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Robust Output Feedback Stabilization and Optimization of Discrete-Time Control Systems *

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1 Introduction

In modern systems theory, continuous-time and discrete-time mathematical models of controlled plants with uncertain elements (parameters, nonlinearities, external disturbances, etc.) are widely used. Robust stabilization, optimization and H_∞-control problems are of prime importance for such systems (see, e.g., [1–6]).

In practice, discrete-time system models have certain advantages over continuous-time ones. In particular, the use of difference equations of motion does not require the study of mathematical problems of the existence and uniqueness of solutions. In addition, difference systems are sufficiently suitable for their direct numerical implementation by computer software. Note, that practical applications of modern methods for both continuous-time and discrete-time control systems design reduce to solving the *linear matrix inequalities* (LMI) [7–9].

In this chapter, we consider linear and nonlinear discrete-time control systems for which closed loop systems may be represented in the

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pseudolinear form

$$x_{t+1} = M(x_t, t)\, x_t, \quad x_t \in \mathbb{R}^n, \quad t \in \mathcal{T} = \{0, 1, 2, \dots\},$$

besides, a matrix function $M(x,t)$ may contain uncertain quantities belonging to certain sets. Intervals, polytopes, affine sets of matrices and other objects may serve as the uncertainty sets. The applied control laws are of the form of static or dynamic output feedback.

Since the theory of linear discrete-time systems very closely parallels the theory of linear continuous-time systems, many of the stated results are similar. For this reason the comments in the text are brief and many proofs are omitted (see [10]).

Our consideration includes the following types of problems: output feedback stabilization of discrete-time control systems (Section 2), robust stabilization and optimization of discrete-time control systems with polyhedral uncertainties (Section 3) and robust stabilization and weighted suppression of perturbations in discrete-time control systems (Section 4).

Throughout the paper, the following notations are used: I_n is the identity $n \times n$ matrix; $0_{n\times m}$ is the $n \times m$ null matrix; $X = X^\top > 0$ (≥ 0) is the symmetric positive definite (semidefinite) matrix X; $\mathrm{i}(X) = \{\mathrm{i}_+, \mathrm{i}_-, \mathrm{i}_0\}$ is the inertia of matrix $X = X^\top$ consisting of the numbers of positive, negative and zero eigenvalues taking into account the multiplicities; $\sigma(A)$ and $\rho(A)$ are the spectrum and the spectral radius of A, respectively; $\lambda_{\max}(X)$ and $\lambda_{\min}(X)$ are the maximum and the minimum eigenvalue of the Hermitian matrix X, respectively; A^+ is the pseudoinverse matrix; W_A is a matrix whose columns make up the bases of the kernel $\operatorname{Ker} A$; $\|x\|$ denotes the Euclidean norm of the vector $x \in \mathbb{R}^n$; $\|w\|_P$ denotes the weighted l_2-norm of a vector sequence w_t, $t \in \mathcal{T}$; $\mathrm{Co}\{A_1, \dots, A_\nu\}$ stands for a polytope in a matrix space described as a convex hull of the set $\{A_1, \dots, A_\nu\}$.

2 Output Feedback Stabilization of Nonlinear Systems

Consider the affine discrete-time control system

$$x_{t+1} = A(x_t)x_t + B(x_t)u_t, \quad y_t = C(x_t)x_t + D(x_t)u_t, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is a state vector, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^l$ are input and output vectors, respectively, $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in some neighborhood $\mathcal{S}_0$ of the zero state $x_t = 0$, $t \in \mathcal{T}$. Assume that $\operatorname{rank} B(x) \equiv m$ and $\operatorname{rank} C(x) \equiv l$ in $\mathcal{S}_0$.

Along with (1), consider the linear system

$$x_{t+1} = Ax_t + Bu_t, \quad y_t = Cx_t + Du_t, \tag{2}$$

with $A = A(0)$, $B = B(0)$, $C = C(0)$ and $D = D(0)$. Let $B^\perp$ and $C^\perp$ be the orthogonal complements of B and C, respectively, i.e. $B^\top B^\perp = 0$, $\det\left[B, B^\perp\right] \neq 0$, $C^\perp C^\top = 0$, $\det\left[C^\top, C^{\perp\top}\right] \neq 0$.

2.1 Static controllers

Formulate stabilizability conditions of the zero state $x_t = 0$ for systems (1) and (2) through the static output-feedback controller

$$u_t = Ky_t, \quad K \in \mathcal{K}_D, \tag{3}$$

where $\mathcal{K}_D = \left\{K \in \mathbb{R}^{m\times l} : \det(I_m - KD) \neq 0\right\}$. Closed loop system (2), (3) has the form

$$x_{t+1} = Mx_t, \quad M = A + B\mathbf{D}(K)C, \tag{4}$$

where $\mathbf{D}(K) = (I_m - KD)^{-1}K \equiv K(I_l - DK)^{-1}$ is a nonlinear operator with the following properties:

- if $K \in \mathcal{K}_D$, then $I_l + D\mathbf{D}(K) \equiv (I_l - DK)^{-1}$;
- if $K_1 \in \mathcal{K}_D$ and $K_2 \in \mathcal{K}_{D_1}$, then $K_1 + K_2 \in \mathcal{K}_D$ and

$$\mathbf{D}(K_1 + K_2) = \mathbf{D}(K_1) + (I_m - K_1D)^{-1}\mathbf{D}_1(K_2)\,(I_l - DK_1)^{-1}, \tag{5}$$

where $\mathbf{D}_1(K_2) = (I_m - K_2D_1)^{-1}K_2$, $D_1 = (I_l - DK_1)^{-1}D$;

- if $-K_0 \in \mathcal{K}_D$, then $K = -\mathbf{D}(-K_0) \in \mathcal{K}_D$ and $\mathbf{D}(K) = K_0$.

Definition 2.1 System (4) is *ρ-stable* if the spectrum $\sigma(M)$ lies inside the circle $\{\lambda : |\lambda| < \rho\}$, where $0 < \rho \leq 1$.

Theorem 2.1 *Let* $\operatorname{rank} B = m < n$ *and* $\operatorname{rank} C = l < n$. *Then the following statements are equivalent:*
1) *There exists a controller* (3) *ensuring* ρ*-stability of system* (4).
2) *There exists a matrix* $X = X^\top > 0$ *satisfying the relations*

$$B^{\perp\top}(AXA^\top - \rho^2 X)B^\perp < 0, \tag{6}$$

$$\mathrm{i}(H) = \{l, m, 0\}, \quad H = \begin{bmatrix} H_0 & H_1^\top \\ H_1 & H_2 \end{bmatrix}, \tag{7}$$

where $H_0 = B^+(L - LRL)B^{+\top}$, $H_1 = CXA^\top(I_n - RL)B^{+\top}$, $H_2 = C(X - XA^\top RAX)C^\top$, $L = AXA^\top - \rho^2 X$, $R = B^\perp S^{-1} B^{\perp\top}$, $S = B^{\perp\top} L B^\perp$;
3) *There exists a matrix* $X = X^\top > 0$ *satisfying* (6) *and*

$$AXA^\top - \rho^2 X < AXC^\top (CXC^\top)^{-1} CXA^\top. \tag{8}$$

4) *There exist mutually inverse matrices* $X = X^\top > 0$ *and* $Y = Y^\top > 0$ *satisfying* (6) *and*

