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Robust Stability and Evaluation of the Quality Functional for Nonlinear Control Systems

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Abstract—We develop new methods of robust stability analysis for equilibrium states and optimization of nonlinear feedback control systems. For a family of nonlinear systems with uncertain matrices of coefficients and measurable output feedback we formulate sufficient stability conditions for the zero state with a general quadratic Lyapunov function. We propose a solution for the general robust stabilization and estimation problem for a quadratic performance index for a family of nonlinear systems. We show an example of a stabilization system for a single-link robot manipulator.

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1. INTRODUCTION

In applied problems of analysis and synthesis of real objects, one often uses systems of differential and difference equations with uncertain parameters and functional structure (see, e.g., [1–3]). For instance, the zero solution $x \equiv 0$ of a system of differential equations with parametric uncertainty

$$\dot{x} = f(x, p, t), \quad f(0, p, t) \equiv 0, \quad x \in \mathbb{R}^n, \quad p \in \mathcal{P}, \quad t \geqslant 0, \tag{1.1}$$

is called *robustly stable* with respect to a given set of parameters $\mathcal{P} \subseteq \mathbb{R}^\nu$ if it is Lyapunov stable for every fixed $p \in \mathcal{P}$. Intervals, polytopes, affine families of matrices and other objects may serve as the parametric uncertainty set $\mathcal{P}$ for system (1.1). In defining uncertainties and robust stability conditions for systems in semiordered spaces one can use cone inequalities and intervals [4, 5]. Numerous works find sufficient stability conditions for linear controllable systems with uncertain matrices of coefficients and feedback with respect to measurable output in terms of linear matrix inequalities (LMI). A survey of problems and known methods of robust stability analysis and stabilization of feedback control systems can be found in [6, 7].

This work is devoted to developing new methods of robust stability analysis for equilibrium states and optimization for a class of nonlinear multidimensional control systems with output feedback. We assume that the measurable output vector contains components of both the system state and the control. The considered nonlinear systems are called pseudolinear due to their vector–matrix representation. One can reduce to a vector–matrix form, for instance, nonlinear motion equations for certain robotic and pendulum systems, flying vehicles etc.

Using the results of [3, 8], we formulate sufficient stability conditions for the zero state of a family of control systems with uncertain matrices of coefficients and static measurable output feedback. We find the general Lyapunov function and an estimate for the quadratic quality functional. As a result, we propose new ways to optimize the considered family of systems. Application of our results reduces to solving systems of differential or algebraic LMI. To solve LMI with constant matrices, one can use a rather efficient procedure in the Matlab suite.

2. NOTATION AND AUXILIARY STATEMENTS

We use the following notation: I_n is the unit $n \times n$ matrix; $X = X^{\mathrm{T}} > 0$ ($\geqslant 0$) is a positive (nonnegative) definite symmetric matrix X; $i(X) = \{i_+(X), i_-(X), i_0(X)\}$ is the inertia of matrix $X = X^{\mathrm{T}}$ composed of the numbers of its positive, negative, and zero eigenvalues with multiplicities; $\lambda_{\max}(X)$ ($\lambda_{\min}(X)$) is the maximal (minimal) eigenvalue of matrix X; $\sigma(A)$ ($\rho(A)$) is the spectrum (spectral radius) of matrix A, $\|x\|$ is the Euclidean norm of vector x; $\mathrm{Co}\{A_1, \ldots, A_\nu\} = \left\{A = \sum_{i=1}^{\nu} \alpha_i A_i : \ \alpha_i \geqslant 0, \ i = \overline{1, \nu}, \ \sum_{i=1}^{\nu} \alpha_i = 1\right\}$ is the convex polyhedron (polytope) with vertices $A_1, \ldots, A_\nu$ in the space of matrices.

Consider a linear control system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad u = Ky, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^l$ are respectively the state, control, and observable object output vectors, A, B, C, and D are constant matrices of corresponding sizes $n \times n$, $n \times m$, $l \times n$, and $l \times m$, and, moreover, $\operatorname{rank} B = m$ and $\operatorname{rank} C = l$. Control system diagram is shown on Fig. 1. Its characteristic feature is that it can use measurements of linear combinations of both the system state vector and the control.

We introduce on the set of matrices $\mathcal{K}_D = \{K : \det(I_m - KD) \neq 0\}$ a nonlinear operator

$$\mathcal{D} : \mathbb{R}^{m \times l} \to \mathbb{R}^{m \times l}, \quad \mathcal{D}(K) = (I_m - KD)^{-1} K.$$

For each feedback matrix $K_* \in \mathcal{K}_D$ the closed-loop control system (2.1) has the form

$$\dot{x} = M_* x, \quad M_* = A + B\mathcal{D}(K_*)C. \tag{2.2}$$

We list the properties of the operator $\mathcal{D}$ without proof:

1) if $K \in \mathcal{K}_D$ then

$$\mathcal{D}(K) \equiv K[I_l + D\mathcal{D}(K)] \equiv K(I_l - DK)^{-1}, \quad I_l + D\mathcal{D}(K) \equiv (I_l - DK)^{-1}; \tag{2.3}$$

2) if $K_1 \in \mathcal{K}_D$ and $K_3 = (I_m - K_1 D)^{-1} K_2 \in \mathcal{K}_D$ then

$$K_1 + K_2 \in \mathcal{K}_D, \quad \mathcal{D}(K_1 + K_2) \equiv \mathcal{D}(K_1) + \mathcal{D}(K_3)\,[I_l + D\mathcal{D}(K_1)]; \tag{2.4}$$

3) if $K \in \mathcal{K}_D$ then

$$K_* = -\mathcal{D}(K) \in \mathcal{K}_D, \quad \mathcal{D}(K_*) = -K. \tag{2.5}$$

According to (2.5), to achieve the desired properties and, in particular, to stabilize system (2.2) it suffices to provide a system with matrix $M_* = A - BKC$ with these properties.

For matrices B and C, that have full rank with respect to columns and rows respectively, we introduce orthogonal complements and pseudoinverse matrices: $B^{\mathrm{T}} B^{\perp} = 0$, $\det\left[B, B^{\perp}\right] \neq 0$, $C^{\perp} C^{\mathrm{T}} = 0$, $\det\left[C^{\mathrm{T}}, C^{\perp\mathrm{T}}\right] \neq 0$, $B^+ = (B^{\mathrm{T}} B)^{-1} B^{\mathrm{T}}$, $C^+ = C^{\mathrm{T}}(CC^{\mathrm{T}})^{-1}$. The following statement, which is proven in the Appendix, shows a way to place the spectrum of matrix $M_* = A - BKC$ with certain properties with respect to the straight line $\operatorname{Re} \lambda = \alpha$, $\lambda \in \mathbb{C}^1$.

