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STABILIZATION BY A MEASURABLE OUTPUT AND ESTIMATION OF THE LEVEL OF ATTENUATION FOR PERTURBATIONS IN CONTROL SYSTEMS

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We establish new criteria for the output stabilization in linear control systems with the help of static and dynamic regulators. It is shown that the stabilization algorithms derived from these criteria can be applied to a certain class of nonlinear control systems. We propose some algorithms for the construction of the regularities of control guaranteeing the required estimates of the weighted level of attenuation of input signals. The obtained results are illustrated by an example of a system stabilizing a one-link robot-manipulator.

1. Introduction

The problem of stabilization of dynamical systems is one of the main problems of control theory. For a class of linear control systems with static feedback by the output

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{1}$$

$$u = Ky, \tag{2}$$

the problem is reduced to the determination of the matrix of amplification coefficients K for which the closed system is asymptotically stable. Here, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^l$ are, respectively, the vectors of state, control, and measurable output of the system and A, B, C, and D are matrices of suitable sizes. The complete solution of this important problem is known only for some special cases (see the surveys [1, 2]). Note that numerous available algorithms of stabilization of the systems are reduced to the solution of linear matrix inequalities with the use of the efficient LMI-Toolbox means of the Matlab computer system (see, e.g., [3–5]).

If a stabilizing static feedback cannot be constructed, then we can consider the possibility of stabilization of system (1) with the help of a dynamic regulator of order $r \leq n$ of the form

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \tag{3}$$

where $\xi \in \mathbb{R}^r$ is the vector of state of the regulator and Z, V, U, and K are the unknown matrices of suitable sizes.

In the present work, we propose some new criteria of stabilization of the linear system (1) with the help of static and dynamic feedbacks. We also present some methods for the construction of regulators guaranteeing the asymptotic stability of the state $x \equiv 0$ of a class of nonlinear systems

$$\dot{x} = A(x)x + B(x)u, \quad y = C(x)x + Du, \tag{4}$$

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where $A(x)$, $B(x)$, and $C(x)$ are matrix functions continuous in the vicinity of the point $x = 0$. In this case, we assume that $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{l\times n}$ are matrices of full rank with $m < n$ and $l < n$, respectively. For the class of linear systems (1), we propose the algorithms of construction of the regularities of control of the forms (2) and (3) guaranteeing the possibility of estimation of a certain criterion of quality, which describes the weighed level of attenuation of the input signals and the robust stabilization relative to a given set of uncertainties. The proposed quality criterion is an analog of the H_∞-norm of the transfer matrix function $H(\lambda)$ of the analyzed control system.

We use the following notation: I_n is the identity matrix of order n; $0_{n\times m}$ is the $n\times m$ zero matrix; $X = X^T > 0$ (≥ 0) is a positive (nonnegative) definite symmetric matrix X; $\mathrm{i}(X) = \{i_+, i_-, i_0\}$ is the inertia of an Hermitian matrix X formed by the numbers of its positive, negative, and zero eigenvalues with regard for their multiplicities; $\lambda_{\max}(X)$ ($\lambda_{\min}(X)$) is the maximum (minimum) eigenvalue of the Hermitian matrix X; A^+ is a pseudoreciprocal matrix; $\|x\|$ is the Euclidean norm of a vector x; $W_L \in \mathbb{R}^{n\times n-\mathrm{rank}\,L}$ is a matrix whose columns form a basis of the kernel of the matrix $L \in \mathbb{R}^{l\times n}$, and $B^\perp$ ($C^\perp$) is the orthogonal supplement of the matrix $B \in \mathbb{R}^{n\times m}$ ($C \in \mathbb{R}^{l\times n}$) of full rank m (l) defined by the relations $B^T B^\perp = 0$ and $\det\left[B, B^\perp\right] \neq 0$ ($C^\perp C^T = 0$ and $\det\left[C^T, C^{\perp T}\right] \neq 0$).

2. Static Stabilization by the Output

First, we consider the linear system (1) with feedback (2). If the matrix of amplification coefficients K belongs to the set

$$\mathcal{K}_D = \{K : \det(I_m - KD) \neq 0\},$$

then the closed system takes the form

$$\dot{x} = Mx, \quad M = A + B\mathcal{D}(K)C. \tag{5}$$

The nonlinear operator $\mathcal{D}(K) = (I_m - KD)^{-1}K$ has the following properties [6]:

1) if $K \in \mathcal{K}_D$, then $\mathcal{D}(K) \equiv K(I_l - DK)^{-1}$ and $I_l + D\mathcal{D}(K) \equiv (I_l - DK)^{-1}$;

2) if $K_1 \in \mathcal{K}_D$ and $K_2 \in \mathcal{K}_{D_1}$, then

$$\mathcal{D}(K_1 + K_2) = \mathcal{D}(K_1) + (I_m - K_1 D)^{-1}\mathcal{D}_1(K_2)\,(I_l - DK_1)^{-1}$$

and $K_1 + K_2 \in \mathcal{K}_D$, where

$$D_1 = (I_l - DK_1)^{-1}D \quad \text{and} \quad \mathcal{D}_1(K_2) = (I_m - K_2 D_1)^{-1}K_2;$$

3) if $-K_0 \in \mathcal{K}_D$, then $K = -\mathcal{D}(-K_0) \in \mathcal{K}_D$ and $\mathcal{D}(K) = K_0$.

By $n_\alpha^-(M)$, $n_\alpha^+(M)$, and $n_\alpha^0(M)$ we denote the numbers of eigenvalues of the matrix $M = A + BK_0C$ with regard for multiplicities that belong to the corresponding sets $\mathbb{C}_\alpha^- = \{\lambda : \mathrm{Re}\,\breve{} + \alpha < 0\}$, $\mathbb{C}_\alpha^+ = \{\lambda : \mathrm{Re}\,\breve{} + \alpha > 0\}$, and $\mathbb{C}_\alpha^0 = \{\lambda : \mathrm{Re}\,\breve{} + \alpha = 0\}$, where $\alpha \in \mathbb{R}$. If $n_\alpha^-(M) = n$, then, for $\alpha \geq 0$, system (5) has a spectral stability margin α.