$$C^\perp (A^\top Y A - \rho^2 Y) C^{\perp\top} < 0. \tag{9}$$

5) *There exists a matrix* $Y = Y^\top > 0$ *satisfying* (9) *and*

$$A^\top Y A - \rho^2 Y < A^\top Y B (B^\top Y B)^{-1} B^\top Y A. \tag{10}$$

When one of the statements 2 – 4 *is true, then the controller*

$$u_t = K y_t, \quad K = -\mathbf{D}(-K_0) \in \mathcal{K}_D, \tag{11}$$

where K_0 *is a solution of one of the equivalent LMI*

$$\begin{aligned} P_1^\top K_0 Q_1 + Q_1^\top K_0^\top P_1 &< \begin{bmatrix} \rho^2 X & AX \\ XA^\top & X \end{bmatrix}, \\ P_2^\top K_0 Q_2 + Q_2^\top K_0^\top P_2 &< \begin{bmatrix} -H_0 & 0 \\ 0 & H_2^{-1} \end{bmatrix}, \end{aligned} \tag{12}$$

with $P_1 = \big[-B^\top, 0\big]$, $Q_1 = \big[0, CX\big]$, $P_2 = \big[I_m, 0\big]$ *and* $Q_2 = \big[H_1, I_l\big]$, *ensures* ρ*-stability of closed loop system* (4).

For the equivalence of the statements 1 and 2 in Theorem 2.1, see [11]. Equivalence of the statements 2 and 3 follows from (see [12, p. 147]) $H = \widehat{H}_0 - \widehat{H}_1^\top \widehat{H}_2^{-1} \widehat{H}_1$, $\mathrm{i}_+(\widehat{H}) = \mathrm{i}_+(H) = \mathrm{i}_+(\Delta)$ and $\mathrm{i}_-(\widehat{H}) = \mathrm{i}_-(H) + n - m = \mathrm{i}_-(\Delta)$, where

$$\widehat{H} = \begin{bmatrix} \widehat{H}_0 & \widehat{H}_1^\top \\ \widehat{H}_1 & \widehat{H}_2 \end{bmatrix} = \left[\begin{array}{cc|c} B^+LB^{+\top} & B^+AXC^\top & B^+LB^\perp \\ CXA^\top B^{+\top} & CXC^\top & CXA^\top B^\perp \\ \hline B^{\perp\top}LB^{+\top} & B^{\perp\top}AXC^\top & S \end{array}\right] = W\Delta W^\top,$$

$$\Delta = \begin{bmatrix} AXA^\top - \rho^2 X & AXC^\top \\ CXA^\top & CXC^\top \end{bmatrix}, \; W^\top = \begin{bmatrix} B^{+\top} & 0 & B^\perp \\ 0 & I_l & 0 \end{bmatrix}, \; \det W \neq 0.$$

For the equivalence of the statements 1 and 4, see also [11, Therem 6.1.2] and [13].

Theorem 2.2 *Let one of the statements* 2 – 4 *of Theorem* 2.1 *hold for linear system* (2)*. Then* (11) *and* (12) *determine a static controller ensuring asymptotic stability of the state* $x \equiv 0$ *and quadratic Lyapunov function* $v(x) = x^\top X^{-1} x$ *of nonlinear closed loop system* (1)*,* (11)*.*

2.2 Dynamic controllers

The dynamic output feedback stabilization problem for system (1) is to find, if possible, a dynamic control law described by

$$\xi_{t+1} = Z\xi_t + Vy_t, \quad u_t = U\xi_t + Ky_t, \quad t \in \mathcal{T}, \tag{13}$$

where $\xi_t \in \mathbb{R}^r$ and $r \leq n$, such that the zero state of closed loop system is asymptotically stable. Equations (1) and (13) may be represented by control system in the extended phase space $\mathbb{R}^{n+r}$ with static controller

$$\widehat{x}_{t+1} = \widehat{A}(\widehat{x}_t)\widehat{x}_t + \widehat{B}(\widehat{x}_t)\widehat{u}_t, \;\; \widehat{y}_t = \widehat{C}(\widehat{x}_t)\widehat{x}_t + \widehat{D}(\widehat{x}_t)\widehat{u}_t, \;\; \widehat{u}_t = \widehat{K}\widehat{y}_t, \tag{14}$$

where

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \quad \widehat{y}_t = \begin{bmatrix} y_t \\ \xi_t \end{bmatrix}, \quad \widehat{u}_t = \begin{bmatrix} u_t \\ \xi_{t+1} \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix},$$

$$\widehat{A}(\widehat{x}) = \begin{bmatrix} A(x) & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{B}(\widehat{x}) = \begin{bmatrix} B(x) & 0 \\ 0 & I_r \end{bmatrix},$$

$$\widehat{C}(\widehat{x}) = \begin{bmatrix} C(x) & 0 \\ 0 & I_r \end{bmatrix}, \quad \widehat{D}(\widehat{x}) = \begin{bmatrix} D(x) & 0 \\ 0 & 0 \end{bmatrix}.$$

If $K \in \mathcal{K}_D$, then linear closed loop system (2), (13) has the form

$$\widehat{x}_{t+1} = \widehat{M}\,\widehat{x}_t, \quad \widehat{M} = \widehat{A} + \widehat{B}\widehat{\mathbf{D}}(\widehat{K})\widehat{C}, \tag{15}$$

where $\widehat{A} = \widehat{A}(0)$, $\widehat{B} = \widehat{B}(0)$, $\widehat{C} = \widehat{C}(0)$, $\widehat{D} = \widehat{D}(0)$, $\widehat{\mathbf{D}}(\widehat{K}) = (I_{m+r} - \widehat{K}\widehat{D})^{-1}\widehat{K}$, and

$$\widehat{\mathbf{D}}(\widehat{K}) = \left[\begin{array}{c|c} \mathbf{D}(K) & (I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1} & Z + VD(I_m - KD)^{-1}U \end{array}\right],$$

$$\widehat{M} = \left[\begin{array}{c|c} M & B(I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1}C & Z + VD(I_m - KD)^{-1}U \end{array}\right].$$

Theorem 2.3 *The following statements are equivalent:*
1) *There exists a dynamic controller* (13) *of order $r \le n$ ensuring ρ-stability of closed loop system* (15).
2) *There exist matrices X and X_0 satisfying* (6) *and*

$$X \ge X_0 > 0, \quad \operatorname{rank}(X - X_0) \le r, \\ AX_0A^\top - \rho^2 X_0 < AX_0C^\top(CX_0C^\top)^{-1}CX_0A^\top. \tag{16}$$

3) *There exist matrices X and Y satisfying* (6), (9) *and*

$$W = \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \ge 0, \quad \operatorname{rank} W \le n + r. \tag{17}$$

Proof of Theorem 2.3 follows from the corresponding statements of Theorem 2.1 taking into account the structure of block matrices in (15) (see [11]).

Remark 2.1 The coefficient matrices of stabilizing controller (13) in Theorem 2.3 may be defined in the form

$$K = (I_m + K_0D)^{-1}K_0, \quad U = (I_m + K_0D)^{-1}U_0, \\ V = V_0(I_l + DK_0)^{-1}, \quad Z = Z_0 - V_0(I_l + DK_0)^{-1}DU_0, \tag{18}$$

using the solution $\widehat{K}_0$ of the LMI

$$\widehat{P}^\top \widehat{K}_0 \widehat{Q} + \widehat{Q}^\top \widehat{K}_0^\top \widehat{P} < \widehat{F}, \tag{19}$$

where $\widehat{P} = \big[-\widehat{B}^\top, 0\big]$, $\widehat{Q} = \big[0, \widehat{C}\widehat{X}\big]$, $X - X_0 = X_1^\top X_2^{-1} X_1 \geq 0$, $K_0 \in \mathcal{K}_D$, $0 < \rho \leq 1$,

$$\widehat{F} = \begin{bmatrix} \rho^2 \widehat{X} & \widehat{A}\widehat{X} \\ \widehat{X}\widehat{A}^\top & \widehat{X} \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}, \quad \widehat{X} = \begin{bmatrix} X & X_1^\top \\ X_1 & X_2 \end{bmatrix} > 0.$$

For example, one can use the decomposition $X - X_0 = X_1^\top X_1 \geq 0$ with $X_2 = I_r$.