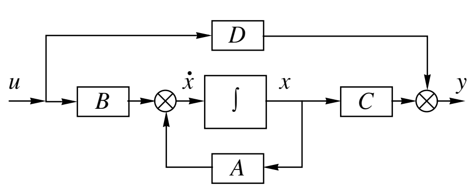

Fig. 1. Control system diagram.

Lemma 1. *There exists a matrix K for which the spectrum $\sigma(M_*)$ consists of p and q points in the corresponding half-planes $\operatorname{Re}\lambda < \alpha$ and $\operatorname{Re}\lambda > \alpha$ if and only if the following system of relations is feasible with respect to $X = X^{\mathrm{T}}$:*

$$S = B^{\perp\mathrm{T}} L B^{\perp} < 0, \quad i(X) = \{p, q, 0\}, \quad i(H) = \{l, m, 0\}, \quad H = \begin{bmatrix} H_0 & H_1^{\mathrm{T}} \\ H_1 & H_2 \end{bmatrix}, \tag{2.6}$$

where $L = AX + XA^{\mathrm{T}} - 2\alpha X$, $H_0 = B^{+}(L - LRL)B^{+\mathrm{T}}$, $H_1 = CX(I_n - RL)B^{+\mathrm{T}}$, $H_2 = -CXRXC^{\mathrm{T}}$, $R = B^{\perp}S^{-1}B^{\perp\mathrm{T}}$. *Under conditions* (2.6) *matrix K can be found by solving one of the following equivalent matrix inequalities*

$$Y_1 = H_0 - KH_1 - H_1^{\mathrm{T}}K^{\mathrm{T}} + KH_2K^{\mathrm{T}} < 0, \tag{2.7}$$

$$Y = L - BKCX - XC^{\mathrm{T}}K^{\mathrm{T}}B^{\mathrm{T}} < 0. \tag{2.8}$$

In particular, if in relations (2.6), (2.7) $X = X^{\mathrm{T}} > 0$ *and* $\alpha \leqslant 0$ *then real parts of all points of the spectrum* $\sigma(M_*)$ *are negative.*

Note that inequality (2.8), which is linear in K, holds if

$$K = B^{\mathrm{T}}X^{-1}C^{+}, \quad AX + XA^{\mathrm{T}} - 2\alpha X < 2\,BB^{\mathrm{T}}, \quad (C^{+}C - I_n)X^{-1}B = 0, \tag{2.9}$$

and in order for inequality (2.7), which is quadratic in K, to hold it suffices that

$$K = \gamma\, B^{\mathrm{T}}X^{-1}C^{+}, \quad \gamma > \lambda_{\max}(H_0)/2, \quad C^{\perp}X^{-1}B = 0, \tag{2.10}$$

and the latter equalities (2.9) and (2.10) are equivalent. Due to (2.5), (2.6), and (2.10), for $\alpha \leqslant 0$ we have sufficient conditions that guarantee asymptotic stability for system (2.2):

$$\begin{gathered} X = X^{\mathrm{T}} > 0, \quad B^{\perp\mathrm{T}}(AX + XA^{\mathrm{T}} - 2\alpha X)\,B^{\perp} < 0, \quad C^{\perp}X^{-1}B = 0, \\ K_* = -\mathcal{D}(K), \quad K = \gamma\, B^{\mathrm{T}}X^{-1}C^{+} \in \mathcal{K}_D, \quad \gamma > \lambda_{\max}(H_0)/2. \end{gathered}$$

We formulate an auxiliary statement that will be used in the proofs of our main results. Consider a nonlinear operator

$$\mathcal{F}(K) = W + U^{\mathrm{T}}\mathcal{D}(K)V + V^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K)U + V^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K)\,R\,\mathcal{D}(K)V$$

and an ellipsoidal set of matrices

$$\mathcal{K} = \left\{K \in \mathbb{R}^{m\times l} :\, K^{\mathrm{T}}PK \leqslant Q\right\}, \tag{2.11}$$

where $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $R = R^{\mathrm{T}} \geqslant 0$, $W = W^{\mathrm{T}} \leqslant 0$, U, V, and D are matrices of suitable sizes. Due to the equivalence of matrix inequalities [9]

$$K^{\mathrm{T}}PK \leqslant Q, \quad \begin{bmatrix} P^{-1} & K \\ K^{\mathrm{T}} & Q \end{bmatrix} \geqslant 0, \quad KQ^{-1}K^{\mathrm{T}} \leqslant P^{-1},$$

the set (2.11) can also be described as $\mathcal{K} = \{K \in \mathbb{R}^{m\times l} :\, KQ^{-1}K^{\mathrm{T}} \leqslant P^{-1}\}$. Here in case $m = 1$ the ellipsoid $\mathcal{K}$ is described with a scalar inequality.

Lemma 2. *Suppose that the following matrix inequalities hold:*

$$D^{\mathrm{T}}QD + R < P, \quad \Omega = \begin{bmatrix} W & U^{\mathrm{T}} & V^{\mathrm{T}} \\ U & R - P & D^{\mathrm{T}} \\ V & D & -Q^{-1} \end{bmatrix} \leqslant 0 \ (< 0). \tag{2.12}$$

Then $\mathcal{F}(K) \leqslant 0$ (< 0) *for every matrix* $K \in \mathcal{K}$.

Proof of Lemma 2 is given in the Appendix.

Note that Lemma 2 is a generalization of the sufficiency statement for an existing criterion known as the Petersen's lemma on matrix uncertainty [10] (see also [11]). According to [10], for every matrix $K \in \mathbb{R}^{m\times l}$ with bounded norm $\|K\| = (\lambda_{\max}(K^{\mathrm{T}}K))^{1/2} \leqslant 1$ the matrix inequality $\mathcal{F}(K) = W + U^{\mathrm{T}}KV + V^{\mathrm{T}}K^{\mathrm{T}}U < 0$ holds if and only if there exists $\varepsilon > 0$ such that $W + \varepsilon^{-1}U^{\mathrm{T}}U + \varepsilon V^{\mathrm{T}}V < 0$. The latter relation can be represented in block form as

$$\Omega = \begin{bmatrix} W & U^{\mathrm{T}} & V^{\mathrm{T}} \\ U & -\varepsilon I_m & 0 \\ V & 0 & -\varepsilon^{-1} I_l \end{bmatrix} < 0,$$

while requirement $\|K\| \leqslant 1$ holds if $K^{\mathrm{T}}K \leqslant I_l$. Letting in Lemma 2 $D = 0$, $R = 0$, $P = \varepsilon I_m$, and $Q = \varepsilon I_l$, where $\varepsilon > 0$ is a certain number, we have the sufficiency statement of Petersen's lemma.