Lemma 1. *There exists a matrix K_0 such that*

$$n_\alpha^-(M) = p, \quad n_\alpha^+(M) = q, \quad n_\alpha^0(M) = 0, \tag{6}$$

iff the following system of relations is solvable with respect to X*:*

$$B^{\perp T}(AX + XA^T + 2\alpha X)\, B^{\perp} < 0, \tag{7}$$

$$\mathrm{i}(X) = \{p, q, 0\}, \quad X = X^T, \tag{8}$$

$$\mathrm{i}(\Delta) = \{l, n, 0\}, \quad \Delta = \begin{bmatrix} AX + XA^T + 2\alpha X & XC^T \\ CX & 0 \end{bmatrix}. \tag{9}$$

If conditions (7)–(9) are satisfied, then the matrix K_0 *guaranteeing the validity of conditions (6) can be found as a solution of the linear matrix inequalities*

$$AX + XA^T + 2\alpha X + BK_0CX + XC^TK_0^TB^T < 0. \tag{10}$$

Proof. According to the theorem on inertia [7], equalities (6) are equivalent to the consistency of the system of relations (8) and (10) with respect to X. In [6], it was shown that the problem of determination of the matrix X satisfying the given system is reduced to the solution of the matrix inequality (7) under the conditions

$$\mathrm{i}(H) = \{l, m, 0\}, \quad H = \begin{bmatrix} B^+(L - LRL)B^{+T} & B^+(I_n - LR)XC^T \\ CX(I_n - RL)B^{+T} & -CXRXC^T \end{bmatrix}, \tag{11}$$

where

$$L = AX + XA^T + 2\alpha X, \quad R = B^{\perp}S^{-1}B^{\perp T}, \quad \text{and} \quad S = B^{\perp T}LB^{\perp}.$$

The block matrix H can be represented in the form $H = \widehat{H}_0 - \widehat{H}_1^T\widehat{H}_2^{-1}\widehat{H}_1$, where

$$\widehat{H} = \begin{bmatrix} \widehat{H}_0 & \widehat{H}_1^T \\ \widehat{H}_1 & \widehat{H}_2 \end{bmatrix} = \left[\begin{array}{cc|c} B^+LB^{+T} & B^+XC^T & B^+LB^{\perp} \\ CXB^{+T} & 0 & CXB^{\perp} \\ \hline B^{\perp T}LB^{+T} & B^{\perp T}XC^T & S \end{array}\right] = W\Delta W^T, \quad W = \begin{bmatrix} B^+ & 0 \\ 0 & I_l \\ B^{\perp T} & 0 \end{bmatrix}.$$

By using the well-known formulas for the indices of inertia of a block matrix [8, p. 147] and the formula $\widehat{H}_2 = S < 0$, we obtain

$$\mathrm{i}_+(\widehat{H}) = \mathrm{i}_+(\widehat{H}_2) + \mathrm{i}_+(H) = \mathrm{i}_+(H), \quad \mathrm{i}_-(\widehat{H}) = \mathrm{i}_-(\widehat{H}_2) + \mathrm{i}_-(H) = \mathrm{i}_-(H) + n - m.$$

Since $W \in \mathbb{R}^{n+l \times n+l}$ is a square nonsingular matrix, we get $\mathrm{i}(\widehat{H}) = \mathrm{i}(\Delta)$. Hence, relations (9) and (11) are equivalent under conditions (7) and (8).

The lemma is proved.

In the open set of solutions of the matrix inequality (10), it is always possible to choose a matrix K_0 such that $-K_0 \in \mathcal{K}_D$. In this case, $M = A + BK_0C$ is a matrix of the closed system (5) (see property 3) of the operator $\mathcal{D}(K)$). Therefore, Lemma 1 yields the following criterion of stabilization of system (1).

Theorem 1. *The linear system (1) can be stabilized with the spectral stability margin $\alpha \geq 0$ with the help of the static feedback (2) iff there exists a matrix $X = X^T > 0$ satisfying relations (7) and (9). In this case, the stabilizing feedback matrix can be defined in the form*

$$K = -\mathcal{D}(-K_0) \in \mathcal{K}_D, \tag{12}$$

where K_0 is a solution of the linear matrix inequalities (10).

Remark 1. Conditions (8) and (9) are equivalent to the matrix inequality

$$C^{\perp}(A^T Y + YA + 2\alpha Y)\, C^{\perp T} < 0, \tag{13}$$

where $Y = X^{-1}$. Indeed, by finding the indices of inertia of the block matrix

$$\Delta_1 = W_1^T \Delta W_1 = \left[\begin{array}{c|cc} C^{\perp} L_1 C^{\perp T} & 0 & C^{\perp} L_1 C^{+} \\ \hline 0 & 0 & I_l \\ C^{+T} L_1 C^{\perp T} & I_l & C^{+T} L_1 C^{+} \end{array}\right],$$

where

$$L_1 = A^T Y + YA + 2\alpha Y, \quad W_1 = \begin{bmatrix} YC^{\perp T} & 0 & YC^{+} \\ 0 & I_l & 0 \end{bmatrix} \in \mathbb{R}^{n+l \times n+l}, \quad \det W_1 \neq 0,$$

we get

$$\mathrm{i}_{\pm}(\Delta_1) = \mathrm{i}_{\pm}(C^{\perp} L_1 C^{\perp T}) + l = \mathrm{i}_{\pm}(\Delta)$$

(see [8, p. 147]). Therefore, equalities (6) hold only under conditions (7) and (13). As a consequence, the criterion of stabilization of system (1) by control (2) in Theorem 1 is reduced to the consistency of two linear matrix inequalities (7) and (13) with respect to the mutually reciprocal positive-definite matrices X and Y (see also [4]).