Remark 2.2 Note, that matrices X and X_0 satisfy statement 2 iff matrices X and $Y = X_0^{-1}$ satisfy statement 3. From (17) it follows that matrices X and Y are positive definite. The rank restriction in (17) always holds in case of full order $r = n$ dynamic controller.

Theorem 2.4 *Let one of the statements* 2 *or* 3 *of Theorem* 2.3 *hold for linear system* (2)*. Then* (18) *and* (19) *determine dynamic controller* (13) *ensuring asymptotic stability of the state* $x \equiv 0$ *and quadratic Lyapunov function* $v(\widehat{x}) = \widehat{x}^\top \widehat{X}^{-1} \widehat{x}$ *of nonlinear closed loop system* (1)*,* (13)*.*

3 Robust Stabilization and Optimization of Nonlinear Systems

The main results of this section are based on the application of an auxiliary statement on matrix uncertainty which generalizes the sufficiency statement of the Petersen's lemma [15]. Consider a nonlinear operator

$$\mathbf{F}(K) = W + U^\top \mathbf{D}(K) V + V^\top \mathbf{D}^\top(K) U + V^\top \mathbf{D}^\top(K)\, R\, \mathbf{D}(K) V \tag{20}$$

with $\mathbf{D}(K) = (I_m - KD)^{-1} K$ and an ellipsoidal set of matrices

$$\mathcal{K} = \big\{K \in \mathbb{R}^{m \times l} : K^\top P K \leq Q\big\}, \tag{21}$$

where $P = P^\top > 0$, $Q = Q^\top > 0$, $R = R^\top \geq 0$, $W = W^\top$, U, V and D are matrices of suitable sizes.

Lemma 3.1 [14] *If the matrix inequalities*

$$D^\top QD + R < P, \quad \begin{bmatrix} W & U^\top & V^\top \\ U & R-P & D^\top \\ V & D & -Q^{-1} \end{bmatrix} \leq 0 \ (< 0) \tag{22}$$

hold, then $\mathbf{F}(K) \leq 0$ (< 0) *for any matrix* $K \in \mathcal{K}$.

Consider a nonlinear control system in the vector-matrix form

$$x_{t+1} = A(x_t,t)x_t + B(x_t,t)u_t, \ \ y_t = C(x_t,t)x_t + D(x_t,t)u_t, \tag{23}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^l$. We construct a set of the static controllers

$$u_t = K(x_t,t)\, y_t, \ \ K(x_t,t) = K_*(x_t,t) + \widetilde{K}(x_t,t), \ \ \widetilde{K}(x_t,t) \in \mathcal{K}, \tag{24}$$

where $\mathcal{K}$ is an ellipsoidal set of matrices of the form (21). We assume that the matrices A, B, C, D, K and K_* depend on x_t and t continuously and the equilibrium state $x_t \equiv 0$ is isolated, i.e., the neighborhood $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \leq h\}$ does not contain other equilibrium states of this system. If $K \in \mathcal{K}_D$, then the closed loop system (23), (24) can be represented as

$$x_{t+1} = M(x_t,t)x_t, \quad M(x_t,t) = A + B\mathbf{D}(K)C. \tag{25}$$

Let the zero state of this system for $K \equiv K_*$ be asymptotically stable. When looking for the stabilizing matrix K_* in the class of autonomous systems (1), one can use Theorem 2.1 and its special cases. The problem is to construct conditions under which the zero state of system (25) is asymptotically stable for every matrix $\widetilde{K}(x_t,t) \in \mathcal{K}$. We find a solution for our problem in terms of a quadratic Lyapunov function (see [11, 14]).

Theorem 3.1 *Let for some matrix functions* $X_t = X_t^\top$ *and* $K_*(x,t)$ *the relations*

$$\varepsilon_1 I_n \leq X_t \leq \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \leq \varepsilon_2, \tag{26}$$

$$\begin{bmatrix} M_*^\top X_{t+1}M_* - X_t + \varepsilon_0 I_n & M_*^\top X_{t+1}B_* & C_*^\top \\ B_*^\top X_{t+1}M_* & B_*^\top X_{t+1}B_* - P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0, \quad (27)$$

hold with $\varepsilon_0 > 0$, $M_* = A + B\mathbf{D}(K_*)C$, $B_* = B(I_m - K_*D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$ *and* $D_* = D(I_m - K_*D)^{-1}$, $x_t = 0$ *and* $t \in \mathcal{T}$. *Then any control* (24) *ensures asymptotic stability of the zero state* $x_t \equiv 0$ *for system* (25) *and a common Lyapunov function* $v(x,t) = x^\top X_t x$.

Consider control system (23) with a quadratic quality functional

$$J_u(x_0) = \sum_{t=0}^{\infty} \left[x_t^\top \ u_t^\top\right] \Phi_t \begin{bmatrix} x_t \\ u_t \end{bmatrix}, \quad \Phi_t = \begin{bmatrix} S & N \\ N^\top & R \end{bmatrix}, \qquad (28)$$

where $S \geq NR^{-1}N^\top + \eta I_n$, $R > 0$ and $\eta > 0$.

Theorem 3.2 *Let for some matrix functions* $X_t = X_t^\top$ *and* $K_*(x,t)$ *the relations* (26) *and*

$$\begin{bmatrix} M_*^\top X_{t+1}M_* - X_t + \Phi_* + \varepsilon_0 I_n & M_*^\top X_{t+1}B_* + N_* + C^\top K_*^\top R_* & C_*^\top \\ B_*^\top X_{t+1}M_* + N_*^\top + R_*K_*C & B_*^\top X_{t+1}B_* + R_* - P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} < 0$$

hold with $\varepsilon_0 > 0$, $\Phi_* = L_*^\top \Phi L_*$, $L_*^\top = \left[I_n, C^\top \mathbf{D}^\top(K_*)\right]$, $R_* = (I_m - K_*D)^{-1\top} R\,(I_m - K_*D)^{-1}$, $N_* = N(I_m - K_*D)^{-1}$, $x_t = 0$ *and* $t \in \mathcal{T}$. *Then any control* (24) *ensures asymptotic stability of the zero state* $x_t \equiv 0$ *for system* (25), *a common Lyapunov function* $v(x,t) = x^\top X_t x$ *and evaluation* $J_u(x_0) \leq v(x_0, 0)$.

