_1 = Q - Q D_{12} R_0^{-1} D_{12}^\top Q$, $Q_2 = C_1^\top (Q_1 + Q_1 D_{11} R_1^{-1} D_{11}^\top Q_1) C_1$, then there exists an r-order dynamic controller (59) *provided the admissibility and evaluation $J < \gamma$ for a closed loop system* (70).

Remark 4.7 The coefficient matrices of dynamic controller (59) in Theorems 4.5 and 4.6 can be constructed in the form (61) by solving the LMI (62) with

$$\widehat{\Omega} = \begin{bmatrix} \widehat{A}^\top \widehat{X} + \widehat{X}^\top \widehat{A} & \widehat{X}^\top \widehat{B}_1 & \widehat{C}_1^\top \\ \widehat{B}_1^\top \widehat{X} & -\gamma^2 P & D_{11}^\top \\ \widehat{C}_1 & D_{11} & -Q^{-1} \end{bmatrix}, \quad \widehat{X} = \begin{bmatrix} X & X_3 \\ X_1 & X_2 \end{bmatrix},$$

$$\Theta = X_3 X_2^{-1} X_3^\top, \quad X_1 = X_3^\top E, \quad X_1 \in \mathbb{R}^{r\times n}, \; X_2 \in \mathbb{R}^{r\times r}, \; X_3 \in \mathbb{R}^{n\times r},$$

wherein the remaining matrix expressions in (62) are the same.

Remark 4.8 In the case $\Theta = 0$, Theorems 4.5 and 4.6 give the conditions for existence of a static controller (54) such that a closed loop system is admissible with $J < \gamma$.

Remark 4.9 Without loss of generality, the matrix X in Lemma 4.6 and Theorems 4.5 and 4.6 can be defined as

$$X = S_1 E + E_0 G_1, \quad 0 < S_1 = S_1^\top < \gamma^2 H,$$

where $E_0 = W_{E^\top} \in \mathbb{R}^{n\times(n-\rho)}$ and $G_1 \in \mathbb{R}^{(n-\rho)\times n}$. Then in (66) we have $0 \le E^\top S_1 E = S \le \gamma^2 X_0$ and $\operatorname{rank}(S - \gamma^2 X_0) = \rho$.

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