Theorem 2. *Assume that the following linear matrix inequalities hold for a matrix $X = X^T > 0$ and some $\alpha \geq 0$:*

$$B_0^{\perp T}(A_0 X + X A_0^T + 2\alpha X) B_0^{\perp} < 0 \tag{14}$$

and, moreover, that one of the relations

$$\mathrm{i}(\Delta) = \{l, n, 0\} \quad \textit{and} \quad C_0^{\perp}(A_0^T Y + Y A_0 + 2\alpha Y)\, C_0^{\perp T} < 0 \tag{15}$$

is true, where

$$A_0 = A(0), \quad B_0 = B(0), \quad C_0 = C(0), \quad Y = X^{-1}, \quad \Delta = \begin{bmatrix} A_0 X + X A_0^T + 2\alpha X & X C_0^T \\ C_0 X & 0 \end{bmatrix}.$$

Then the static regulator (2) with matrix (12), where K_0 is a solution of the linear matrix inequalities

$$A_0 X + X A_0^T + 2\alpha X + B_0 K_0 C_0 X + X C_0^T K_0^T B_0^T < 0, \tag{16}$$

guarantees the asymptotic stability of the state $x \equiv 0$ *of the nonlinear system (4) and the quadratic Lyapunov function* $v(x) = x^T Yx$.

Proof. Conditions (14) and (15) ensure the solvability of the linear matrix inequality (16) with respect to K_0. In this case, by the continuity of the matrix functions $A(x)$, $B(x)$, and $C(x)$, the following relations are true for some $h > 0$:

$$M(x)X + XM^T(x) + 2\alpha X < 0, \quad \dot{v}(x) < -2\alpha\, v(x) \le 0, \quad x \in \mathcal{S}_0,$$

where $M(x) = A(x) + B(x)K_0C(x)$, $\mathcal{S}_0 = \{x : \|x\| < h\}$, and $\dot{v}(x)$ is the derivative of the function $v(x)$ by virtue of the closed system (4), (2), (12). Therefore, Theorem 2 is a consequence of Theorem 1 and the Lyapunov theorem on asymptotic stability [9]. In this case, $-K_0 \in \mathcal{K}_D$, $K \in \mathcal{K}_D$, $\mathcal{D}(K) = K_0$, and the spectrum of the matrix $M(x)$ is located in the half plane $\mathbb{C}_\alpha^-$ for $x \in \mathcal{S}_0$.

The theorem is proved.

Remark 2. In Theorems 1 and 2, the matrix of stabilizing feedback K is determined as a result of the solution of the corresponding linear matrix inequalities (10) and (16). Under additional restrictions, the sizes of the resolved matrix inequalities can be decreased. Thus, if, in the set of solutions of the linear matrix inequalities (7), one can find a matrix X such that $C^\perp X^{-1} B = 0$, then, for sufficiently large $\gamma > 0$, the feedback matrix (12) in which

$$-K_0 = \gamma\, B^T X^{-1} C^+ \in \mathcal{K}_D$$

guarantees the asymptotic stability of the closed system (5) with the spectral stability margin α. In this case, it suffices to take $\gamma > \lambda_{\max}(H_0)/2$, where $H_0 = B^+(L - LRL)B^{+T}$ [6].

3. Dynamic Regulators

The control system (1) with dynamic feedback (3) of order $r \neq 0$ is equivalent to a control system with static feedback in the extended phase space $\mathbb{R}^{n+r}$:

$$\dot{\widehat{x}} = \widehat{A}\widehat{x} + \widehat{B}\widehat{u}, \quad \widehat{y} = \widehat{C}\widehat{x} + \widehat{D}\widehat{u}, \quad \widehat{u} = \widehat{K}\widehat{y}, \tag{17}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{y} = \begin{bmatrix} y \\ \xi \end{bmatrix}, \quad \widehat{u} = \begin{bmatrix} u \\ \dot{\xi} \end{bmatrix}, \quad \widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix},$$

$$\widehat{A} = \begin{bmatrix} A & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B & 0_{n\times r} \\ 0_{r\times m} & I_r \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} C & 0_{l\times r} \\ 0_{r\times n} & I_r \end{bmatrix}, \quad \widehat{D} = \begin{bmatrix} D & 0_{l\times r} \\ 0_{r\times m} & 0_{r\times r} \end{bmatrix}.$$

For the matrix coefficients $\widehat{B}$ and $\widehat{C}$ of full rank, the formulas for the orthogonal supplements and pseudoreciprocal matrices are as follows:

$$\widehat{B}^\perp = \begin{bmatrix} B^\perp \\ 0_{r\times(n-m)} \end{bmatrix}, \quad \widehat{B}^+ = \begin{bmatrix} B^+ & 0_{m\times r} \\ 0_{r\times n} & I_r \end{bmatrix}, \quad \widehat{C}^\perp = \begin{bmatrix} C^\perp, 0_{(n-l)\times r} \end{bmatrix}, \quad \widehat{C}^+ = \begin{bmatrix} C^+ & 0_{n\times r} \\ 0_{r\times l} & I_r \end{bmatrix}.$$

Under the condition $K \in \mathcal{K}_D$, the closed system (17) can be represented in the form

$$\dot{\widehat{x}} = \widehat{M}\,\widehat{x}, \quad \widehat{M} = \widehat{A} + \widehat{B}\widehat{D}(\widehat{K})\widehat{C}, \tag{18}$$

where

$$\widehat{M} = \left[\begin{array}{c|c} M & B(I_m - KD)^{-1}U \\ \hline V(I_l - DK)^{-1}C & Z + VD(I_m - KD)^{-1}U \end{array}\right], \quad M = A + B\mathcal{D}(K)C.$$

Theorem 3. *The following assertions are equivalent:*

(i) there exists a dynamic regulator (3) of order $r \leq n$ guaranteeing the asymptotic stability of the closed system (18) with spectral margin $\alpha \geq 0$;