Corollary 3.1 *Let for some matrices* $X = X^\top > 0$ *and* K_* *the matrix inequalities*

$$\begin{bmatrix} M_{ijk}^\top X_{t+1}M_{ijk} - X_t + \Phi_k + \varepsilon_0 I_n & M_{ijk}^\top X_{t+1}B_{*j} + N_* + C_k^\top K_*^\top R_* & C_{*k}^\top \\ B_{*j}^\top X_{t+1}M_{ijk} + N_*^\top + R_*K_*C_k & B_{*j}^\top X_{t+1}B_{*j} + R_* - P & D_*^\top \\ C_{*k} & D_* & -Q^{-1} \end{bmatrix} < 0$$

hold with $\varepsilon_0 > 0$, $M_{ijk} = A_i + B_j\mathbf{D}(K_*)C_k$, $B_{*j} = B_j(I_m - K_*D)^{-1}$,

$\Phi_k = L_k^\top \Phi L_k$, $L_k^\top = \left[I_n, C_k^\top \mathbf{D}^\top(K_*)\right]$, $C_{*k} = (I_l - DK_*)^{-1} C_k$, $i = \overline{1,\alpha}$, $j = \overline{1,\beta}$, $k = \overline{1,\gamma}$, $x_t = 0$, $t \in \mathcal{T}$. Then any control (24) *ensures asymptotic stability of the zero state $x_t \equiv 0$ for system* (25) *with uncertainties $A(0,t) \in \mathrm{Co}\{A_1, \dots, A_\alpha\}$, $B(0,t) \in \mathrm{Co}\{B_1, \dots, B_\beta\}$ and $C(0,t) \in \mathrm{Co}\{C_1, \dots, C_\gamma\}$, a common Lyapunov function $v(x,t) = x^\top X x$ and evaluation $J_u(x_0) \le v(x_0, 0)$.*

Note that the proofs of Theorems 3.1 and 3.2 follow directly from Lemma 3.1 and the Lyapunov theorem on asymptotic stability taking into account representation of the first difference of Lyapunov function $v(x,t)$ with respect to system (25) in the form of a quadratic function with a matrix of the form (20) and application of formula (5) (see [11, 14]).

4 Generalized H_∞-control

4.1 Weighted level of perturbation suppression

Consider a dynamical system with external perturbations

$$x_{t+1} = f(x_t, w_t, t), \quad y_t = g(x_t, w_t, t), \quad t \in \mathcal{T}, \tag{29}$$

where $x_t \in \mathbb{R}^n$, $w_t \in \mathbb{R}^s$ and $y_t \in \mathbb{R}^l$ are the state, the l_2-norm-limited external perturbations and the output vector, respectively.

Definition 4.1 The dynamical system (29) is called *nonexpansive* if for any square-summable sequence w_t and $\tau > 0$

$$\sum_{t=0}^{\tau} y_t^\top Q y_t \le \sum_{t=0}^{\tau} w_t^\top P w_t + x_0^\top X_0 x_0,$$

where Q, P and X_0 are weight symmetric positive definite matrices.

We introduce the performance criterion of system (29) with respect to output y:

$$J = \sup_{0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty} \frac{\|y\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \tag{30}$$

where $\|y\|_Q$ and $\|w\|_P$ are *weighted* l_2*-norms* of y_t and w_t ($t \in \mathcal{T}$), respectively, i.e. $\|y\|_Q^2 = \sum_{t=0}^{\infty} y_t^\top Q y_t$ and $\|w\|_P^2 = \sum_{t=0}^{\infty} w_t^\top P w_t$. In case of $x_0 = 0$, we denote J by J_0. It is obvious that $J_0 \leq J$ and $J \leq 1$ for a nonexpansive system. The value J describes the weighted level of external and initial perturbation suppression in system (29). A pair (w, x_0) is the *worst* for system (29) with respect to the performance criterion J, if in (30) a supremum is reached. If $P = I_s$, $Q = I_l$ and $X_0 = \rho I_n$, then J and J_0 coincide with known performance criteria of discrete-time systems [16].

Consider the class of linear systems

$$x_{t+1} = Ax_t + Bw_t, \quad y_t = Cx_t + Dw_t, \quad t \in \mathcal{T}. \tag{31}$$

Lemma 4.1 *Let* $\rho(A) < 1$*. Then an evaluation* $J_0 < \gamma$ *for system* (31) *holds iff the LMI*

$$\Psi = \begin{bmatrix} A^\top XA - X + C^\top QC & A^\top XB + C^\top QD \\ B^\top XA + D^\top QC & B^\top XB + D^\top QD - \gamma^2 P \end{bmatrix} < 0 \tag{32}$$

has a solution $X = X^\top > 0$*. Moreover,* $J < \gamma$ *iff the LMI* (32) *has a solution* X *such that*

$$0 < X < \gamma^2 X_0. \tag{33}$$

The sufficiency assertion of Lemma 4.1 follows from the relation

$$\Delta v(x_t) + y_t^\top Q y_t - \gamma^2 w_t^\top P w_t = \left[x_t^\top, w_t^\top\right] \Psi \begin{bmatrix} x_t \\ w_t \end{bmatrix} < 0,$$

where $\Delta v(x_t) = v(x_{t+1}) - v(x_t)$ is the first difference of Lyapunov function $v(x) = x^\top X x$ with respect to system (31). The necessity assertion of Lemma 4.1 may be established via representation of functional $\varphi(w, x_0)$ by similar expression with the identity weight matrices (see the proof of Lemma 5.1.1 in [11] and [16]).

Remark 4.1 If $\Psi < 0$, then system (31) with a structurally uncertain input

$$w_t = \frac{1}{\gamma}\,\Theta\, y_t, \quad \Theta^\top P \Theta \leq Q, \quad t \in \mathcal{T}, \tag{34}$$

is robust stable and has a common Lyapunov function $v(x) = x^\top Xx$ (see Theorem 3.1). The functional $\varphi(w, x_0)$ on a set of the functions (34) takes the minimum value if $\Theta^\top P\Theta = Q$.

It follows from Lemma 4.1 that the performance criteria J and J_0 of system (31) may be computed as the solutions of the corresponding optimization problems:

$$J_0 = \inf\big\{\gamma : \Psi < 0,\ X > 0\big\},\quad J = \inf\big\{\gamma : \Psi < 0,\ 0 < X < \gamma^2 X_0\big\}.$$

Consider the *affine system* with external perturbations

$$x_{t+1} = A(x_t)x_t + B(x_t)w_t,\quad y_t = C(x_t)x_t + D(x_t)w_t,\quad t \in \mathcal{T},\ (35)$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are continuous matrix functions in $\mathcal{S}_0$. We can formulate the following statement.

Lemma 4.2 *Suppose that there exists a matrix* $X = X^\top > 0$ *satisfying the matrix inequality*

$$\begin{bmatrix} A^\top(x)XA(x) - X + C^\top(x)QC(x) & A^\top(x)XB(x) + C^\top(x)QD(x) \\ B^\top(x)XA(x) + D^\top(x)QC(x) & B^\top(x)XB(x) + D^\top(x)QD(x) - \gamma^2 P \end{bmatrix} < 0$$

for all $x \in \mathcal{S}_0$. *Then* $J_0 \leq \gamma$ *and the zero state* $x_t \equiv 0$ *of system* (35) *with a structured uncertainty* (34) *is robust stable with a common Lyapunov function* $v(x) = x^\top Xx$. *In addition, if* $0 < X \leq \gamma^2 X_0$, *then* $J \leq \gamma$.

4.2 Static controllers with perturbations

Consider control systems (1), (2) and the performance criteria J and J_0 of the form (30). We are interested in control laws that ensure nonexpansivity property of closed loop system and minimize J and J_0. A control law is said to be *J-optimal* if the corresponding closed loop system has minimum performance criteria J.

