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Stability, Oscillations and Optimization of Systems
A Series of Monographs, Textbooks and Lecture Notes

Volume 2

MATRIX EQUATIONS, SPECTRAL PROBLEMS AND STABILITY OF DYNAMIC SYSTEMS

A.G. Mazko

Matrix equations, Spectral problems and Stability of Dynamic Systems
The volume contains the methods for localization of eigen values of matrices and matrix functions, based on the construction and study of the generalized Lyapunov equation. The theory of linear equations and operators in a matrix space is developed and the known theorems on the inertia of Hermitian solutions of matrix equations are generalized. The author develops new algebraic methods for stability analysis, and an evaluation of spectrum and representation of solutions of linear arbitrary order differential and difference systems.
This monograph is intended for researchers, engineers, and post-graduates interested in the theory of stability and stabilization of dynamic systems, matrix analysis and its applications.

About the Author

A.G.Mazko is Leading researcher, Institute of Mathematics, National Academy of Sciences of Ukraine, Kiev. He is the author or coauthor of more than 75 research papers, including the monograph Localization of Spectrum and Stability of Dynamic Systems (Institute of Mathematics, National Academy of Sciences of Ukraine, 1999). Dr. Mazko received a PhD degree in physics and mathematics from the Institute of Mathematics, National Academy of Sciences of Ukraine, Kiev in 1995.

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Matrix Equations, Spectral Problems and Stability of Dynamic Systems

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CONTENTS

INTRODUCTION TO THE SERIES

Modern stability theory, oscillations and optimization of nonlinear systems have developed in response to the practical problems of celestial mechanics and applied engineering has become an integral part of human activity and development at the end of the 20th century.

For a process or a phenomenon, such as atom oscillations or a supernova explosion, if a mathematical model is constructed in the form of a system of differential equations, then the investigation is possible either by a direct (numerical as a rule) integration of the equations or by analysis by qualitative methods.

In the 20th century, the fundamental works by Euler (1707 – 1783), Lagrange (1736 – 1813), Poincaré (1854 – 1912), Liapunov (1857 – 1918) and others have been thoroughly developed and applied in the investigation of stability and oscillations of natural phenomena and the solution of many problems of technology.

In particular, the problems of piloted space flights and those of astrodynamics were solved due to the modern achievements of stability theory and motion control. The Poincaré and Liapunov methods of qualitative investigation of solutions to nonlinear systems of differential equations in macroworld study have been refined to a great extent though not completed. Also modelling and establishing stability conditions for microprocesses are still at the stage of accumulating ideas and facts and forming the principles; for examples, the stability problem of thermonuclear synthesis.

Obviously, this is one of the areas for the application of stability and control theory in the 21th century. The development of efficient methods and algorithms in this area requires the interaction and publication of ideas and results of various mathematical theories as well as the co-operation of scientists specializing in different areas of mathematics and engineering.

The mathematical theory of optimal control (of moving objects, water resources, global processes in world economy, etc.) is being developed in terms of basic ideas and results obtained in 1956 – 1961 and formulated in Pontryagin's principle of optimality and Bellman's principle of dynamical programming. The efforts of many scholars and engineers in the framework of these ideas resulted in the efficient methods of control for many concrete systems and technological processes.

Thus, the development of classical ideas and results of stability and control theory remains the principle direction for scholars and experts modern stage of the mathematical theories. The aim of the International book series; **Stability, Oscillations and Optimization of Systems** is to provide a medium for the rapid publication of high quality original monographs in the following areas:

Development of the theory and methods of stability analysis:

a. Nonlinear Systems (ordinary differential equations, partial differential equations, stochastic differential equations, functional differential equations, integral equations, difference equations, etc.)
b. Nonlinear operators (bifurcations and singularity, critical point theory, polystability, etc.)

Development of up-to-date methods of the theory of nonlinear oscillations:

a. Analytical methods.
b. Qualitative methods.
c. Topological methods.
d. Numerical and computational methods, etc.

Development of the theory and up-to-date methods of optimization of systems:

a. Optimal control of systems involving ODE, PDE, integral equations, equations with retarded argument, etc.
b. Nonsmooth analysis.
c. Necessary and sufficient conditions for optimality.
d. Hamilton-Jacobi theories.
e. Methods of successive approximations, etc.

Applications:

a. Physical sciences (classical mechanics, including fluid and solid mechanics, quantum and statistical mechanics, plasma physics, astrophysics, etc.).

b. Engineering (mechanical engineering, aeronautical engineering, electrical engineering, chemical engineering).
c. Mathematical biology and life sciences (molecular biology, population dynamics, theoretical ecology).
d. Social sciences (economics, philosophy, sociology).

In the forthcoming publications of the series the readers will find fundamental results and survey papers by international experts presenting the results of investigations in many directions of stability and control theory including uncertain systems and systems with chaotic behaviour of trajectories.

It is in this spirit that we see the importance of the "Stability, Oscillations and Optimization of Systems" series, and we are would like to thank Cambridge Scientific Publishers, Ltd. for their interest and cooperation in publishing this series.

PREFACE

In modern applied mathematics, spectral and algebraic techniques of the study of dynamic systems is being rapidly developed based on the use of matrix equations and inequalities. The Lyapunov matrix equation presents a constructive method of research, which is being successfully used, not only in problems of stability analysis, but also in the design of controllable systems with prescribed quality.

The constantly growing requirements for the quality of designed objects lead to the use of complex mathematical models and the necessity of construction of analogues of the Lyapunov equation as techniques of analysis and synthesis of the corresponding classes of systems. The most essential results in these directions have been obtained in the last two decades and published in scientific journals.

This book is dedicated to the development of new, and systematization of the known, methods of research of dynamic systems of various types, based on the construction and the study of the generalized Lyapunov equation. The main attention is given to the problems of localization of the spectrum and estimation of stability of wide classes of linear differential systems most often occurring in applications. The mathematical rationale of the described methods is the theorems on solvability and inertia of Hermitian solutions of matrix equations, as well as the theory of linear equations with positively invertible operators in a partially ordered space.

In Chapters 1 and 2, methods for construction of analogues of the Lyapunov equations for matrices and matrix functions are proposed. The properties of solutions of such equations are formulated as generalized Lyapunov and Ostrowsky–Schneider theorems describing the location of eigenvalues with respect to relatively wide classes of analytic curves. Systems of spectrum splitting and solutions of generalized block spectral problems are determined and used.

The method of generalized Lyapunov equations underlies the proposed algebraic criteria and the sufficient stability conditions for some classes of differential, difference, differential-difference, and stochastic systems (Chapter 3). A general technique for construction of solutions of differential and difference systems based on the solution of the respective block spectral problem is also proposed.

In Chapter 4 the theory of Sylvester matrix equations of the general form is set forth, which includes new and already known methods of transformations, analysis of solvability conditions and construction of solutions. The main results of this chapter are several theorems generalizing Hill's and Schneider's theorems on inertia of Hermitian solutions of transformable matrix equations.

Chapter 5 is devoted to the study of the stability conditions for the dynamic systems in a partially ordered Banach space. Classes of positive and monotone systems with respect to given cones of a phase space are determined. The main results of the research are the criteria of asymptotic stability of linear positive systems, stated in terms of positive operators, methods for the stability investigation of nonlinear monotone systems, as well as the development of methods of comparison of systems in a partially ordered space. These results, taking into consideration the supplements, allow us to consider the earlier studied matrix problems and the stability problem from the general positions of the theory of operators in a partially ordered space.

In the supplementary Chapter 6 representations of linear operators acting in matrix and other partially ordered spaces are studied. Special attention is given to the description of classes of positive and positively invertible operators with respect to a given cone, in particular, the set of nonnegative definite matrices.

The book mainly contains the results of the author's works published in periodicals. I hope that it will be useful to many researchers developing the methods of analysis and synthesis of dynamic systems, and also to engineers, post-graduates and students of higher technical educational institutions.

I hereby express my deep gratitude to Professor A. A. Martynyuk for his advice and comments which contributed to the improvement of the contents of the book. I would like to thank my colleagues from

the Institute of Mathematics of the National Academy of Sciences of Ukraine for useful discussions and technical support.

A. G. Mazko

NOTATION

$$\left.\begin{array}{l} R^n \text{ — real} \\ C^n \text{ — complex} \end{array}\right\} \; n \text{ dimensional vector space;}$$

$$\left.\begin{array}{l} R^{n\times m} \text{ — space of real} \\ C^{n\times m} \text{ — space of complex} \end{array}\right\} \; n \times m \text{ matrices;}$$

$$A = ||a_{ij}||_{i,j=1}^{n,m} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} \text{ — } \begin{array}{l} n \times m \text{ matrix} \\ \text{with entries } a_{ij}; \end{array}$$

$I\,(I_n)$ — unit matrix (of order n);

0 — zero matrix, zero scalar or zero vector;

$$\left.\begin{array}{l} A^T \text{ — transposed} \\ A^* \text{ — conjugate} \\ A^{-1} \text{ — inverse} \\ A^- \text{ — semi-inverse} \\ A^+ \text{ — pseudoinverse} \end{array}\right\} \; \text{matrix to } A;$$

$f(A)$ — analytic function of matrix A;

Λ_f^0 — analytic curve described by the equation $f(\lambda, \bar{\lambda}) = 0$;

$\Lambda_f^{\pm}$ — open domains bounded by the curve Λ_f^0 ;

$\lambda_{\max}(A)\,(\lambda_{\min}(A))$ — maximal (minimal) eigenvalue of the Hermitian matrix $A = A^*$;

$$\left.\begin{array}{l} \sigma(A) \text{ — spectrum} \\ \det A \text{ — determinant} \\ \operatorname{tr} A \text{ — trace} \\ i(A) \text{ — inertia} \\ \operatorname{rank} A \text{ — rank} \\ \operatorname{sign} A \text{ — signature} \end{array}\right\} \text{ of matrix } A;$$

$i_+(A)$, $i_-(A)$, $i_0(A)$ — number of positive, negative, and zero eigenvalues of the matrix A, taking into account the multiplicities;

$i_f^+(A)$, $i_f^-(A)$, $i_f^0(A)$ — number of points of the spectrum $\sigma(A)$, belonging to the respective sets Λ_f^+, Λ_f^-, Λ_f^0;

$$\left.\begin{array}{l} A \otimes B \text{ — Kronecker product} \\ A \odot B \text{ — Schur product} \end{array}\right\} \text{ of matrices } A \text{ and } B;$$

$\oint\limits_{\omega}$ — Cauchy type integral over a closed contour ω;

$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{X}_{pq}$, $\mathcal{Y}_{pq}$ — sets in a matrix space;

$\mathcal{H}_n$ ($\mathcal{K}_n$) — set of Hermitian (nonnegative definite) $n \times n$ matrices;

LX, $L_f X$, $M_f X$ — linear operators (transformation) of X;

E — identity operator;

$\rho(L)$ — spectral radius of operator (matrix);

$\ker L$ — kernel of operator (matrix);

$\mathcal{K}$ — cone in a partially ordered space $\mathcal{E}$;

$\mathcal{K}_0$ — set of inner points of the cone $\mathcal{K}$;

$X \overset{\mathcal{K}}{\geq} Y$, $X \overset{\mathcal{K}}{\leq} Y$, $X \overset{\mathcal{K}}{>} Y$, $X \overset{\mathcal{K}}{<} Y$ — inequalities between elements X and Y, generated by the cone $\mathcal{K}$;

$$\left.\begin{array}{l} r(X) \text{ — rank} \\ s(X) \text{ — signature} \\ i(X) \text{ — inertia} \end{array}\right\} \quad \text{of } X \in \mathcal{E};$$

$\begin{pmatrix} q \\ p \end{pmatrix} = C_p^q$ — number of combinations from p elements on q, equal to $\dfrac{p!}{q!(p-q)!}$.

0

PRELIMINARIES

0.1 The Object and Review of the Book

During the study and creation of controllable physical objects (transport, electromechanical, extraterrestrial, etc.) there occur problems of stability and quality of the systems that describe their motion. Modern methods of solving such problems are based on application of the state space procedure and matrix analysis and oriented at the use of the scope of computation engineering.

The dynamics of many real objects is adequately modelled by differential or difference systems of the form

$$F(D)\,x(t) = g(t), \quad t \geq t_0, \tag{0.1.1}$$

where $F(\lambda)$ is an analytic matrix function, x is a state vector of the system, D is an operator of time differentiation or a shift operator, g is a vector of external forces (operating controls, random perturbations, etc.). The main subclasses of systems (0.1.1) are determined by the structure and properties of the matrix function $F(\lambda)$:

$F(\lambda) = \lambda I - A$ — systems of equations of Cauchy type;

$F(\lambda) = A - \lambda B$ — systems of equations not solved with respect to derivatives or iterations (descriptor systems);

$F(\lambda) = A + \lambda B + \lambda^2 C$ — systems of second-order differential or difference equations;

$F(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^s A_s$ — systems of s–th order differential or difference equations;

$F(\lambda) = A_0 + \lambda A_1 + \sum\limits_{k>1} e^{-\lambda\tau_{k-1}} A_k$ — delay systems.

Numerous problems of mechanics, mathematical physics and analysis lead to systems of the form (0.1.1). In the study of mechanical systems, a particular role is played by linear and quadratic pencils whose coefficients are formed on the basis of expressions for kinetic and potential energies and dissipative functions. Matrix polynomials of higher degrees occur, e.g., in the control problems with dynamic state-feedback.

Stability conditions and quality indices of systems of the type (0.1.1) are described by location of some algebraic or transcendent equations, i. e. eigenvalues of the matrix function $F(\lambda)$, composing its spectrum $\sigma(F)$ in the complex plane. E.g., the number

$$\alpha = -\max\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(F)\}$$

determines the *stability factor* and the estimate for the time of the transient process $t \le 1/\alpha$. Localization of the spectrum inside a vertical band is used at construction of majorants and minorants of the transient process. Localization of the spectrum of a stable system between two beams passing through the point of origin secures the oscillation of the system, to not exceed the prescribed one. The requirement for aperiodicity of the system in terms of spectrum means that all eigenvalues are located on the real axis. In applications there occur systems with more complex limitations on the domain of the possible location of the spectrum.

Stability of the motion of some classes of nonstationary systems is also related to the problem of spectrum location. Thus, the condition of the asymptotic stability of linear systems of the Cauchy type with periodic coefficients is the location of the spectrum of the *monodromy matrix* inside a unit disk with its center in the point of origin. Spectral methods of analysis and synthesis of systems of the type (0.1.1) are successfully used in the study of more complex classes of nonlinear nonstationary systems.

The increasing requirements for the quality of designed systems make it necessary to study the general problems of eigenvalues location of matrix polynomials and functions. These problems are the natural generalization of the *Routh–Hurwitz problem* formulated for the roots of characteristic polynomials in the stability theory of autonomous systems.

The basics of classical methods of solving the Routh–Hurwitz problem for simple domains are laid out in the works of Cauchy, Sturm, Hermite, Routh, Hurwitz, Lienard, Shepard, Schur, and others (see references in Gantmacher [1] and Krein, Naimark [1]). One of the principal methods of solving this problem is the famous *Lyapunov theorem*. According to it, the spectrum of a matrix A is located in the open left half-plane if and only if for any given positive definite Hermitian matrix $Y = Y^* > 0$ the linear algebraic matrix equation

$$-AX - XA^* = Y \tag{0.1.2}$$

has the unique solution $X = X^* > 0$.

The Lyapunov criterion, as against the determinant conditions of Routh–Hurwitz, is written in terms of the system coefficients without direct calculation of its characteristic polynomial. This is above all the reason for its wide application, not only in the study of practical stability problems, but also in problems of synthesis of controllable systems with given properties, as well as in other areas of applied mathematics and physics.

For the *Lyapunov equation* (0.1.2) the *inertia theorem* is proved which describes the location of the spectrum of a matrix A with respect to an imaginary axis in terms of inertia indices of the solution X (*Ostrowsky–Schneider and Tausski theorems*). The Lyapunov theorem and inertia theorem are applied to some classes of algebraic domains used in the study of the matrix spectrum. At the same time respective analogues of the Lyapunov matrix equation are constructed.

Substantial interest is now shown in the construction and application of analogues of the Lyapunov equation for different classes of dynamic systems. Such equations are presented in the form

$$\sum_{i,j} c_{ij} A_i X A_j^* = Y, \tag{0.1.3}$$

where c_{ij} are scalar coefficients, A_i is a set of given matrices, and X and Y are Hermitian matrices subject to determination.

For the class of equations of the type (0.1.3) with simultaneously triangulable matrix coefficients, inertia theory has been developed which determines spectral conditions of solvability and inertial properties of Hermitian solutions in the form of generalization of Lyapunov, Ostrowsky–Schneider and Tausski theorems. In particular, in this theory families of functions of matrix can be used, as well as commutative and quasi-commutative sets of matrices.

Matrix coefficients of the known analogues of the Lyapunov equation for differential–difference and stochastic systems are not constrained by any limitations and do not satisfy the conditions of simultaneous reducibility to triangular form. Therefore for such equations a more general inertia theory is required, which does not use such limitations on matrix coefficients.

The aim of this monograph is to get the reader acquainted with general methods of construction, study and application of matrix equations acting as analogues of the Lyapunov equation in problems of stability and spectrum location of the dynamic systems. Such methods can be used while solving practical problems of analysis of stability and synthesis of controllable objects. Their efficiency is proved by the new opportunities arising due to the application of computational technologies. The main advantage of the method of the generalized Lyapunov equation is that the analysis of the quality of the studied object adds up to solving algebraic equations and does not require calculation of the spectrum of matrix functions.

Chapter 1 deals with the problem of distribution of the spectrum of a matrix A with respect to the sets Λ_f^+, Λ_f^-, and Λ_f^0 consisting of those points $\lambda \in C^1$ for which the values of the Hermitian function $f(\lambda, \bar{\lambda})$ are respectively positive, negative, and zero. It is required to estimate the numbers i_f^+, i_f^-, and i_f^0 equal to the number of points of the spectrum $\sigma(A)$, taking into account the multiplicities, belonging respectively to Λ_f^+, Λ_f^-, and Λ_f^0. In particular, we are interested in the criteria of the belonging of the whole spectrum to each of the mentioned sets. The problem is studied by the method of the matrix equation

$$L_f X = Y$$

with the *Krein–Daletskii operator* L_f. Spectral and algebraic properties of operators of the type L_f are studied. Proceeding from eigenvalues, components and eigenvectors of the matrix A, methods of determination of eigenelements of the operator L_f are proposed. One of the main results of the first chapter is the generalized Lyapunov theorem derived on the basis of auxiliary statements on positivity and positive invertibility of the operator L_f with respect to a cone of nonnegative definite matrices. The class of domains for which the criterion of inclusion $\sigma(A) \subset \Lambda_f^+$ is obtained for $f \in \mathcal{H}_0^m$ is maximum allowable (within the framework of matrix equations). This class contains all known domains for which the Lyapunov theorem is generalized. In the generalized inertia theorem the numbers $i_f^\pm$ are determined in terms of inertia of solutions of the inequality $L_f X > 0$, where $f \in \mathcal{H}_2^m$ is some class of functions. For estimation of i_f^0 the solutions of the homogeneous equation $L_f X = 0$ are used. The technique for extension of the sets of matrices X and Y used for localization of the spectrum in the generalized Lyapunov equation is described. The author also proposes a modification of the known method of spectrum localization which adds up to determination of the characteristic polynomial of some λ–matrix.

Developed in Chapter 2 are methods of the generalized Lyapunov equation in the study of spectral properties of matrix functions $F(\lambda)$. Linear, quadratic and polynomial pencils of matrices are considered, as well as analytic matrix functions allowing regular factorization. In the general case, some subset of spectrum $\sigma_0(F)$ is separated which consists of r eigenvalues, taking into account the multiplicities, and the quantities of whose points belong to the given sets Λ_f^+, Λ_f^-, and Λ_f^0 are determined. The possibilities of solution of the eigenvalues localization problem are studied with the use of the matrix equation

$$M_f X = Y,$$

where M_f is the generalization of the operator L_f. Proposed are various techniques for construction of the operator M_f, connected with methods of spectrum splitting. The most general theorem on eigenvalues location are formulated on the basis of the introduced notions of right and left pairs of the matrix function $F(\lambda)$. Proposed are sufficient conditions for localization of the spectrum $\sigma(F)$ in some domains, that are based on the solution of systems of matrix equations and inequalities.

Chapter 3 is dedicated to the application of the results of the first two chapters to the analysis of linear dynamic systems, most frequently occurring in applications. Proposed for a linear controllable object is a method for construction of a controller ensuring the best value of the averaged quality functional and location of a system spectrum in a given domain. This method generalizes the known algorithms of suboptimal stabilization and allows not only the carrying out of the minimization of the functional, but at the same time effective provision of quality performance of the controlled system by way of solving the generalized Lyapunov equation. Formulated for linear descriptor systems, as well as for second-order differential and difference systems, are the new criteria of asymptotic stability and methods of Lyapunov functions construction, that are based on solving matrix equations. A technique of analysis of stability of differential-difference and stochastic systems is described, which is based on solving the Sylvester equation. For the analysis and computational construction of solutions of systems of the type (0.1.1) a general technique is proposed which uses the properties of the right pairs of matrix functions.

In Chapter 4, the theory of linear matrix equations of the general form is described. The inertial properties of the solutions of symmetric equations of the form (0.1.3) are studied, which are transformable to a special form. The main results of this chapter are given in the form of generalized theorems of Hill and Schneider. During the study of matrix equations a number of new facts are found related to the determination of the rank and signature of matrices. The methods for computational and analytic construction of solutions of linear matrix equations are described, following from the general theory. Criteria of solvability of the Sylvester equation with arbitrary matrix coefficients are formulated, as well as the criterion of stability of the class of linear positive systems, ensuing from the integral representation of solutions of operator equations.

The results described in Chapters 1 to 4 show that for each dynamic system of the type (0.1.1) it is possible to construct matrix equations (0.1.3) with properties of their solutions connected with the stability conditions and the location of the spectrum of the given system. Conversely, to each matrix equation (0.1.3) some class of linear systems of the form (0.1.1) corresponds, with their dynamic properties described in the form of inertia theorems for a given equation. This is the main reason for the great attention given to the theory of linear equations and operators in the space of matrices.

In Chapter 5 linear and nonlinear dynamic systems in a partially ordered Banach space are considered. Classes of positive and monotone systems with respect to given cones of a phase space are determined. The main results of the study are the criteria of asymptotic stability of linear positive systems, formulated in terms of positive and positively invertible operators, as well as the development of the methods of analysis of robust stability and the known principle of comparison of systems in a partially ordered space.

In the supplementary Chapter 6 methods of representation of linear operators in the space of matrices are given, as well as the basic properties of the latter. The main attention is paid to the classes of positive and positively invertible operators with respect to a cone of nonnegative definite matrices. The results of the study of the class of linear equations

$$LX - PX = Y, \tag{0.1.4}$$

are given, where X and Y are elements of some partially ordered space $\mathcal{E}$ with a cone $\mathcal{K}$, and L and P are given operators satisfying the condition $P\mathcal{K} \subseteq L\mathcal{K}$. In particular, it is supposed that the operator P is *positive*, and the operator L *is positively invertible*, i.e. $P\mathcal{K} \subseteq \mathcal{K} \subseteq L\mathcal{K}$. In wide assumptions, equations with linear operators occurring in applications are representable in the form (0.1.4). In particular, the class of generalized matrix equations of Lyapunov and Sylvester studied in Chapters 1 to 4 can be represented in the form (0.1.4). Here $\mathcal{K}$ is a cone of Hermitian nonnegative definite matrices. The properties of operators L and P are used in the study of the iteration process

$$X_0 = G, \quad LX_{k+1} = PX_k + Y, \quad k = 0, 1, \dots,$$

as a method of monotone approximation to the solutions of the equation (0.1.4).

The main substance of the book is the results of the authors published works. The known results are used in logical constructs or for comparison. The list of literature does not pretend to be complete and only contains the publications that were most available for the author.

0.2 Notes and References

Classes of dynamic systems of the form (0.1.1) occur in problems of analysis and synthesis of controllable objects. Solutions, stability property and quality indices of such systems are described in terms of spectral characteristics of a matrix $F(\lambda)$. The study of spectral theory of matrix and operator pencils is described in Markus [1], Gohberg, Sigal [1], Rezvan [1], Krein, Langer [1], Gantmacher [1], Kublanovskaya [1], and others.