3. ROBUST STABILIZATION OF NONLINEAR CONTROL SYSTEMS

Consider a nonlinear control system in vector-matrix form

$$\dot{x} = A(x,t)\,x + B(x,t)\,u, \quad y = C(x,t)\,x + D(x,t)\,u, \quad t \geqslant 0, \tag{3.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^l$ are state, control, and observable object output vectors respectively. We control the system with output feedback:

$$u = K(x,t)\,y, \quad K(x,t) = K_*(x,t) + \widetilde{K}(x,t), \quad \widetilde{K}(x,t) \in \mathcal{K}, \tag{3.2}$$

where $\mathcal{K}$ is an ellipsoidal set of matrices of the form (2.11) in the space $\mathbb{R}^{m\times l}$ defined by symmetric positive definite matrices P and Q. We assume that the matrices in question A, B, C, D, K_*, and K depend on x and t continuously and will omit it for brevity. We assume matrices P and Q to be constant, although in what follows they may also be functions of x and t.

According to (2.11), (3.1), and (3.2), the following inequality must hold:

$$[x^{\mathrm{T}}, u^{\mathrm{T}}] \begin{bmatrix} C^{\mathrm{T}}QC - C^{\mathrm{T}}K_*^{\mathrm{T}}PK_*C & C^{\mathrm{T}}QD + C^{\mathrm{T}}K_*^{\mathrm{T}}PG \\ D^{\mathrm{T}}QC + G^{\mathrm{T}}PK_*C & \Delta \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \geqslant 0,$$

where $\Delta = D^{\mathrm{T}}QD - G^{\mathrm{T}}PG$, $G = I_m - K_*D$. We assume that

$$\Delta(x,t) < 0, \quad x \in \mathcal{S}_0, \ t \geqslant 0, \tag{3.3}$$

where $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \leqslant h\}$ is a neighborhood of the point $x = 0$. Then $x = 0$ implies $u = 0$, and $x \equiv 0$ is an equilibrium state for the system. In what follows we assume that this equilibrium state is isolated, i.e., the neighborhood $\mathcal{S}_0$ does not contain other equilibrium states of this system.

The problem is to construct conditions under which the zero state of the closed-loop control system (3.1) and (3.2) is Lyapunov asymptotically stable for every matrix $\widetilde{K} \in \mathcal{K}$. Matrix K_* is chosen for the purposes of stabilization, e.g., in case when the zero state of the system without control ($u = 0$) is unstable. When looking for the stabilizing matrix K_* in the class of linear autonomous systems (2.1), one can use Lemma 1 and its special cases (see also [3, 6, 7]).

Under assumption (3.3) matrix G must be nondegenerate. Therefore for every $x \in \mathcal{S}_0$ and $t \geqslant 0$ values of the operator $\mathcal{D}(K_*) = (I_m - K_*D)^{-1}K_*$ are defined. If $\widetilde{K} \in \mathcal{K}$ then values of $\mathcal{D}(K)$ and $\mathcal{D}(\widehat{K})$ are also defined, where $\widehat{K} = G^{-1}\widetilde{K}$. Indeed, under conditions (3.2) and (3.3) we have

$$D^{\mathrm{T}}\widetilde{K}^{\mathrm{T}}P\widetilde{K}D \leqslant D^{\mathrm{T}}QD < G^{\mathrm{T}}PG, \quad F^{\mathrm{T}}PF < P,$$

where $F = \widetilde{K}DG^{-1}$ and $P > 0$. Therefore $\rho(F) < 1$, and matrix $I_m - F$ is nondegenerate, and hence matrices $I_m - KD = (I_m - F)G$ and $I_m - \widehat{K}D = G^{-1}(I_m - KD)$ are nondegenerate as well.

So, the closed-loop system (3.1), (3.2) under constraint (3.3) can be represented as

$$\dot{x} = M(x,t)x, \quad M(x,t) = A + B\mathcal{D}(K)C. \tag{3.4}$$

We assume that the zero state of this system for $K \equiv K_*$ is asymptotically stable. We find a solution for our problem for system (3.4) with a quadratic Lyapunov function $v(x,t) = x^{\mathrm{T}}X(t)x$, where $X(t)$ is a continuously differentiable symmetric matrix that satisfies conditions

$$\varepsilon_1 I_n \leqslant X(t) \leqslant \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \leqslant \varepsilon_2, \quad t \geqslant 0. \tag{3.5}$$

Theorem 1. *Suppose that for some $\varepsilon_i > 0$ $(i = 0,1,2)$ and for $x = 0$ the following matrix inequalities hold:* (3.3), (3.5), *and*

$$\Omega(t) = \begin{bmatrix} \dot{X} + M_*^{\mathrm{T}}X + XM_* + \varepsilon_0 I_n & XB & C_*^{\mathrm{T}} \\ B^{\mathrm{T}}X & -G^{\mathrm{T}}PG & D^{\mathrm{T}} \\ C_* & D & -Q^{-1} \end{bmatrix} \leqslant 0, \quad t \geqslant 0, \tag{3.6}$$

where $M_* = A + B\mathcal{D}(K_*)C$, $C_* = C + D\mathcal{D}(K_*)C$. Then any control (3.2) *ensures asymptotic stability of the zero state for system* (3.1) *and the general Lyapunov function $v(x,t) = x^{\mathrm{T}}X(t)\,x$.*

Proof of Theorem 1 is given in the Appendix.

Note that in [3], based on the so-called non-inferiority property for the S-procedure, the authors obtain a similar statement with constant matrix X in case $P = I_m$ and $Q = \mu I_l$, where μ is the stability radius for feedback matrices K for the linear autonomous system (2.1). Note that (3.3) follows from the strict inequality (3.6), while matrices P and $Q_1 = Q^{-1}$ occur in the expression (3.6) linearly. Therefore, together with X they can be treated as unknowns and found with an efficient procedure implemented in the Matlab suite. This extends the capabilities of the quadratic stabilization method [3] even to the class of systems (2.1).