Primarily, we consider the static output-feedback controller

$$u_t = K_* y_t + w_t,\quad t \in \mathcal{T}, \tag{36}$$

where $w_t \in \mathbb{R}^m$ is a vector of l_2-bounded perturbations and $K_* \in \mathcal{K}_D$ is an unknown matrix. Assuming that det $\big[I_m - K_* D(x)\big] \neq 0$, $x \in \mathcal{S}_0$, we rewrite the corresponding closed loop systems in the form

$$x_{t+1} = A_*(x_t)x_t + B_*(x_t)w_t, \quad y_t = C_*(x_t)x_t + D_*(x_t)w_t, \tag{37}$$

$$x_{t+1} = A_* x_t + B_* w_t, \quad y_t = C_* x_t + D_* w_t, \tag{38}$$

where $A_*(x) = A(x) + B(x)\big[I_m - K_* D(x)\big]^{-1} K_* C(x)$, $B_*(x) = B(x)\big[I_m - K_* D(x)\big]^{-1}$, $C_*(x) = \big[I_l - D(x)K_*\big]^{-1} C(x)$, $D_*(x) = \big[I_l - D(x)K_*\big]^{-1} D(x)$, $A_* = A_*(0)$, $B_* = B_*(0)$, $C_* = C_*(0)$, $D_* = D_*(0)$.

Theorem 4.1 *For linear system* (2)*, there exists a controller* (36) *such that* $J < \gamma$ *iff the following relations are feasible:*

$$W_R^\top \begin{bmatrix} A^\top XA - X + C^\top QC & A^\top XB + C^\top QD \\ B^\top XA + D^\top QC & B^\top XB + D^\top QD - \gamma^2 P \end{bmatrix} W_R < 0, \tag{39}$$

$$W_L^\top \begin{bmatrix} AYA^\top - Y + BP^{-1}B^\top & AYC^\top + BP^{-1}D^\top \\ CYA^\top + DP^{-1}B^\top & CYC^\top + DP^{-1}D^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{40}$$

$$0 < X < \gamma^2 X_0, \quad XY = \gamma^2 I_n, \tag{41}$$

where $R = [C, D]$, $L = [B^\top, D^\top]$. *The gain matrix* K_* *of the controller may be constructed in the form* $K_* = K_0(I_l + DK_0)^{-1}$, *where* K_0 *is a solution of the LMI*

$$L_0^\top K_0 R_0 + R_0^\top K_0^\top L_0 + \Omega < 0 \tag{42}$$

with

$$\Omega = \begin{bmatrix} -X & 0 & A^\top & C^\top \\ 0 & -\gamma^2 P & B^\top & D^\top \\ A & B & -X^{-1} & 0 \\ C & D & 0 & -Q^{-1} \end{bmatrix}, \quad R_0^\top = \begin{bmatrix} C^\top \\ D^\top \\ 0 \\ 0 \end{bmatrix}, \quad L_0^\top = \begin{bmatrix} 0 \\ 0 \\ B \\ D \end{bmatrix}.$$

LMI (42) has a solution K_0 if and only if

$$W_{L_0}^\top \Omega W_{L_0} < 0, \quad W_{R_0}^\top \Omega W_{R_0} < 0, \tag{43}$$

where W_{L_0} (W_{R_0}) is a matrix whose columns make up the bases of Ker L_0 (Ker R_0) (Projection Lemma [9]).

4.3 Dynamic controllers with perturbations

Consider control systems (1) and (2) with the dynamic output-feedback controller

$$\xi_{t+1} = Z\xi_t + Vy_t, \quad u_t = U\xi_t + Ky_t + w_t, \quad t \in \mathcal{T}, \tag{44}$$

where $\xi_0 = 0$, $w_t \in \mathbb{R}^m$ is a vector of bounded perturbations, Z, V, U and K are unknown coefficient matrices. If $K \in \mathcal{K}_D$, then linear closed loop system (2), (44) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_*\widehat{x}_t + \widehat{B}_*w_t, \quad y_t = \widehat{C}_*\widehat{x}_t + \widehat{D}_*w_t, \tag{45}$$

where $\widehat{A}_* = \widehat{A} + \widehat{B}\widehat{K}_0\widehat{C}$, $\widehat{B}_* = \widehat{B}_1 + \widehat{B}\widehat{K}_0\widehat{D}_1$, $\widehat{C}_* = \widehat{C}_1 + \widehat{D}_2\widehat{K}_0\widehat{C}$, $\widehat{D}_* = D + \widehat{D}_2\widehat{K}_0\widehat{D}_1$, $K_0 = \mathbf{D}(K)$,

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \ \widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ \widehat{B} = \begin{bmatrix} B & 0 \\ 0 & I_r \end{bmatrix}, \ \widehat{C} = \begin{bmatrix} C & 0 \\ 0 & I_r \end{bmatrix},$$

$$\widehat{B}_1 = \begin{bmatrix} B \\ 0 \end{bmatrix}, \widehat{C}_1 = \begin{bmatrix} C & 0 \end{bmatrix}, \widehat{D}_1 = \begin{bmatrix} D \\ 0 \end{bmatrix}, \widehat{D}_2 = \begin{bmatrix} D & 0 \end{bmatrix}, \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix},$$

$U_0 = (I_m - KD)^{-1}U, V_0 = V(I_l - DK)^{-1}, Z_0 = Z + VD(I_m - KD)^{-1}U.$

We give the following auxiliary statement (see also [17] in the case $\gamma = 1$).

Lemma 4.3 *Given the matrices $X > 0$, $Y > 0$ and the number $\gamma > 0$, there are matrices $X_1 \in \mathbb{R}^{r\times n}$, $X_2 \in \mathbb{R}^{r\times r}$, $Y_1 \in \mathbb{R}^{r\times n}$ and $Y_2 \in \mathbb{R}^{r\times r}$ such that*

$$\widehat{X} = \begin{bmatrix} X & X_1^\top \\ X_1 & X_2 \end{bmatrix} > 0, \ \widehat{Y} = \begin{bmatrix} Y & Y_1^\top \\ Y_1 & Y_2 \end{bmatrix} > 0, \ \widehat{X}\widehat{Y} = \gamma^2 I_{n+r}, \tag{46}$$

if and only if

$$W = \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} \geq 0, \quad \operatorname{rank} W \leq n + r. \tag{47}$$

Applying 4.3, Projection Lemma and Theorem 4.1 to system (45), we get the following result.

Theorem 4.2 *There exists a dynamic controller* (44) *such that the evaluation* $J < \gamma$ *holds for linear system* (45), *iff the LMI system* (33), (39), (40) *and* (47) *is solvable with respect to* $X = X^\top > 0$ *and* $Y = Y^\top > 0$.

Remark 4.2 The coefficient matrices of dynamic controller (44) in Theorem 4.2 may be constructed in the form (18) by solving LMI with respect to $\widehat{K}_0$:

$$\widehat{L}^\top \widehat{K}_0 \widehat{R} + \widehat{R}^\top \widehat{K}_0^\top \widehat{L} + \widehat{\Omega} < 0, \tag{48}$$

where

$$\widehat{\Omega} = \begin{bmatrix} -\widehat{X} & 0 & \widehat{A}^\top & \widehat{C}_1^\top \\ 0 & -\gamma^2 P & \widehat{B}_1^\top & D^\top \\ \widehat{A} & \widehat{B}_1 & -\widehat{X}^{-1} & 0 \\ \widehat{C}_1 & D & 0 & -Q^{-1} \end{bmatrix}, \ \widehat{R}^\top = \begin{bmatrix} \widehat{C}^\top \\ \widehat{D}_1^\top \\ 0 \\ 0 \end{bmatrix}, \ \widehat{L}^\top = \begin{bmatrix} 0 \\ 0 \\ \widehat{B} \\ \widehat{D}_2 \end{bmatrix}.$$

Here $\widehat{X}$ is a block matrix determined in Lemma 4.3 for X and Y satisfying Theorem 4.2.