In the systems theory and applications the methods of spectrum localization in the complex plane are very important. Classical methods for estimation of polynomial roots are described in Gantmacher [1], Krein, Naimark [1], Postnikov [1], Cebotarev, Meiman [1], Parodi [1], Jury [1], and others.

The matrix equation (0.1.2) is known as the Lyapunov equation for continuous systems (see Lyapunov [1] and Gantmacher [1]). Its solutions have unique properties and are widely used in various problems of analysis and synthesis of systems (see, e.g., Andreev [1], Afanasiev, Kolmanovskii, Nosov [1], Ikramov [1], Zubov [1], Kuntsevich, Lychak [1], Aliev, Larin [1], Anderson, Moor [1], Boyd, Ghaoui, Feron, Balakrishman [1], Martynyuk [1]).

The inertia theorem for the Lyapunov equation has been obtained in Ostrowsky, Schneider [1] and Taussky [1]. Generalizations of the Lyapunov theorem and the inertia theorem for some classes of equations of the form (0.1.3) have been proved by Jury [1], Kalman [1], Mazko [1, 2, 5, 6, 7], Kharitonov [1], Gutman, Chojnowski [2], Carlson, Hill [1], Wimmer [1,2], and others.

Analogues of the Lyapunov equation for matrix polynomials and functions have been constructed by Mazko [8, 11–13, 15, 25, 28].

Obtained in Schneider [1] and Hill [1] were inertia theorems for equations of the form (0.1.3) with simultaneously triangulable matrix coefficients. In Mazko [19–21, 23] these results were extended to more general classes of equations, using the notions of collectives and transformations.

Matrix equations of the form (0.1.3) occur in problems of stability of differential-difference and stochastic systems (see Korenevskii, Mazko [1, 2], Korenevskii [1], Zelentzovsky [1], Valeev, Karelova, Gorelov [1], Skorodinskii [1]).

The study of linear equations of the form (0.1.4) and operators in partially ordered spaces leads to the natural generalization of some facts of the theory of matrix equations and methods of spectrum localization (see e. g. Krasnoselskii, Lifshits, Sobolev [1], Mazko [30–32], Schneider [1]).

1

LOCATION OF MATRIX SPECTRUM WITH RESPECT TO PLANE CURVES

1.0 Introduction

The Gershgorin theorem and its generalizations give descriptions of domains in the complex plane, containing the eigenvalues of a given matrix. In this chapter the reader will find the description of an alternative approach to the study of spectral properties of a matrix. We assume that the domain of possible location of the spectrum is given, and look for the conditions of the desired clustering of eigenvalues with respect to the domain boundary. In particular we are interested in the criteria of belonging of the whole spectrum to a given domain. The main results in this direction are related to the computation of inertia indices of Hermitian solutions of linear matrix equations (the generalized Lyapunov equation).

Recall that the *inertia* of the Hermitian matrix $X = X^*$ is represented by the ordered triple of numbers

$$i(X) = \{i_+(X), i_-(X), i_0(X)\},$$

which is determined by the numbers of its positive (i_+), negative (i_-), and zero (i_0) eigenvalues, taking into account the multiplicities.

In Section 1.1 the technique of description of curves and domains in the complex plane by using Hermitian functions is described. A general problem of distribution of eigenvalues of an arbitrary complex matrix with respect to plane curves is defined.

Section 1.2 is devoted to the study of the class of integral operators L_f of Cauchy type acting in the space of matrices and being the generalization of the Lyapunov operator $LX = AX + XA^*$. Spectral and algebraic properties of operators L_f are described, as well as the representations of their eigenelements.

Described in Section 1.3 is the full proof of the generalized Lyapunov theorem which is the criterion of belonging of the matrix spectrum to an arbitrary domain from some maximum allowed class. There is also a number of auxiliary propositions on positivity and positive invertibility of the operator L_f with respect to a cone of nonnegative definite matrices, and on the solvability of the generalized Lyapunov equation $L_f X = Y$.

Section 1.4 is devoted to the study of the maximum class of Hermitian functions $\mathcal{H}_0^m$ satisfying the generalized Lyapunov theory. Its main subclasses are distinguished, including the known ones, and several examples of domains are given that are bounded by remarkable algebraic and transcendental curves.

In Section 1.5 the maximally generalized inertia theorem is studied, presenting a method for distribution of a matrix spectrum with respect to analytic curves in terms of inertia of solutions of matrix equations or inequalities.

Formulated in Section 1.6 are the conditions of location of matrix eigenvalues on plane curves with the use of solutions of the homogeneous matrix equation $L_f X = 0$.

Estimates and methods of localization of a matrix spectrum are described in Section 1.7. A modification of the known method of spectrum localization is also proposed, which adds up to the determination of the characteristic polynomial of some λ matrix.

In Section 1.8 the technique of extension of the sets of Hermitian matrices X and Y is described that are used for spectrum localization in the generalized Lyapunov equation. The concept of controllability of a pair of matrices is used, as well as some of its generalizations.

1.1 Description of Domains of the Complex Plane

Let $f(\lambda, \mu)$ be a complex function of two variables, uniquely determined in some domain and satisfying the identity

$$f(\lambda, \mu) \equiv \overline{f(\bar{\mu}, \bar{\lambda})}, \quad \lambda \in C^1, \quad \mu \in C^1. \tag{1.1.1}$$

All values of this function for $\mu = \bar{\lambda}$ are real. In the plane C^1 determine the sets

$$\Lambda_f^+ = \left\{\lambda \colon\, f(\lambda, \bar{\lambda}) > 0\right\}, \tag{1.1.2}$$

$$\Lambda_f^- = \left\{\lambda \colon\, f(\lambda, \bar{\lambda}) < 0\right\}, \tag{1.1.3}$$

$$\Lambda_f^0 = \left\{\lambda \colon\, f(\lambda, \bar{\lambda}) = 0\right\}. \tag{1.1.4}$$

These sets are described by the real function $g(x, y) = f(\lambda, \bar{\lambda})$, where $x = \operatorname{Re}\lambda$, $y = \operatorname{Im}\lambda$. Conversely, if some real function $g(x, y)$ is given, then the sets of points on which it takes on positive, negative, and zero values can be described as (1.1.2)–(1.1.4), assuming

$$f(\lambda, \bar{\lambda}) = g\left(\frac{\lambda + \bar{\lambda}}{2}, \frac{\lambda - \bar{\lambda}}{2i}\right), \quad \lambda = x + iy.$$

Here the function f is *Hermitian*, i.e. it satisfies the identity (1.1.1).

Assign a class of Hermitian functions with separable variables

$$f(\lambda, \bar{\mu}) = \sum_{p,q} \gamma_{pq} f_p(\lambda)\, \overline{f_q(\mu)} \equiv z_\lambda \Gamma z_\mu^*, \tag{1.1.5}$$

where γ_{pq} are entries of the matrix Γ, and $f_p(\lambda)$ are components of the vector function $z_\lambda = [f_1(\lambda), \ldots, f_k(\lambda)]$. If the matrix Γ is Hermitian, then the function (1.1.5) satisfies the identity (1.1.1). The converse is true when the functions $f_1(\lambda), \ldots, f_k(\lambda)$ are linearly independent.

The locus of the type (1.1.4) can be considered as a curve separating the domains $\Lambda_f^\pm$ in the complex plane. The boundary $\partial\Lambda_f^+$ ($\partial\Lambda_f^-$) of the domain Λ_f^+ (Λ_f^-) may not coincide with Λ_f^0. In particular, if $z_\lambda = \left[1, \lambda, \ldots, \lambda^{k-1}\right]$, then the function (1.1.5) describes the algebraic curve Λ_f^0 of order $r \leq 2k - 2$, for which the inclusions $\partial\Lambda_f^+ \subset \Lambda_f^0$ and $\partial\Lambda_f^- \subset \Lambda_f^0$ hold true.

If we have several curves of the form (1.1.4), then for the description of various domains separated by those curves in the complex plane, one can use the properties of so-called R-functions. For example, for the intersection and union of domains of the form (1.1.2), corresponding to the functions f and φ, the following relations are true:

$$\Lambda_f^+ \cap \Lambda_\varphi^+ = \Lambda_u^+, \qquad \Lambda_f^+ \cup \Lambda_\varphi^+ = \Lambda_v^+, \tag{1.1.6}$$

where

$$u(\lambda,\bar\lambda) = f(\lambda,\bar\lambda) + \varphi(\lambda,\bar\lambda) - \sqrt{f^2(\lambda,\bar\lambda) + \varphi^2(\lambda,\bar\lambda)} = \\ = z_\lambda \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} z_\lambda^*,$$

$$v(\lambda,\bar\lambda) = f(\lambda,\bar\lambda) + \varphi(\lambda,\bar\lambda) + \sqrt{f^2(\lambda,\bar\lambda) + \varphi^2(\lambda,\bar\lambda)} = \\ = z_\lambda \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} z_\lambda^*,$$

$$z_\lambda = \left[(1+i)/2, \psi_\lambda, \sqrt{\psi_\lambda}\right], \quad \psi_\lambda = f(\lambda,\bar\lambda) + i\varphi(\lambda,\bar\lambda).$$

If the real functions $g(x,y) = f(\lambda,\bar\lambda)$ and $h(x,y) = \varphi(\lambda,\bar\lambda)$ for $\lambda = x+iy \in \Lambda_v^+$ are continuous and satisfy the Cauchy–Riemann conditions

$$\frac{\partial g(x,y)}{\partial x} = \frac{\partial h(x,y)}{\partial y}, \qquad \frac{\partial g(x,y)}{\partial y} = -\frac{\partial h(x,y)}{\partial x},$$

then ψ_λ is an analytic function of λ, and the functions u and v describing the sets (1.1.6) are representable in the form (1.1.5).

Let an arbitrary matrix $A \in C^{n\times n}$ be given. Its *spectrum* $\sigma(A) = \{\sigma_1,\dots,\sigma_n\}$ is composed of n eigenvalues, taking into account the multiplicities. If $\lambda_1,\dots,\lambda_\alpha$ are all pairwise distinct points $\sigma(A)$, then the *characteristic and minimal polynomials* of the matrix A have the form

$$\chi(\lambda) = (\lambda-\lambda_1)^{n_1} \dots (\lambda-\lambda_\alpha)^{n_\alpha},$$

$$\Theta(\lambda) = (\lambda-\lambda_1)^{m_1} \dots (\lambda-\lambda_\alpha)^{m_\alpha},$$

where

$$m_t \le n_t, \quad m \le n, \quad m = \sum_t m_t, \quad n = \sum_t n_t, \quad t = \overline{1,\alpha}.$$

Determine the quantity of the points of the spectrum $\sigma(A)$, belonging to each of the sets (1.1.2)–(1.1.4):

$$i_f^+(A) = \sum_{\lambda_t \in \Lambda_f^+} n_t, \quad i_f^-(A) = \sum_{\lambda_t \in \Lambda_f^-} n_t, \quad i_f^0(A) = \sum_{\lambda_t \in \Lambda_f^0} n_t. \tag{1.1.7}$$

It is required to construct classes of functions f and matrices A on which the functionals (1.1.7) have prescribed properties. In particular, we are interested in the conditions under which the equality $i_f^+(A) = n$ equivalent to the location of all eigenvalues of the matrix A in the domain (1.1.2) would hold true.

The problem makes sense if at least the values of the function $f(\lambda_t, \bar{\lambda}_t)$ are determined. Let $\mathcal{H}$ denote a set of Hermitian functions f for which all the partial derivatives

$$f_{ij}(\lambda_t, \bar{\lambda}_\tau) = \frac{\partial^{i+j-2}}{\partial \lambda_t^{i-1} \partial \bar{\lambda}_\tau^{j-1}} f(\lambda_t, \bar{\lambda}_\tau),$$

are determined that compose the block Hermitian $m \times m$ matrix

$$\Gamma_f \begin{pmatrix} m_1 & \cdots & m_\alpha \\ \lambda_1 & \cdots & \lambda_\alpha \end{pmatrix} \triangleq \begin{bmatrix} F_{11} & \cdots & F_{1\alpha} \\ \cdots & \cdots & \cdots \\ F_{\alpha 1} & \cdots & F_{\alpha\alpha} \end{bmatrix}, \tag{1.1.8}$$

where

$$F_{t\tau} = \begin{bmatrix} f_{11}(\lambda_t, \bar{\lambda}_\tau) & \cdots & f_{1m_\tau}(\lambda_t, \bar{\lambda}_\tau) \\ \cdots & \cdots & \cdots \\ f_{m_t 1}(\lambda_t, \bar{\lambda}_\tau) & \cdots & f_{m_t m_\tau}(\lambda_t, \bar{\lambda}_\tau) \end{bmatrix}, \quad t, \tau = \overline{1, \alpha}.$$

If $m_t = m_\tau = 1$, then the respective block $F_{t\tau}$ represents the value of the function $f(\lambda_t, \bar{\lambda}_\tau)$. For the functions (1.1.5) the matrix (1.1.8) is representable as

$$\Gamma_f \begin{pmatrix} m_1 & \cdots & m_\alpha \\ \lambda_1 & \cdots & \lambda_\alpha \end{pmatrix} = Z \Gamma Z^*, \tag{1.1.9}$$

where Z is a rectangular matrix whose rows are the derivatives $z_\lambda^{(i-1)}$ for $\lambda = \lambda_t$, $i = \overline{1, m_t}$, $t = \overline{1, \alpha}$.

Hermitian matrices of the type (1.1.8), (1.1.9) in particular, will be used in the further study related to finding the admissible subclasses of functions of the class $\mathcal{H}$ in the process of solving the set problem.

1.2 Operator L_f

In the matrix space consider the linear operator L_f, defined by

$$L_f X = -\frac{1}{4\pi^2} \oint\limits_{\omega_1} \oint\limits_{\omega_2} f(\lambda, \mu)(A - \lambda I)^{-1} X (B - \mu I)^{-1} d\lambda\, d\mu, \quad (1.2.1)$$

where $A \in C^{n\times n}$ and $B \in C^{s\times s}$ are given square matrices, I is a unit matrix of appropriate dimensions, ω_1 (ω_2) is a simple closed contour containing the spectrum $\sigma(A)$ ($\sigma(B)$) and separating in the complex plane the closed domain Ω_1 (Ω_2), and f is a single-valued function which has no singularity in the domain $\Omega_1 \times \Omega_2$.

Assign a family of operators of the form (1.2.1) that preserve the set of Hermitian matrices:

$$L_f X = -\frac{1}{4\pi^2} \oint\limits_{\omega_1} \oint\limits_{\omega_2} f(\lambda, \bar{\mu})(A - \lambda I)^{-1} X (A - \mu I)^{-1^*} d\lambda\, d\bar{\mu}. \quad (1.2.2)$$

In this case $B = A^*$, and the function $f \in \mathcal{H}$ satisfies the identity (1.1.1). In particular, for the function (1.1.5) the operator (1.2.2) reduces to the form

$$L_f X = \sum_{p,q} \gamma_{pq} f_p(A) X f_q(A)^*, \quad (1.2.3)$$

where the matrix coefficients represent analytic *functions of matrix* A:

$$f_p(A) = -\frac{1}{2\pi i} \oint\limits_{\omega_1} f_p(\lambda)(A - \lambda I)^{-1} d\lambda.$$

The operators (1.2.2) and (1.2.3) underlie our investigations related to the solution of the problem of distribution of the spectrum $\sigma(A)$.

Operators of the form (1.2.1), in particular (1.2.2) and (1.2.3), have interesting algebraic properties. If the conditions

$$f(\lambda, \mu) \neq 0, \quad \lambda \in \sigma(A), \quad \mu \in \sigma(B), \quad (1.2.4)$$

hold true, then the operator (1.2.1) is invertible, and the inverse operator has the same form:

$$L_f^{-1} = L_\varphi, \quad \varphi(\lambda, \mu) = \frac{1}{f(\lambda, \mu)}. \quad (1.2.5)$$

The spectrum $\sigma(L_f)$ of the operator (1.2.1) is formed by ns values of the function f, determined in (1.2.4). Any operators L_{f_1} and L_{f_2} of the form (1.2.1) commute and satisfy the relations

$$L_{f_1}L_{f_2} = L_{f_1 f_2}, \quad c_1 L_{f_1} + c_2 L_{f_2} = L_{c_1 f_1 + c_2 f_2}, \tag{1.2.6}$$

where c_1 and c_2 are arbitrary constants. Moreover, if we have a given set of operators $L_{f_1}, \ldots, L_{f_k}$ and a given function $g(z_1, \ldots, z_k)$, single-valued and analytic in the neighbourhood of the set $\sigma(L_{f_1}) \times \cdots \times \sigma(L_{f_k})$, then

$$g(L_{f_1}, \ldots, L_{f_k}) = L_{g(f_1, \ldots, f_k)}. \tag{1.2.7}$$

Here the function of the family of commuting operators has the form

$$g(L_{f_1}, \ldots, L_{f_k}) = \nu_k \oint_{\sigma_1} \cdots \oint_{\sigma_k} g(z_1, \ldots, z_k) \prod_{j=1}^{k} (z_j E - L_{f_j})^{-1} dz,$$

where $\nu_k = 1/(2\pi i)^k$, $dz = dz_1 \ldots dz_k$, E is an identity operator, σ_j is a closed contour containing the spectrum of the operator L_{f_j}. The formula (1.2.7) is proved on the basis of the relations (1.2.5) and (1.2.6).

We will show that each operator of the type (1.2.1) is representable as

$$L_f X = \sum_{p=0}^{m-1} \sum_{q=0}^{r-1} \gamma_{pq} A^p X B^q, \tag{1.2.8}$$

where γ_{pq} are some coefficients and $m(r)$ is the degree of the minimal polynomial of the matrix $A(B)$. Let $\lambda_1, \ldots, \lambda_\alpha$ $(\mu_1, \ldots, \mu_\beta)$ denote all pairwise distinct points of the spectrum $\sigma(A)$ $(\sigma(B))$ with the respective indices $m_1, \ldots, m_\alpha$ $(r_1, \ldots, r_\beta)$. Use the expansions of resolvents

$$(\lambda I - A)^{-1} = \sum_{t=1}^{\alpha} \sum_{i=1}^{m_t} \frac{(i-1)!}{(\lambda - \lambda_t)^i} A_{ti},$$

$$(\mu I - B)^{-1} = \sum_{\tau=1}^{\beta} \sum_{j=1}^{r_\tau} \frac{(j-1)!}{(\mu - \mu_\tau)^j} B_{\tau j}.$$

The matrices A_{ti} ($B_{\tau j}$) are linearly independent and called *components of the matrix* $A(B)$. Substituting these relations into (1.2.1) and calculating the derivatives of integrals of Cauchy type, we obtain

$$L_f X = \sum_{t=1}^{\alpha}\sum_{\tau=1}^{\beta}\sum_{i=1}^{m_t}\sum_{j=1}^{r_\tau} f_{ij}(\lambda_t,\mu_\tau)A_{ti}XB_{\tau j}, \tag{1.2.9}$$

where

$$f_{ij}(\lambda_t,\mu_\tau) = -\frac{(i-1)!(j-1)!}{4\pi^2}\oint\limits_{\omega_1}\oint\limits_{\omega_2}\frac{f(\lambda,\mu)\,d\lambda d\mu}{(\lambda-\lambda_t)^i(\mu-\mu_\tau)^j} =$$

$$= \frac{\partial^{i+j-2}}{\partial\lambda_t^{i-1}\partial\mu_\tau^{j-1}}\,f(\lambda_t,\mu_\tau).$$

The components $A_{ti} = \alpha_{ti}(A)$ ($B_{\tau_j} = \beta_{\tau j}(B)$) are scalar polynomials of A (B) with their degrees not exceeding $m-1$ ($r-1$). Therefore the expression (1.2.9) for the operator (1.2.1) is reducible to the form (1.2.8).

Note that the properties (1.2.5)–(1.2.7) of the class of operators (1.2.1) can be found proceeding from the representation (1.2.9), by calculation of higher order partial derivatives for sums and products of the respective functions, and by using the properties of matrix components as well. In particular, the pairwise commuting components of the matrix A have the following properties:

$$A_{t1}^2 = A_{t1}, \quad \sum_{t=1}^{\alpha} A_{t1} = I, \quad A_{ti} = \frac{1}{(i-1)!}\,(A-\lambda_t I)^{i-1}\,A_{t1},$$

$$A_{ti}A_{\tau j} = \begin{cases} 0, & t\neq\tau \quad \text{or} \quad k>m_t, \\ \begin{pmatrix} i-1 \\ k-1 \end{pmatrix} A_{tk}, & t=\tau \quad \text{and} \quad k\le m_t, \end{cases} \tag{1.2.10}$$

$$k = i+j-1, \quad i=\overline{1,m_t}, \quad j=\overline{1,m_\tau}, \quad t,\tau=\overline{1,\alpha}.$$

Components of the matrix B have similar properties.

In the expansion (1.2.9) isolate the terms whose left (right) matrix coefficients form the spectral projectors A_{t1} ($B_{\tau 1}$) of the matrix

$A(B)$:

$$L_f X = \sum_{t=1}^{\alpha} \sum_{\tau=1}^{\beta} \left[f(\lambda_t, \mu_t) A_{t1} X B_{\tau 1} + N_{t\tau} X \right], \tag{1.2.11}$$

where

$$N_{t\tau} X = \sum_{i+j>2} f_{ij}(\lambda_t, \lambda_\tau) A_{ti} X B_{\tau j}, \quad t = \overline{1, \alpha}, \quad \tau = \overline{1, \beta}.$$

Using the properties of matrix components, one can find that all operators $N_{t\tau}$ in (1.2.11) are nilpotent.

We will study the spectral properties of the operator (1.2.1), proceeding from its representation (1.2.11). In particular, we are interested in the structure of eigenelements (eigenvectors) of the operator (1.2.1) and the conditions of existence of nonnegative definite matrices acting as eigenelements of this operator.

Consider the homogeneous matrix equation

$$L_f W = w W. \tag{1.2.12}$$

Each nonzero solution $W = W(w)$ of this equation is an *eigenelement* of the operator (1.2.1), corresponding to the *eigenvalue*

$$w \in \left\{ f(\lambda_t, \mu_\tau) \colon t = \overline{1, \alpha},\, \tau = \overline{1, \beta} \right\}.$$

Lemma 1.2.1 *Matrix W is an eigenelement of the operator (1.2.1), corresponding to the eigenvalue w if and only if it is representable in the form*

$$W = \sum_{(t,\tau) \in \Theta_w} A_{t1} H_{t\tau} B_{\tau 1} \neq 0, \tag{1.2.13}$$

where $\Theta_w \neq \emptyset$ is a set of pairs (t, τ), for which $f(\lambda_t, \mu_\tau) = w$, $H_{t\tau}$ are some matrices satisfying the conditions

$$N_{t\tau} H_{t\tau} = 0, \quad (t, \tau) \in \Theta_w. \tag{1.2.14}$$

Proof. Let $\Theta_w \neq \emptyset$. Then any matrix of the form (1.2.13) under the conditions (1.2.14) satisfies the equation (1.2.12). This is proved directly by substituting (1.2.13) into (1.2.12), using the properties of matrix coefficients of the expansion (1.2.11). Thus, assuming $H_{t\tau} = A_{tm_t} C_{t\tau} B_{\tau r_\tau}$, $C_{t\tau} \in C^{n\times s}$, one can find that the conditions (1.2.14) hold true. In this case we have the eigenelements of the operator (1.2.1) of the form

$$W = \sum_{(t,\tau)\in\Theta_w} A_{tm_t} C_{t\tau} B_{\tau r_\tau} \neq 0.$$

We will show that if $W \neq 0$ is a solution of the equation (1.2.12), then $\Theta_w \neq \emptyset$, and for some block matrix

$$H = \begin{bmatrix} H_{11} & \cdots & H_{1\beta} \\ \cdots & \cdots & \cdots \\ H_{\alpha 1} & \cdots & H_{\alpha\beta} \end{bmatrix}$$

the relations (1.2.13)–(1.2.14) hold true. Let $H_{t\tau} = A_{t1} W B_{\tau 1}$. Then, multiplying (1.2.12) from the left (right) by A_{t1} ($B_{\tau 1}$), taking into consideration (1.2.10) and (1.2.11), we obtain the system

$$[w - f(\lambda_t, \mu_\tau)]\, H_{t\tau} = N_{t\tau} H_{t\tau}, \quad t = \overline{1,\alpha}, \quad \tau = \overline{1,\beta}, \tag{1.2.15}$$

which is equivalent to the input equation (1.2.12). If $w = f(\lambda_t, \mu_\tau)$, then the conditions (1.2.14) hold true. If $w \neq f(\lambda_t, \mu_\tau)$, then from the nilpotency of the operator $N_{t\tau}$ follows $H_{t\tau} = 0$. Taking into consideration the properties of projectors A_{t1} and $B_{\tau 1}$, we have the relation between the matrices W and H in the form

$$\begin{gathered} H = \begin{bmatrix} A_{11} \\ \vdots \\ A_{\alpha 1} \end{bmatrix} W \begin{bmatrix} B_{11} & \cdots & B_{\beta 1} \end{bmatrix}, \\ W = \begin{bmatrix} A_{11} & \cdots & A_{\alpha 1} \end{bmatrix} H \begin{bmatrix} B_{11} \\ \vdots \\ B_{\beta 1} \end{bmatrix}. \end{gathered} \tag{1.2.16}$$

Consequently, (1.2.13) and (1.2.14) hold true. At the same time, $\Theta_w \neq \emptyset$. Otherwise, all the blocks $H_{t\tau}$ are zero, and according to (1.2.16), $W = 0$, which is contrary to the assumptions.