(ii) there exist matrices X and X_0 satisfying relations (7) and such that

$$\mathrm{i}(\Delta_0) = \{l, n, 0\}, \quad X \geq X_0 > 0, \quad \mathrm{rank}\,(X - X_0) \leq r, \tag{19}$$

where

$$\Delta_0 = \begin{bmatrix} AX_0 + X_0A^T + 2\alpha X_0 & X_0C^T \\ CX_0 & 0 \end{bmatrix};$$

(iii) there exist matrices X and Y satisfying relations (7) and (13) and such that

$$W = \begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \geq 0, \quad \mathrm{rank}\, W \leq n + r. \tag{20}$$

Proof. According to Theorem 1, the criterion of stabilization of system (17) with the help of a static regulator has the form

$$\widehat{B}^{\perp T}\left(\widehat{A}\widehat{X} + \widehat{X}\widehat{A}^T + 2\alpha\widehat{X}\right)\widehat{B}^{\perp} < 0, \quad \mathrm{i}(\widehat{\Delta}) = \{l + r, n + r, 0\}, \tag{21}$$

where

$$\widehat{\Delta} = \begin{bmatrix} \widehat{A}\widehat{X} + \widehat{X}\widehat{A}^T + 2\alpha\widehat{X} & \widehat{X}\widehat{C}^T \\ \widehat{C}\widehat{X} & 0 \end{bmatrix}, \quad \widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad \alpha \geq 0.$$

In view of the structure of the block matrices, the first relation in (21) coincides with the matrix inequality (7) with respect to X. We now use the congruent transformation of the matrix $\widehat{\Delta}$:

$$\widehat{L}\widehat{\Delta}\widehat{L}^T = \begin{bmatrix} \Delta_0 & 0 \\ 0 & \Delta_1 \end{bmatrix}, \tag{22}$$

where

$$\widehat{L} = \left[\begin{array}{cc|cc} I_n & -X_1^T X_2^{-1} & 0 & -AX_1^T X_2^{-1} \\ 0 & 0 & I_l & -CX_1^T X_2^{-1} \\ \hline 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & I_r \end{array}\right], \quad \Delta_1 = \begin{bmatrix} 2\alpha X_2 & X_2 \\ X_2 & 0 \end{bmatrix}.$$

Here, the diagonal block Δ_0 is defined in (19) for $X_0 = X - X_1^T X_2^{-1} X_1$. In this case,

$$\mathrm{i}(\Delta_1) = \{r, r, 0\}, \quad \mathrm{rank}\,(X - X_0) = \mathrm{rank}\,(X_1^T X_2^{-1} X_1) \leq r, \quad \text{and} \quad X \geq X_0.$$

Hence, relation (21) yields relations (7) and (19) for some positive-definite matrices X and X_0. Conversely, if the system of relations (7) and (19) is solvable with respect to $X = X^T > 0$ and $X_0 = X_0^T > 0$, then, in view of (22), one can always find a block matrix $\widehat{X} > 0$ satisfying relations (21). In this case, the matrix X must be its first diagonal block. As X_1 and X_2, we can choose, e.g., the multiplier of the expansion $X - X_0 = X_1^T X_1 \geq 0$ and the identity matrix I_r, respectively.

The equivalence of Assertions 1 and 3 is established with regard for Remark 1 and the block structure of the analyzed matrices. It is worth noting that the matrices X and X_0 satisfy Assertion 2 iff the matrices X and $Y = X_0^{-1}$ satisfy Assertion 3. In order that relations (20) be true, it is necessary that the matrices X and Y be positive definite. The rank restrictions in relations (19) and (20) are always satisfied for the dynamic regulator of the full order $r = n$.

The theorem is proved.

Assertion 2 of Theorem 3 yields the following algorithm of construction of the stabilizing dynamic regulator (3) of order $r \leq n$ for system (1):

Algorithm 1.

1. Determination of the matrices $X = X^T > 0$ and $X_0 = X_0^T > 0$ satisfying relations (7) and (19).

2. Decomposition of the nonnegative-definite matrix

$$X - X_0 = X_1^T X_1 \geq 0, \quad X_1 \in \mathbb{R}^{r \times n}, \quad \mathrm{rank}\, X_1 \leq r.$$

3. Solution of the linear matrix inequalities

$$\widehat{A}\widehat{X} + \widehat{X}\widehat{A}^T + 2\alpha \widehat{X} + \widehat{B}\widehat{K}_0\widehat{C}\widehat{X} + \widehat{X}\widehat{C}^T\widehat{K}_0^T\widehat{B}^T < 0$$

with respect to $\widehat{K}_0$ under the restrictions $\det\,(I_m + K_0 D) \neq 0$ and $\alpha \geq 0$, where

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & I_r \end{bmatrix} > 0, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}.$$

4. Determination of the matrices of regulator (3) by the formulas

$$K = (I_m + K_0 D)^{-1} K_0, \quad U = (I_m + K_0 D)^{-1} U_0,$$
$$V = V_0 (I_l + D K_0)^{-1}, \quad Z = Z_0 - V_0 D (I_m + K_0 D)^{-1} U_0. \tag{23}$$

By using relations (17) and (18), we can now formulate sufficient conditions for the existence of the dynamic regulator (3) and the methods for its construction guaranteeing the asymptotic stability of the state $x \equiv 0$ of the nonlinear system (4) (see Theorem 2).

4. H_∞-Control by the Output

Consider system (1) with the trivial initial vector $x(0) = 0$ and a class of controls

$$u = K_* y + w, \quad K_* \in \mathcal{K}_D, \tag{24}$$

where K_* is the stabilizing matrix of static feedback. As the input w, we can take a vector of external perturbations or a new control. We can represent system (1) with control (24) in the form

$$\dot{x} = A_* x + B_* w, \quad y = C_* x + D_* w, \quad x(0) = 0, \tag{25}$$

where $A_* = A + B\mathcal{D}(K_*)C$, $B_* = B(I_m - K_* D)^{-1}$, $C_* = (I_l - DK_*)^{-1}C$, $D_* = (I_l - DK_*)^{-1}D$, and $\mathcal{D}(K_*) = (I_m - K_* D)^{-1} K_*$.