If $K \in \mathcal{K}_D$, then $\det \big[I_m - KD(x)\big] \neq 0$ for all $x \in \mathcal{S}_0$, where $\mathcal{S}_0$ is some neighbourhood of $x = 0$, and nonlinear closed loop system (1), (44) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_*(\widehat{x}_t)\widehat{x}_t + \widehat{B}_*(\widehat{x}_t)w_t, \quad y_t = \widehat{C}_*(\widehat{x}_t)\widehat{x}_t + \widehat{D}_*(\widehat{x}_t)w_t, \tag{49}$$

where all coefficient matrices are continuous in $\mathcal{S}_0$. Therefore, the dynamic controller (44), (18) ensures robust stability of the zero state $\widehat{x}_t \equiv 0$ of system (49) with uncertainty (34) and a common Lyapunov function $v(\widehat{x}) = \widehat{x}^\top \widehat{X} \widehat{x}$. To evaluate characteristics J_0 and J of system (49), we can apply Lemma 4.2.

4.4 Control systems with controlled and observed outputs

Consider the linear control system

$$\begin{aligned} x_{t+1} &= Ax_t + B_1 w_t + B_2 u_t, \\ z_t &= C_1 x_t + D_{11} w_t + D_{12} u_t, \\ y_t &= C_2 x_t + D_{21} w_t + D_{22} u_t, \end{aligned} \tag{50}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $w_t \in \mathbb{R}^s$, $z_t \in \mathbb{R}^k$ and $y_t \in \mathbb{R}^l$ are the state, the control, the norm-limited external perturbations, the controlled and observed outputs, respectively, and $t \in \mathcal{T}$. We are interested in static and dynamic control laws that ensure nonexpansivity property of closed loop system and minimize the performance criteria J and J_0 with respect to controlled output z of the form (30).

4.4.1 Static controllers

If we use the static output feedback controller

$$u_t = Ky_t, \quad \det\big(I_m - KD_{22}\big) \neq 0, \quad t \in \mathcal{T}, \tag{51}$$

then closed loop system (50), (51) has the form

$$x_{t+1} = A_* x_t + B_* w_t, \quad z_t = C_* x_t + D_* w_t, \tag{52}$$

where $A_* = A + B_2K_0C_2$, $B_* = B_1 + B_2K_0D_{21}$, $C_* = C_1 + D_{12}K_0C_2$, $D_* = D_{11} + D_{12}K_0D_{21}$ and $K_0 = (I_m - KD_{22})^{-1}K$. To formulate an analog of Theorem 4.1 we construct the following LMI

$$W_R^\top \begin{bmatrix} A^\top XA - X + C_1^\top QC_1 & A^\top XB_1 + C_1^\top QD_{11} \\ B_1^\top XA + D_{11}^\top QC_1 & B_1^\top XB_1 + D_{11}^\top QD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{53}$$

$$W_L^\top \begin{bmatrix} AYA^\top - Y + B_1P^{-1}B_1^\top & AYC_1^\top + B_1P^{-1}D_{11}^\top \\ C_1YA^\top + D_{11}P^{-1}B_1^\top & C_1YC_1^\top + D_{11}P^{-1}D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{54}$$

where $R = \big[\, C_2, D_{21} \,\big]$, $L = \big[\, B_2^\top, D_{12}^\top \,\big]$.

Theorem 4.3 *For system* (50)*, there exists a controller* (51) *such that* $J < \gamma$ *iff the matrix system* (41), (53) *and* (54) *is feasible.*

If we use a static state feedback $u_t = Kx_t$, then $C_2 = I_n$, $D_{21} = 0$ and $D_{22} = 0$. In this case (41) and (53) can be written as

$$\begin{bmatrix} X_0 & I_n \\ I_n & Y \end{bmatrix} > 0, \quad \begin{bmatrix} P - \gamma^{-2} D_{11}^\top QD_{11} & B_1^\top \\ B_1 & Y \end{bmatrix} > 0. \tag{55}$$

Corollary 4.1 *For system* (50)*, there exists a state feedback controller* $u_t = Kx_t$ *such that* $J < \gamma$ *iff the LMI system* (54) *and* (55) *is solvable for some matrix* $Y = Y^\top > 0$.

Remark 4.3 The gain matrix K in Theorem 4.3 and Corollary 4.1 may be constructed as $K = K_0(I_l + D_{22}K_0)^{-1}$, where K_0 is an arbitrary solution of the LMI

$$L_0^\top K_0 R_0 + R_0^\top K_0^\top L_0 + \Omega < 0,$$

$$\Omega = \begin{bmatrix} -X & 0 & A^\top & C_1^\top \\ 0 & -\gamma^2 P & B_1^\top & D_{11}^\top \\ A & B_1 & -X^{-1} & 0 \\ C_1 & D_{11} & 0 & -Q^{-1} \end{bmatrix}, \; R_0^\top = \begin{bmatrix} C_2^\top \\ D_{21}^\top \\ 0 \\ 0 \end{bmatrix}, \; L_0^\top = \begin{bmatrix} 0 \\ 0 \\ B_2 \\ D_{12} \end{bmatrix}.$$

4.4.2 Dynamic controllers

If we use the dynamic output feedback

$$\xi_{t+1} = Z\xi_t + Vy_t, \quad u_t = U\xi_t + Ky_t, \quad t \in \mathcal{T}, \tag{56}$$

with $\xi_0 = 0$ and $\det\big(I_m - KD_{22}\big) \neq 0$, then closed loop system (50), (56) has the form

$$\widehat{x}_{t+1} = \widehat{A}_*\widehat{x}_t + \widehat{B}_*w_t, \quad z_t = \widehat{C}_*\widehat{x}_t + \widehat{D}_*w_t, \tag{57}$$

where $\widehat{A}_* = \widehat{A} + \widehat{B}_2\widehat{K}_0\widehat{C}_2$, $\widehat{B}_* = \widehat{B}_1 + \widehat{B}_2\widehat{K}_0\widehat{D}_{21}$, $\widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_0\widehat{C}_2$, $\widehat{D}_* = D_{11} + \widehat{D}_{12}\widehat{K}_0\widehat{D}_{21}$,

$$\widehat{x}_t = \begin{bmatrix} x_t \\ \xi_t \end{bmatrix}, \; \widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \; \widehat{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I_r \end{bmatrix}, \; \widehat{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & I_r \end{bmatrix},$$

$$\widehat{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \widehat{C}_1 = \big[C_1, 0\big], \quad \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix},$$

$$\widehat{D}_{12} = \big[D_{12}, 0\big], \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}.$$

Here the blocks of matrix $\widehat{K}_0$

$$K_0 = (I_m - KD_{22})^{-1}K, \quad U_0 = (I_m - KD_{22})^{-1}U,$$
$$V_0 = V(I_l - D_{22}K)^{-1}, \quad Z_0 = Z + VD_{22}(I_m - KD_{22})^{-1}U,$$

are unknown, and

$$K = (I_m + K_0 D_{22})^{-1} K_0, \quad U = (I_m + K_0 D_{22})^{-1} U_0,$$
$$V = V_0 (I_l + D_{22} K_0)^{-1}, \quad Z = Z_0 - V_0 D_{22} (I_m + K_0 D_{22})^{-1} U_0. \tag{58}$$

Applying Lemmas 4.3, Projection Lemma and Theorem 4.1 to system (57), we get the following result.