The lemma is proved.

Lemma 1.2.2 *The nonnegative definite matrix* $W = W^* \geq 0$ *is an eigenelement of the operator (1.2.2), corresponding to the eigenvalue* w*, if and only if it is representable in the form*

$$W = \sum_{(t,\tau)\in\Theta_w^*} A_{t1} H_{t\tau} A_{\tau 1}^* \neq 0, \tag{1.2.17}$$

where $\Theta_w^* \neq \emptyset$ *is a set of pairs* (t,τ) *for which*

$$f(\lambda_t, \bar{\lambda}_\tau) = f(\lambda_t, \bar{\lambda}_t) = f(\lambda_\tau, \bar{\lambda}_\tau) = w,$$

$H_{t\tau}$ *are blocks of a Hermitian nonnegative definite matrix, satisfying the conditions*

$$N_{t\tau} H_{t\tau} = 0, \quad (t,\tau) \in \Theta_w^*. \tag{1.2.18}$$

Proof. Use the proof of Lemma 1.2.1 in the case of $A = B$. If $\Theta_w \neq \emptyset$, then under the conditions (1.2.18) the matrix (1.2.17) satisfies the equation (1.2.12). In particular, one can assume that

$$W = \sum_{(t,\tau)\in\Theta_w^*} A_{tm_t} C_{t\tau} A_{\tau m_\tau}^* \geq 0,$$

where $C_{t\tau}$ are blocks of an arbitrary matrix $C = C^* > 0$.

If the nonzero matrix $W = W^* \geq 0$ satisfies the equation (1.2.12), then, according to (1.2.16), $H = H^* \geq 0$, and $H \neq 0$. All columns and rows of a nonnegative definite matrix that intersect on zero diagonal elements are equal to zero. Therefore if $w \neq f(\lambda_t, \bar{\lambda}_t)$, then, according to (1.2.15), $H_{tt} = 0$, and with any τ the off-diagonal blocks $H_{t\tau}$ and $H_{\tau t}$ are zero. Consequently, (1.2.13) and (1.2.14) are reducible to the form (1.2.17) and (1.2.18), where $\Theta_w^* \neq \emptyset$.

The lemma is proved.

Note that if the matrices A and B have a simple structure, then the operators $N_{t\tau}$ in (1.2.11) are zero, and in the expressions (1.2.13) and (1.2.17) there are no constraints on blocks $H_{t\tau}$ of the form (1.2.14) and (1.2.18).

Find the relation between the solutions of the equation (1.2.12) and the eigenvectors of the matrices A and B. From the right (left) eigenvectors of the matrix $A(B)$, corresponding to the eigenvalue λ_t (μ_τ), construct the matrices $U_t \in C^{n\times u_t}$ $(V_\tau \in C^{v_\tau \times s})$. These matrices satisfy the relations

$$AU_t = \lambda_t U_t, \qquad V_\tau B = \mu_\tau V_\tau.$$

Let U_t (V_τ) be a matrix of full rank u_t (v_τ) equal to the geometric multiplicity of the eigenvalue λ_t (μ_τ). Proceeding from the basic interpolation properties of the polynomials α_{ti} and $\beta_{\tau j}$, obtain the relations

$$\begin{gathered}A_{ti}U_p = \begin{cases} U_t, & t = p \quad \text{and} \quad i = 1, \\ 0, & t \neq p \quad \text{or} \quad i > 1, \end{cases} \\ V_q B_{\tau j} = \begin{cases} V_\tau, & \tau = q \quad \text{and} \quad j = 1, \\ 0, & \tau \neq q \quad \text{or} \quad j > 1. \end{cases}\end{gathered} \tag{1.2.19}$$

From (1.2.19) it follows that any matrices of the form $H_{t\tau} = U_t S_{t\tau} V_\tau$ satisfy the conditions (1.2.14).

Thus, if $\Theta_w \neq \emptyset$, then according to Lemma 1.2.1, the expression

$$W = \sum_{(t,\tau)\in\Theta_w} U_t S_{t\tau} V_\tau \neq 0 \tag{1.2.20}$$

is an eigenelement of the operator (1.2.1), corresponding to the eigenvalue w. Under the conditions $A = B$ and $\Theta_w^* \neq \emptyset$ the matrix (1.2.20) is reducible to the form

$$W = \sum_{(t,\tau)\in\Theta_w^*} U_t S_{t\tau} U_\tau^* \geq 0 \quad (\neq 0) \tag{1.2.21}$$

and serves as an eigenelement of the operator (1.2.2), corresponding to the eigenvalue w. It is always possible to select the blocks $S_{t\tau}$ so that the rank of the matrix (1.2.21) would take on any value from the interval

$$1 \leq \operatorname{rank} W \leq \sum_{(t,t)\in\Theta_w^*} u_t,$$

where u_t is the geometric multiplicity of the eigenvalue $\lambda_t \in \sigma(A)$.

The described techniques of solving the equation (1.2.12) can be used in the study of a more intricate problem of construction of

Jordan sequences of elements for the operator (1.2.1). The number of such elements for each eigenvalue w of the operator (1.2.1) equals the algebraic multiplicity calculated as

$$k(w) = \sum_{(t,\tau)\in\Theta_w} n_t\, s_\tau,$$

where n_t (s_τ) is the algebraic multiplicity of the eigenvalue λ_t (μ_τ) of the matrix $A(B)$.

1.3 The Generalized Lyapunov Theorem

Consider the linear matrix equation

$$L_f X = Y, \tag{1.3.1}$$

where L_f is an operator of the form (1.2.2), constructed for the given matrix $A \in C^{n\times n}$ and function $f \in \mathcal{H}$. For any matrix $Y \in C^{n\times n}$ the equation (1.3.1) has the unique solution $X \in C^{n\times n}$ if and only if the following inequalities hold true:

$$f(\lambda_t, \bar{\lambda}_\tau) \neq 0, \quad t = \overline{1, \alpha}, \quad \tau = \overline{1, \alpha}, \tag{1.3.2}$$

where $\lambda_1, \ldots, \lambda_\alpha$ are pairwise distinct points of the spectrum $\sigma(A)$. If Y is a Hermitian matrix, then under the conditions (1.1.1) the solution X is also Hermitian.

Let $\mathcal{K}_0$ ($\mathcal{K}$) denote a set of Hermitian positive (nonnegative) definite matrices of order n. The set $\mathcal{K}$ is a reproducing cone of the space $C^{n\times n}$. If $L_f\mathcal{K} \subseteq \mathcal{K}$, then $\mathcal{K}$ is an *invariant set* of the operator L_f.

Study the relation between the spectral properties of the matrix A and the solvability conditions of the equation (1.3.1) in $\mathcal{K}_0$ and $\mathcal{K}$. From Lemma 1.2.2 it follows in particular that under the condition $\sigma(A) \cap \Lambda_f^+ \neq \emptyset$ there exist matrices $X \geq 0$ and $Y \geq 0$ that satisfy the equation (1.3.1).

Lemma 1.3.1 *If for some matrix $Y > 0$ the equation (1.3.1) has a solution $X \geq 0$, then all eigenvalues of the matrix A are located in the domain (1.1.2), i.e.*

$$f(\lambda_1, \bar{\lambda}_1) > 0, \ldots, f(\lambda_\alpha, \bar{\lambda}_\alpha) > 0. \tag{1.3.3}$$

Conversely, if the inequalities (1.3.3) hold true, then there exist matrices $X > 0$ and $Y > 0$ that satisfy the equation (1.3.1).

Proof. Multiplying (1.3.1) from the left (right) by A_{tm_t} ($A^*_{\tau m_\tau}$), taking into account (1.2.10) and (1.2.11) gives

$$f(\lambda_t, \bar{\lambda}_\tau) A_{tm_t} X A^*_{\tau m_\tau} = A_{tm_t} Y A^*_{\tau m_\tau}, \quad t = \overline{1, \alpha}, \quad \tau = \overline{1, \alpha}. \qquad (1.3.4)$$

If $Y > 0$, then in the right-hand side of (1.3.4) for $t = \tau$ there are nonzero nonnegative definite matrices. If, in addition, $X \geq 0$, then the inequalities (1.3.3) hold true. From (1.3.4) it also follows that with $X > 0$ and $Y \geq 0$ all values of the function $f(\lambda_t, \bar{\lambda}_t)$ are nonzero.

We will show that under the conditions (1.3.3) it is possible to construct matrices $X > 0$ and $Y > 0$ satisfying the equation (1.3.1). Let $J = TAT^{-1}$ be the left Jordan form of the matrix A. Transform the equation (1.3.1) to the form

$$\sum_{t=1}^{\alpha} \sum_{\tau=1}^{\alpha} \sum_{i=1}^{m_t} \sum_{j=1}^{m_\tau} f_{ij}(\lambda_t, \bar{\lambda}_\tau)\, \alpha_{ti}(J) H \alpha_{\tau j}(J)^* = G, \qquad (1.3.5)$$

where $H = TXT^*$, $G = TYT^*$. Use some properties of the matrices $\alpha_{ti}(J)$, following from (1.2.10). All elements of the matrices $\alpha_{t1}(J)$ are zero except n_t diagonal elements $\alpha_{t1}(\lambda_t) = 1$. All nonzero elements of the matrices $\alpha_{ti}(J)$ for $i > 1$ are located below the leading diagonal.

Let H_k and G_k be sequential leading submatrices of order k of the corresponding matrices H and G, $k = \overline{1, n}$. Then, according to (1.3.5)

$$H_k = \begin{bmatrix} H_{k-1} & u_k \\ u_k^* & h_{kk} \end{bmatrix}, \quad G_k = \begin{bmatrix} G_{k-1} & v_k \\ v_k^* & f(\sigma_k, \bar{\sigma}_k)\, h_{kk} + w_k \end{bmatrix}, \qquad (1.3.6)$$

where G_{k-1}, v_k, and w_k do not depend on the entry h_{kk} of the matrix H, $k = \overline{2, n}$. Under the conditions (1.3.3) we have the recurrent algorithm of finding a matrix H for which $G > 0$. Apparently, $G_1 = f(\sigma_1, \bar{\sigma}_1) H_1 > 0$ for $H_1 = h_{11} > 0$. If $H_{k-1} > 0$ and $G_{k-1} > 0$, then the inequalities $H_k > 0$ and $G_k > 0$ are achieved by increasing the diagonal entry h_{kk}. Thus, if

$$f(\sigma_k, \bar{\sigma}_k)\, h_{kk} + w_k > v_k^*\, G_{k-1}^{-1}\, v_k, \qquad (1.3.7)$$

then $G_k > 0$. To off-diagonal elements H the arbitrary values $h_{ks} = \bar{h}_{sk}$ can be assigned. Using this technique, one can construct a diagonal matrix $H > 0$ satisfying the inequalities (1.3.7). The sought matrices $X > 0$ and $Y > 0$ for the input equation (1.3.1) are determined in (1.3.5).

The lemma is proved.

Lemma 1.3.2 *$\mathcal{K}$ is an invariant cone of the operator (1.2.2) if and only if the following condition holds true:*

$$\Gamma_f \begin{pmatrix} m_1 & \dots & m_\alpha \\ \lambda_1 & \dots & \lambda_\alpha \end{pmatrix} \geq 0. \tag{1.3.8}$$

The operator (1.2.2) preserves the set of positive definite matrices $\mathcal{K}_0$ if and only if the system of inequalities (1.3.3) and (1.3.8) holds true.

Proof. The inclusion $L_f\mathcal{K} \subseteq \mathcal{K}$ means that in (1.3.1) $X \geq 0$ implies $Y \geq 0$. For each vector $c \in C^n$, introduce a set of pairs of indices $\Delta_c = \{(t,i) \,|\, g_{ti} \neq 0\}$, where $g_{ti} = A_{ti}^* c$, $t = \overline{1,\alpha}$, $i = \overline{1,m_t}$. Using (1.2.9) and (1.3.1), calculate the Hermitian form

$$c^* Y c = \sum_{(t,i)\in\Delta_c} \sum_{(\tau,j)\in\Delta_c} f_{ij}(\lambda_t, \bar{\lambda}_\tau) g_{ti}^* X g_{\tau j} = \operatorname{tr}(Q_c X), \tag{1.3.9}$$

where $Q_c = G_c F_c^T G_c^*$, and F_c is the principal submatrix in (1.1.8) with the entries $f_{ij}(\lambda_t, \bar{\lambda}_\tau)$ for $(t,i) \in \Delta_c$, $(\tau,j) \in \Delta_c$, and the matrix G_c is formed by the column vectors g_{ti} with $(t,i) \in \Delta_c$.

If $c \neq 0$, then $\Delta_c \neq \emptyset$, and all nonzero columns g_{ti} are linearly independent. Indeed, assuming $l_t = \max\{i \,|\, (t,i) \in \Delta_c\}$ and multiplying the linear combination

$$\sum_{(t,i)\in\Delta_c} d_{ti}\, g_{ti} = 0$$

from the left consecutively by $A_{\tau j}^*$ $(j = l_\tau - 1, l_\tau - 2, \dots)$, taking into consideration (1.2.10), obtain $d_{\tau j} = 0$ with $(t,j) \in \Delta_c$. Here it is also taken into account that $(\tau,j) \notin \Delta_c$ implies $(\tau,q) \notin \Delta_c$ with $q > j$.

Since the matrix G_c has full rank with respect to its columns for $c \neq 0$, then the inequalities $F_c \geq 0$ and $Q_c \geq 0$ are equivalent. If the inequalities (1.3.3) hold, then $Q_c \neq 0$. We will show that there exists a vector c for which G_c has the maximal rank m.

Consider the linear combination

$$\sum_{t=1}^{\alpha}\sum_{i=1}^{m_t} d_{ti}\, g_{ti} = \sum_{t=1}^{\alpha}\sum_{i=1}^{m_t} d_{ti}\, \alpha_{ti}(A)^* c = z(A)^* c = 0, \tag{1.3.10}$$

where $z(\lambda) = \sum\limits_{t,i} \bar{d}_{ti}\, \alpha_{ti}(\lambda)$ is a polynomial of order $p < m$. Let c_0 be a vector whose minimal annihilating polynomial with respect to the matrix A^* coincides with the minimal polynomial of order m for this matrix. Then, assuming in (1.3.10) $c = c_0$, we have $z(\lambda) \equiv 0$ and, owing to the independence of the polynomials α_{ti}, all coefficients $d_{ti} = 0$. This means that G_c is a matrix of rank m. For the above mentioned vector c, the principal submatrix F_c coincides with the whole matrix (1.1.8).

According to the Feyer theorem, the inequality $\mathrm{tr}(Q_c X) \geq 0$ holds for any matrix $X \geq 0$ if and only if $Q_c \geq 0$. For any matrix $X > 0$, the strict inequality $\mathrm{tr}(Q_c X) > 0$ holds if and only if $Q_c \geq 0$ and $Q_c \neq 0$. Using these criteria and the determined properties of the matrices Q_c, F_c, and G_c in the relation (1.3.9) for the vector c running through the whole space, we arrive at the propositions of the lemma.

The lemma is proved.

Note that if there exists a matrix $X > 0$ for which $L_f X > 0$, then the inclusions $L_f \mathcal{K}_0 \subseteq \mathcal{K}_0$ and $L_f \mathcal{K} \subseteq \mathcal{K}$ are equivalent.

Lemma 1.3.3 *For any positive definite matrix Y the equation (1.3.1) has a positive definite solution X ($\mathcal{K}_0 \subseteq L_f \mathcal{K}_0$) if and only if the inequalities (1.3.2) hold true, as well as the matrix inequality*

$$\Gamma_\varphi \begin{pmatrix} m_1 & \cdots & m_\alpha \\ \lambda_1 & \cdots & \lambda_\alpha \end{pmatrix} \geq 0, \tag{1.3.11}$$

where $\varphi(\lambda, \bar{\mu}) \triangleq 1/f(\lambda, \bar{\mu})$.

Proof. If the equation (1.3.1) is solvable for any matrix $Y > 0$, then the inequalities (1.3.2) hold true and the operator (1.2.2) is invertible. Indeed, any matrix Y is representable in the form of a linear combination of positive definite matrices Y_k, and the solution X corresponds to it. In particular, one can assume

$$Y = Y_1 - Y_2 + i(Y_3 - Y_4), \quad X = X_1 - X_2 + i(X_3 - X_4),$$

where $X_k > 0$ is a solution of (1.3.1), corresponding to $Y_k > 0$. The inequalities (1.3.2) follow from (1.3.4), as the right-hand sides of these relations, with the proper choice of the matrix Y, can be nonzero.

According to (1.2.5) and (1.2.9), the unique solution of (1.3.1) under the conditions (1.3.2) has the form

$$X = L_\varphi Y = \sum_{t=1}^{\alpha} \sum_{\tau=1}^{\alpha} \sum_{i=1}^{m_t} \sum_{j=1}^{m_\tau} \varphi_{ij}(\lambda_t, \bar{\lambda}_\tau) A_{ti} Y A_{\tau j}^*. \tag{1.3.12}$$

The matrix (1.3.11) is composed of the coefficients of this expression

$$\varphi_{ij}(\lambda_t, \bar{\lambda}_\tau) = \frac{\partial^{i+j-2}}{\partial \lambda_t^{i-1} \partial \bar{\lambda}_\tau^{j-1}} \varphi(\lambda_t, \bar{\lambda}_\tau).$$

Applying Lemma 1.3.2 to the operator (1.3.12), we arrive at the proposition of Lemma 1.3.3.

The lemma is proved.

Lemma 1.3.4 *Let the matrix (1.1.8) have exactly one positive eigenvalue:*

$$i_+ \left(\Gamma_f \begin{pmatrix} m_1 & \dots & m_\alpha \\ \lambda_1 & \dots & \lambda_\alpha \end{pmatrix} \right) = 1. \tag{1.3.13}$$

Then the system of inequalities (1.3.2) and (1.3.11) is equivalent to the inequalities (1.3.3).

Proof. Values of the function $\varphi(\lambda_t, \bar{\lambda}_t)$ are located on the leading diagonal of the matrix (1.3.11). Therefore the inequalities (1.3.3) follow from (1.3.11). We will prove the converse proposition. First of all, under the conditions (1.3.3) and (1.3.13) the inequalities (1.3.2) hold true. Otherwise, in (1.1.8) there is a principal submatrix of the form

$$\begin{bmatrix} f(\lambda_t, \bar{\lambda}_t) & f(\lambda_t, \bar{\lambda}_\tau) \\ f(\lambda_\tau, \bar{\lambda}_t) & f(\lambda_\tau, \bar{\lambda}_\tau) \end{bmatrix} > 0,$$

which is contrary to the condition (1.3.13).

Consider the algebraic function

$$g(\lambda,\bar\mu)=\sum_{t,\tau=1}^{\alpha}\sum_{i=1}^{m_t}\sum_{j=1}^{m_\tau} f_{ij}(\lambda_t,\bar\lambda_\tau)\,\alpha_{ti}(\lambda)\,\overline{\alpha_{\tau j}(\mu)}\equiv \tag{1.3.14}$$

$$\equiv a_\lambda\,\Gamma_f\begin{pmatrix} m_1 & \dots & m_\alpha\\ \lambda_1 & \dots & \lambda_\alpha\end{pmatrix} a_\mu^*.$$

Here the elements of the row vector $a_\lambda \in C^m$ are the polynomials $\alpha_{ti}(\lambda)$ that determine the components $A_{ti}=\alpha_{ti}(A)$ of the matrix A and satisfy the following interpolation conditions:

$$\frac{d^{k-1}}{d\lambda^{k-1}}\,\alpha_{ti}(\lambda)\bigg|_{\lambda=\lambda_\tau}=\begin{cases}1, & t=\tau \quad\text{and}\quad k=i,\\ 0, & t\neq\tau \quad\text{or}\quad k\neq i.\end{cases}$$

Using these conditions, obtain

$$f_{ij}(\lambda_t,\bar\lambda_\tau)=\frac{\partial^{i+j-2}}{\partial\lambda_t^{i-1}\partial\bar\lambda_\tau^{j-1}}\,g(\lambda_t,\bar\lambda_\tau),$$

$$\varphi_{ij}(\lambda_t,\bar\lambda_\tau)=\frac{\partial^{i+j-2}}{\partial\lambda_t^{i-1}\partial\bar\lambda_\tau^{j-1}}\,\psi(\lambda_t,\bar\lambda_\tau),$$

where $\psi(\lambda,\bar\mu)=1/g(\lambda,\bar\mu)$. This means that the matrix Γ_f (Γ_φ) does not change if $g(\psi)$ is used instead of $f(\varphi)$.

Transforming the matrix (1.1.8) to diagonal form under the condition (1.3.13) and using Cauchys inequality, we arrive at relations true in the neighbourhood of the points $(\lambda_t,\bar\lambda_\tau)$, of the form

$$g(\lambda,\bar\mu)=u(\lambda)\,\overline{u(\mu)}\,[1-v(\lambda,\bar\mu)]\neq 0,\quad v(\lambda,\bar\mu)=\sum_s v_s(\lambda)\,\overline{v_s(\mu)},$$

$$|v(\lambda,\bar\mu)|^2\le v(\lambda,\bar\lambda)\,v(\mu,\bar\mu)<1,\quad v(\lambda,\bar\lambda)<1,\quad v(\mu,\bar\mu)<1,$$

$$\psi(\lambda,\bar\mu)=\frac{1}{u(\lambda)\,\overline{u(\mu)}}\sum_{k=0}^{\infty}v^k(\lambda,\bar\mu)=\sum_{k=0}^{\infty}w_k(\lambda)\,\overline{w_k(\mu)},$$

where u, v_s, and w_k are some rational functions constructed from the polynomials α_{ti}. As a result, obtain a matrix inequality

$$\Gamma_\varphi\begin{pmatrix} m_1 & \dots & m_\alpha\\ \lambda_1 & \dots & \lambda_\alpha\end{pmatrix}=\sum_{k=0}^{\infty}d_k\,d_k^*\ge 0,$$

where

$$d_k = [\delta_{k1}(\lambda_1), \dots, \delta_{km_1}(\lambda_1), \dots, \delta_{k1}(\lambda_\alpha), \dots, \delta_{km_\alpha}(\lambda_\alpha)]^T,$$

$$\delta_{ki}(\lambda) = \frac{d^{i-1}}{d\lambda^{i-1}} w_k(\lambda).$$

The lemma is proved.

For a given function f the conditions (1.3.11) and (1.3.13) can be verified if the eigenvalues λ_t of the matrix A and their indices m_t are known. For a diagonalizable matrix, the condition (1.3.11) is represented as

$$\Gamma_\varphi \begin{pmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_m \end{pmatrix} = \begin{bmatrix} \dfrac{1}{f(\lambda_1, \overline{\lambda}_1)} & \cdots & \dfrac{1}{f(\lambda_1, \overline{\lambda}_m)} \\ \cdots & \cdots & \cdots \\ \dfrac{1}{f(\lambda_m, \overline{\lambda}_1)} & \cdots & \dfrac{1}{f(\lambda_m, \overline{\lambda}_m)} \end{bmatrix} \geq 0. \quad (1.3.15)$$

Let $\mathcal{H}_0^m$ be a class of functions $f \in \mathcal{H}$ satisfying the condition (1.3.15) for any set of points $\lambda_1, \dots, \lambda_m$ from the domain (1.1.2).