For system (25), we define the quality criterion as follows:

$$J_{P,Q} = \sup_{0<\|w\|_P<\infty} J(w), \tag{26}$$

where

$$J(w) = \frac{\|y\|_Q}{\|w\|_P}, \quad \|y\|_Q^2 = \int_0^\infty y^T Q y dt, \quad \|w\|_P^2 = \int_0^\infty w^T P w dt,$$

$Q = Q^T > 0$ and $P = P^T > 0$ are positive-definite matrices specifying the weighed L_2-norms $\|y\|_Q$ and $\|w\|_P$. In this case, the following two-sided inequality is true:

$$\gamma_1 J \leq J_{P,Q} \leq \gamma_2 J, \quad \gamma_1 = \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}, \quad \gamma_2 = \frac{\lambda_{\max}(Q)}{\lambda_{\min}(P)},$$

where $J = J_{I_m, I_l}$ coincides with the H_∞-norm of the transfer matrix function of system (1):

$$\|H\|_\infty = \sup_{\omega \in \mathbb{R}} \sqrt{\lambda_{\max}(H^T(-i\omega)H(i\omega))}, \quad H(\lambda) = C(\lambda I_n - A)^{-1} B + D.$$

The quantity J characterizes the *level of attenuation of the input signals* in the system, i.e., the "output–input" energy ratio [3]. In the solution of various control problems, it is desirable to get the minimum values of this ratio. The quality criterion (26), is called the *weighed level of attenuation of the input signals* in system (25).

Lemma 2. *Assume that, for some matrix $K_* \in \mathcal{K}_D$, the matrix A_* is the Hurwitz matrix. Then $J_{P,Q} < 1$ iff the following linear matrix inequality holds for some matrix $X = X^T > 0$:*

$$\begin{bmatrix} A_*^T X + X A_* + C_*^T Q C_* & X B_* + C_*^T Q D_* \\ B_*^T X + D_*^T Q C_* & D_*^T Q D_* - P \end{bmatrix} < 0. \tag{27}$$

In this case, the closed system (1), (24) with uncertainty

$$w = \Theta y, \quad \Theta^T P \Theta \le Q, \tag{28}$$

is robust stable with the general Lyapunov function $v(x) = x^T X x$.

Proof. By using the expansions of the positive-definite matrices $P = \widehat{P}^T \widehat{P}$ and $Q = \widehat{Q}^T \widehat{Q}$, we arrive at the system

$$\dot{x} = A_* x + \widehat{B}_* \widehat{w}, \quad \widehat{y} = \widehat{C}_* x + \widehat{D}_* \widehat{w}, \quad x(0) = 0,$$

where $\widehat{y} = \widehat{Q} y$, $\widehat{w} = \widehat{P} w$, $\widehat{B}_* = B_* \widehat{P}^{-1}$, $\widehat{C}_* = \widehat{Q} C_*$, and $\widehat{D}_* = \widehat{Q} D_* \widehat{P}^{-1}$. In this case, the vector $\widehat{w}$ is regarded as the input of the analyzed system with a quality criterion of the form J. Hence, the estimate $J_{P,Q} < \gamma$ holds iff the following linear matrix inequality is satisfied for some matrix $X = X^T > 0$ [10, 11]:

$$\widehat{\Omega}_\gamma = \begin{bmatrix} A_*^T X + X A_* & X \widehat{B}_* & \widehat{C}_*^T \\ \widehat{B}_*^T X & -\gamma I_m & \widehat{D}_*^T \\ \widehat{C}_* & \widehat{D}_* & -\gamma I_l \end{bmatrix} < 0.$$

For $K_* = 0$, $P = I_m$, and $Q = I_l$, this estimate is equivalent to the frequency inequality

$$H^T(-i\omega) H(i\omega) < \gamma^2 I_m, \quad \omega \in \mathbb{R}.$$

The obtained matrix inequality can be represented in the form

$$\Omega_\gamma = G^T \widehat{\Omega}_\gamma G = \begin{bmatrix} A_*^T X + X A_* & X B_* & C_*^T \\ B_*^T X & -\gamma P & D_*^T \\ C_* & D_* & -\gamma Q^{-1} \end{bmatrix} < 0, \tag{29}$$

where $G = \mathrm{diag}\{I_n, \widehat{P}, \widehat{Q}^{-1T}\}$. It is clear that, under condition (29), the matrix A_* must be the Hurwitz matrix.

Setting $\gamma = 1$ and applying the Schur lemma, we get a criterion of validity of the estimate $J_{P,Q} < 1$ in the form of the matrix inequality (27). The asymptotic stability of the closed system (1), (24) for any vector (28) (i.e., the robust stability) with the general Lyapunov function $v(x) = x^T X x$ is a consequence of Theorem 1 [12].

The theorem is proved.

Note that characteristic (26) is determined as a result of the solution of the following optimization problem with respect to X and K_*:

$$J_{P,Q} = \inf \left\{ \gamma : \Omega_\gamma < 0, X = X^T > 0,\ K_* \in \mathcal{K}_D \right\}.$$

As the parameters of optimization, parallel with X and K_*, we can also take the positive-definite matrices P and Q,

We now establish a criterion of existence of the matrix K_* satisfying Lemma 2.