We assume that system (3.1) for $x = 0$ has unknown coefficients:

$$A \in \mathrm{Co}\{A_1,\ldots,A_{\nu_a}\}, \quad B \in \mathrm{Co}\{B_1,\ldots,B_{\nu_b}\}, \quad C \in \mathrm{Co}\{C_1,\ldots,C_{\nu_c}\}, \ \ x = 0, \quad t \geqslant 0, \tag{3.7}$$

where given tuples of constant matrices A_i, B_j and C_k are vertices of certain polytopes in the corresponding spaces $\mathbb{R}^{n\times n}$, $\mathbb{R}^{n\times m}$ and $\mathbb{R}^{l\times n}$. Then matrix inequality (3.6), due to a linear dependence of the block expression Ω on these coefficients, follows from a system of similar inequalities

$$\begin{bmatrix} \dot{X} + M_{ijk}^{\mathrm{T}}X + XM_{ijk} + \varepsilon_0 I_n & XB_j & C_k^{\mathrm{T}} + C_k^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K_*)D^{\mathrm{T}} \\ B_j^{\mathrm{T}}X & -G^{\mathrm{T}}PG & D^{\mathrm{T}} \\ C_k + D\mathcal{D}(K_*)C_k & D & -Q^{-1} \end{bmatrix} \leqslant 0, \tag{3.8}$$

where $M_{ijk} = A_i + B_j\mathcal{D}(K_*)C_k$, $i = \overline{1,\nu_a}$, $j = \overline{1,\nu_b}$, $k = \overline{1,\nu_c}$, $x = 0$, $t \geqslant 0$. Indeed, due to (3.7), after multiplying matrix inequalities (3.8) by unknown parameters of convex linear combinations of vertices of polytopes A_i, B_j and C_k and summing them up respectively over i, j, and k we get matrix inequality (3.6). Consequently, the statement of Theorem 1 holds for the family of systems (3.1) and (3.7) if instead of (3.6) we use the system of matrix inequalities (3.8). Here strict inequalities (3.8) ensure that condition (3.3) of Theorem 1 holds.

Suppose that together with (3.7) it holds that

$$K_* \equiv 0, \quad D \in \mathrm{Co}\{D_1,\ldots,D_{\nu_d}\}, \quad x = 0, \ \ t \geqslant 0. \tag{3.9}$$

Then $\mathcal{D}(K_*) = 0$, $M_{ijk} = A_i$, and $G = I_m$ in (3.8). If, in addition, matrix $X > 0$ is constant then we can let $\varepsilon_i = 0$ $(i = 0,1,2)$. Thus, under stronger assumptions than Theorem 1 the following statement holds.

Corollary 1. *Suppose that the system of LMI with constant matrices*

$$X > 0, \quad \begin{bmatrix} A_i^{\mathrm{T}} X + X A_i & X B_j & C_k^{\mathrm{T}} \\ B_j^{\mathrm{T}} X & -P & D_s^{\mathrm{T}} \\ C_k & D_s & -Q^{-1} \end{bmatrix} < 0, \quad i = \overline{1, \nu_a}, \quad j = \overline{1, \nu_b}, \quad k = \overline{1, \nu_c}, \quad s = \overline{1, \nu_d} \tag{3.10}$$

is feasible. Then any control (3.2) *provides asymptotic stability for the zero state in the family of systems* (3.1), (3.7), (3.9), *and the general quadratic Lyapunov function* $v(x) = x^{\mathrm{T}} X x$.

Remark. Systems of matrix inequalities (3.8) and (3.10) can be used to solve inverse robust stabilization problems. For instance, for a given matrix $X > 0$ under the assumptions of Corollary 1 one can construct a family of stabilization systems defined by certain polytopes of matrix coefficients (3.7) and (3.9) and by the ellipsoid of feedback matrices (2.11). In that problem, vertices of polytopes A_i, B_j, C_k, and D_s will serve as unknowns together with positive definite matrices P and Q that define the ellipsoid in question.

4. BOUNDS ON THE QUADRATIC QUALITY CRITERION FOR A FAMILY OF SYSTEMS

Consider a control system (3.1) with quadratic quality functional

$$J(u, x_0) = \int_0^\infty \varphi(x, u, t)\, dt, \quad \varphi(x, u, t) = \left[x^{\mathrm{T}}, u^{\mathrm{T}}\right] \Phi(t) \begin{bmatrix} x \\ u \end{bmatrix}, \qquad \Phi(t) = \begin{bmatrix} S & N \\ N^{\mathrm{T}} & R \end{bmatrix}, \tag{4.1}$$

where $x_0 = x(0)$, and the blocks of symmetric matrix $\Phi(t)$ for some $\delta > 0$ satisfy conditions

$$S \geqslant N R^{-1} N^{\mathrm{T}} + \delta\, I_n, \quad R > 0, \quad t \geqslant 0. \tag{4.2}$$

We need to describe the set of controls (3.2) that would provide asymptotic stability for the state $x \equiv 0$ of system (3.1) and a bound

$$J(u, x_0) \leqslant \omega, \tag{4.3}$$

where ω is some maximal admissible value of the functional. When solving this problem, we still use the Lyapunov function $v(x, t) = x^{\mathrm{T}} X(t) x$ with a continuously differentiable matrix $X(t)$ that satisfies

$$x_0^{\mathrm{T}} X(0)\, x_0 \leqslant \omega, \quad \varepsilon_1 I_n \leqslant X(t) \leqslant \varepsilon_2 I_n, \quad 0 < \varepsilon_1 \leqslant \varepsilon_2, \quad t \geqslant 0. \tag{4.4}$$

Under assumptions (3.2) and (3.3) values of $\mathcal{D}(K)$, $\mathcal{D}(K_*)$, and $\mathcal{D}(\widehat{K})$ are defined, where $\widehat{K} = G^{-1}\widetilde{K}$ (see Section 3). Here the closed-loop system can be represented as (3.4), and the derivative of function $v(x, t)$ due to system (3.4) and the expression under the integral in (4.1) have the form

$$\dot{v}(x, t) = x^{\mathrm{T}} (\dot{X} + M^{\mathrm{T}} X + X M)\, x, \quad \varphi(x, u, t) = x^{\mathrm{T}} L^{\mathrm{T}} \Phi L\, x,$$

where $M = A + B\mathcal{D}(K)C$, $L^{\mathrm{T}} = [I_n, C^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K)]$, $K = K_* + \widetilde{K}$.