Consider the case $m = 2$. The class $\mathcal{H}_0^2$ is determined by the inequality

$$\Delta(\lambda, \bar{\mu}) = |f(\lambda, \bar{\mu})|^2 - f(\lambda, \bar{\lambda}) f(\mu, \bar{\mu}) \geq 0, \quad \forall \lambda, \mu \in \Lambda_f^+.$$

The matrix inequality (1.3.11) for $m_1 = m$ and $\lambda_1 = \lambda \in \Lambda_f^+$ reduces to the form

$$\delta(\lambda, \bar{\lambda}) = \frac{\partial f(\lambda, \bar{\lambda})}{\partial \lambda} \frac{\partial f(\lambda, \bar{\lambda})}{\partial \bar{\lambda}} - f(\lambda, \bar{\lambda}) \frac{\partial^2 f(\lambda, \bar{\lambda})}{\partial \lambda \partial \bar{\lambda}} \geq 0. \quad (1.3.16)$$

Proceeding to the limit for $\mu \to \lambda$, we have

$$\psi(\mu, \lambda, \bar{\lambda}) = \frac{f(\mu, \bar{\lambda}) - f(\lambda, \bar{\lambda})}{\mu - \lambda} \quad \to \quad \frac{\partial f(\lambda, \bar{\lambda})}{\partial \lambda},$$

$$\frac{\Delta(\lambda, \bar{\mu})}{|\mu - \lambda|^2} = |\psi(\mu, \lambda, \bar{\lambda})|^2 - f(\lambda, \bar{\lambda}) \frac{\psi(\mu, \lambda, \bar{\mu}) - \psi(\mu, \lambda, \bar{\lambda})}{\bar{\mu} - \bar{\lambda}} \quad \to \quad \delta(\lambda, \bar{\lambda}).$$

Consequently, for the functions $f \in \mathcal{H}_0^2$ the inequality (1.3.16) holds true. A more general statement follows.

Lemma 1.3.5 *If $f \in \mathcal{H}_0^m$, then the inequality (1.3.11) holds true for any sets of points $\lambda_1, \dots, \lambda_\alpha$ from the domain Λ_f^+ and the real numbers $m_1, \dots, m_\alpha$ with their sum not exceeding m.*

Proof. Let $\max\{m_1, \dots, m_\alpha\} \geq 2$. In the neighbourhood of each point $\lambda_t \in \Lambda_f^+$ for $m_t \geq 2$ determine a set of pairwise distinct points $\lambda_{t1}, \dots, \lambda_{tm_t} \in \Lambda_f^+$ and construct a block matrix

$$\Phi = \begin{bmatrix} \Phi_{11} & \cdots & \Phi_{1\alpha} \\ \Phi_{\alpha 1} & \cdots & \Phi_{\alpha\alpha} \end{bmatrix} = \qquad (1.3.17)$$

$$= \Gamma_\varphi \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ \lambda_{11} & \cdots & \lambda_{1m_1} & \cdots & \lambda_{\alpha 1} & \cdots & \lambda_{\alpha m_\alpha} \end{pmatrix},$$

where

$$\Phi_{t\tau} = \begin{bmatrix} \varphi(\lambda_{t1}, \bar{\lambda}_{\tau 1}) & \cdots & \varphi(\lambda_{t1}, \bar{\lambda}_{\tau m_\tau}) \\ \cdots & \cdots & \cdots \\ \varphi(\lambda_{tm_t}, \bar{\lambda}_{\tau 1}) & \cdots & \varphi(\lambda_{tm_t}, \bar{\lambda}_{\tau m_\tau}) \end{bmatrix}, \quad t, \tau = \overline{1, \alpha}.$$

Assume also that $\lambda_t = \lambda_{t1}$ and $\lambda_{t2} - \lambda_{t1} = \cdots = \lambda_{tm_t} - \lambda_{tm_t - 1} = \delta_t$. Using recurrent formulae for approximate computation of higher order derivatives of the function, for $\delta_t \to 0$ and $\delta_\tau \to 0$ obtain

$$\delta_t^{1-i} \delta_\tau^{1-j} \sum_{p=1}^{i} \sum_{q=1}^{j} (-1)^{i+j-p-q} \binom{i-p}{i-1} \binom{j-q}{j-1} \varphi(\lambda_{tp}, \bar{\lambda}_{\tau q}) \ \to$$

$$\to \ \varphi_{ij}(\lambda_t, \bar{\lambda}_\tau).$$

Hence

$$U \Phi U^* \to \Phi_0 = \Gamma_\varphi \begin{pmatrix} m_1 & \dots & m_\alpha \\ \lambda_1 & \dots & \lambda_\alpha \end{pmatrix}, \qquad (1.3.18)$$

where

$$U = \begin{bmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_\alpha \end{bmatrix}, \quad U_t = \begin{bmatrix} u_{t1}^{(1)} & & 0 \\ \vdots & \ddots & \\ u_{t1}^{(m_t)} & \dots & u_{tm_t}^{(m_t)} \end{bmatrix},$$

$$u_{tp}^{(i)} = (-1)^{i-p} \begin{pmatrix} i-p \\ i-1 \end{pmatrix} \delta_t^{1-i}.$$

On the basis of (1.3.17) and (1.3.18) one can construct a sequence of Hermitian matrices Φ_k converging in norm to Φ_0 and such that $i(\Phi_k) = i(\Phi)$, $k = 1, 2, \ldots$. If $f \in \mathcal{H}_0^m$, then $\Phi \geq 0$ and $\Phi_k \geq 0$. Owing to the closedness of the cone of nonnegative definite matrices, we have $\Phi_0 \geq 0$, i.e. the inequality (1.3.11) holds true.

The lemma is proved.

Theorem 1.3.1 *Let a matrix $A \in C^{n\times n}$, a function $f \in \mathcal{H}_0^m$ and an arbitrary positive definite matrix $Y \in \mathcal{K}_0$ be given. Then the spectrum $\sigma(A)$ is located in the domain Λ_f^+ if and only if the equation (1.3.1) has the unique positive definite solution $X \in \mathcal{K}_0$.*

Proof. If for some matrix $Y > 0$ the equation (1.3.1) has a solution $X > 0$, then, according to Lemma 1.3.1, $\sigma(A) \subset \Lambda_f^+$. Here f may be an arbitrary function of the class $\mathcal{H}$.

Let $f \in \mathcal{H}_0^m$ and the inequalities (1.3.3) be true. Then, according to Lemma 1.3.5, the system of inequalities (1.3.2) and (1.3.11) holds true. From Lemma 1.3.3 it follows that for any matrix $Y > 0$ the equation (1.3.1) has a unique solution $X > 0$ of the form (1.3.12).

This proposition can be proved without using Lemma 1.3.5, proceeding from the following reasoning about continuity. An arbitrary matrix A, with the use of infinitesimal perturbations of its entries, can be transformed into a matrix A_ε of a simple structure. Specifically, assuming

$$A_\varepsilon = A + D, \quad D = \operatorname{diag}\{\varepsilon_1, \ldots, \varepsilon_n\},$$

one can choose arbitrary small numbers ε_k so that all eigenvalues of the matrix A_ε will be different and, under the conditions (1.3.3), will belong to the domain (1.1.2). In this case, according to Lemma 1.3.3, for any matrix $Y > 0$ the equation (1.3.1) has a solution $X_\varepsilon > 0$. If $D \to 0$, then $X_\varepsilon \to X \geq 0$, where X is a solution of the equation (1.3.1), corresponding to the matrices A and Y. We will show that $X > 0$. According to Lemma 1.3.1, there exist matrices $X_0 > 0$ and $Y_0 > 0$ that satisfy the equation (1.3.1). Choose a small number $\delta > 0$ so that the inequality $Y - \delta Y_0 > 0$ is true. Then, in accordance with the above proved, for this matrix the equation (1.3.1) has the solution $X - \delta X_0 \geq 0$. Hence, $X > 0$.

The theorem is proved.

Theorem 1.3.1 gives the criterion of the inclusion $\sigma A) \subset \Lambda_f^+$ and is a generalization of the Lyapunov theorem. The used class of functions $f \in \mathcal{H}_0^m$ is in a certain sense the maximum allowed. Indeed, if the condition (1.3.15) is violated for some $\lambda_t \in \Lambda_f^+$, then, according to Lemma 1.3.3, the criterion does not hold true for any matrix A with eigenvalues λ_t, $t = \overline{1,m}$. If the degree of the minimal polynomial m of the matrix A is unknown, then in the conditions of Theorem 1.3.1 one can assume $f \in \mathcal{H}_0^n$.

1.4 Hermitian Functions of the Class $\mathcal{H}_0^m$

Using Theorem 1.3.1, it is necessary to solve the question of belonging of a given function f to the class $\mathcal{H}_0^m$, i.e. to verify that the matrix inequality (1.3.15) holds for any $\lambda_1, \dots, \lambda_m \in \Lambda_f^+$. In $\mathcal{H}_0^m$ choose the important subclasses of Hermitian functions, determined by relations simpler than the inequalities (1.3.15) and containing some known classes of functions. Here we will assume that each of the sets (1.1.2)–(1.1.4) is nonempty.

First of all, note that if a Hermitian function is representable in the form $f(\lambda,\mu) = u(\lambda,\mu) - v(\lambda,\mu)$, and $\Lambda_f^+ \subset \Lambda_u^+$, and the functions u and v for $\forall \lambda, \mu \in \Lambda_f^+$ satisfy the inequalities

$$|u(\lambda,\bar{\mu})|^2 \geq u(\lambda,\bar{\lambda})\, u(\mu,\bar{\mu}), \quad |v(\lambda,\bar{\mu})|^2 \leq v(\lambda,\bar{\lambda})\, v(\mu,\bar{\mu}),$$

then $u(\lambda,\bar{\mu}) \neq 0$, $f(\lambda,\bar{\mu}) \neq 0$ and the expansion

$$\frac{1}{f(\lambda,\bar{\mu})} = \frac{1}{u(\lambda,\bar{\mu})} \sum_{k=0}^{\infty} w^k(\lambda,\bar{\mu}), \tag{1.4.1}$$

holds true, where $w(\lambda,\mu) = v(\lambda,\mu)/u(\lambda,\mu)$, $|w(\lambda,\bar{\mu})| < 1$, $\forall\, \lambda,\mu \in \Lambda_f^+$. If the functions u and v are such that

$$U = \left\| \frac{1}{u(\mu_i,\bar{\mu}_j)} \right\|_1^m \geq 0, \quad V = \|v(\mu_i,\bar{\mu}_j)\|_1^m \geq 0, \tag{1.4.2}$$

for $\forall\ \mu_1, \dots, \mu_m \in \Lambda_f^+$, and $m \geq 2$, then they satisfy the mentioned requirements and, furthermore, the matrix inequality

$$\left\| \frac{1}{f(\mu_i,\bar{\mu}_j)} \right\|_1^m = U + U \odot U \odot V + U \odot U \odot U \odot V \odot V + \cdots \geq 0,$$

is true, which is the consequence of the expansion (1.4.1) and the known properties of the Schur product $\odot$. If this inequality holds for $\forall\, \mu_1, \dots, \mu_m \in \Lambda_f^+$, it means that $f \in \mathcal{H}_0^m$.

Let $\mathcal{H}_0$ be a class of Hermitian functions f for which the following relations hold true:

$$f(\lambda, \bar{\mu}) \neq 0, \ \ \frac{1}{f(\lambda, \bar{\mu})} = \sum_k \varphi_k(\lambda)\, \overline{\varphi_k(\mu)}, \quad \forall\, \lambda, \mu \in \Lambda_f^+, \tag{1.4.3}$$

where φ_k are functions analytical in the domain Λ_f^+. Then $\mathcal{H}_0 \subset \mathcal{H}_0^m$ for any natural m (see the proof of Lemma 1.3.4). The series (1.4.3) can be constructed, proceeding from (1.4.1), for subclasses of functions $\mathcal{H}_1 \subset \mathcal{H}_0^m$ and $\mathcal{H}_2 \subset \mathcal{H}_0^m$ determined by the respective relations

$$f = u - v, \quad u = f_1(\lambda)\, \overline{f_1(\mu)}, \quad v = \sum_{k>1} f_k(\lambda)\, \overline{f_k(\mu)}, \tag{1.4.4}$$

$$f = u - v, \quad u = f_1(\lambda)\, \overline{f_1(\mu)}, \quad v = f_2(\lambda)\, \overline{f_2(\mu)}. \tag{1.4.5}$$

For the functions u and v the matrix inequalities (1.4.2) hold true.

Under the condition

$$i_+(\Gamma) = 1 \tag{1.4.6}$$

each function (1.1.5) is representable in the form (1.4.4) by transforming the matrix Γ to diagonal form. The class $\mathcal{H}_1$ is therefore determined by (1.1.5) and (1.4.6). Similarly, the class $\mathcal{H}_2$ is composed by the functions (1.1.5), for which

$$i_\pm(\Gamma) \le 1. \tag{1.4.7}$$

If $\Lambda_f^+ \neq \emptyset$, then the equality (1.4.6) must hold true.

Let $\mathcal{H}_1^m$ and $\mathcal{H}_2^m$ denote the classes of Hermitian functions satisfying the respective conditions

$$i_+\left(\|f(\mu_i, \bar{\mu}_j)\|_1^m\right) = 1, \quad \forall\, \mu_1, \dots, \mu_m \in \Lambda_f^+;$$

$$i_\pm\left(\|f(\mu_i, \bar{\mu}_j)\|_1^m\right) \le 1, \quad \forall\, \mu_1, \dots, \mu_m \notin \Lambda_f^0.$$

Lemma 1.4.4 directly implies $\mathcal{H}_1^m \subset \mathcal{H}_0^m$, and from the formulae (1.1.5), (1.1.9), and (1.4.6) it follows that $\mathcal{H}_1 \subset \mathcal{H}_1^m$. Similarly, in our assumptions $\mathcal{H}_2^m$ is a subclass in $\mathcal{H}_0^m$ and contains $\mathcal{H}_2$.

Thus, for $m \geq 1$ we have the following inclusions:

$$\begin{array}{ccccccc} \mathcal{H}_2 & \subset & \mathcal{H}_1 & \subset & \mathcal{H}_0 & & \\ \cap & & \cap & & \cap & & \\ \mathcal{H}_2^m & \subset & \mathcal{H}_1^m & \subset & \mathcal{H}_0^m & \subset & \mathcal{H} \end{array}$$

All the described subclasses of Hermitian functions are used in Theorem 1.3.1. If $f \in \mathcal{H}_2^m$, then Theorem 1.3.1 is applicable for the both domains Λ_f^+ and Λ_f^-.

Studying the algebraic functions (1.1.5) for $f_k(\lambda) = \lambda^k$, the limitations have been constructed in the form of nonnegative definiteness of the matrix (see Mazko [2])

$$S_\lambda = \Gamma\, z_\lambda^* z_\lambda\, \Gamma - f(\lambda, \bar{\lambda})\, \Gamma \geq 0, \tag{1.4.8}$$

where λ is an arbitrary point of the domain (1.1.2), and also in terms of rank and signature of the matrix Γ (see Kharitonov [1])

$$\operatorname{rank}\Gamma + \operatorname{sign}\Gamma = 2. \tag{1.4.9}$$

The conditions (1.4.6), (1.4.8), and (1.4.9) are equivalent (see Section 4.2). In the conditions (1.4.7) one can assume

$$\operatorname{rank}\Gamma = 2, \quad \operatorname{sign}\Gamma = 0. \tag{1.4.10}$$

Note that the equality (1.4.6) is equivalent to each of the conditions

$$S = \Gamma\, z_0^* z_0\, \Gamma - z_0 \Gamma z_0^*\, \Gamma \geq 0, \quad z_0 \in Z, \tag{1.4.11}$$

$$G = \left\| \frac{1}{z_i \Gamma z_j^*} \right\|_{i,j=1}^{n} \geq 0, \quad \forall z_1, \ldots, z_m \in Z, \tag{1.4.12}$$

where $z_0, \ldots, z_m$ are vectors from the set $Z = \{z : z\, \Gamma\, z^* > 0\} \neq \emptyset$.

The proof of the equivalence of the relations (1.4.6) and (1.4.11) follows from the more general results of Chapter 4. The equivalence of the inequalities (1.4.11) and (1.4.12) is proved in the process of reduction of the matrix G to diagonal form by using elementary transformations. All the forms of the limitations (1.4.3)–(1.4.12) on the matrix Γ can be used in Theorem 1.3.1 for the analytic functions (1.1.5) with the vectors $z_\lambda, \lambda \in \Lambda_f^+$. The numerical verification of the conditions (1.4.8), (1.4.11), or (1.4.12) is based on the application of the known criteria of sign definiteness of Hermitian matrices. The conditions (1.4.6), (1.4.7), (1.4.9), and (1.4.10) are related to the calculation of the matrix inertia.

Take a subclass of functions (1.1.5) corresponding to the family of second-order algebraic curves, assuming

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & 0 \\ \gamma_{31} & 0 & 0 \end{bmatrix}, \quad z_\lambda = \left[1, \lambda, \lambda^2\right].$$

Calculating the entries s_{ij} of the matrix (1.4.8), obtain:

$$s_{11} = (\gamma_{12}\gamma_{21} - \gamma_{11}\gamma_{22})\,\lambda\bar{\lambda} + \gamma_{12}\gamma_{31}\lambda^2\bar{\lambda} + \gamma_{13}\gamma_{21}\lambda\bar{\lambda}^2 + \gamma_{13}\gamma_{31}\lambda^2\bar{\lambda}^2,$$

$$s_{12} = \bar{s}_{21} = (\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21})\,\lambda + \gamma_{13}\gamma_{22}\lambda\bar{\lambda}^2 - \gamma_{12}\gamma_{31}\lambda^2,$$

$$s_{13} = \bar{s}_{31} = -\gamma_{13}\gamma_{21}\lambda - \gamma_{13}\gamma_{22}\lambda\bar{\lambda} - \gamma_{13}\gamma_{31}\lambda^2,$$

$$s_{22} = \gamma_{21}\gamma_{12} - \gamma_{11}\gamma_{22} - \gamma_{22}\gamma_{13}\lambda^2 - \gamma_{22}\gamma_{31}\lambda^2,$$

$$s_{23} = \bar{s}_{32} = \gamma_{21}\gamma_{13} + \gamma_{22}\gamma_{13}\bar{\lambda},$$

$$s_{33} = \gamma_{13}\gamma_{31}.$$

If $\gamma_{22} \leq 0$ and $\lambda \in \Lambda_f^+$, then all principal minors of the matrix (1.4.8) are nonnegative:

$$s_{11} \geq |\lambda|^2|\gamma_{12} + \gamma_{13} + \gamma_{22}\lambda|^2 \geq 0, \qquad s_{22} \geq |\gamma_{21} + \gamma_{22}\bar{\lambda}|^2 \geq 0,$$

$$s_{33} = |\gamma_{13}|^2 \geq 0, \qquad \det S_\lambda \equiv 0,$$

$$\det\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = -\gamma_{22}|\lambda|^4|\gamma_{13}|^2 f(\lambda,\bar{\lambda}) \geq 0,$$

$$\det\begin{bmatrix} s_{11} & s_{13} \\ s_{31} & s_{33} \end{bmatrix} = -\gamma_{22}|\lambda|^2|\gamma_{13}|^2 f(\lambda,\bar{\lambda}) \geq 0,$$

$$\det\begin{bmatrix} s_{22} & s_{23} \\ s_{32} & s_{33} \end{bmatrix} = -\gamma_{22}|\gamma_{13}|^2 f(\lambda,\bar{\lambda}) \geq 0.$$

Consequently, $f \in \mathcal{H}_1$ if $\gamma_{22} \leq 0$. In this case the equality (1.4.6) holds true.

Examples.

Here are the examples of algebraic and transcendental domains in the complex plane of the form

$$\Lambda_f^+ = \{\lambda\colon f(\lambda, \bar{\lambda}) = z_\lambda \, \Gamma \, z_\lambda^* > 0\},$$

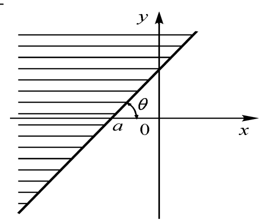
the geometric properties of those domains can be used in the problems of analysis and control of the quality of systems. All functions f describing those domains belong to the class $\mathcal{H}_1$ and, consequently, satisfy the generalized Lyapunov theorem. In addition, in Examples 1, 2, 13, 14, 16, 20–22 the functions $f \in \mathcal{H}_2$ satisfy the conditions of the inertia theorem (see Section 1.5). Unfortunately, general geometric regularities of the domains Λ_f^+ corresponding to the class of functions $f \in \mathcal{H}_0^m$ have not been found yet.

The list of functions below can be considerably extended. When making it out, the author used equations of major algebraic curves of order $p \le 6$, and also of some transcendent curves of the form $\varphi(x, y) = 0$, $x = \operatorname{Re}\lambda$, $y = \operatorname{Im}\lambda$. In the pictures the hatched part of the plane C^1 corresponds to each domain Λ_f^+.

1. Straight line $y \cos\theta = (x - a) sin\theta,\ 0 \le \theta \le \pi/2$.

$$\Gamma = \begin{bmatrix} 2a \sin\theta & -\sin\theta + i\cos\theta \\ -\sin\theta - i\cos\theta & 0 \end{bmatrix},$$

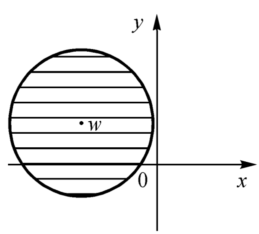
$z_\lambda = [1, \lambda]$.



2. Circle $(x - a)^2 + (y - b)^2 = r^2,\ w = a + ib$.

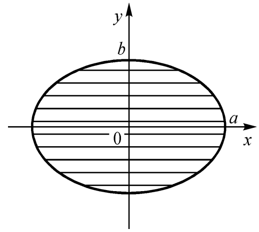
$$\Gamma = \begin{bmatrix} r^2 - |w|^2 & w \\ \bar{w} & -1 \end{bmatrix},$$

$z_\lambda = [1, \lambda]$.



3. Ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > 0$, $b > 0$.

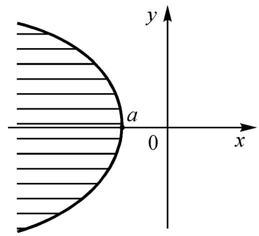
$$\Gamma = \begin{bmatrix} 4a^2b^2 & 0 & a^2-b^2 \\ 0 & -2(a^2+b^2) & 0 \\ a^2-b^2 & 0 & 0 \end{bmatrix},$$



$$z_\lambda = [1, \lambda, \lambda^2].$$

4. Parabola $x = a - by^2$, $a < 0$, $b > 0$.

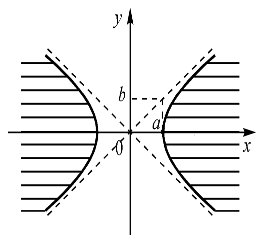
$$\Gamma = \begin{bmatrix} 2a & -1 & b/2 \\ -1 & -b & 0 \\ b/2 & 0 & 0 \end{bmatrix},$$



$$z_\lambda = [1, \lambda, \lambda^2].$$

5. Hyperbola $x^2/a^2 - y^2/b^2 = 1$, $0 < b \leq a$.

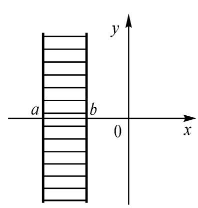
$$\Gamma = \begin{bmatrix} -4a^2b^2 & 0 & a^2+b^2 \\ 0 & 2(b^2-a^2) & 0 \\ a^2+b^2 & 0 & 0 \end{bmatrix},$$



$$z_\lambda = [1, \lambda, \lambda^2].$$

6. Vertical straight lines $(x - a)(b - x) = 0$, $a < b$.

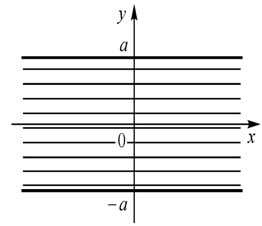
$$\Gamma = \begin{bmatrix} -2ab & a+b & -1/2 \\ a+b & -1 & 0 \\ -1/2 & 0 & 0 \end{bmatrix},$$



$$z_\lambda = [1, \lambda, \lambda^2].$$

7. Horizontal straight lines $y^2 = a^2$, $a > 0$.

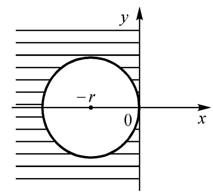
$$\Gamma = \begin{bmatrix} 4a^2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$



$z_\lambda = [1, \lambda, \lambda^2]$.