Let $K_0 = \mathcal{D}(K_*)$. Then $A_* = A + BK_0C$, $B_* = B(I_m + K_0D)$, and $C_* = (I_l + DK_0)C$, $D_* = (I_l + DK_0)D$. For $\gamma = 1$, the matrix inequality (29) takes the form

$$L^T K_0 R + R^T K_0^T L + S < 0, \tag{30}$$

where

$$R = [C, D, 0_{l\times l}], \quad L = \left[B^T, D^T, 0_{m\times m}\right]\widehat{X},$$

$$\widehat{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_m & 0 \end{bmatrix}, \quad S = \begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -P & D^T \\ C & D & -Q^{-1} \end{bmatrix}.$$

This inequality is solvable with respect to K_0 iff

$$W_R^T S W_R < 0, \quad W_L^T S W_L < 0, \tag{31}$$

where W_L and W_R are matrices whose columns form the bases of the corresponding kernels $\ker L$ and $\ker R$ [11]. Since

$$W_R = \begin{bmatrix} W_{[C,D]} & 0 \\ 0 & I_l \end{bmatrix}, \quad W_L = \widehat{X}^{-1}\begin{bmatrix} W_{[B^T,D^T]} & 0 \\ 0 & I_m \end{bmatrix},$$

conditions (31) can be reduced, with regard for the Schur lemma, to the form

$$W_{[C,D]}^T \begin{bmatrix} A^T X + XA + C^T QC & XB + C^T QD \\ B^T X + D^T QC & D^T QD - P \end{bmatrix} W_{[C,D]} < 0, \tag{32}$$

$$W_{[B^T,D^T]}^T \begin{bmatrix} AY + YA^T + BP^{-1}B^T & YC^T + BP^{-1}D^T \\ CY + DP^{-1}B^T & DP^{-1}D^T - Q^{-1} \end{bmatrix} W_{[B^T,D^T]} < 0, \tag{33}$$

where $Y = X^{-1}$. If the matrix inequality (30) is solvable, then it is always possible to choose its solution K_0 such that the matrix $I_l + DK_0$ is nonsingular. In this case,

$$I_l + DK_0 = (I_l - DK_*)^{-1}$$

and

$$K_* = K_0(I_l + DK_0)^{-1}. \tag{34}$$

Theorem 4. *There exists a matrix K_* for which $J_{P,Q} < 1$ iff the system of linear matrix inequalities (32) and (33) is solvable with respect to mutually reciprocal matrices $X = X^T > 0$ and $Y = Y^T > 0$. In this case, the closed system (1), (24) with uncertainty (28) is robust stable with the general Lyapunov function $v(x) = x^T Xx$.*

The algorithm of finding the matrix K_* satisfying Theorem 4 is based on the solution of linear matrix inequalities under certain additional restrictions.

Algorithm 2.

1. Determination of the matrices $W_{[C,D]}$ and $W_{[B^T,D^T]}$.

2. Determination of the matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying conditions (32) and (33) and $XY = I_n$.

3. Solution of the linear matrix inequalities (30) with respect to K_0 under the restriction $\det(I_l + DK_0) \neq 0$.

4. Determination of the matrix K_* according to formula (34).

In Lemma 2 and Theorem 4, the matrices P and Q are given. However, we can assume that, in the presented algorithm of robust stabilization, they are unknown and determine them together with the positive definite matrices X and Y. Moreover, we can take into account the uncertainty

$$A \in \mathrm{Co}\{A_1, \dots, A_\alpha\} \triangleq \left\{ \sum_{i=1}^{\nu} \alpha_i A_i : \alpha_i \geq 0, i = \overline{1, \nu}, \sum_{i=1}^{\nu} \alpha_i = 1 \right\}.$$

In this case, it is necessary to solve a system of 2α linear matrix inequalities of the form (32) and (33) for each vertex A_i of the given polytope. In Lemma 2, we can also consider the uncertainties $B \in \mathrm{Co}\{B_1, \dots, B_\beta\}$ and $C \in \mathrm{Co}\{C_1, \dots, C_\gamma\}$ with the use of the corresponding systems of linear matrix inequalities.

For system (1) with the trivial initial vector, we now consider the quality criterion (26) and a class of dynamic regulators

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky + w, \quad \xi(0) = 0, \tag{35}$$

where $w \in \mathbb{R}^m$ is the vector of input signals. Under the condition that $K \in \mathcal{K}_D$, the combined system is reduced to the form

$$\dot{\widehat{x}} = \widehat{M}\widehat{x} + \widehat{N}w, \quad y = \widehat{F}\widehat{x} + \widehat{G}w, \quad \widehat{x}(0) = 0, \tag{36}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \quad \widehat{M} = \begin{bmatrix} A + BK_0C & BU_0 \\ V_0C & Z_0 \end{bmatrix}, \quad \widehat{N} = \begin{bmatrix} B + BK_0D \\ V_0D \end{bmatrix},$$

$$\widehat{F} = [C + DK_0C, DU_0], \quad \widehat{G} = D + DK_0D,$$

$$K_0 = \mathcal{D}(K), \quad U_0 = (I_m - KD)^{-1}U, \quad V_0 = V(I_l - DK)^{-1}, \quad Z_0 = Z + VD(I_m - KD)^{-1}U.$$

If $\widehat{M}$ is the Hurwitz matrix, then, by Lemma 2, $J_{P,Q} < 1$ if and only if the following linear matrix inequality is satisfied for a matrix $\widehat{X} = \widehat{X}^T > 0$:

$$\begin{bmatrix} \widehat{M}^T\widehat{X} + \widehat{X}\widehat{M} + \widehat{F}^TQ\widehat{F} & \widehat{X}\widehat{N} + \widehat{F}^TQ\widehat{G} \\ \widehat{N}^T\widehat{X} + \widehat{G}^TQ\widehat{F} & \widehat{G}^TQ\widehat{G} - P \end{bmatrix} < 0. \tag{37}$$

In this case, system (36) with uncertainty (28) is robust stable with the general Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$. Representing relation (37) in the form of a linear matrix inequality for the unknown K_0, U_0, V_0, and Z_0, we find

$$\widehat{L}^T \widehat{K}_0 \widehat{R} + \widehat{R}^T \widehat{K}_0^T \widehat{L} + \widehat{S} < 0, \tag{38}$$

where

$$\widehat{S} = \begin{bmatrix} A^T X + XA & A^T X_1^T & XB & C^T \\ X_1 A & 0 & X_1 B & 0 \\ B^T X & B^T X_1^T & -P & D^T \\ C & 0 & D & -Q^{-1} \end{bmatrix}, \quad \widehat{L}^T = \begin{bmatrix} XB & X_1^T \\ X_1 B & X_2 \\ 0 & 0 \\ D & 0 \end{bmatrix},$$

$$\widehat{R} = \begin{bmatrix} C & 0 & D & 0 \\ 0 & I_r & 0 & 0 \end{bmatrix}, \quad \widehat{K}_0 = \begin{bmatrix} K_0 & U_0 \\ V_0 & Z_0 \end{bmatrix}.$$

In this case, the matrices of regulator (35) and blocks of the matrix $\widehat{K}_0$ are connected by relations (23).