We now require that together with (3.3) and (4.4) the following inequalities hold:

$$\dot{v}(x, t) \leqslant -\varphi(x, u, t) \leqslant -\delta\, \|x\|^2, \quad x \in \mathcal{S}_0, \ t \geqslant 0, \tag{4.5}$$

where $\mathcal{S}_0$ is a neighborhood of the point $x = 0$ containing x_0. For this it suffices that matrix inequalities (4.2) hold, and that (see the proof of Theorem 1)

$$\dot{X} + M_0^{\mathrm{T}} X + X M_0 + L_0^{\mathrm{T}} \Phi L_0 \leqslant -\varepsilon_0 I_n, \quad t \geqslant 0, \tag{4.6}$$

where

$$\varepsilon_0 > 0, \quad M_0 = A_0 + B_0\widehat{\mathcal{D}}(K_0)C_0, \quad L_0^{\mathrm{T}} = [I_n, C_0^{\mathrm{T}}\widehat{\mathcal{D}}^{\mathrm{T}}(K_0)],$$
$$\widehat{\mathcal{D}}(K_0) = (I_m - K_0D_0)^{-1}K_0, \quad K_0 = K_{*0} + \widetilde{K}_0.$$

Here the zero index of each matrix indicates its value for $x = 0$ and $t \geqslant 0$. Then the zero solution of system (3.4) is asymptotically stable and together with (4.4) and (4.5) we get an upper bound on the functional (4.1):

$$J(u, x_0) \leqslant -\int_0^\infty \frac{d}{dt}v(x,t)\,dt = x_0^{\mathrm{T}}X(0)\,x_0 \leqslant \omega. \tag{4.7}$$

Using property (2.4) of operator $\widehat{\mathcal{D}}$, we rewrite inequality (4.6) as

$$\begin{gathered}\mathcal{F}(\widehat{K}) = W + U^{\mathrm{T}}\mathcal{D}(\widehat{K})V + V^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(\widehat{K})U + V^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(\widehat{K})R\mathcal{D}(\widehat{K})V \leqslant 0,\\ x = 0, \quad t \geqslant 0,\end{gathered} \tag{4.8}$$

where

$$W = \dot{X} + M_*^{\mathrm{T}}X + XM_* + \Phi_* + \varepsilon I_n, \quad \Phi_* = L_*^{\mathrm{T}}\Phi L_*,$$
$$U = B^{\mathrm{T}}X + N^{\mathrm{T}} + R\mathcal{D}(K_*)C, \quad V = C_*, \quad L_*^{\mathrm{T}} = [I_n, C^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K_*)].$$

Here

$$\widetilde{K} \in \mathcal{K} \Longleftrightarrow \widehat{K} \in \hat{\mathcal{K}} = \{K : K^{\mathrm{T}}\widehat{P}K \leqslant Q\},$$

where $\widehat{K} = G^{-1}\widetilde{K}$, $\widehat{P} = G^{\mathrm{T}}PG$.

Applying Lemma 2 and relations (4.4)–(4.8), we arrive at the following result.

Theorem 2. *Suppose that for some $\varepsilon_i > 0$ $(i = 0, 1, 2)$ and for $x = 0$ the system of matrix inequalities* (4.4) *holds, and*

$$G^{\mathrm{T}}PG - D^{\mathrm{T}}QD > R, \quad t \geqslant 0, \tag{4.9}$$

$$\begin{bmatrix} \dot{X} + M_*^{\mathrm{T}}X + XM_* + \Phi_* + \varepsilon_0 I_n & XB + N + C^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K_*)R & C_*^{\mathrm{T}} \\ B^{\mathrm{T}}X + N^{\mathrm{T}} + R\mathcal{D}(K_*)C & R - G^{\mathrm{T}}PG & D^{\mathrm{T}} \\ C_* & D & -Q^{-1} \end{bmatrix} \leqslant 0, \quad t \geqslant 0. \tag{4.10}$$

Then any control (3.2) *provides asymptotic stability for the zero state of system* (3.1)*, general Lyapunov function $v(x,t) = x^{\mathrm{T}}X(t)\,x$, and a bound on the functional* (4.3)*.*

The statement of Theorem 2 holds for the family of systems (3.1), (3.7) if instead of (4.10) we use the system of matrix inequalities

$$\begin{bmatrix} \dot{X} + M_{ijk}^{\mathrm{T}}X + XM_{ijk} + \Phi_k + \varepsilon_0 I_n & XB_j + N + C_k^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K_*)R & C_{*k}^{\mathrm{T}} \\ B_j^{\mathrm{T}}X + N^{\mathrm{T}} + R\mathcal{D}(K_*)C_k & R - G^{\mathrm{T}}PG & D^{\mathrm{T}} \\ C_{*k} & D & -Q^{-1} \end{bmatrix} \leqslant 0, \tag{4.11}$$

where

$$M_{ijk} = A_i + B_j\mathcal{D}(K_*)C_k, \quad \Phi_k = L_k^{\mathrm{T}}\Phi L_k, \quad L_k^{\mathrm{T}} = [I_n, C_k^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(K_*)],$$
$$C_{*k} = C_k + D\mathcal{D}(K_*)\,C_k, \quad i = \overline{1,\nu_a}, \quad j = \overline{1,\nu_b}, \quad k = \overline{1,\nu_c}, \quad x = 0,\ t \geqslant 0.$$

We formulate a corollary of Theorem 2 under stronger assumptions.

Corollary 2. *Suppose that the following system of LMI with constant matrices:*

$$P - D_s^T Q D_s > R, \quad \begin{bmatrix} A_i^{\mathrm{T}} X + X A_i + S & X B_j + N & C_k^{\mathrm{T}} \\ B_j^{\mathrm{T}} X + N^{\mathrm{T}} & R - P & D_s^{\mathrm{T}} \\ C_k & D_s & -Q^{-1} \end{bmatrix} \leqslant 0, \quad X > 0, \tag{4.12}$$

is feasible, where $i = \overline{1, \nu_a}$, $j = \overline{1, \nu_b}$, $k = \overline{1, \nu_c}$, $s = \overline{1, \nu_d}$. *Then any control* (3.2) *provides asymptotic stability for the zero state in the family of systems* (3.1), (3.7), (3.9), *general Lyapunov function* $v(x) = x^{\mathrm{T}} X\, x$ *and bound of the functional* (4.3).

Based on Theorem 2 and its corollaries, we can formulate the following optimization problems for system (3.1) and families of systems (3.1), (3.7) and (3.1), (3.7), (3.9):

(1) *minimize* $\omega > 0$ *under constraints* (4.4), (4.9) *and* (4.10);

(2) *minimize* $\omega > 0$ *under constraints* (4.4), (4.9) *and* (4.11);

(3) *minimize* $\omega > 0$ *under constraints* (4.12) *and* $x_0^{\mathrm{T}} X x_0 \leqslant \omega$.

To solve these problems, in the case of constant matrices one can use various methods of mathematical programming. As optimization parameters one can use positive definite matrices, quadratic Lyapunov function (X), coefficients of the feedback ellipsoid (P and Q), and the quality functional (Φ). Here results of the computations depend on the initial vector x_0.