8. Straight line and circle $x(x^2 + y^2 + 2rx) = 0$, $r \geq 0$.

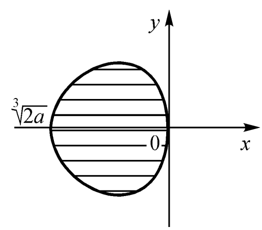
$$\Gamma = \begin{bmatrix} 0 & 0 & -r \\ 0 & -2r & -1 \\ -r & -1 & 0 \end{bmatrix},$$



$z_\lambda = [1, \lambda, \lambda^2]$.

9. Curve $2ax - (x^2 + y^2)^2 = 0$, $a < 0$.

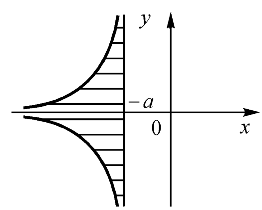
$$\Gamma = \begin{bmatrix} 0 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$



$z_\lambda = [1, \lambda, \lambda^2]$.

10. Curve $(x + a)[y^2(x + a) + b] = 0$, $a > 0$, $b > 0$.

$$\Gamma = \begin{bmatrix} -4ab & -2b & a^2 & a & 1/4 \\ -2b & -2a^2 & -a & 0 & 0 \\ a^2 & -a & -1/2 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 \end{bmatrix},$$

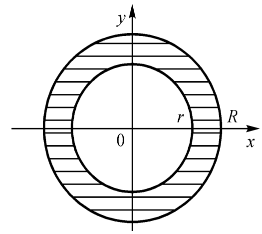


$z_\lambda = [1, \lambda, \lambda^2, \lambda^3, \lambda^4]$.

11. Circles $(x^2+y^2-r^2)(R^2-x^2-y^2)=0,\ 0<r<R.$

$$\Gamma = \begin{bmatrix} -r^2R^2 & 0 & 0 \\ 0 & r^2+R^2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

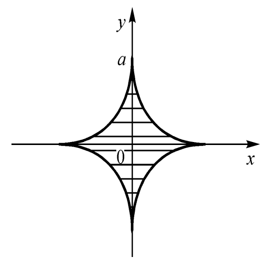
$z_\lambda = [1, \lambda, \lambda^2].$



12. Astroid $(x^2+y^2-a^2)^3+27a^2x^2y^2=0,\ a>0.$

$$\Gamma = \begin{bmatrix} 16a^6 & 0 & 0 & 0 & 27a^2 \\ 0 & -48a^4 & 0 & 0 & 0 \\ 0 & 0 & -6a^2 & 0 & 0 \\ 0 & 0 & 0 & -16 & 0 \\ 27a^2 & 0 & 0 & 0 & 0 \end{bmatrix},$$

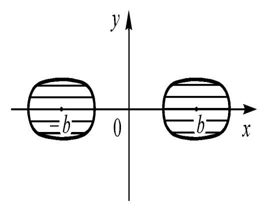
$z_\lambda = [1, \lambda, \lambda^2, \lambda^3, \lambda^4].$



13. Cassini ovals $(x^2+y^2)^2-2b^2(x^2-y^2)=a^4-b^4,\ 0<a<b.$

$$\Gamma = \begin{bmatrix} a^4-b^4 & b^2 \\ b^2 & -1 \end{bmatrix},$$

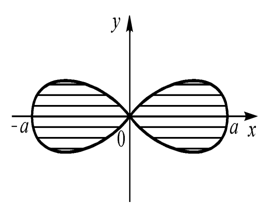
$z_\lambda = [1, \lambda^2].$



14. Bernoulli lemniscate $(x^2+y^2)^2=2a^2(x^2-y^2),\ a>0.$

$$\Gamma = \begin{bmatrix} 0 & a^2 \\ a^2 & -1 \end{bmatrix},$$

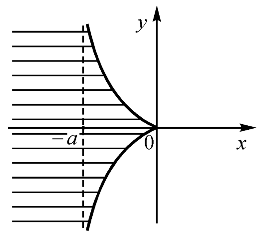
$z_\lambda = [1, \lambda^2].$



15. Cissoid of Diocles $x^3 + xy^2 + ay^2 = 0$, $a > 0$.

$$\Gamma = \begin{bmatrix} 0 & 0 & a/2 \\ 0 & -a & -1 \\ a/2 & -1 & 0 \end{bmatrix},$$

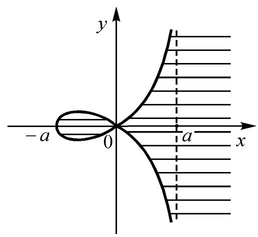
$$z_\lambda = [1, \lambda, \lambda^2].$$



16. Strophoid $(x + a)x^2 + (x - a)y^2 = 0$, $a > 0$.

$$\Gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

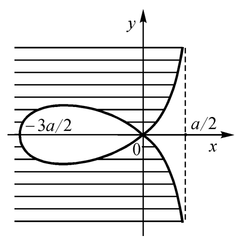
$$z_\lambda = [\lambda + a, \lambda^2].$$



17. Maclaurin trisector $2x(x^2 + y^2) = a(y^2 - 3x^2)$, $a > 0$.

$$\Gamma = \begin{bmatrix} 0 & 0 & -a \\ 0 & -a & -1 \\ -a & -1 & 0 \end{bmatrix},$$

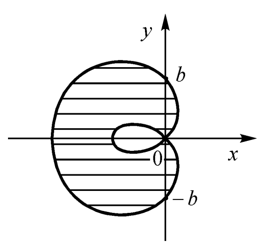
$$z_\lambda = [1, \lambda, \lambda^2].$$



18. Pascals limacon $(x^2 + y^2 + 2ax)^2 = b^2(x^2 + y^2)$, $0 < b \le \sqrt{2}a$.

$$\Gamma = \begin{bmatrix} 0 & 0 & -a^2 \\ 0 & b^2 - 2a^2 & -2a \\ -a^2 & -2a & -1 \end{bmatrix},$$

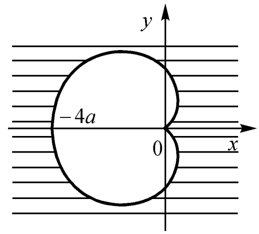
$$z_\lambda = [1, \lambda, \lambda^2].$$



19. Cardioid $(x^2+y^2+2ax)^2 = 4a^2(x^2+y^2)$, $a>0$.

$$\Gamma = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & -2a^2 & 2a \\ a^2 & 2a & 1 \end{bmatrix},$$

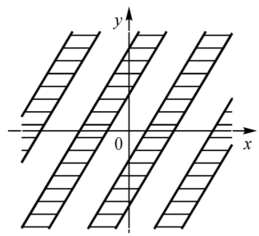
$$z_\lambda = [1, \lambda, \lambda^2].$$



20. Family of lines $\cos[2(ax-by+c)] = 0$, $w = a+ib$.

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

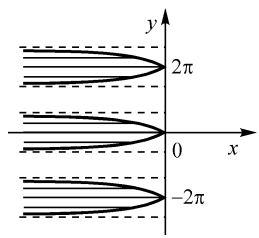
$$z_\lambda = [\cos(w\lambda + c), \sin(w\lambda + c)].$$



21. Curve $\cos y = e^x$.

$$\Gamma = \begin{bmatrix} 0 & 1/2 \\ 1/2 & -1 \end{bmatrix},$$

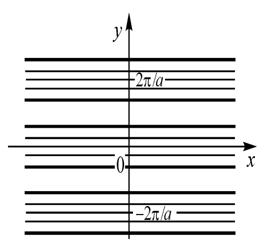
$$z_\lambda = [1, e^\lambda].$$



22. Family of lines $\cos(ay) = 0$, $a>0$.

$$\Gamma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

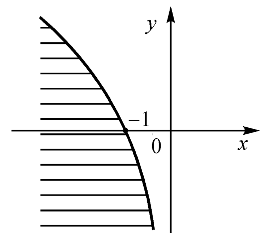
$$z_\lambda = [1, e^{a\lambda}].$$



23. Exponential curve $x = -a^y$, $a > 1$.

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

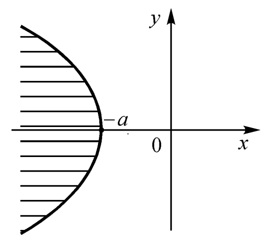
$$z_\lambda = [1, -\lambda/2, a^{\lambda/2i}].$$



24. Catenary $x = -a\,\mathrm{ch}(y/a)$, $a > 0$.

$$\Gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -a \end{bmatrix},$$

$$z_\lambda = [1, -\lambda, e^{\lambda/(2ai)}, e^{-\lambda/(2ai)}].$$



1.5 Inertia Theorem

It is known that the matrix A does not have purely imaginary eigenvalues if and only if the matrix inequality

$$AX + XA^* > 0$$

is solvable with respect to $X = X^*$. The number of eigenvalues of the matrix A, which have positive and negative real parts, taking into account the multiplicities, coincides respectively with $i_+(X)$ and $i_-(X)$. This statement gives the distribution of the spectrum $\sigma(A)$ with respect to an imaginary axis in terms of inertia of Hermitian forms (*theorems of Ostrowsky–Schneider and Taussky*).

Along with the equation (1.3.1) consider the matrix inequality

$$L_f X > 0, \tag{1.5.1}$$

where L_f is the operator (1.2.2) constructed for the given matrix $A \in C^{n\times n}$ and function $f \in \mathcal{H}$. An arbitrary solution of the equation (1.3.1) for $Y > 0$ at the same time is a solution of the inequality (1.5.1).

Lemma 1.5.1 *Let a function $f \in \mathcal{H}$ satisfy the inequalities*

$$i_\pm \left(\Gamma_f \begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_m \end{pmatrix}\right) \le p_\pm, \quad \forall\, \mu_1, \ldots, \mu_m \in \Lambda,$$

where $p_\pm \ge 0$ are integers, $\Lambda \subset C^1$ is some open set. Then for any sets of points $\lambda_1, \ldots, \lambda_\alpha \in \Lambda$ and natural numbers $m_1, \ldots, m_\alpha$ whose sum does not exceed m the following inequalities hold true:

$$i_\pm \left(\Gamma_f \begin{pmatrix} m_1 & \cdots & m_\alpha \\ \lambda_1 & \cdots & \lambda_\alpha \end{pmatrix}\right) \le p_\pm. \tag{1.5.2}$$

Proof. Use the proof technique of Lemma 1.3.5 and construct a sequence of matrices F_k such that $i_\pm(F_k) \le p_\pm$, $F_k - F_0 = \Delta_k \to 0$, $k \to \infty$, where F_0 is the matrix (1.1.8) composed of partial derivatives of the function f.

If $p_+ = 0$, then $F_k \le 0$ and $F_0 \le 0$. Let $p_+ \ne 0$ and $VF_0V^* = D > 0$, where V is a matrix composed of all the left eigenvectors of the matrix F_0, corresponding to the positive eigenvalues. Then for sufficiently large k the following relations hold true:

$$VF_kV^* = D + V\Delta_k V^* > 0, \;\; i_+(F_0) = i_+(VF_kV^*) \le i_+(F_k) \le p_+.$$

Similarly, $i_-(F_0) \le p_-$, i.e. the inequalities (1.5.2) hold true.

The lemma is proved.

Using the class of functions $\mathcal{H}_2^m$(see Section 1.4), formulate the following proposition.

Theorem 1.5.1 *The matrix inequality (1.5.1) has a solution if and only if the following conditions hold true:*

$$f(\lambda_t, \bar{\lambda}_t) \ne 0, \quad t = \overline{1, \alpha}. \tag{1.5.3}$$

Under the conditions (1.5.3) there exists a solution X satisfying the equalities

$$i_f^+(A) = i_+(X), \quad i_f^-(A) = i_-(X), \quad i_0(X) = 0. \tag{1.5.4}$$

If X is a solution of the inequality (1.5.1) for $f \in \mathcal{H}_2^m$, then the relations (1.5.3) and (1.5.4) hold true.

Proof. If in the equation (1.3.1) $Y > 0$, then according to (1.3.4), the inequalities (1.5.3) hold true. Use the recurrent algorithm of search of the matrices X and Y, which follows from (1.3.5)–(1.3.7) and represents the solution of the equation (1.5.1) under the conditions (1.5.3). At each step of this algorithm the inequality (1.3.7) must hold, i.e. $G > 0$. In addition, if $f(\sigma_k, \bar{\sigma}_k) < 0$, then h_{kk} follows the inequality

$$\delta_k = \frac{\det H_k}{\det H_{k-1}} = h_{kk} - u_k^* \, H_{k-1}^{-1} \, u_k < 0.$$

If $f(\sigma_k, \bar{\sigma}_k) > 0$, then the strict inequality $\delta_k > 0$ must hold true. Taking into account the Yacobi theorem for the matrix H and Sylvesters law of inertia, obtain the relations (1.5.4).

Let X be a solution of the inequality (1.5.1) and $f \in \mathcal{H}_2^m$. Then, according to Lemma 1.6.1, the conditions (1.5.2) hold true for $p_\pm = 1$, and the function (1.3.14) is representable in the form

$$g(\lambda, \bar{\mu}) = p(\lambda)\,\overline{p(\mu)} - q(\lambda)\,\overline{q(\mu)},$$

where p and q are some polynomials. The equation (1.5.1), in consideration of (1.3.5), is reducible to the form

$$[p(J), q(J)] \begin{bmatrix} H & 0 \\ 0 & -H \end{bmatrix} \begin{bmatrix} p(J)^* \\ q(J)^* \end{bmatrix} = G > 0. \tag{1.5.5}$$

This implies that H is a nonsingular matrix, since

$$\operatorname{rank} H = i_+(H) + i_-(H) \geq i_+(G) = n.$$

Moreover, taking into account the triangular structure of $p(J)$ and $q(J)$, one can find that all successive principal minors of the matrix H are nonzero: $h_k = \det H_k \neq 0$. Therefore there exists an expansion

$$H = LDL^*, \qquad D = \operatorname{diag}\{d_1, \ldots, d_n\}, \tag{1.5.6}$$

where $d_k = h_k/h_{k-1}$, $k = \overline{1,n}$, $h_0 = 1$, L is the lower triangular matrix with a unit diagonal. Taking the successive principal submatrices G_k in (1.5.5) in consideration of (1.5.6), obtain the relations

$$G_1 = f(\sigma_1, \bar{\sigma}_1)\, d_1 > 0, \;\; G_k = C_k \Delta_k C_k^* + R_k > 0, \;\; k = \overline{2,n}, \quad (1.5.7)$$

where

$$C_k = U_k \left[p(J)\, V_k,\; q(J)\, V_k\right], \quad U_k = \left[I_k,\; 0\right], \quad V_k = L\, \left[I_{k-1},\; 0\right]^T,$$

$$\Delta_k = \begin{bmatrix} d_1 & \cdots & \cdots & & 0 \\ & \ddots & & & \\ \vdots & & d_{k-1} & & \vdots \\ \vdots & & & -d_1 & \vdots \\ & & & \ddots & \\ 0 & \cdots & \cdots & & -d_{k-1} \end{bmatrix},$$

$$R_k = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\sigma_k, \bar{\sigma}_k)\, d_k \end{bmatrix}.$$

Using the properties of the indices of inertia in (1.5.7), obtain

$$i_+(G_k) = k \le i_+(\Delta_k) + i_+(R_k) = k - 1 + i_+(R_k).$$

Hence $i_+(R_k) = 1$, and the inequalities

$$f(\sigma_k, \bar{\sigma}_k)\, d_k > 0$$

hold true which, taking into account (1.5.6) and the law of inertia, are equivalent to the equalities (1.5.4).

The theorem is proved.

Formulate a corollary of Theorem 1.5.1 for the matrix equation

$$\sum_{i,j} \gamma_{ij} f_i(A) X f_j(A)^* = Y. \quad (1.5.8)$$

Corollary 1.5.1 *Let $f \in \mathcal{H}_2$, $Y > 0$ and X be a solution of the equation (1.5.8). Then the curve (1.1.4) does not intersect with the spectrum $\sigma(A)$, and in the domains (1.1.2) and (1.1.3) respectively $i_+(X)$ and $i_-(X)$ eigenvalues of the matrix A, taking into account the multiplicities, are located.*

1.6 Location of Eigenvalues on Plane Curves

Consider the problem of belonging of eigenvalues of the matrix A to the plane curve (1.1.4). In particular, we are interested in the estimates for the number $i_f^0(A)$ and in the criteria of location of all eigenvalues on the curve (1.1.4). Similar problems occur e.g. during the study of conditions of stability and aperiodicity of some mechanical systems.

Use the properties of the homogeneous matrix equation

$$L_f X = 0, \tag{1.6.1}$$

where L_f is the operator (1.2.2), in particular (1.2.3). This equation can be regarded as a problem of determination of eigenelements of the operator (1.2.2), corresponding to its zero eigenvalue.

Theorem 1.6.1 *If the equation (1.6.1) has a nonzero nonnegative definite solution $X \geq 0$, then on the curve (1.1.4) there are at least* $\operatorname{rank} X$ *eigenvalues of the matrix A:*

$$i_f^0(A) \geq \operatorname{rank} X. \tag{1.6.2}$$

In particular, for $X > 0$ the whole spectrum of the matrix A is located on the curve (1.1.4). Conversely, if $i_f^0(A) \neq 0$, then the equation (1.6.1) has a nonnegative definite solution of any rank from the interval

$$0 < \operatorname{rank} X \leq \sum_{\lambda_t \in \Lambda_f^0} \xi_t, \tag{1.6.3}$$

where ξ_t is the geometric multiplicity of the eigenvalue $\lambda_t \in \sigma(A)$.

Proof. Let $X \geq 0$ be a solution of the equation (1.6.1). Then, according to Lemma 1.2.2, it can be determined by using the relations (1.2.17) and (1.2.18) with $w = 0$. Transform the expression (1.2.17), using the expansions of idempotent components

$$A_{t1} = A_t\, A_t^+, \quad A_t^+ A_t = I_{n_t}, \quad t = \overline{1, \alpha}.$$

The multipliers A_t (A_t^+) determine the right (left) Jordan vector trains of the matrix A, corresponding to the eigenvalue λ_t with algebraic multiplicity n_t. For the solution X we have the expression

$$X = \sum_{(t,\tau)\in\Theta_0^*} A_t\, C_{t\tau}\, A_\tau^* \geq 0, \tag{1.6.4}$$

where $C_{t\tau}$ are some matrices of dimensions $n_t \times n_\tau$. If $\lambda_t \in \Lambda_f^0$, then $(t,t) \in \Theta_0^*$. Applying the Sylvester inequality to the expression (1.6.4), obtain

$$\operatorname{rank} X \le \sum_{\lambda_t \in \Lambda_f^0} \operatorname{rank} A_t = \sum_{\lambda_t \in \Lambda_f^0} n_t = i_f^0(A).$$

Consequently, the estimate (1.6.2) holds true. If $X > 0$, then $i_f^0(A) = n$.

If on the curve (1.1.4) there are eigenvalues of the matrix A, then $\Theta_0^* \neq \emptyset$, and according to (1.2.21) the arbitrary nonnegative definite matrix

$$X = \sum_{(t,\tau)\in\Theta_0^*} U_t\, S_{t\tau}\, U_\tau^* \ge 0 \tag{1.6.5}$$

is a solution of the equation (1.6.1). Since the multipliers U_k have the full rank ξ_t, then free parameters of the matrices $S_{t\tau}$ of dimensions $\xi_t \times \xi_\tau$ can be chosen from any given value of the rank of the solution from the interval (1.6.3).

The theorem is proved.

Corollary 1.6.1 *If a matrix A has a simple structure, then the inequality $i_f^0(A) \ge \rho$ holds true if and only if the equation (1.6.1) has a nonnegative definite solution of the rank ρ.*

This criterion follows from Theorem 1.6.1 under the conditions $n_t = \xi_t$. In this case the formula (1.6.5) determines the general form of the nonnegative definite solution of the equation (1.6.1).

Note that if the function f has the property

$$\lambda, \mu \in \Lambda_f^0, \quad \lambda \neq \mu \quad \Longrightarrow \quad f(\lambda, \bar\mu) \neq 0, \tag{1.6.6}$$

then the expressions (1.6.4) and (1.6.5) can be simplified. Thus, under the conditions (1.6.6) the equation (1.6.1) is satisfied by any matrix X representable in the form

$$X = \sum_{\lambda_t \in \Lambda_f^0} U_t\, S_t\, U_t^* \ge 0,$$

where S_t are matrices of dimensions $\xi_t \times \xi_t$.

1.7 Estimates and Localization of Eigenvalues

1. Consider the relation (1.3.1) under the conditions $f \in \mathcal{H}$, $A \in C^{n\times n}$, and $X \in \mathcal{K}_0$. As X one can select any positive definite matrix and calculate the relation Y. There is no need to solve the equation (1.3.1).

If $X > 0$, then there exist numbers ε_1 and ε_2 satisfying the inequalities

$$\varepsilon_1 X \leq Y \leq \varepsilon_2 X, \qquad \varepsilon_1 \leq \varepsilon_2. \tag{1.7.1}$$

Using the proof of Lemma 1.3.1, obtain an estimate for the domain containing the spectrum of the matrix A:

$$\varepsilon_1 \leq f(\lambda, \bar{\lambda}) \leq \varepsilon_2, \qquad \lambda \in \sigma(A). \tag{1.7.2}$$

The interval $[\varepsilon_1, \varepsilon_2]$ is determined by solving the system of inequalities (1.7.1) with respect to unknown parameters ε_1 and ε_2. Reducing this interval, we thereby reduce the domain (1.7.2) containing the spectrum $\sigma(A)$. It can be proved that the values of the parameters ε_1 and ε_2, calculated by using the relations

$$\varepsilon_1 = \min\{\varepsilon\colon \det(Y - \varepsilon X) = 0\} = \min_{\|z\|=1} \frac{z^* Y z}{z^* X z},$$

$$\varepsilon_2 = \max\{\varepsilon\colon \det(Y - \varepsilon X) = 0\} = \max_{\|z\|=1} \frac{z^* Y z}{z^* X z},$$

represent one of the solutions of the system of inequalities (1.7.1). The corresponding interval $[\varepsilon_1, \varepsilon_2]$ is minimal.

Note that in the particular case $\varepsilon_1 = \varepsilon_2$ all points of the spectrum $\sigma(A)$ belong to the set $\Lambda^0_{f-\varepsilon_1}$.

2. In the relation (1.6.1) let the matrices A and $X > 0$ be given, and $f \in \mathcal{H}$ be an unknown function. If, proceeding from (1.6.1), one succeeds in finding some function f, then according to Theorem 1.6.1 the spectrum of the matrix A is located on the respective curve (1.1.4). If we have two such functions f and g, then each eigenvalue of the matrix A is an intersection of curves Λ^0_f and Λ^0_g of the form (1.1.4).

We will confine ourselves to the class of algebraic curves and consider a homogeneous equation

$$\sum_{k,s=0}^{r-1} \gamma_{ks}\, A^k X A^{sT} = 0,$$

where $A \in R^{n\times n}$ and $X = X^T > 0$ are given real matrices, and $\gamma_{ks} = \gamma_{sk}$ are unknown coefficients. This equation reduces to the system of algebraic equations

$$G(X)\,\gamma = 0, \tag{1.7.3}$$

where $G(X)$ is some $p\times q$ matrix, γ is a vector of unknown coefficients of order q, and $p = n(n+1)/2$, $q = r(r+1)/2$. The system (1.7.3) has a nontrivial solution if $\operatorname{rank} G(X) < q$. It is advisable to select the matrix X so that this inequality would hold true for the minimum possible value r.