Repeating the proof of Theorem 4 for system (36), we arrive at the following proposition.

Theorem 5. *There exists a dynamic regulator (35) for which $J_{P,Q} < 1$ if and only if the system of relations (20), (32), and (33) is solvable with respect to the matrices $X = X^T > 0$ and $Y = Y^T > 0$. In this case, the closed system (1), (35) with uncertainty (28) is robust stable with the general Lyapunov function $v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}$, where $\widehat{X}$ is a solution of the linear matrix inequality (37).*

We now present the algorithm of construction of the dynamic regulator (35) satisfying Theorem 5.

Algorithm 3.

1. Determination of the matrices $W_{[C,D]}$ and $W_{[B^T,D^T]}$.

2. Determination of the matrices $X = X^T > 0$ and $Y = Y^T > 0$ satisfying relations (20), (32), and (33).

3. Formation of the block mutually reciprocal matrices

$$\widehat{X} = \begin{bmatrix} X & X_1^T \\ X_1 & X_2 \end{bmatrix} > 0, \quad \widehat{Y} = \begin{bmatrix} Y & Y_1^T \\ Y_1 & Y_2 \end{bmatrix} > 0, \quad \widehat{X}\widehat{Y} = I_{n+r}.$$

4. Solution of the linear matrix inequality (38) with respect to $\widehat{K}_0$ under the restriction $\det(I_l + DK_0) \neq 0$.

5. Determination of the matrices of regulator (35) from relations (23).

In item 3 of the proposed algorithm, one can use the Frobenius formula for the inversion of the block matrices [13]. According to this formula,

$$X = Y^{-1} + Y^{-1} Y_1^T H^{-1} Y_1 Y^{-1}, \quad X_1 = -H^{-1} Y_1 Y^{-1}, \quad X_2 = H^{-1},$$

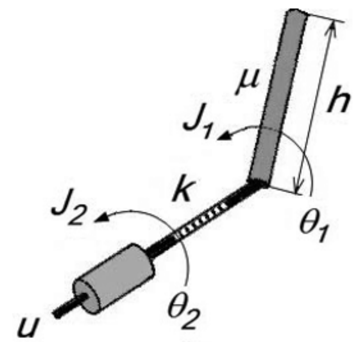

Fig. 1. One-link robot-manipulator.

where

$$H = Y_2 - Y_1 Y^{-1} Y_1^T.$$

If some matrices X_1 and H satisfy the relations

$$X - Y^{-1} = X_1^T H X_1 \geq 0, \quad H = H^T > 0, \quad \text{and} \quad \operatorname{rank} X_1 \leq r,$$

then we can set $X_2 = H^{-1}$, $Y_1 = -HX_1Y$, and $Y_2 = H + HX_1YX_1^TH$. In particular, under the conditions $r = n$ and $X > Y^{-1}$, we can take $X_1 = X_2 = X - Y^{-1}$ and $H = (X - Y^{-1})^{-1}$.

Example 1. Consider a system of control of a one-link robot-manipulator. In this system, the rotational motion of the link around one of the ends is realized with the help of a flexible joint of the link and a driving mechanism (Fig. 1).

This system is described by two nonlinear second-order differential equations guaranteeing the mechanical balance of the driving mechanism (shaft of an electric motor) and the link of the robot-manipulator in the absence of friction forces and external perturbations [14]. The equations of motion of the system can be represented in the vector-matrix form (4), where

$$x = \begin{bmatrix} \theta_1 \\ \dot\theta_1 \\ \theta_2 \\ \dot\theta_2 \end{bmatrix}, \quad A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -[\mu g h \varphi(\theta_1) + k]/J_1 & 0 & k/J_1 & 0 \\ 0 & 0 & 0 & 1 \\ k/J_2 & 0 & -k/J_2 & -d/J_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/J_2 \end{bmatrix},$$

x_1 and x_2 are angular coordinates of the link of manipulator and the shaft of the motor, respectively, u is the controlling moment created by the motor, J_1 and J_2 are the moments of inertia of the link of manipulator and the shaft of the motor, respectively, k is the stiffness of the connecting mechanism, d is the damping coefficient, μ is the mass of the link of manipulator, h is its length, g is the gravitational acceleration, $\mu g h \sin\theta_1$ is the moment of the gravity forces acting upon the link of manipulator, and $\varphi(\theta) = (\sin\theta)/\theta$ is a continuous function.

Let $\mu g h = 5$, $d = 0.1$, $k = 100$, $J_1 = 1$, and $J_2 = 0.3$. Assume that we measure the output vector

$$y = Cx + Du = \begin{bmatrix} \theta_1 \\ \dot\theta_2 + u \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By using Algorithm 1, we get the following matrices of the dynamic regulator (3) guaranteeing the asymptotic stability of the linear system (18) with the spectral stability margin $\alpha = 0.3$:

$$K = \begin{bmatrix} 0.41138 & 1.02011 \end{bmatrix}, \quad U = \begin{bmatrix} -0.01041 & 1.26562 & 5.01012 & 5.63158 \end{bmatrix},$$

$$V = \begin{bmatrix} 40.87656 & 12.66551 \\ -82.39505 & 2.64321 \\ 1.12738 & 1.01236 \\ 0.43389 & 0.45288 \end{bmatrix},$$

$$Z = \begin{bmatrix} -221.68319 & -73.24345 & 435.55701 & -16.56658 \\ -5.86971 & -7.50591 & -13.13094 & -3.32791 \\ -122.55623 & -99.16994 & -54.17854 & -48.20623 \\ 726.00834 & 651.89123 & 284.48100 & -0.45528 \end{bmatrix}.$$

The trivial solution of the closed nonlinear system (3), (4) is also asymptotically stable.