Note that instead of (4.1) one can use the quadratic functional

$$J_0(u) = \int\limits_{\mathcal{S}_0} \mu(x_0)\, J(u, x_0)\, dx_0, \tag{4.13}$$

averaged over initial conditions, where $\mu(x_0) \geqslant 0$ is a given distribution density function for the vector x_0 on a certain set $\mathcal{S}_0 \subseteq \mathbb{R}^n$, e.g., on a ball $\mathcal{S}_0 = \{x \in \mathbb{R}^n : \|x\| \leqslant h\}$. Under assumption (4.5) we have upper bounds for the functional (4.13):

$$J_0(u) \leqslant \operatorname{tr}(\Sigma\, X(0)) \leqslant \mu_0\, \lambda_{\max}(X(0)),$$
$$\Sigma = \int\limits_{\mathcal{S}_0} \mu(x_0)\, x_0 x_0^{\mathrm{T}}\, dx_0, \quad \mu_0 = \int\limits_{\mathcal{S}_0} \mu(x_0) \|x_0\|^2\, dx_0.$$

Therefore in the formulated optimization problems (1)–(3) instead of the first condition of (4.4) we can use inequalities $\operatorname{tr}(\Sigma\, X(0)) \leqslant \omega$ or $\mu_0\, \lambda_{\max}(X(0)) \leqslant \omega$.

Example. Consider a control system for a single-link robot manipulator whose link's circular motion from one end to another is done with a flexible connection of the link and the executive mechanism (Fig. 2).

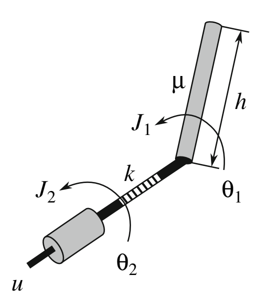

Fig. 2. A single-link robot manipulator.

A linear torsion spring is located between the executive mechanism and the end of a link. This system is defined with two nonlinear differential equations of order two that follow from the mechanical balance of the executive mechanism (motor shaft) and the manipulator link discarding the friction and external disturbances, or, in vector-matrix form [12],

$$\dot{x} = A(x)\, x + Bu, \tag{4.14}$$

where

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(\mu gh\, \varphi(\theta_1) + k)/J_1 & 0 & k/J_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/J_2 & 0 & -k/J_2 & -d/J_2 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J_2 \end{bmatrix}, \quad x = \begin{bmatrix} \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix},$$

where θ_1 and θ_2 are angular coordinates of the manipulator link and motor shaft respectively, u is the controlling moment produced by the electric drive, J_1 and J_2 are moments of inertia respectively for the manipulator link and the electric drive, k is the rigidity of the transmission gear, d is the damping coefficient, μ is the manipulator link's mass, h is the manipulator link's length, g is the gravitational acceleration, and $\varphi(\theta) = (\sin\theta)/\theta$ is a continuous function.

Let $\mu gh = 5$, $d = 0.1$, $k = 100$, and let J_1 and J_2 be unknown parameters that take values on intervals

$$0.5 \leqslant J_1 \leqslant 1.5, \quad 0.1 \leqslant J_2 \leqslant 0.5. \tag{4.15}$$

We assume that the output vector

$$y = Cx + Du = \begin{bmatrix} x_1 + 0.1\, u \\ x_4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}$$

can be measured. Solving two LMIs (2.6) and (2.8) for $\alpha = -0.1$, $J_1 = 1$, and $J_2 = 0.3$, we find the matrix $X = X^{\mathrm{T}} > 0$, vector $K = [-0.6799 \quad -9.0603]$, and the corresponding control

$$u = K_* y, \quad K_* = -\mathcal{D}(K) = [-0.7295 \quad -9.7213] \tag{4.16}$$

that provides asymptotic stability for the linear system

$$\dot{x} = M_* x, \quad M_* = A(0) + BKC, \quad K = \mathcal{D}(K_*).$$

Here the spectrum equals $\sigma(M_*) = \{-0.6449;\ -15.0004;\ -7.4445 \pm 11.8447i\}$, $i(H) = \{2, 1, 0\}$ (see Lemma 1), and the zero state of the original nonlinear system (4.14) is asymptotically stable as well.

We define a matrix functional (4.1):

$$S = 0.5\, I_4, \quad R = 0.2, \quad N = 0.1\, [1\ 0\ 0\ 1]^{\mathrm{T}}.$$

The system of relations (4.11) consists of four matrix inequalities that correspond to possible values of the pair (J_1, J_2) at the ends of intervals (4.15). Using the Matlab suite, we find $P = 2.33$ and

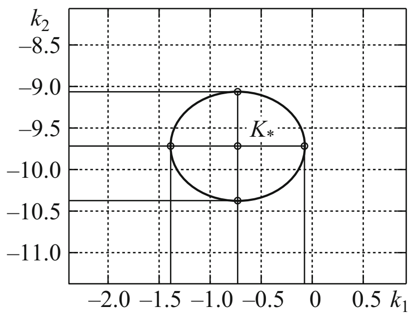

Fig. 3. Region of feedback amplification coefficients $(K - K_*)Q^{-1}(K - K_*)^{\mathrm{T}} \leqslant P^{-1}$.

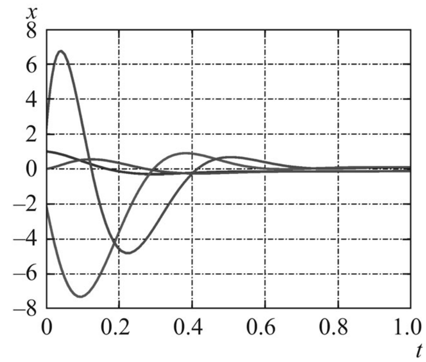

Fig. 4. System behavior with control $u = K_* y$.

positive definite matrices

$$Q = \begin{bmatrix} 1.0013 & 0.0013 \\ 0.0013 & 1.0013 \end{bmatrix}, \quad X = \begin{bmatrix} 955.4267 & -20.1682 & -936.1927 & -31.1040 \\ -20.1682 & 5.2221 & 21.7147 & -0.1949 \\ -936.1927 & 21.7147 & 926.8484 & 31.0357 \\ -31.1040 & -0.1949 & 31.0357 & 2.9214 \end{bmatrix}$$

that satisfy the above system of strict inequalities for $\varepsilon_0 = 0$.

Thus, for all values of the moments of inertia (4.15) and the vector of feedback amplification coefficients $K = K_* + \widetilde{K}$ from a closed region bounded by the ellipse $(K - K_*)Q^{-1}(K - K_*)^{\mathrm{T}} = P^{-1}$ (Fig. 3), the motion of the manipulator robot in a neighborhood of the zero state is asymptotically stable. Here $v(x) = x^{\mathrm{T}}Xx$ is a general Lyapunov function, and the value of the given quality functional does not exceed $v(x_0) = 945.8169$. The behavior of solutions of system (4.14) with control $u = K_* y$ and initial vector $x_0 = [1\ \ -2\ \ 0\ \ 2]^{\mathrm{T}}$ is shown on Fig. 4.