Corresponding to the subspace of solutions of the system (1.7.3) is a family of algebraic curves Λ_f^0 with

$$f(\lambda,\mu) = \sum_{k,s=0}^{r-1} \gamma_{ks}\lambda^k\mu^s.$$

Each of these curves crosses α different points $\lambda_t \in \sigma(A)$ and has an order not exceeding $2r-2$. If one manages to find two solutions of the system (1.7.3) so that the corresponding curves have α different intersections, then each of those points is an eigenvalue of the matrix A.

3. Let matrices A, $Y = Y^* > 0$ and a function $f \in \mathcal{H}$ be given. Consider the curve (1.1.4) and its neighbourhood $\Lambda^+_{f_\varepsilon}$, where $f_\varepsilon = 1 - \varepsilon^2 f^2$, $\varepsilon > 0$ is a numeric parameter. Obviously $\Lambda_f^0 \subset \Lambda^+_{f_\varepsilon}$, and the domain $\Lambda^+_{f_\varepsilon}$ degenerates into (1.1.4) while $\varepsilon \to \infty$. If $f_\varepsilon \in \mathcal{H}_0^m$, then, according to Theorem 1.3.1, the matrix equation

$$X - \varepsilon L_{f^2} X = Y \tag{1.7.4}$$

has a unique positive definite solution $X = X(\varepsilon)$ if and only if the inclusion $\sigma(A) \subset \Lambda^+_{f_\varepsilon}$ holds true. This proposition, for sufficiently large value ε, can be used for the estimation of the location of the spectrum $\sigma(A)$ near the curve (1.1.4) with some desirable accuracy.

If there is a limit

$$\lim_{\varepsilon\to\infty} X(\varepsilon) = X_\infty > 0. \tag{1.7.5}$$

where $X(\varepsilon)$ is a solution of the equation (1.7.4), then the matrix X_∞ satisfies the homogeneous equation $L_{f^2} X_\infty = 0$, and according to Theorem 1.6.1, the inclusion $\sigma(A) \subset \Lambda_f^0$ holds true. The converse proposition is proved within additional limitations. Thus, if the matrix A has a simple structure, $\sigma(A) \subset \Lambda_f^0$, $f_\varepsilon \in \mathcal{H}_0^m$ and the conditions (1.6.6) hold true, then the limiting value (1.7.5) for the solution of the equation (1.7.4) is a positive definite matrix.

4. Set out in a generalized form the known technique of localization of a spectrum $\sigma(A)$ (see Gutman, Chojnowski [1]) for the class of domains

$$\Lambda = \Lambda_{f_0}^+ \cap \Lambda_{f_1}^+ \cap \cdots \cap \Lambda_{f_s}^+,$$

where $f_k(\lambda,\mu)$ are prescribed Hermitian functions, $k = \overline{0,s}$, $s \geq 1$. Introduce the following notation:

$$f(\lambda,\mu,z) = \sum_{k=0}^{s} f_k(\lambda,\mu)\, z^k, \qquad F(z) = \sum_{k=0}^{s} F_k\, z^k,$$

$$F_k = -\frac{1}{4\pi^2} \oint_\omega \oint_{\bar\omega} f_k(\lambda,\mu)(A-\lambda I)^{-1} \otimes \left(\bar A - \mu I\right)^{-1} d\lambda d\mu,$$

where $\otimes$ is the sign of the Kronecker product of matrices, ω $(\bar\omega)$ is a closed contour enclosing the spectrum $\sigma(A)$ $(\sigma(\bar A))$.

Theorem 1.7.1 *Let the following conditions hold true:*
1) $\Lambda_{f_0}^- \cap \Lambda_{f_1}^- \cap \cdots \cap \Lambda_{f_s}^- = \emptyset$;
2) $f(\lambda,\bar\lambda,z) = 0 \implies z \in R^1$;
3) $r_k(\lambda,\mu) = \sum\limits_{p+q=k} f_p(\lambda,\bar\mu)\, f_q(\mu,\bar\lambda) > 0,\ \forall\, \lambda,\mu \in \Lambda,\ k = \overline{0,2s}.$

Then the spectrum of a matrix A is located in the domain Λ if and only if all coefficients of the polynomial

$$\det F(z) = a_0 + a_1 z + \cdots + a_N\, z^N, \quad N = s\, n^2,$$

are positive.

Proof. Each $z \in C^1$ of the function f has a corresponding operator L_f of the type (1.2.1) whose action in the space C^{n^2} is described by the matrix $F(z)$, i.e.

$$L_f X = Y \iff F(z)\, x = y,\ x = [x_{1*}, \ldots, x_{n*}]^T,\ y = [y_{1*}, \ldots, y_{n*}]^T,$$

where x_{i*} is the i-th row of the matrix X. The spectrum of the operator L_f consists of n^2 eigenvalues $f(\sigma_i, \bar{\sigma}_j, z)$, where $\sigma_i \in \sigma(A)$, whose product gives the expression

$$\det F(z) \equiv d_\sigma(z) = \left(\prod_i \sum_{k=0}^{s} f_k(\sigma_i, \bar{\sigma}_i)\, z^k\right)\left(\prod_{i<j} \sum_{k=0}^{2s} r_k(\sigma_i, \sigma_j)\, z^k\right).$$

If $\sigma(A) \subset \Lambda$ and conditions 3) hold true, then all coefficients of the polynomial multipliers in this expression are positive, and hence the polynomial $d_\sigma(z)$ has degree N and positive coefficients. The converse statement is the consequence of the described relations and conditions 1) and 2). Indeed, all real roots of the polynomial $d_\sigma(z)$ of degree N with positive coefficients are negative. In particular, the polynomials $f(\sigma_i, \bar{\sigma}_i, z)$ must have real negative roots only, and therefore the positive coefficients $f_k(\sigma_i, \bar{\sigma}_i) > 0$, which means $\sigma(A) \subset \Lambda$.

The theorem is proved.

Theorem 1.7.1 holds true if instead of condition 3) we require that for any set of points $\sigma_1, \ldots, \sigma_n \in \Lambda$ all coefficients of the polynomial $d_\sigma(z)$ of degree N must be positive. The expressions for the coefficients a_k as functions of $\sigma_1, \ldots, \sigma_n$ are determined as a result of multiplication of all polynomials $f(\sigma_i, \bar{\sigma}_j, z)$. For example, if $f(\lambda, \mu, z) = f_0(\lambda, \mu) + z$, then these expressions can be constructed by using Vieta's formulae in the form

$$a_{N-p} = \sum_{ni_1 + j_1 < \cdots < ni_p + j_p} f_0(\sigma_{i_1}, \bar{\sigma}_{j_1}) \cdots f_0(\sigma_{i_p}, \bar{\sigma}_{j_p}), \quad p = \overline{1, N}.$$

The class of domains Λ used in Theorem 1.7.1 is sufficiently wide. Each domain of the form (1.1.2), corresponding to the class of functions $\mathcal{H}_1$ (see Section 1.4), can be described as an intersection of some domain so that the conditions of Theorem 1.7.1 hold true. Theorem 1.7.1 is satisfied by the class of I-transformable domains (see Gutman, Chojnowski [1]), which in the case of algebraic polynomials f_k was determined by using the inequalities $f_s(\lambda,\bar{\mu}) \neq 0$ $(\forall\lambda,\mu \in \Lambda)$, $f_s(\lambda,\bar{\lambda}) \geq 0$ $(\forall\lambda \in C^1)$ instead of condition 1), and by using the requirement of stability of the family of polynomials $f(\lambda,\bar{\mu},z)$ $(\lambda,\mu \in \Lambda)$ of degree s instead of condition 3). The last limitation is sufficient for condition 3) to hold true. If all domains $\Lambda^+_{f_k}$ are simple, i.e. $\Lambda^0_{f_k} = \partial\Lambda^+_{f_k}$, then given the conditions of Theorem 1.8.1, the following criterion holds true:

$$\sigma(A) \subset \bar{\Lambda} \iff a_i \geq 0, \quad i = \overline{0,N},$$

where $\bar{\Lambda}$ is the closure of the domain Λ, and a_i are the coefficients of the polynomial $\det F(z)$.

Consider the case $s = 1$. Here the application of Theorem 1.7.1 adds up to the computation of coefficients of the characteristic polynomial of the linear pencil of matrices $F(z) = F_0 + zF_1$ of order n^2. Condition 2) of the theorem follows from the fact that the functions f_0 and f_1 must be Hermitian. Under condition 1) the domain Λ coincides with Λ^+_g, where $g = f_0 f_1$. Condition 3) means that $g \in \mathcal{H}_*$, where $\mathcal{H}_*$ is a class of Hermitian functions with the following property:

$$g(\lambda,\bar{\lambda}) > 0,\ g(\mu,\bar{\mu}) > 0 \implies \operatorname{Re} g(\lambda,\bar{\mu}) > 0.$$

Let $f_0 = u - v$ and $f_1 = 1/u$, where u and v are Hermitian functions such that for $\forall\ \lambda,\mu \in \Lambda$

$$|u(\lambda,\bar{\mu})|^2 \geq u(\lambda,\bar{\lambda})\, u(\mu,\bar{\mu}), \quad |v(\lambda,\bar{\mu})|^2 \leq v(\lambda,\bar{\lambda})\, v(\mu,\bar{\mu}).$$

Then for the function $w = v/u$ the following inequalities hold true:

$$\operatorname{Re} w(\lambda,\bar{\mu}) \leq |w(\lambda,\bar{\mu})| \leq \sqrt{w(\lambda,\bar{\lambda})\, w(\mu,\bar{\mu})} < 1,\ \ \forall\ \lambda,\mu \in \Lambda,$$

and hence it follows that $g = 1 - w \in \mathcal{H}_*$. If we assume that

$$U = \left\| \frac{1}{u(\mu_i,\bar{\mu}_j)} \right\|_1^m \geq 0,\ \ V = \|v(\mu_i,\bar{\mu}_j)\|_1^m \geq 0,\ \ \forall\ \mu_1,\ldots,\mu_m \in \Lambda,$$

then for $\mu_1, \ldots, \mu_m \in \Lambda$ the following relations are true (see Section 1.4)

$$W = U \odot V = \|w(\mu_i, \bar{\mu}_j)\|_1^m \geq 0,$$

$$\left\| \frac{1}{g(\mu_i, \bar{\mu}_j)} \right\|_1^m = E + W + W \odot W + W \odot W \odot W + \cdots \geq 0,$$

where $\odot$ is the sign of the Schur product, E a matrix with all its elements equal to 1, and therefore $g \in \mathcal{H}_* \cap \mathcal{H}_0^m$.

1.8 Controllability Conditions for the Generalized Lyapunov Equation

Hereinabove we have established the relation between the indices of inertia of the Hermitian matrices X and Y satisfying the equation (1.3.1), and the location of the spectrum of the matrix A with respect to the sets (1.1.2)–(1.1.4). Now we will enlarge the sets of the matrices X and Y used for solving the spectrum localization problem. We will need the concept of controllability of a pair of matrices, which emerged in the controllable system theory.

Let A and R be matrices of dimensions $n \times n$ and $n \times r$ respectively. Construct a sequence of block matrices

$$P_k(A, R) = \left[R, AR, \ldots, A^{k-1}R\right], \quad k = 1, 2, \ldots .$$

The pair (A, R) is said to be *controllable*, if for some k the matrix $P_k(A, R)$ has full rank n.

Lemma 1.8.1 *If $Z = RR^*$, then the following statements are equivalent:*
(a) the pair of matrices (A, R) is controllable;
(b) there exists a function $\varphi \in \mathcal{H}$ such that $L_\varphi Z > 0$;
(c) the pair of matrices (A, Z) is controllable.

Proof. The expression (1.2.8) for the operator L_φ is reducible to the form

$$L_\varphi Z = P_m(A, R)\,(\Gamma \otimes I)\,P_m(A, R)^*,$$

where $\otimes$ is a Kronecker product of matrices. If this expression is nonsingular – positive definite in particular – matrix, then the multiplier $P_m(A, R)$ has full rank and the pair (A, R) is controllable ((b) $\Rightarrow$ (a)).

Consider the sequence of matrices

$$Z_k = P_k(A, R)\, P_k(A, R)^* \geq 0, \quad k = 1, 2, \ldots . \tag{1.8.1}$$

If the pair (A, R) is controllable, then, starting from some number $k = q$, all matrices of this sequence are positive definite. Statements (b) and (c) hold true, since

$$Z_k = L_{\varphi_k} Z = P_k(A, Z)\, P_k(A, I)^*,$$

where $\varphi_k(\lambda, \bar{\mu}) = 1 + \lambda\bar{\mu} + \cdots + \lambda^{k-1}\bar{\mu}^{k-1}$.

The fact that (c) implies (a) follows from the Sylvester inequality for the rank of matrix product and the relations

$$P_k(A, Z) = P_k(A, R)\ \mathrm{diag}\,\{R^*, \ldots, R^*\}\,, \quad k = 1, 2, \ldots .$$

The lemma is proved.

Lemma 1.8.2 *Let a matrix sequence*

$$Z_1 \geq 0, \quad Z_{k+1} = Z_1 + LZ_k, \quad k = 1, 2, \ldots , \tag{1.8.2}$$

be given, where L is a linear operator preserving the cone of nonnegative definite matrices invariant ($L\mathcal{K} \subseteq \mathcal{K}$). Then the following relations hold true:

$$r_1 < r_2 < \cdots < r_q = r_{q+1} = \cdots = r, \tag{1.8.3}$$

where $q \leq n - r_1 + 1$, $r_k = \mathrm{rank}\, Z_k$, $k = 1, 2, \ldots$.

Represent the sequence (1.8.1) in the form (1.8.2), assuming $Z_1 = RR^*$, $LZ = AZA^*$. Then, according to Lemma 1.8.2, the controllability condition for the pair (A, R) means that $r_q = n$, where q is the minimum value of the index k, for which the sequence of ranks r_k in (1.8.3) reaches the maximum value $r = n$. In this case the following estimate is true:

$$q \leq \min\{m, n - r_1 + 1\}\,,$$

where m is the degree of the minimal polynomial of the matrix A.

Theorem 1.8.1 *Let the matrices X and Y satisfy the equation (1.3.1). Then the following statements hold true:*
1) the controllability of the pair (A,Y) implies the controllability of the pair (A,X);
2) if $X \geq 0$ and $i_f^0(A) = 0$, then the controllability of the pair (A,X) implies the controllability of the pair (A,Y);
3) if $Y \geq 0$ and the pair (A,Y) is controllable, then $i_f^0(A) = 0$;
4) if $X \geq 0$, $Y \geq 0$ and the pair (A,Y) is controllable, then $i_f^+(A) = n$;
5) if $X \geq 0$, $Y \geq 0$ and the pair (A,X) is controllable, then $i_f^-(A) = 0$;
6) if $X \geq 0$, $Y = 0$ and the pair (A,X) is controllable, then $i_f^0(A) = n$.

Proof. The controllability of the pair (A,R) is equivalent to the conditions

$$\operatorname{rank}[A - \lambda I, R] = n, \qquad \lambda \in \sigma(A).$$

These conditions do not hold true if and only if there exists a left eigenvector v_t^* of the matrix A, corresponding to the eigenvalue λ_t, for which $v_t^* R = 0$. If $v_t^* X = 0$, then according to (1.2.19),

$$v_t^* Y = \sum_{\tau=1}^{\alpha} \sum_{j=1}^{m_\tau} f_{1j}(\lambda_t, \bar{\lambda}_\tau)\, v_t^* X A_{\tau j}^* = 0.$$

Therefore the pair (A,X) is controllable if such is the pair (A,Y).

Given $X \geq 0$ the equalities $v_t^* X = 0$ and $v_t^* X v_t = 0$ are equivalent. If $v_t^* X \neq 0$ and $f(\lambda_t, \bar{\lambda}_t) \neq 0$, then $v_t^* Y \neq 0$, since

$$f(\lambda_t, \bar{\lambda}_t)\, v_t^* X v_t = v_t^* Y v_t, \quad t = 1, \ldots, \alpha. \tag{1.8.4}$$

Thereby statements 1) and 2) are proved.

The equalities (1.8.4) are similarly used for derivation of statements 3)–6). If $Y \geq 0$ and $v_t^* Y \neq 0$, then $f(\lambda_t, \bar{\lambda}_t) \neq 0$. If $X \geq 0$, $Y \geq 0$, then the inequality $v_t^* X \neq 0$ ($v_t^* Y \neq 0$) implies $f(\lambda_t, \bar{\lambda}_t) \geq 0$ ($f(\lambda_t, \bar{\lambda}_t) > 0$). In the case $Y = 0$ $v_t^* X \neq 0$ implies $f(\lambda_t, \bar{\lambda}_t) = 0$. Statements 3) – 6) can be also determined by using Lemma 1.8.1.

The theorem is proved.

If some operator L commutes with the operator (1.2.2) and the matrices X and Y satisfy the equation (1.3.1), then the expressions $\widehat{X} = LX$ and $\widehat{Y} = LY$ also satisfy this equation and therefore can be used in Theorems 1.3.1, 1.5.1, 1.6.1, and 1.8.1. The respective limitations on the inertial properties of the parent matrices X and Y are canceled. In particular, in Theorem 1.8.1 the inequalities $X \geq 0$ or $Y \geq 0$ are not required. If the operator L determines the nonempty sets of matrices

$$\mathcal{L}(L) = \{Z \colon LZ > 0\}, \quad \bar{\mathcal{L}}(L) = \{Z \colon LZ \geq 0\},$$

then for the inclusion $\sigma(A) \subset \Lambda_f^+$ it is sufficient that the equation (1.3.1) has a solution $X \in \bar{\mathcal{L}}(L)$ for some matrix $Y \in \mathcal{L}(L)$. If $f \in \mathcal{H}_0^m$, then the inclusions $\sigma(A) \subset \Lambda_f^+$ and $\mathcal{L}(L) \subseteq L_f\mathcal{L}(L)$ are equivalent. The latter statement is an analogue of Theorem 1.3.1.

Let L_φ and L_ψ be operators of the type (1.2.2), describing the nonempty sets of matrices

$$\mathcal{L}(L_\varphi), \;\; \mathcal{L}(L_\psi), \;\; L_\varphi\mathcal{K}, \;\; L_\psi\mathcal{K}, \;\; \bar{\mathcal{L}}(L_\varphi), \;\; \bar{\mathcal{L}}(L_\psi), \;\; L_\varphi\mathcal{K}_0, \;\; L_\psi\mathcal{K}_0.$$

It is possible to construct different conditions for localization of the spectrum $\sigma(A)$ in terms of solutions of the equation (1.3.1), belonging to one of those sets. Here the functions φ and ψ must have some additional properties.

In the equation (1.3.1) let $X \in L_\varphi\mathcal{K}$ and $Y \in \mathcal{L}(L_\psi)$. Then the following relations hold true:

$$L_g\widehat{X} = \widehat{Y}, \quad X = L_\varphi\widehat{X}, \quad L_\psi Y = \widehat{Y}, \tag{1.8.5}$$

where $\widehat{X} \geq 0$, $\widehat{Y} > 0$ are some matrices, $g = f\varphi\psi$ is a product of functions. If the functions φ and ψ are selected so that

$$\Lambda_f^- \cap \Lambda_{\varphi\psi}^- = \emptyset, \tag{1.8.6}$$

then, according to Lemma 1.3.1, the spectrum $\sigma(A)$ belongs to the domain (1.1.2). If the following inequalities hold true

$$\Gamma_{1/g}\begin{pmatrix} m_1 & \cdots & m_\alpha \\ \lambda_1 & \cdots & \lambda_\alpha \end{pmatrix} \geq 0, \quad g(\lambda_t, \bar{\lambda}_\tau) \neq 0, \quad t, \tau = \overline{1, \alpha},$$

then, according to (1.8.5) and Lemma 1.4.3, the equation (1.3.1) has a solution $X \in L_\varphi \mathcal{K}$ for any matrix $Y \in \mathcal{L}(L_\psi)$. If we assume that along with (1.8.6) the following conditions hold true

$$\mu_1, \dots, \mu_m \in \Lambda_f^+ \implies \Gamma_{1/g}\begin{pmatrix} 1 & \cdots & 1 \\ \mu_1 & \cdots & \mu_m \end{pmatrix} \geq 0, \qquad g(\mu_t, \bar{\mu}_\tau) \neq 0, \quad t, \tau = \overline{1, m}, \tag{1.8.7}$$

then the inclusions $\sigma(A) \subset \Lambda_f^+$ and $\mathcal{L}(L_\psi) \subseteq L_f L_\varphi \mathcal{K}$ are equivalent. This criterion adds up to Theorem 1.3.1 in the particular case $\varphi = \psi \equiv 1$. The conditions (1.8.7) hold true if, e.g., $g \in \mathcal{H}_0^m$ and $\Lambda_f^+ \subseteq \Lambda_{\varphi\psi}^+$.

See the consequences of the described approach for the class of functions $f \in \mathcal{H}_1$, in particular $f \in \mathcal{H}_2$.

Theorem 1.8.2 *Let the functions f and ψ be represented as*

$$f = f_+ - f_-, \quad \psi = \sum_{j=0}^{s} f_+^{s-j} f_-^j, \tag{1.8.8}$$

where

$$f_+(\lambda, \bar{\mu}) = f_1(\lambda)\, \overline{f_1(\mu)}, \quad f_-(\lambda, \bar{\mu}) = \sum_{i>1} f_i(\lambda)\, \overline{f_i(\mu)}, \quad s \geq 1.$$

Then the equation (1.3.1) has a unique solution $X > 0$ under the conditions

$$\sigma(A) \subset \Lambda_f^+, \qquad L_\psi Y > 0. \tag{1.8.9}$$

If $Y \geq 0$, $s = n - \operatorname{rank} Y$ and $X > 0$ is the unique solution of the equation (1.3.1), then the conditions (1.8.9) hold true.

Proof. Since $g = f\psi = f_+^{s+1} - f_-^{s+1} \in \mathcal{H}_1$, then the conditions (1.8.6) and (1.8.7) hold true. In this case $\varphi \equiv 1$. If $\sigma(A) \subset \Lambda_f^+$, then the operators L_f, L_ψ, and L_g are invertible. From Theorem 1.3.1 and the relations (1.8.7) and (1.8.9) it follows that the equation (1.3.1) has the unique solution $X > 0$.

Let for some matrix $Y \geq 0$ the equation (1.3.1) have the unique solution $X > 0$. The invertibility of the operator L_f and Lemma 1.3.1

imply the inequalities (1.3.2), (1.3.3) and the relations

$$f_1(\lambda_t) \neq 0, \qquad \frac{1}{f(\lambda_t, \bar{\lambda}_\tau)} = \frac{1 + \delta(\lambda_t, \bar{\lambda}_\tau) + \delta^2(\lambda_t, \bar{\lambda}_\tau) + \cdots}{f_1(\lambda_t)\overline{f_1(\lambda_\tau)}},$$

$$|\delta(\lambda_t, \bar{\lambda}_\tau)|^2 \leq \delta(\lambda_t, \bar{\lambda}_t)\, \delta(\lambda_\tau, \bar{\lambda}_\tau) < 1, \quad t, \tau = \overline{1, \alpha}.$$

Here Cauchy's inequality for the function $\delta = f_-/f_+$ was used. Taking into consideration the formula (1.2.7), we obtain the expansion of the inverse operator

$$L_f^{-1} = L_{f_+}^{-1} \left(E + L_\delta + L_\delta^2 + \cdots\right) = L_{f_+}^{-s-1} L_\psi + \Delta_s,$$

where E is an identity operator, $\Delta_s \to 0$ $(s \to \infty)$. The matrix sequence

$$X_1 = L_{f_+}^{-1} Y, \quad X_{s+1} = X_s + L_\delta X_s = f_1(A)^{-s-1} \left(L_\psi Y\right) f_1(A)^{-s-1*},$$

converges to a positive definite solution X and satisfies the conditions of Lemma 1.8.2. Consequently, for some s we arrive at the strict inequality $L_\psi Y > 0$. In particular, we can put $s = n - \operatorname{rank} Y$.

The theorem is proved.