Theorem 4.4 *For linear system* (50)*, there exists a dynamic controller* (56) *such that* $J < \gamma$ *iff the matrix system* (33)*,* (47)*,* (53) *and* (54) *is feasible.*

Remark 4.4 The coefficient matrices of dynamic controller (56) in Theorem 4.4 may be constructed in the form (58) by solving the LMI

$$\widehat{L}^\top \widehat{K}_0 \widehat{R} + \widehat{R}^\top \widehat{K}_0^\top \widehat{L} + \widehat{\Omega} < 0, \tag{59}$$

where

$$\widehat{\Omega} = \begin{bmatrix} -\widehat{X} & 0 & \widehat{A}^\top & \widehat{C}_1^\top \\ 0 & -\gamma^2 P & \widehat{B}_1^\top & D_{11}^\top \\ \widehat{A} & \widehat{B}_1 & -\widehat{X}^{-1} & 0 \\ \widehat{C}_1 & D_{11} & 0 & -Q^{-1} \end{bmatrix}, \ \widehat{R}^\top = \begin{bmatrix} \widehat{C}_2^\top \\ \widehat{D}_{21}^\top \\ 0 \\ 0 \end{bmatrix}, \ \widehat{L}^\top = \begin{bmatrix} 0 \\ 0 \\ \widehat{B}_2 \\ \widehat{D}_{12} \end{bmatrix}.$$

Here $\widehat{X}$ is a block matrix determined in Lemma 4.3 for X and Y satisfying Theorem 4.4.

We give the following algorithm for constructing stabilizing dynamic controller (56) satisfying Theorem 4.4.

Algorithm 4.1 1) calculate the matrices W_R and W_L, where $R = \left[\, C_2, D_{21} \,\right]$ and $L = \left[\, B_2^\top, D_{12}^\top \,\right]$;

2) find the matrices $X = X^\top > 0$ and $Y = Y^\top > 0$ satisfying (33), (47), (53) and (54);

3) construct decomposition $Z = Y - \gamma^2 X^{-1} = S^\top S$, $S \in \mathbb{R}^{r \times n}$, $\ker S = \ker Z$ and form the block matrix

$$\widehat{X} = \begin{bmatrix} X & X_1^\top \\ X_1 & X_2 \end{bmatrix} > 0, \quad X_1 = \frac{1}{\gamma} S X, \quad X_2 = \frac{1}{\gamma^2} S X S^\top + I_r;$$

4) solve the LMI (59) under restriction $\det(I_m + K_0 D_{22}) \neq 0$;

5) calculate the coefficient matrices of dynamic controller (56) by formula (58).

Static and dynamic output-feedback controllers (51) and (56) may be applied to a class of affine systems

$$\begin{aligned} x_{t+1} &= A(x_t)x_t + B_1(x_t)w_t + B_2(x_t)u_t, \\ z_t &= C_1(x_t)x_t + D_{11}(x_t)w_t + D_{12}(x_t)u_t, \\ y_t &= C_2(x_t)x_t + D_{21}(x_t)w_t + D_{22}(x_t)u_t. \end{aligned} \tag{60}$$

So, closed loop system (56), (60) reduces to the form

$$\widehat{x}_{t+1} = \widehat{A}_*(\widehat{x}_t)\widehat{x}_t + \widehat{B}_*(\widehat{x}_t)w_t, \quad z_t = \widehat{C}_*(\widehat{x}_t)\widehat{x}_t + \widehat{D}_*(\widehat{x}_t)w_t. \tag{61}$$

To evaluate characteristics J_0 and J of system (61), we can apply Lemma 4.2.

Remark 4.5 Note that we have necessary and sufficient conditions for an evaluation $J_0 < \gamma$ represented by the corresponding statements of Theorems 4.1 – 4.4 without using additional restriction $X < \gamma^2 X_0$. With the use of static state feedback or full order dynamic controllers the problems under consideration are reduced to solving LMI systems. We can formulate analogs of Theorems 4.1 – 4.4 for the corresponding control systems with a polyhedral uncertainties of the matrices A, B_1, C_1 and D_{11}. In addition, sufficient statements of these theorems may be generalized for the corresponding affine control systems with continuous coefficient matrices (see Lemma 4.2).

4.5 H_∞-Control problem for descriptor systems

Consider a linear discrete-time *descriptor system* with bounded perturbations

$$Ex_{t+1} = Ax_t + Bw_t, \quad z_t = Cx_t + Dw_t, \quad t \in \mathcal{T} = \{0, 1, \dots\}, \tag{62}$$

where $x_t \in \mathbb{R}^n$, $w_t \in \mathbb{R}^m$, $z_t \in \mathbb{R}^l$ and $\operatorname{rank} E = \rho \leq n$.

Definition 4.2 A matrix pair (E, A) is said to be *admissible* if it is *regular*, *causal* and *stable*, i.e. $\det F(\lambda) \not\equiv 0$, $\deg F(\lambda) = \rho$ and $\sigma(F) \subset \{\lambda \in \mathbb{C} : \|\lambda\| < 1\}$, respectively, where $F(\lambda) = A - \lambda E$. Descriptor system (62) with admissible pair (E, A) is *admissible*.

Lemma 4.4 [18] *System* (62) *is admissible, if and only if there exists matrix* $X = X^\top$ *such that* $A^\top XA - E^\top XE < 0$ *and* $E^\top XE \geq 0$.

We introduce an analog of the performance (30) for system (62):

$$J = \sup_{(w,x_0)\in\mathcal{W}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \tag{63}$$

where $P > 0$, $Q > 0$ and $X_0 \geq 0$ are weight matrices, $\mathcal{W}$ is a set of pairs (w, x_0) such that system (62) has a solution and $0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty$. To formulate the following analog of the Bounded Real Lemma for system (62), we suppose that $X_0 = E^\top HE \geq 0$, where $H = H^\top > 0$.

Lemma 4.5 [19] *Given* $\gamma > 0$, *the descriptor system* (62) *is admissible and satisfies* $J < \gamma$ *if and only if there exists matrix* $X = X^\top$ *such that*

$$0 \leq E^\top XE \leq \gamma^2 X_0, \quad \operatorname{rank}(E^\top XE - \gamma^2 X_0) = \rho, \tag{64}$$

$$\begin{bmatrix} A^\top XA - E^\top XE + C^\top QC & A^\top XB + C^\top QD \\ B^\top XA + D^\top QC & B^\top XB + D^\top QD - \gamma^2 P \end{bmatrix} < 0. \tag{65}$$