5. CONCLUSION

In this work, we have proposed new methods of robust stability analysis for equilibrium states and optimization of nonlinear control systems with static output feedback. Here values of unknown matrix coefficients may belong to given polytopes, in particular, to matrix intervals, while the measurable output vector contains components of both the system state and the control. Practical implementation of the proposed methods is related to solving differential or algebraic LMIs. To solve algebraic LMIs, one can use an efficient procedure already implemented in Matlab. An important characteristic feature that distinguishes LMIs that we have found from known ones is the possibility to construct an ellipsoid of stabilizing matrices for the feedback amplification coefficients, general quadratic Lyapunov function, and also bounds on the quadratic quality functional for nonlinear control systems with the considered uncertainties.

Results of this work are based on a generalization of the sufficiency statement of Petersen's lemma on matrix uncertainty. This generalization provides new possibilities in the robust stability analysis problems for control systems with structured uncertainty, in particular to construct a set of stabilizing feedback matrices as an ellipsoid, to bound the quadratic quality functional of the control system, and also to solve similar robust stabilization problems for a class of nonlinear discrete systems [13].

Unfortunately, conditions of Theorems 1 and 2 obtained from Lemma 2 and the second Lyapunov's theorem, in the general case are rather theoretical. Their practical use in the robust stabilization problems based on constructing quadratic Lyapunov functions with non-constant matrices requires one to develop new methods for solving differential matrix inequalities. This remains an important problem for further study.

APPENDIX

Proof of Lemma 1. According to the inertia theorem [14], matrix $M_* = A - BKC$ has p and q $(p+q=n)$ eigenvalues, counting multiplicities, in half-planes $\mathrm{Re}\lambda < \alpha$ and $\mathrm{Re}\lambda > \alpha$ respectively only in case when the matrix inequality (2.8) has a solution $X = X^{\mathrm{T}}$ with inertia $i(X) = \{p, q, 0\}$. If, moreover, $X > 0$ and $\alpha \leqslant 0$ then real parts of all points of the spectrum of $\sigma(M_*)$ are negative.

Suppose that matrix inequality (2.8) has a nondegenerate solution $X = X^{\mathrm{T}}$. Let us show that relations (2.6) hold. Applying Schur's lemma [9] to block matrix $T^{\mathrm{T}}YT < 0$, where $T = [B^{+\mathrm{T}}, B^{\perp}]$, $\det T \neq 0$, we obtain a system of inequalities $S < 0$ and $Y_1 < 0$ which is equivalent to (2.8). Inertias of matrices H and

$$\begin{bmatrix} I_m & -K \\ 0 & I_l \end{bmatrix} H \begin{bmatrix} I_m & 0 \\ -K^{\mathrm{T}} & I_l \end{bmatrix} = \begin{bmatrix} Y_1 & H_3^{\mathrm{T}} \\ H_3 & H_2 \end{bmatrix}$$

are the same, so $i_\pm(H) = i_\pm(Y_1) + i_\pm(H_4)$, where $H_4 = H_2 - H_3 Y_1^{-1} H_3^{\mathrm{T}}$, $H_3 = H_1 - H_2 K^{\mathrm{T}}$ (see [4, Corollary 4.2.6]). Under conditions $S < 0$ and $Y_1 < 0$, due to the structure of the blocks in matrix H we have

$$i_-(H) = m \quad \text{and} \quad i_+(H) = i_+(H_4) = \mathrm{rank}\,[CXB^{\perp}, H_3] = \mathrm{rank}(CXT\Psi) = l,$$

where Ψ is a nondegenerate matrix of the form

$$\Psi = \begin{bmatrix} 0 & I_m \\ I_{n-m} & S^{-1}B^{\perp\mathrm{T}}(XC^{\mathrm{T}}K^{\mathrm{T}} - LB^{+\mathrm{T}}) \end{bmatrix}.$$

Let us show that under conditions (2.6) matrix inequalities (2.7) and (2.8) are feasible with respect to K. Using the spectral decomposition of nondegenerate symmetric matrix H, we get

$$H = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \begin{bmatrix} U_1^{\mathrm{T}} & U_2^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{bmatrix} V_1^{\mathrm{T}} & V_2^{\mathrm{T}} \end{bmatrix}, \quad \mathrm{rank} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = l, \quad \mathrm{rank} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = m.$$

Here $\det U_2 \neq 0$. Indeed, $U_2 U_2^{\mathrm{T}} - V_2 V_2^{\mathrm{T}} = H_2 \geqslant 0$ and hence (see [4, Lemma 6.1.1]) $V_2 = U_2 G$, where G is some $l \times m$ matrix such that $GG^{\mathrm{T}} \leqslant I_l$. But then $\mathrm{rank}\,[U_2, V_2] = \mathrm{rank}\, U_2 = l$.

Let us show that there exists a matrix K for which $\det(V_1 - KV_2) \neq 0$ and

$$Y_1 = (U_1 - KU_2)(U_1 - KU_2)^{\mathrm{T}} - (V_1 - KV_2)(V_1 - KV_2)^{\mathrm{T}} < 0.$$

The latter inequality holds if we let $U_1 - KU_2 = (V_1 - KV_2)F$ or $KU_2(I_l - GF) = U_1 - V_1 F$, where F is such an $m \times l$ matrix that $FF^{\mathrm{T}} < I_m$. Then, taking into account that $GG^{\mathrm{T}} \leqslant I_l$, we have $GFF^{\mathrm{T}}G^{\mathrm{T}} < I_l$ and $\rho(GF) < 1$. Consequently, under conditions (2.6) matrix $K = (U_1 - V_1 F)(I_l - GF)^{-1} U_2^{-1}$ satisfies relations (2.7) and (2.8). Here matrix $V_1 - KV_2 = N(I_m - FG)^{-1}$ is nondegenerate because the following matrices are nondegenerate:

$$\begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix} = \begin{bmatrix} -N & U_1 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} 0 & -I_m \\ I_l & U_2^{-1} V_2 \end{bmatrix}, \quad N = V_1 - U_1 U_2^{-1} V_2.$$

This completes the proof of the lemma.

Proof of Lemma 2. Let $K \in \mathcal{K}$. Since $D^{\mathrm{T}}K^{\mathrm{T}}PKD \leqslant D^{\mathrm{T}}QD \leqslant D^{\mathrm{T}}QD + R < P$ then $\rho(KD) < 1$, $K \in \mathcal{K}_D$ and operator $\mathcal{D}(K)$ is defined.