Theorem 1.8.3 *Let the matrices $A, Y = RR^* \geq 0$ and the function $f \in \mathcal{H}_2$ of the form (1.4.5) satisfy the condition*

$$\operatorname{rank}\,[F_0 R, \ldots, F_s R] = n, \tag{1.8.10}$$

where $F_k = f_1^{s-k}(A) f_2^k(A)$, $k = \overline{0, s}$, $s = n - \operatorname{rank} Y$. Then if X is a solution of the equation

$$f_1(A) X f_1(A)^* - f_2(A) X f_2(A)^* = Y, \tag{1.8.11}$$

then the curve (1.1.4) does not intersect with the spectrum $\sigma(A)$, and in the domain Λ_f^+ (Λ_f^-) there are exactly $i_+(X)$ $(i_-(X))$ eigenvalues of the matrix A, taking into account the multiplicities.

Proof. Represent the functions f and ψ in the form (1.8.8). Under the condition (1.8.10) we have the inequality $L_\psi Y = PP^* > 0$, where P is the block matrix determined in (1.8.10). Act on both parts of the equation (1.8.11) by the operator L_ψ. As a result, arrive at the inequality $L_g X > 0$, where $g = f\psi = f_+^{s+1} - f_-^{s+1} \in \mathcal{H}_2$. Taking into account the corollary of Theorem 1.5.1, obtain (1.5.4).

The theorem is proved.

The statement of Theory 1.8.3 holds under the controllability conditions of the pair of matrices (A, Y), and the limitations (see Carlson, Hill [1])

$$h(\lambda_t) \neq h(\lambda_\tau) \quad (t \neq \tau), \quad h'(\lambda_t) \neq 0 \quad (m_t > 1), \tag{1.8.12}$$

where

$$h(\lambda) = \frac{f_1(\lambda) + f_2(\lambda)}{f_1(\lambda) - f_2(\lambda)}.$$

These limitations are equivalent to the coincidence of the geometric multiplicities of the eigenvalues λ_t of the matrix A with the corresponding geometric multiplicities of the eigenvalues $h(\lambda_t)$ of the matrix $h(A)$. When using the limitation (1.8.10), unlike (1.8.12), no information on the spectrum $\sigma(A)$ is required.

Note that for the function $f(\lambda, \bar{\mu}) = 1 - \lambda\, \bar{\mu}$ describing the unit circle Λ_f^0, the equality (1.8.10) coincides with the controllability condition of the matrix pair (A, R).

1.9 Notes and References

1.1 The description of sets in the complex plane by means of Hermitian functions in the form (1.1.2)–(1.1.4) is used in eigenvalue location problems (see e.g. Gutman, Jury [1], Howland [1], Kalman [1], Carlson, Hill [1], Barnett, Saraton [1], Gutman, Chojnowski [1, 2], Mazko [1–15, 20, 22, 24–31], Kharitonov [1], and others). For the description of uniting, intersection, and other operations with given sets, R–functions can be used (see Rvachev [1]).

1.2 The operators (1.2.1) and (1.2.2) are taken from Daletskii, Krein [1] and Daletskii [1]. Their representation (1.2.9) and expressions for eigenvalues in Lemmas 1.1.1 and 1.1.2 were obtained in Mazko [22]. The used properties of functions and components of matrix are taken from Lancaster [1] and Gantmacher [1].

1.3 Lemmas 1.3.1–1.3.5 and Theorem 1.3.1 are proved in Mazko [21–23, 25]. In the proof of Lemmas 1.3.1 and 1.3.2 the known facts of the theory of matrices were used, taken from Gantmacher [1] and Horn, Johnson [1]. The description of the structure of the matrices $\alpha_{ti}(J)$ is available in Lancaster [1].

The statement of Theorem 1.3.1 without the use of Lemma 1.3.5 in the case of algebraic domains is also formulated by Mazko [6] and Gutman, Chojnowski [2]. Maximal classes of algebraic and transcendent domains described by using matrices of the type (1.1.8) in Theorem 1.3.1 were for the first time determined in Mazko [1, 2, 5].

1.4 The classes of algebraic and analytic functions of the type $\mathcal{H}_1$, described in terms of the matrix (1.4.8) in the generalized Lyapunov theorem were introduced in Mazko [2, 5]. The statement of this theorem is proved for algebraic domains with the limitation (1.4.9) (see Kharitonov [1]). The equivalence of the conditions (1.4.8) and (1.4.9) was proved in Mazko, Kharitonov [1]. Limitations on the matrix Γ of the form (1.4.10) were used by Jury [1], Kalman [1], and others.

The examples of algebraic and transcendent curves were taken from Savelov [1].

1.5 The results of this Section were proved in Mazko [22, 23, 25]. Theorem 1.5.1 is a generalization of the known inertia theorems (see Ostrowsky, Schneider [1] and Taussky [1]). The justification of the expansion (1.5.6) is available in Gantmacher [1].

1.6 The known techniques for solution of problems of belonging of polynomial roots to some curves are described, as well as the technical applications related to them (see Jury [1] and Postnikov [1]). Theorem 1.6.1 and its corollaries give the conditions of location of matrix eigenvalues in terms of solutions of a homogeneous matrix equation (see Mazko [7, 8, 10]).

1.7 At construction of the minimal interval $[\varepsilon_1, \varepsilon_2]$ in (1.6.2) one can use, e.g., statements 12.57 and 12.60 from Voevodin, Kuznetsov [1]. The matrix equation (1.7.4) in a problem of spectrum localization was used in Mazko [7, 10]. Theorem 1.7.1 was formulated by Mazko [39] on the basis of the technique proposed by Gutman, Chojnowski [1].

1.8 The concept of a controllable matrix pair emerged in controllable system theory (see e.g. Andreev [1], Wonham [1], Simon, Mitter [1], and others). The Lyapunov equation with a nonnegative definite right-hand side was studied in Bakhilina, Lerner [1], Carlson, Schneider [1], Snyders, Zakai [1], Carlson, Hill [1], Wimmer [1,2], and others. Lemmas 1.8.1 and 1.8.2 and Theorems 1.8.1–1.8.3 were obtained in Mazko [13, 30]. The proposition of Theorem 1.8.3 with the limitations (1.8.12) was proved in Carlson, Hill [1].

2

ANALOGUES OF THE LYAPUNOV EQUATION FOR MATRIX FUNCTIONS

2.0 Introduction

This chapter deals with the analysis of spectral properties of matrix polynomials and functions. We propose classes of linear operators and respective matrix equations playing the role of the generalized Lyapunov equation in problems of stability and localization of eigenvalues. Here we use different methods of spectrum splitting that are based on the construction of contour integrals of Cauchy type, regular factorization of matrix functions, solution of special algebraic systems and definition of right and left pairs of matrix functions.

In Section 2.1 a class of the linear operators M_f is defined which generalize a class of the operators L_f studied in Chapter 1. Operators M_f are used in construction of analogues of the Lyapunov equations for matrix polynomials and functions. Using series expansions of the multiplicative derivative of the investigated matrix function F and the analytic function f which defines the domains for distribution of the selected subset of the spectrum $\sigma_0(F)$ in the complex plane, spectral and algebraic properties of the operator M_f are determined.

In Section 2.2 it is supposed that the matrix function F admits regular factorization with a regular linear multiplier whose spectrum coincides with the subset $\sigma_0(F)$. The operator M_f can be reduced to a special form which, taking into consideration the inertia law, allows us to generalize well-known theorems of eigenvalues localization (Lyapunov, Ostrowsky-Schneider, etc.).

In Section 2.3 the spectrum of a matrix polynomial is studied with the use of its linear accompanying form. In this case the operator M_f reduces to a standard form where matrix coefficients are determined in the form of integrals of Cauchy type or calculated by expansion of the resolvent $F^{-1}(\lambda)$ in Laurent series in the neighbourhood of an infinite point. The definition of the main theorems of eigenvalues localization is similar to the statements given in Section 2.2.

In Section 2.4 the procedure of construction of coefficients of the generalized Lyapunov equation for a matrix polynomial is described which is based on solving auxiliary algebraic matrix systems. Every nontrivial solution of such systems defines some subset of the spectrum of the matrix polynomial and the corresponding operator M_f satisfying the eigenvalues localization theorems stated in Section 2.2.

In Section 2.5 the notions of right and left pairs of a matrix function are introduced, generalizing the solutions of respectively the right and left problems for eigenvalues. Each right (left) pair of a matrix function defines some subset of the spectrum, with the number of its points, taking into account the multiplicities, equal to the observability (controllability) index of the given pair of matrices. A generalized spectral problem for a matrix polynomial is stated in the form of a corresponding algebraic matrix equation.

The analogues of the Lyapunov equation that are constructed in Section 2.6 by means of the right and left pairs of a matrix function are used in the study of a selected subset of the spectrum of this matrix function. The stated theorems on eigenvalue localization are the most general with respect to the main results described in Sections 2.2–2.4 and, within the framework of the method of matrix equations, are unique in the published books on this subject.

In Section 2.7 an effective simplified technique of the study of spectral properties of matrix polynomials and functions is described. Sufficient conditions of location of the spectrum of a matrix function in a specified domain are proposed which add up to solution of linear matrix equations and inequalities. Stated as a corollary are the conditions of stability of matrix quasi-polynomials used in description of differential-difference dynamic systems.

We hope that the generalized Lyapunov equation as a new direction of research in stability theory will be widely adopted both in analysis problems and in those of synthesis of dynamic systems with prescribed properties.

2.1 Operator M_f

Let $F(\lambda)$ be an $n \times n$ matrix composed of single-valued analytic functions and satisfying the condition

$$\chi(\lambda) = \det F(\lambda) \not\equiv 0, \quad \lambda \in C^1. \tag{2.1.1}$$

Zeros of the function $\chi(\lambda)$ are the eigenvalues of the matrix $F(\lambda)$ and form its spectrum $\sigma(F)$. Let us select some subset of the spectrum

$$\sigma_0(F) = \{\lambda_1, \ldots, \lambda_1; \ldots; \lambda_\alpha, \ldots, \lambda_\alpha\}, \tag{2.1.2}$$

where $\lambda_1, \ldots, \lambda_\alpha$ are pairwise different eigenvalues with the corresponding multiplicities $n_1, \ldots, n_\alpha$. For a matrix polynomial the whole spectrum $\sigma(F)$ consisting of l eigenvalues can be considered as $\sigma_0(F)$. Henceforth we will construct the subset (2.1.2) by using the methods of spectrum splitting. Construct a linear operator

$$M_f X = -\frac{1}{4\pi^2} \oint\limits_{\omega} \oint\limits_{\bar{\omega}} f(\lambda, \bar{\mu})\, R_\lambda X R_\mu^*\, d\lambda\, d\bar{\mu}, \tag{2.1.3}$$

where f is a given Hermitian function, $R_\lambda = F'(\lambda)F^{-1}(\lambda)$ is a *multiplicative derivative* of matrix $F(\lambda)$, and ω $(\bar{\omega})$ is a simple closed contour enclosing the points λ_t $(\bar{\lambda}_t)$, $t = 1, \ldots, \alpha$. The operator M_f is the generalization of the operator L_f studied in Chapter 1. In the case of $f \equiv 1$ the expression (2.1.3) reduces to

$$MX = \Delta X \Delta^*, \quad \Delta = \frac{1}{2\pi i} \oint\limits_{\omega} R_\lambda\, d\lambda, \tag{2.1.4}$$

where Δ is a matrix analogue of the logarithmic residue of a function with respect to the set of points $\sigma_0(F)$,

$$\operatorname{tr} R_\lambda \equiv \frac{\chi'(\lambda)}{\chi(\lambda)}, \quad \lambda \notin \sigma(F), \quad \operatorname{tr} \Delta = n_1 + \cdots + n_\alpha = r.$$

For functions f with separable variables we will use the representation

$$M_f X = \sum_{p,q} \gamma_{pq} \, F_p X F_q^*, \quad F_p = \frac{1}{2\pi i} \oint_\omega f_p(\lambda) \, R_\lambda \, d\lambda. \tag{2.1.5}$$

We will obtain an analogue of the formula (1.2.9) for the operator M_f. Each eigenvalue $\lambda_t \in \sigma_0(F)$ is a pole of order m_t of the matrix function $R_\lambda (m_1 + \cdots + m_\alpha = m)$. In some neighbourhood of the point λ_t the following expansion holds true:

$$R_\lambda = \sum_{i=1}^{m_t} \frac{(i-1)!}{(\lambda - \lambda_t)^i} \, A_{ti} + S_t(\lambda), \tag{2.1.6}$$

where A_{ti} are constant matrices, $S_t(\lambda)$ is a matrix function analytical in the given neighbourhood. We will use *Taylor series* for the function f in the neighbourhood of $(\lambda_t, \bar{\lambda}_\tau)$:

$$f(\lambda, \bar{\mu}) = \sum_{i,j=1}^{\infty} f_{ij}(\lambda_t, \bar{\lambda}_\tau) \, \frac{(\lambda - \lambda_t)^{i-1} (\bar{\mu} - \bar{\lambda}_\tau)^{j-1}}{(i-1)!(j-1)!},$$

$$f_{ij}(\lambda_t, \bar{\lambda}_\tau) = \frac{\partial^{i+j-2}}{\partial \lambda_t^{i-1} \partial \bar{\lambda}_\tau^{j-1}} \, f(\lambda_t, \bar{\lambda}_\tau).$$

Calculating integrals in (2.1.3) with the help of the main theorem of residues, we come to the following representation:

$$M_f X = \sum_{t,\tau=1}^{\alpha} \sum_{i,j=1}^{m_t, m_\tau} f_{ij}(\lambda_t, \bar{\lambda}_\tau) \, A_{ti} X A_{\tau j}^*. \tag{2.1.7}$$

The matrix coefficients in (2.1.5) and (2.1.7) satisfy the relations

$$\Delta = \sum_{t=1}^{\alpha} A_{t1}, \quad F_p = \sum_{t=1}^{\alpha} \sum_{i=1}^{m_t} \frac{d^{i-1} f_p(\lambda_t)}{d\lambda_t^{i-1}} \, A_{ti},$$

$$\operatorname{tr} A_{ti} = \begin{cases} n_t, & i = 1 \\ 0, & i > 1 \end{cases}, \quad \operatorname{tr} F_p = \sum_{t=1}^{\alpha} n_t f_p(\lambda_t).$$

Consider the case of the linear pencil $F(\lambda) = A - \lambda B$. On the assumption of *regularity* (2.1.1) there exist nonsingular matrices P and Q reducing to the *canonical form*

$$P(A - \lambda B)Q \equiv \begin{bmatrix} J - \lambda I & 0 \\ 0 & I - \lambda N \end{bmatrix}, \tag{2.1.8}$$

where J and N are square matrices of order r and $n - r$ respectively such that $\sigma(J) = \sigma(F)$, $N^\nu = 0$, I is a unit matrix of appropriate dimensions. The set of *finite elementary divisors* of the pencil $F(\lambda)$ consists of the elementary divisors of the matrix J. The nilpotency index ν of the matrix N is determined by the maximum power of *infinite elementary divisors* of the pencil $F(\lambda)$. Taking into account the identity (2.1.8), obtain the relation

$$R_\lambda = -B(A - \lambda B)^{-1} = -P^{-1} \begin{bmatrix} (J - \lambda I)^{-1} & 0 \\ 0 & N(I - \lambda N)^{-1} \end{bmatrix} P, \tag{2.1.9}$$

where

$$(J - \lambda I)^{-1} = -\sum_{t=1}^{\alpha} \sum_{i=1}^{m_t} \frac{(i-1)!}{(\lambda - \lambda_t)^i} J_{ti}, \quad (I - \lambda N)^{-1} = \sum_{i=0}^{\nu - 1} \lambda^i N^i,$$

$J_{ti} = \alpha_{ti}(J)$ are components of the matrix J, corresponding to the eigenvalues λ_t. Consequently, for the evaluation of the coefficients in (2.1.5) and (2.1.7), in accordance with (2.1.6) and (2.1.9), we obtain the relations

$$F_p = \Delta f_p(\Theta), \quad A_{ti} = \Delta\, \alpha_{ti}(\Theta), \tag{2.1.10}$$

$$\begin{aligned} \Delta = BZ = P^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P, \\ \Theta = AZ = P^{-1} \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} P, \end{aligned} \tag{2.1.11}$$

$$Z = -\frac{1}{2\pi i} \oint_\omega (A - \lambda B)^{-1} d\lambda = Q \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} P. \tag{2.1.12}$$

If the functions $f_p(\lambda)$ and $\alpha_{ti}(\lambda)$ in the point $\lambda = 0$ take on zero values, then the multiplier Δ in (2.1.10) can be omitted. For numerical evaluation of the matrices Δ and Θ one can use the following *Laurent expansion* ensuing from (2.1.9)

$$R_\lambda = \lambda^{\nu-2}K^{\nu-1} + \cdots + \lambda K^2 + K + \frac{1}{\lambda}\Delta + \frac{1}{\lambda^2}\Theta + \frac{1}{\lambda^3}\Theta^2 + \cdots,$$

where K is some nilpotent matrix. The matrices A_{ti} in (2.1.10) commute pairwise and satisfy the relations (see Section 1.2)

$$A_{t1}^2 = A_{t1}, \quad \sum_{t=1}^{\alpha} A_{t1} = \Delta, \quad A_{t1}A_{ti} = A_{ti}A_{t1} = A_{ti},$$

$$A_{ti}A_{\tau j} = 0 \quad (t \neq \tau), \quad A_{ti} = \frac{1}{(i-1)!}\,(\Theta - \lambda_t\Delta)^{i-1}A_{t1}.$$

Some algebraic and spectral properties of the operators L_f apply to the class of operators M_f in the case of a regular pencil of matrices. In particular, we have the relations

$$M_{f_1}M_{f_2} = M_{f_2}M_{f_1} = M_{f_1 f_2}, \quad c_1M_{f_1} + c_2M_{f_2} = M_{c_1f_1+c_2f_2},$$

$$M\, g\,(M_{f_1}, \ldots, M_{f_s}) = M_{g(f_1,\ldots,f_s)},$$

$$M_fW_{t\tau} = f(\lambda_t, \bar{\lambda}_\tau)\,W_{t\tau}, \quad W_{t\tau} = A_{tm_t}C_{t\tau}A^*_{\tau m_\tau} \neq 0,$$

where c_1 and c_2 are arbitrary constants, $g, f_1, \ldots, f_s$ are given functions, and $C_{t\tau}$ are some matrices. If w is an eigenvalue of the operator M_f with the multiplicity q, then either $w = f(\lambda_t, \bar{\lambda}_\tau)$ and $q \geq n_tn_\tau$, or $w = 0$ and $q \geq n^2 - r^2$. Using the results of Chapter 1, one can obtain a general representation of the eigenelements of the operator M_f.

2.2 Matrix Functions Admitting Regular Factorization

We study the properties of the operator M_f, assuming that the matrix function $F(\lambda)$ admits a *regular factorization*

$$\begin{gathered} F(\lambda) = C(\lambda)\,D(\lambda), \quad D(\lambda) = A - \lambda B, \\ \sigma(D) = \sigma_0(F), \quad \sigma(C) \cap \sigma(D) = \emptyset. \end{gathered} \tag{2.2.1}$$

Lemma 2.2.1 *Let ω be a closed contour enclosing the spectrum of the matrix pencil $A - \lambda B$ and separating a closed domain Ω in the complex plane, and $C_1(\lambda)$ and $C_2(\lambda)$ be analytic in Ω matrix functions. Then the following equality holds true:*

$$2\pi i \oint_{\omega} C_1(\lambda) S_\lambda C_2(\lambda)\, d\lambda = \oint_{\omega} C_1(\lambda) S_\lambda\, d\lambda \cdot \oint_{\omega} S_\lambda C_2(\lambda)\, d\lambda, \quad (2.2.2)$$

where $S_\lambda = -B(A - \lambda B)^{-1}$.

Proof. Represent the right-hand side of the equality (2.2.2) in the form of

$$C = \oint_{\omega} \oint_{\widehat{\omega}} C_1(\lambda) S_\lambda S_\mu C_2(\mu)\, d\lambda d\mu \quad (\lambda \in \omega,\ \mu \in \widehat{\omega}).$$

Here the closed contour $\widehat{\omega}$ entirely encloses and does not cross ω. It is easy to find that the following identity holds true:

$$S_\lambda - S_\mu \equiv (\mu - \lambda) S_\lambda S_\mu, \quad \lambda, \mu \notin \sigma(F). \quad (2.2.3)$$

If $B = I$, then (2.2.3) coincides with the *Gilbert identity* for the resolvent S_λ. Like in the resolvent case, we have

$$C = \oint_{\omega} C_1(\lambda) S_\lambda \left(\oint_{\widehat{\omega}} \frac{C_2(\mu)}{\mu - \lambda}\, d\mu \right) d\lambda - \oint_{\widehat{\omega}} \left(\oint_{\omega} \frac{C_1(\lambda)}{\mu - \lambda} d\lambda \right) S_\mu C_2(\mu)\, d\mu =$$

$$= 2\pi i \oint_{\omega} C_1(\lambda) S_\lambda C_2(\lambda)\, d\lambda.$$

Here the Cauchy formula and an integral theorem were used, giving as a result

$$\oint_{\widehat{\omega}} \frac{C_2(\mu)}{\mu - \lambda}\, d\mu = 2\pi i C_2(\lambda) \quad (\lambda \in \omega), \qquad \oint_{\omega} \frac{C_1(\lambda)}{\mu - \lambda}\, d\lambda = 0 \quad (\mu \in \widehat{\omega}).$$

The lemma is proved.

Lemma 2.2.2 *The operator (2.1.3) on the assumption of (2.2.1) is represented in the form*

$$M_f X = G\, L_f(HXH^*)\, G^*, \tag{2.2.4}$$

where $G \in C^{n\times r}$, $H \in C^{r\times n}$, L_f is an operator defined in (1.2.2) for some matrix $J \in C^{r\times r}$ with the spectrum $\sigma(J) = \sigma_0(F)$.

Proof. Calculate the multiplicative derivative of the factorized matrix (2.2.1):

$$F'(\lambda)F^{-1}(\lambda) = C'(\lambda)C^{-1}(\lambda) + C(\lambda)S_\lambda C^{-1}(\lambda).$$

Here the first summand does not have singularities within the contour ω. Applying Lemma 2.2.1 separately to each integral in (2.1.3), we obtain

$$M_f X = -\frac{1}{4\pi^2}\oint\limits_\omega \oint\limits_{\bar\omega} f(\lambda,\bar\mu)\, US_\lambda VXV^*\, S_\mu^*\, U^*\, d\lambda\, d\bar\mu,$$

where

$$U = \frac{1}{2\pi i}\oint\limits_\omega C(\lambda)S_\lambda\, d\lambda, \quad V = \frac{1}{2\pi i}\oint\limits_\omega S_\lambda C^{-1}(\lambda)\, d\lambda,$$

Consequently, taking into account (2.1.9) we have the representation (2.2.4). Note that

$$L_f\widehat{X} = -\frac{1}{4\pi^2}\oint\limits_\omega \oint\limits_{\bar\omega} f(\lambda,\bar\mu)(J-\lambda I)^{-1}\widehat{X}(J-\mu I)^{-1^*} d\lambda\, d\bar\mu,$$

$$U = [G, 0]\, P, \quad G = \frac{1}{2\pi i}\oint\limits_\omega C(\lambda)P^{-1}\begin{bmatrix} (\lambda I - J)^{-1} \\ 0 \end{bmatrix} d\lambda,$$

$$V = P^{-1}\begin{bmatrix} H \\ 0 \end{bmatrix}, \quad H = \frac{1}{2\pi i}\oint\limits_\omega \left[(\lambda I - J)^{-1}, 0\right] PC^{-1}(\lambda)\, d\lambda.$$

The lemma is proved.