In addition, for $P = 1$ and $Q = 0.01 I_2$, by using Algorithm 3, we construct the dynamic regulator (35) with the matrices

$$K = \begin{bmatrix} 29.27198 & 17.72540 \end{bmatrix}, \quad U = \begin{bmatrix} -136.98479 & 1.68417 & 159.99785 & -4.63821 \end{bmatrix},$$

$$V = \begin{bmatrix} -8.92308 & 29.89135 \\ 8.78891 & -134.01536 \\ -0.46761 & -4.38040 \\ -0.03049 & 10.43232 \end{bmatrix},$$

$$Z = \begin{bmatrix} -4.25434 & -31.38366 & 9.09675 & 61.90396 \\ 0.95272 & -0.44282 & -0.00310 & 1.39153 \\ -4.49349 & 57.91671 & -0.29637 & -201.07858 \\ 0.15745 & 1.47507 & 1.01022 & -8.30203 \end{bmatrix}$$

guaranteeing the following estimate of the quality criterion: $J_{P,Q} < 1$. We also determine the matrices

$$X = \begin{bmatrix} 504.20760 & -107.19979 & -103.12295 & -17.22386 \\ -107.19979 & 168.16951 & 133.38328 & 50.16634 \\ -103.12295 & 133.38328 & 684.59654 & 40.47486 \\ -17.22386 & 50.16634 & 40.47486 & 16.68031 \end{bmatrix},$$

$$Y = \begin{bmatrix} 50.08075 & -31.18632 & 50.37249 & -25.00599 \\ -31.18632 & 246.76214 & -33.75284 & 40.51271 \\ 50.37249 & -33.75284 & 52.75269 & -32.09252 \\ -25.00599 & 40.51271 & -32.09252 & 707.83260 \end{bmatrix}$$

satisfying the system of linear matrix inequalities (20), (32), and (33). As the supplementing blocks X_1 and X_2, we take $X - Y^{-1}$. In this case, the trivial solution of the closed nonlinear system (4), (35) with uncertainty (28) is

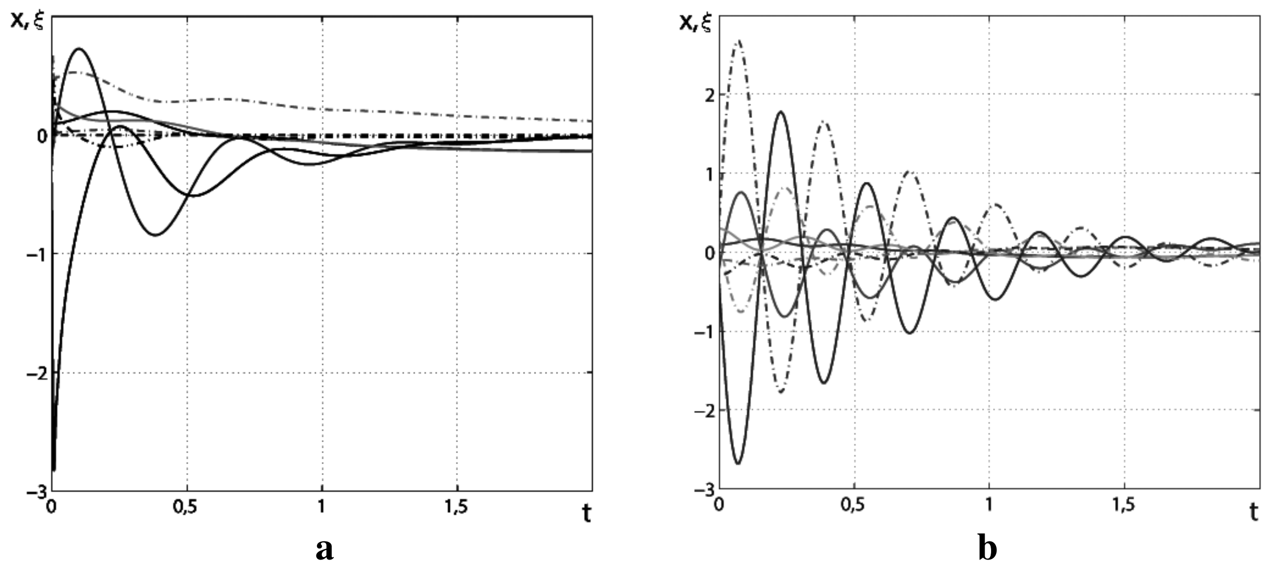

Fig. 2. Behavior of a closed control system: (a) algorithm 1; (b) algorithm 3.

robust stable and the analyzed system possesses the general Lyapunov function

$$v(\widehat{x}) = \widehat{x}^T \widehat{X} \widehat{x}.$$

In Fig. 2, we present the behavior of solutions of the closed nonlinear control system (3), (4) with an initial vector

$$\widehat{x}_0 = [0.1, -0.2, 0.3, -0.4, -0.1, 0.2, -0.3, 0.4]^T$$

in the presence of a dynamic regulator of the total order $r = 4$ with the matrices K, U, V, and Z obtained by using Algorithms 1 and 3. The continuous and dash-dotted lines show the trajectories of the system $x_i(t)$ and the regulator $\xi_i(t)$, $i = \overline{1, 4}$, respectively.

5. Conclusions

We establish new criteria of stabilization of the linear systems with the help of static and dynamic feedbacks by the measurable output and propose new methods for the construction of regulators guaranteeing the asymptotic stability of the equilibrium state of a certain class of nonlinear control systems. For the class of linear systems, we develop the algorithms of construction of the regularities of control guaranteeing the estimation of the quality criterion used to describe the weighed level of attenuation of the input signals and the robust stabilization of the system with respect to a given set of uncertainties. The numerical realization of the proposed methods of construction of the stabilizing regulators is reduced to the solution of systems of linear matrix inequalities. For this purpose, it is possible to use fairly efficient tools of the Matlab computer system.

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