We use the Frobenius' formula to invert the block matrix:

$$\begin{bmatrix} R-P & D^{\mathrm{T}} \\ D & -Q^{-1} \end{bmatrix}^{-1} = \begin{bmatrix} \Delta^{-1} & \Delta^{-1}D^{\mathrm{T}}Q \\ QD\Delta^{-1} & QD\Delta^{-1}D^{\mathrm{T}}Q - Q \end{bmatrix},$$

where $\Delta = D^{\mathrm{T}}QD + R - P$, and reduce matrix inequality $\Omega \leqslant 0\ (< 0)$ to the form

$$[U^{\mathrm{T}}, V^{\mathrm{T}}] \begin{bmatrix} \Delta^{-1} & \Delta^{-1}D^{\mathrm{T}}Q \\ QD\Delta^{-1} & QD\Delta^{-1}D^{\mathrm{T}}Q - Q \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \geqslant W\ (> W). \tag{A.1}$$

Here we have also used the following well-known criterion (Schur's lemma [9]): if $\det S_3 \neq 0$ then

$$\begin{bmatrix} S_1 & S_2^{\mathrm{T}} \\ S_2 & S_3 \end{bmatrix} \leqslant 0\ (< 0) \iff S_3 < 0, \quad S_1 - S_2^{\mathrm{T}}S_3^{-1}S_2 \leqslant 0\ (< 0).$$

Due to (A.1) we see that matrix inequality $\mathcal{F}(K) \leqslant 0\ (< 0)$, representable as

$$[U^{\mathrm{T}}, V^{\mathrm{T}}] \begin{bmatrix} 0 & -\mathcal{D}(K) \\ -\mathcal{D}^{\mathrm{T}}(K) & -\mathcal{D}^{\mathrm{T}}(K)R\mathcal{D}(K) \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} \geqslant W\ (> W),$$

holds if

$$\begin{bmatrix} \Delta^{-1} & \mathcal{D}(K) + \Delta^{-1}D^{\mathrm{T}}Q \\ \mathcal{D}^{\mathrm{T}}(K) + QD\Delta^{-1} & \mathcal{D}^{\mathrm{T}}(K)R\mathcal{D}(K) + QD\Delta^{-1}D^{\mathrm{T}}Q - Q \end{bmatrix} \leqslant 0.$$

Applying Schur's lemma to this expression in case $\det S_1 \neq 0$, we get

$$\begin{aligned} &\mathcal{D}^{\mathrm{T}}(K)R\mathcal{D}(K) + QD\Delta^{-1}D^{\mathrm{T}}Q - Q - [\mathcal{D}^{\mathrm{T}}(K) + QD\Delta^{-1}]\Delta[\mathcal{D}(K) + \Delta^{-1}D^{\mathrm{T}}Q] \\ &= \mathcal{D}^{\mathrm{T}}(K)P\mathcal{D}(K) - Q - \mathcal{D}^{\mathrm{T}}(K)D^{\mathrm{T}}QD\mathcal{D}(K) - QD\mathcal{D}(K) - \mathcal{D}^{\mathrm{T}}(K)DQ \\ &= \mathcal{D}^{\mathrm{T}}(K)P\mathcal{D}(K) - [I_l + \mathcal{D}^{\mathrm{T}}(K)D^{\mathrm{T}}]Q[I_l + D\mathcal{D}(K)] \leqslant 0. \end{aligned}$$

The latter inequality, due to the properties (2.3) of operator $\mathcal{D}$ and the law of inertia, reduces to the form $K^{\mathrm{T}}PK \leqslant Q$, i.e., to condition $K \in \mathcal{K}$.

Note that the stronger assumption $\Omega < 0$ ensures that strict inequality $\mathcal{F}(K) < 0$ holds for every matrix $K \in \mathcal{K}$.

This completes the proof of the lemma.

Proof of Theorem 1. We construct the Lyapunov function for the closed-loop system (3.4) as $v(x,t) = x^{\mathrm{T}}X(t)x$. Under conditions (3.5) it holds that $\varepsilon_1\|x\|^2 \leqslant v(x,t) \leqslant \varepsilon_2\|x\|^2$, $t \geqslant 0$. In order for the derivative of function $v(x,t)$ with respect to system (3.4) in some neighborhood $\mathcal{S}_0$ of the point $x = 0$ to satisfy $\dot{v}(x,t) \leqslant -\varepsilon\|x\|^2$, where $\varepsilon > 0$, it suffices that the following matrix inequality holds:

$$\dot{X} + M^{\mathrm{T}}X + XM + \varepsilon I_n \leqslant 0, \quad x \in \mathcal{S}_0, \quad t \geqslant 0. \tag{A.2}$$

Here according to the second Lyapunov's theorem the state $x \equiv 0$ of this system is uniformly asymptotically stable. Condition (A.2) means that $\sup_{t \geqslant 0, x \in \mathcal{S}_0} \omega(x,t) \leqslant -\varepsilon$, where $\omega(x,t) = \lambda_{\max}(\dot{X} + M^{\mathrm{T}}X + XM)$.

Together with (A.2) we consider condition $\sup_{t \geqslant 0} \omega(0,t) \leqslant -\varepsilon_0$, i.e.,

$$\dot{X} + M_0^{\mathrm{T}}X + XM_0 + \varepsilon_0 I_n \leqslant 0, \quad t \geqslant 0, \tag{A.3}$$

where $M_0 = M(0,t)$, $\varepsilon_0 > \varepsilon$. By continuity it is clear that there exists a neighborhood $\mathcal{S}_0$ of the point $x = 0$ where (A.2) follows from (A.3).

Using property (2.4) of operator $\widehat{\mathcal{D}}(K) = (I_m - KD_0)^{-1}K$, we rewrite inequality (A.3) as

$$\mathcal{F}(\widehat{K}) = W + U^{\mathrm{T}}\mathcal{D}(\widehat{K})V + V^{\mathrm{T}}\mathcal{D}^{\mathrm{T}}(\widehat{K})U \leqslant 0, \quad x = 0, \quad t \geqslant 0,$$

where $W = \dot{X} + M_*^{\mathrm{T}}X + XM_* + \varepsilon I_n$, $U = B^{\mathrm{T}}X$, $V = C_* = C + D\mathcal{D}(K_*)C$, $\widehat{K} = G^{-1}\widetilde{K}$. Here

$$\widetilde{K} \in \mathcal{K} \iff \widehat{K} \in \widehat{\mathcal{K}} = \{K : K^{\mathrm{T}}\widehat{P}K \leqslant Q\},$$

where $\widehat{P} = G^{\mathrm{T}}PG$. Applying Lemma 2 with $R = 0$, we get conditions of the form (3.3) and (3.6) under which inequality (A.3) and, consequently, (A.2) hold for every matrix $\widetilde{K} \in \mathcal{K}$. These conditions together with inequalities (3.5) ensure asymptotic stability for the zero state of the closed-loop system (3.4).

This completes the proof of the theorem.

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