Consider the descriptor control system

$$\begin{aligned} Ex_{t+1} &= Ax_t + B_1 w_t + B_2 u_t, \\ z_t &= C_1 x_t + D_{11} w_t + D_{12} u_t, \\ y_t &= C_2 x_t + D_{21} w_t + D_{22} u_, \end{aligned} \tag{66}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$, $w_t \in \mathbb{R}^s$, $z_t \in \mathbb{R}^k$ and $y_t \in \mathbb{R}^l$. Using the static output feedback controller (51) a closed loop system has the form (see (52))

$$Ex_{t+1} = A_* x_t + B_* w_t, \quad z_t = C_* x_t + D_* w_t. \tag{67}$$

We represent the matrix inequality (65) for system (67) in the form

$$W + U^\top K_0 V + V^\top K_0^\top U + V^\top K_0^\top R K_0 V < 0, \tag{68}$$

where $K_0 = (I_m - KD_{22})^{-1}K, \quad R(X) = B_2^\top X B_2 + D_{12}^\top Q D_{12},$

$$W(X) = \begin{bmatrix} A^\top XA - E^\top XE + C_1^\top QC_1 & A^\top XB_1 + C_1^\top QD_{11} \\ B_1^\top XA + D_{11}^\top QC_1 & B_1^\top XB_1 + D_{11}^\top QD_{11} - \gamma^2 P \end{bmatrix},$$

$$U(X) = \big[B_2^\top XA + D_{12}^\top QC_1,\ B_2^\top XB_1 + D_{12}^\top QD_{11}\big],\ V = \big[C_2,\ D_{21}\big].$$

Lemma 4.6 *Quadratic matrix inequality* (68) *has a solution* K_0 *if and only if* $W_V^\top W W_V < 0$ *and one of the following conditions holds:*
(a) $R = 0,\ W_U^\top W W_U < 0$;
(b) $R > 0,\ W < U^\top R^{-1} U$;
(c) $R \geq 0,\ \operatorname{rank} R < m,\ W_{U_0}^\top\big(W - U^\top R^+ U\big)W_{U_0} < 0,\ U_0 = W_R^\top U.$

Based on Lemmas 3.1 and 4.6, we can state the following results.

Theorem 4.5 *Let there exists a matrix* $X = X^\top$ *that satisfies* (64) *and*

$$R(X) > 0, \quad W(X) < U^\top(X) R^{-1}(X) U(X), \quad W_V^\top W W_V < 0. \tag{69}$$

Then there exists a static output feedback controller (51) *provided the admissibility and evaluation* $J < \gamma$ *for system* (67). *The coefficient matrix of the controller can be defined as* $K = K_0(I_l + D_{22}K_0)^{-1}$, *where* K_0 *is a solution of* (68).

Theorem 4.6 *If there exist matrices* $X = X^\top$, $P_0 = P_0^\top > 0$ *and* $Q_0 = Q_0^\top > 0$, *that satisfy the LMI*

$$\Omega(X, P_0, Q_0) = \begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 \\ \Omega_2^\top & \Omega_4 & \Omega_5 \\ \Omega_3^\top & \Omega_5^\top & \Omega_6 \end{bmatrix} < 0, \quad E^\top XE \geq 0, \tag{70}$$

where

$$\Omega_1 = A^\top XA - E^\top XE + C_1^\top QC_1 + C_2^\top Q_0 C_2,$$
$$\Omega_2 = A^\top XB_1 + C_1^\top QD_{11} + C_2^\top Q_0 D_{21},$$

$$\Omega_3 = A^\top X B_2 + C_1^\top Q D_{12} + C_2^\top Q_0 D_{22},$$

$$\Omega_4 = B_1^\top X B_1 + D_{11}^\top Q D_{11} + D_{21}^\top Q_0 D_{21} - \gamma^2 P,$$

$$\Omega_5 = B_1^\top X B_2 + D_{11}^\top Q D_{12} + D_{21}^\top Q_0 D_{22},$$

$$\Omega_6 = B_2^\top X B_2 + D_{12}^\top Q D_{12} + D_{22}^\top Q_0 D_{22} - P_0,$$

then any controller (51) *with* $K \in \mathcal{K}_0 = \{K : K^\top P_0 K \le Q_0\}$ *provides the admissibility and evaluation* $J_0 < \gamma$ *for system* (67)*. In addition,* $J < \gamma$ *if* (64) *and* (70) *are satisfied.*

References

[1] Zhou, K., Doyle, J. C. and Glover, K. *Robust and Optimal Control.* Englewood: Prentice Hall, 1996.

[2] Dullerud, G. E. and Paganini, F. G. *A Course in Robust Control Theory. A Convex Approach.* Berlin: Springer-Verlag, 2000.

[3] Sanchez-Pena, R. S. and Sznaier, M. *Robust Systems Theory and Applications.* Wiley, New York, 1998.

[4] Polyak, B. T. and Scherbakov, P. S. *Robust Stability and Control.* Moskow: Nauka, 2002. [Russian]

[5] Belov, A. A., Andrianova, O. G. and Kurdyukov, A. P. *Control of Discrete-Time Descriptor Systems. An Anisotropy-Based Approach.* Springer, 2018.

[6] Feng, Yu. and Yagoubi, M. *Robust Control of Linear Descriptor Systems. Studies in Systems, Decision and Control*, Vol. 102. Springer Nature Singapore Pte Ltd.

[7] Boyd, S., Ghaoui, L. El, Feron, E. and Balakrishman, V. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, **15**. Philadelphia: PA, 1994.

[8] Gahinet, P., Nemirovski, A., Laub, A. J. and Chilali, M. *The LMI Control Toolbox. For Use with Matlab. User's Guide.* Natick, MA: The MathWorks, Inc., 1995.

[9] Gahinet, P. and Apkarian, P. A Linear Matrix Inequality Approach to H_∞ Control. *Intern. J. of Robust and Nonlinear Control* **4** (1994) 421–448.

[10] Mazko, A. G. Robust Output Feedback Stabilization and Optimization of Discrete-Time Control Systems. *Nonlinear Dynamics and Systems Theory* **18** (1) (2018) 92–106.

[11] Mazko, A. G. *Robust Stability and Stabilization of Dynamic Systems. Methods of Matrix and Cone Inequalities.* Proceedings of the Institute of Mathematics of NAS of Ukraine, Vol. 102. Kyiv, 2016. [Russian]

[12] Mazko, A. G. *Matrix Equations, Spectral Problems and Stability of Dynamic Systems.* "Stability, Oscillations and Optimization of Systems", Vol. 2 (Eds A. A. Martynyuk, P. Borne and C. Cruz-Hernandez). Cambridge: Cambridge Sci. Publ., 2008.

[13] Balandin, D. V. and Kogan, M. M. *Synthesis of Control Laws Based on Linear Matrix Inequalities.* Moscow: Fizmatlit, 2007.

[14] Mazko, A. G. Robust Stability and Evaluation of the Quality Functional for Nonlinear Control Systems. *Automation and Remote Control* **76** (2) (2015) 251–263.

[15] Petersen, I. A Stabilization Algorithm for a Class of Uncertain Linear Systems. *Syst. Control Lett.* **8** (3) (1987) 351–357.

[16] Balandin, D. V., Kogan, M. M., L. N. Krivdina and A. A. Fedyukov. Design of Generalized Discrete-time H_∞-Optimal Control over Finite and Infinite Intervals // *Automation and Remote Control* **75** (1) (2014) 1–17.

[17] Balandin, D. V. and Kogan, M. M. Generalized H_∞-Optimal Control as a Trade-off Between the H_∞-Optimal and γ-Optimal Controls. *Automation and Remote Control* **71** (6) (2010) 993–1010.

[18] Hsiung, K.-L. and Lee, L. Lyapunov inequality and bounded real lemma for discrete-time descriptor systems. *IEE Proc.– Control Theory Appl.* **146** (4) (1999) 327–331.

[19] Mazko, A. G. Weighted evaluation for the damping level of bounded disturbances in discrete-time descriptor systems. In: *Zb. prac. Inst. mat. NAN Ukr.* **16** (2) (2019) 85–100. [Ukrainian]