Consider the matrix equation

$$M_f X = Y, \tag{2.2.5}$$

where M_f is a linear operator (2.1.3) constructed for the matrix function $F(\lambda)$, f is a Hermitian function describing the nonempty sets

$$\Lambda_f^+ = \{\lambda\colon\, f(\lambda,\bar\lambda) > 0\},$$
$$\Lambda_f^- = \{\lambda\colon\, f(\lambda,\bar\lambda) < 0\},$$
$$\Lambda_f^0 = \{\lambda\colon\, f(\lambda,\bar\lambda) = 0\}.$$

We will look for the matrices X and Y in (2.2.5) in the sets of Hermitian matrices formed by using the operator M of the form (2.1.4):

$$\mathcal{X} = \bigcup_{p=0}^{r} \mathcal{X}_{p0}, \quad \mathcal{Y} = \bigcup_{p=0}^{r} \mathcal{Y}_{p0}, \tag{2.2.6}$$

where

$$\mathcal{X}_{pq} = \Big\{X : \widehat{X} = MX,\ i_+(\widehat{X}) = p,\ i_-(\widehat{X}) = q\Big\},$$
$$\mathcal{Y}_{pq} = \Big\{Y : Y = M\widehat{Y},\ i_+(Y) = p,\ i_-(Y) = q\Big\}.$$

Here we assume that the multipliers G and H in (2.2.4) have full rank. The last limitation is equivalent to the equality $\operatorname{rank}\Delta = r$, from which it follows in particular that $\mathcal{X}_{r0} \neq \emptyset$ and $\mathcal{Y}_{r0} \neq \emptyset$. Determine the quantity of eigenvalues of the subset of the spectrum (2.1.2), belonging to the respective sets Λ_f^+, Λ_f^- and Λ_f^0:

$$r_+ = \sum_{\lambda_t \in \Lambda_f^+} n_t, \qquad r_- = \sum_{\lambda_t \in \Lambda_f^-} n_t, \qquad r_0 = \sum_{\lambda_t \in \Lambda_f^0} n_t.$$

From Lemmas 1.3.1–1.3.3 and the formulas (2.2.4)–(2.2.6) the following statements arise.

Lemma 2.2.3 *If for some matrix $Y \in \mathcal{Y}_{r0}$ the equation (2.2.5) has a solution $X \in \mathcal{X}$, then the subset of the spectrum (2.1.2) is located in the domain Λ_f^+. Conversely, if $\sigma_0(F) \subset \Lambda_f^+$, then there exist matrices $X \in \mathcal{X}_{r0}$ and $Y \in \mathcal{Y}_{r0}$ that satisfy the equation (2.2.5).*

Lemma 2.2.4 *The inclusion $M_f\mathcal{X}_{r0} \subseteq \mathcal{Y}_{r0}$ is equivalent to the relations*

$$f(\lambda_t,\bar\lambda_t) > 0, \quad t = \overline{1,\alpha}; \quad \Gamma_f \begin{pmatrix} m_1 \dots m_\alpha \\ \lambda_1 \dots \lambda_\alpha \end{pmatrix} \geq 0.$$

The last inequality is true if and only if $M_f\mathcal{X} \subseteq \mathcal{Y}$.

Lemma 2.2.5 *For any matrix* $Y \in \mathcal{Y}_{r0}$ *the equation (2.2.5) has the solution* $X \in \mathcal{X}_{r0}$ *if and only if*

$$f(\lambda_t, \bar{\lambda}_\tau) \neq 0, \quad t = \overline{1, \alpha},\ \tau = \overline{1, \alpha}; \quad \Gamma_{1/f}\begin{pmatrix} m_1 \dots m_\alpha \\ \lambda_1 \dots \lambda_\alpha \end{pmatrix} \geq 0.$$

Formulate analogues of Theorems 1.3.1, 1.5.1, and 1.6.1 for the matrix functions (2.2.1).

Theorem 2.2.1 *Let* $f \in \mathcal{H}_0^r$. *Then the inclusion* $\sigma_0(F) \subset \Lambda_f^+$ *is satisfied if and only if for any matrix* $Y \in \mathcal{Y}_{r0}$ *the equation (2.2.5) has the solution* $X \in \mathcal{X}_{r0}$.

Theorem 2.2.2 *If* $M_fX \in \mathcal{Y}_{r0}$, *then* $r_0 = 0$. *If* $r_0 = 0$, *then there exists a matrix* $X \in \mathcal{X}_{pq}$ *such that* $M_fX \in \mathcal{Y}_{r0}$ *and the equalities*

$$r_+ = p, \qquad r_- = q, \qquad p + q = r. \tag{2.2.7}$$

hold true. If some matrices $X \in \mathcal{X}_{pq}$ *and* $Y \in \mathcal{Y}_{r0}$ *satisfy the equation (2.2.5) with* $f \in \mathcal{H}_2^r$, *then the equalities (2.2.7) hold true.*

Theorem 2.2.3 *If a homogeneous matrix equation*

$$M_fX = 0 \tag{2.2.8}$$

has the solution $X \in \mathcal{X}_{p0}$, *then the estimate* $r_0 \geq p$ *holds true. Specifically, for* $X \in \mathcal{X}_{r0}$ *all eigenvalues* $\lambda_t \in \sigma_0(F)$ *are located on the curve* Λ_f^0. *Conversely, if* $r_0 \neq 0$, *then the equation (2.2.8) has the solution* $X \in \mathcal{X}_{p0}$ *for any* p *from the interval*

$$0 < p \leq \sum_{\lambda_t \in \Lambda_f^0} \xi_t,$$

where ξ_t *is the geometric multiplicity of the eigenvalue* $\lambda_t \in \sigma_0(F)$.

The results of Chapter 1 related to the expansion of sets of Hermitian matrices in the problem of localization of matrix spectrum can be generalized for the matrix function (2.2.1). Specifically, in construction of analogues of Theorem 1.8.1 we use the concept of controllability of linear systems not solved for derivatives.

2.3 Matrix Polynomial and its Accompanying Linear Form

When studying the spectral properties of the matrix polynomial

$$F(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^s A_s \in C^{n\times n}, \quad \det F(\lambda) \not\equiv 0, \qquad (2.3.1)$$

one can use the linear *accompanying pencils of matrices* with the spectrum $\sigma(F)$. Different methods of construction of such pencils are known. We will use the following block matrices of the dimensions $ns \times ns$:

$$A = \begin{bmatrix} -A_0 & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & I \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_{s-1} & 0 & \cdots & I \\ A_s & 0 & \cdots & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} A_1 & \cdots & A_{s-1} & A_s \\ I & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & I & 0 \end{bmatrix}, \quad S_1 = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & A_s \\ \cdots & \cdots & \cdots & \cdots \\ 0 & A_s & \cdots & 0 \end{bmatrix}, \qquad (2.3.2)$$

$$S_2 = \begin{bmatrix} A_1 & A_2 & \cdots & A_s \\ A_2 & A_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_s & 0 & \cdots & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} -A_0 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & A_s \\ \cdots & \cdots & \cdots & \cdots \\ 0 & A_s & \cdots & 0 \end{bmatrix}.$$

All scalar spectral characteristics (eigenvalues, finite and infinite elementary divisors) of the linear pencils $D(\lambda) = A - \lambda B$ and $L(\lambda) = A - \lambda C$ coincide and fully determine the corresponding spectral characteristics of the matrix polynomial $F(\lambda)$.

Let $\sigma_0(F)$ be some subset of spectrum (2.1.2) of the matrix polynomial (2.3.1), separated in the complex plane by a contour ω. Construct an operator M_f for the accompanying pencil $D(\lambda)$.

Applying Frobenius's formula for inversion of block matrices, for $\lambda \notin \sigma(F)$ we get

$$\begin{aligned} -D^{-1}(\lambda) &= S_1 W(\lambda) + F_1(\lambda), \\ D'(\lambda) D^{-1}(\lambda) &= S_2 W(\lambda) + F_2(\lambda), \qquad (2.3.3) \\ \lambda D'(\lambda) D^{-1}(\lambda) &= S_3 W(\lambda) + F_3(\lambda), \end{aligned}$$

where $F_1(\lambda)$, $F_2(\lambda)$, $F_3(\lambda)$ are some polynomial matrices, $W(\lambda)$ is a matrix which has the block Hankel structure

$$W(\lambda) = \begin{bmatrix} F^{-1}(\lambda) & \lambda F^{-1}(\lambda) & \cdots & \lambda^{s-1}F^{-1}(\lambda) \\ \lambda F^{-1}(\lambda) & \lambda^2 F^{-1}(\lambda) & \cdots & \lambda^s F^{-1}(\lambda) \\ \dots & \dots & \dots & \dots \\ \lambda^{s-1}F^{-1}(\lambda) & \lambda^s F^{-1}(\lambda) & \cdots & \lambda^{2s-2}F^{-1}(\lambda) \end{bmatrix}.$$

Integrating the equalities (2.3.3) along the closed contour ω enclosing all the points $\sigma_0(F)$, we obtain the relations

$$Z = S_1 H, \quad \Delta = BZ = S_2 H, \quad \Theta = AZ = S_3 H, \tag{2.3.4}$$

where

$$H = \begin{bmatrix} H_1 & H_2 & \cdots & H_s \\ H_2 & H_3 & \cdots & H_{s+1} \\ \dots & \dots & \dots & \dots \\ H_s & H_{s+1} & \cdots & H_{2s-1} \end{bmatrix},$$

$$H_p = \frac{1}{2\pi i} \oint\limits_{\omega} \lambda^{p-1} F^{-1}(\lambda)\, d\lambda, \quad p = 1, 2, \dots .$$

Integrals H_p satisfy the system of matrix equations

$$\begin{gathered} A_0 H_1 + A_1 H_2 + \cdots + A_s H_{s+1} = 0, \\ A_0 H_2 + A_1 H_3 + \cdots + A_s H_{s+2} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ A_0 H_p + A_1 H_{p+1} + \cdots + A_s H_{s+p} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{gathered}$$

It follows that the matrices (2.3.4) are expressed through the first s integrals $H_1, \dots, H_s$. Indeed, the blocks Z_{pq} of the matrix Z have the form

$$Z_{pq} = \begin{cases} \sum\limits_{j=p}^{s} A_j H_{j+q-p+1}, & q < p, \\ H_q, & p = 1, \\ -\sum\limits_{j=0}^{p-1} A_j H_{j+q-p+1}, & q \geq p > 1. \end{cases} \tag{2.3.5}$$

Here there are no matrices H_p for $p > s$. Note that if the contour ω encloses the whole spectrum $\sigma(F)$, then the matrices H_p coincide with the coefficients of the main part of Laurent's expansion of resolvent in the neighbourhood of an infinite point

$$F^{-1}(\lambda) = H_0(\lambda) + \frac{1}{\lambda}H_1 + \frac{1}{\lambda^2}H_2 + \cdots + \frac{1}{\lambda^s}H_s + \cdots. \tag{2.3.6}$$

Operator M_f for the pencil of matrices $D(\lambda)$ is represented in the form (2.1.7) where the coefficients A_{ti} are determined by (2.1.10)–(2.1.12). For the class of functions f with separable variables we will use the operator

$$M_f X = \sum_{p,q} \gamma_{pq}\, F_p X F_q^*, \tag{2.3.7}$$

where

$$F_p = \begin{cases} f_p(\Theta), & f_p(0) = 0, \\ \Delta f_p(\Theta), & f_p(0) \neq 0. \end{cases}$$

The matrices Δ and Θ in (2.3.4) and (2.3.7) have the following properties (see Section 2.1):

$$\begin{aligned} \operatorname{rank}\Delta = r, &\qquad \Delta^2 = \Delta, \\ \Delta\Theta = \Theta\Delta = \Theta, &\qquad \sigma_0(F) \subseteq \sigma(\Theta). \end{aligned} \tag{2.3.8}$$

Thus, if s integrals $H_1, \ldots, H_s$ in the relations (2.3.4) and (2.3.8) are known, then the properties of the operator M_f and the location r of the eigenvalues $\lambda_t \in \sigma_0(F)$ of the matrix polynomial $F(\lambda)$ with respect to the sets Λ_f^+, Λ_f^- and Λ_f^0 can be described by using Lemmas 2.2.2–2.2.5 and Theorems 2.2.1–2.2.3. Later we will construct a system of algebraic relations satisfied by the matrices $H_1, \ldots, H_s$.

2.4 Algebraic Systems of Spectrum Splitting

Each nontrivial solution Z of the algebraic system

$$AZ = ZA, \qquad Z = Z^2, \tag{2.4.1}$$

determines *a projector of matrix* A and some separation of spectrum $\sigma(A)$ into two subsets. Projectors of a matrix can be used for splitting and localization of its spectrum. We will construct analogues of the system (2.4.1) for the pencils of matrices satisfying the condition (2.1.1), and study the potentialities of their usage in Theorems 2.2.1–2.2.3.

Consider the combined equations

$$AZB = BZA, \qquad Z = ZBZ, \tag{2.4.2}$$

where A and B are matrices of the regular pencil $D(\lambda) = A - \lambda B$, Z is an unknown matrix. The general solution of the system (2.4.2) will be sought in the form

$$Z = Q \begin{bmatrix} Z_0 & Z_1 \\ Z_2 & Z_3 \end{bmatrix} P,$$

where P and Q are nonsingular matrices of the transform (2.1.8). To find the blocks Z_j obtain the system of equations

$$JZ_0 = Z_0 J, \quad NZ_3 = Z_3 N,$$

$$Z_1 = JZ_1 N, \quad Z_2 = NZ_2 J,$$

$$Z_0 = Z_0^2 + Z_1 N Z_2, \quad Z_1 = Z_0 Z_1 + Z_1 N Z_3,$$

$$Z_2 = Z_2 Z_0 + Z_3 N Z_2, \quad Z_3 = Z_2 Z_1 + Z_3 N Z_3.$$

Since N is a nilpotent matrix, then the equalities

$$Z_1 = JZ_1 N = J^2 Z_1 N^2 = \cdots = 0$$

hold true. The blocks Z_2 and Z_3 must be zero as well. Therefore the general solution of the system (2.4.2) is found in the form

$$Z = Q \begin{bmatrix} Z_0 & 0 \\ 0 & 0 \end{bmatrix} P, \quad JZ_0 = Z_0 J, \quad Z_0 = Z_0^2, \tag{2.4.3}$$

where Z_0 is an arbitrary projector of the matrix J. If $r \neq 0$ is the rank of the matrix Z_0, then for some nonsingular matrix S the following relations hold true:

$$Z_0 = S^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \qquad SJS^{-1} = \begin{bmatrix} J_0 & 0 \\ 0 & J_1 \end{bmatrix}, \tag{2.4.4}$$

where $J_0 \in C^{r\times r}$. It follows that the system of equations (2.4.2) has a nonempty set of solutions of rank r only if r is equal to the sum of orders of Jordan blocks of the matrix J, corresponding to some combination of elementary divisors of the pencil $D(\lambda)$. If S is an identity matrix, then the solution (2.4.3) of the system (2.4.2) is representable in an integral form:

$$Z = -\frac{1}{2\pi i}\oint\limits_{\omega}(A-\lambda B)^{-1}d\lambda, \qquad \operatorname{rank} Z = r.$$

Here the contour ω encloses a part of the spectrum $\sigma_0(D)$, consisting of r eigenvalues, taking into account the multiplicities. In the case when $\sigma_0(D)$ is the whole spectrum, we find the solution of the system (2.4.2) of the type (2.1.12) of the maximum rank equal to the total quantity of eigenvalues of the pencil $D(\lambda)$.

If Z is a solution of the system (2.4.2) of rank r, then, in accordance with (2.4.3) and (2.4.4), in the relations (2.1.5), (2.1.7), and (2.1.10) one can use the matrices

$$\begin{aligned} \Delta &= BZ = G^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} G, \\ \Theta &= AZ = G^{-1}\begin{bmatrix} J_0 & 0 \\ 0 & 0 \end{bmatrix} G, \end{aligned} \tag{2.4.5}$$

where

$$G = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} P, \qquad \sigma(J_0) \subseteq \sigma(D).$$

Here a subset of the spectrum $\sigma_0(D)$ is determined, which coincides with $\sigma(J_0)$, and the matrices (2.4.5) have the following properties (see Section 2.1):

$$\begin{aligned} &\operatorname{rank}\Delta = r, \qquad \Delta^2 = \Delta, \\ &\Delta\Theta = \Theta\Delta = \Theta, \qquad \sigma_0(D) \subseteq \sigma(\Theta). \end{aligned} \tag{2.4.6}$$

Consequently, if the operators (2.1.5) and (2.1.7) are determined by using the relations (2.1.10) and (2.4.5) for an arbitrary solution Z of the system (2.4.2) of rank r, then the location r of eigenvalues of the matrix pencil $D(\lambda)$ with respect to the given sets Λ_f^+, Λ_f^- and Λ_f^0 can be described by using the statements of Theorems 2.2.1–2.2.3. The system of matrix relations (2.2.5) and (2.4.2) is an analogue of the Lyapunov equation for a regular matrix pencil $D(\lambda)$.

Note that to each nonzero solution Z of the system (2.4.2) a subset of spectrum $\sigma_0(D) = \sigma(J_0)$ corresponds, which coincides with the spectrum $r \times r$ matrix $\Theta_0 = R^*AL$, where L and R^* are multipliers of the *skeleton expansion*

$$Z = LR^*, \quad L \in C^{n\times r}, \quad R \in C^{n\times r}. \tag{2.4.7}$$

The solutions of linear equation in (2.4.2) can be used to lower the dimension in problems of estimation and location of eigenvalues. Thus, if the matrix (2.4.7) satisfies the equation

$$AZB = BZA \tag{2.4.8}$$

and $a \notin \sigma(D)$, then the spectra of regular linear pencils of the matrices

$$U(\lambda) = R^*D(\lambda)D^*(a)R, \qquad V(\lambda) = L^*D^*(a)D(\lambda)L$$

coincide and form some subset of the spectrum of the initial pencil $D(\lambda)$. Here the right (left) eigenvectors of the pencil $D(\lambda)$, corresponding to the given subset of the spectrum, are determined in the form of linear combinations of columns (rows) of the multiplier $L(R^*)$ in the expansion (2.4.7). Using the solutions of the equation (2.4.8), one can construct regular pencils of matrices of the type $U(\lambda)$ and $V(\lambda)$, which are not unimodular under additional rank limitations on Z.

The equation (2.4.8) is equivalent to the identity

$$D(\lambda)ZD(a) \equiv D(a)ZD(\lambda), \quad \lambda \in C^1.$$

The matrices (2.4.7) for which this identity holds true have properties similar to those described above, even if $D(\lambda)$ is a regular matrix function.

Let $F(\lambda)$ be a matrix polynomial of the form (2.3.1), and $D(\lambda) = A - \lambda B$ be its accompanying pencil whose matrices are determined in (2.3.2) and $\sigma(F) = \sigma(D)$. Represent the system of equations (2.4.2) in the form of $2s$ matrix equations with respect to unknown $T_1, \ldots, T_s$.

First consider the case $s = 2$, assuming

$$A = \begin{bmatrix} -A_0 & 0 \\ 0 & I \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & I \\ A_2 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} T_1 & T_2 \\ G_1 & G_2 \end{bmatrix}.$$

In accordance with (2.4.2), we arrive at the system of four matrix equations with respect to T_1 and T_2:

$$\begin{gathered} A_0T_1A_1 - A_1T_1A_0 = A_2T_2A_0 - A_0T_2A_2, \\ A_0T_1A_2 - A_2T_1A_0 = A_2T_2A_1 - A_1T_2A_2, \\ T_1 = T_1A_1T_1 + T_1A_2T_2 + T_2A_2T_1, \\ T_2 = T_2A_2T_2 - T_1A_0T_1. \end{gathered} \tag{2.4.9}$$

The blocks G_1 and G_2 of the unknown matrix Z in the system (2.4.2) are expressed through T_1 and T_2:

$$G_1 = A_2T_2, \quad G_2 = -A_0T_1 - A_1T_2. \tag{2.4.10}$$

In addition, in accordance with the second equation (2.4.2), the following equalities must hold true:

$$G_1 = G_1A_1T_1 + G_2A_2T_1 + G_1^2,$$

$$G_2 = G_1A_1T_2 + G_2A_2T_2 + G_1G_2.$$

However one can find out that these equalities are the consequence of the relations (2.4.9) and (2.4.10). Therefore, if T_1 and T_2 are the solution of the system (2.4.9), then the matrices

$$\begin{gathered} \Delta = BZ = \left[\begin{array}{c|c} A_1T_1 + A_2T_2 & -A_0T_1 \\ \hline A_2T_1 & A_2T_2 \end{array}\right], \\ \Theta = AZ = \left[\begin{array}{c|c} -A_0T_1 & -A_0T_2 \\ \hline A_2T_2 & -A_0T_1 - A_1T_2 \end{array}\right], \end{gathered} \tag{2.4.11}$$

satisfy the relations (2.4.6) and can be used in construction of analogues of the Lyapunov equation in Theorems 2.2.1–2.2.3 for the quadratic matrix pencil $F(\lambda) = A_0 + \lambda A_1 + \lambda^2 A_2$ (see (2.1.10) and (2.4.5)).

Let us generalize the system (2.4.9) for a matrix polynomial of power $s \geq 2$. Represent the matrices A, B, and Z in the form

$$A = \left[A^{(0)} \,\middle|\, A^{(1)}\right] = \left[\begin{array}{c|ccc} -A_0 & 0 & \cdots & 0 \\ 0 & I & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & I \end{array}\right],$$

$$B = \left[B^{(0)} \,\middle|\, B^{(1)}\right] = \left[\begin{array}{c|ccc} A_1 & I & & 0 \\ A_2 & & \ddots & \\ \vdots & 0 & & I \\ A_s & 0 & \cdots & 0 \end{array}\right],$$

$$Z = \left[\begin{array}{c} T \\ \hline G \end{array}\right] = \left[\begin{array}{cccc} T_1 & T_2 & \cdots & T_s \\ \hline G_{11} & G_{12} & \cdots & G_{1s} \\ \cdots & \cdots & \cdots & \cdots \\ G_{s-11} & G_{s-12} & \cdots & G_{s-1s} \end{array}\right].$$

Then the system (2.4.2) is equivalent to the relations

$$A^{(0)}TB^{(0)} - B^{(0)}TA^{(0)} = B^{(1)}GA^{(0)} - A^{(1)}GB^{(0)}, \tag{2.4.12}$$

$$WG \triangleq A^{(1)}GB^{(1)} - B^{(1)}GA^{(1)} = B^{(0)}TA^{(1)} - A^{(0)}TB^{(1)}, \tag{2.4.13}$$

$$T = TB^{(0)}T + TB^{(1)}G, \tag{2.4.14}$$

$$G = GB^{(0)}T + GB^{(1)}G. \tag{2.4.15}$$

Note that the equality (2.4.13) gives explicit representation of G through T:

$$G_{pq} = \begin{cases} -\sum\limits_{j=0}^{p} A_j T_{q-p+j}, & p < q, \\ \sum\limits_{j=p+1}^{s} A_j T_{q-p+j}, & p \geq q. \end{cases} \tag{2.4.16}$$

The operator W determined in the right-hand side of the equality (2.4.13) is invertible. Applying the operator W to both sides of the equality (2.4.15), one can find out that the equality (2.4.15) is the consequence of the relations (2.4.12), (2.4.14), and (2.4.16). Substituting (2.4.16) into (2.4.12) and (2.4.14), we arrive at the following system $2s$ of matrix equations with respect to the unknown $T_1, \ldots, T_s$:

$$\begin{gathered} \sum_{i=0}^{p} \sum_{j=p+1}^{s} (A_i T_{i+j-p} A_j - A_j T_{i+j-p} A_i) = 0, \\ T_q = \sum_{i=q}^{s} \sum_{j=i}^{s} T_i A_j T_{q+j-i} - \sum_{i=0}^{q-1} \sum_{j=-1}^{i-1} T_i A_j T_{q+j-i}, \end{gathered} \tag{2.4.17}$$

where $T_0 = A_{-1} = 0$, $p = \overline{0, s-1}$, $q = \overline{1, s}$.

Construct the block matrices $\Delta = BZ$ and $\Theta = AZ$ of the form

$$\Delta = \begin{bmatrix} \Delta_{11} & \cdots & \Delta_{1s} \\ \cdots & \cdots & \cdots \\ \Delta_{s1} & \cdots & \Delta_{ss} \end{bmatrix}, \quad \Theta = \begin{bmatrix} \Theta_{11} & \cdots & \Theta_{1s} \\ \cdots & \cdots & \cdots \\ \Theta_{s1} & \cdots & \Theta_{ss} \end{bmatrix}, \tag{2.4.18}$$

where

$$\Delta_{pq} = \begin{cases} -\sum\limits_{j=0}^{p-1} A_j T_{q-p+j}, & p < q, \\ \sum\limits_{j=p}^{s} A_j T_{q-p+j}, & p \geq q, \end{cases}$$

$$\Theta_{pq} = \begin{cases} -\sum\limits_{j=0}^{p-1} A_j T_{q-p+j+1}, & p \leq q, \\ \sum\limits_{j=p}^{s} A_j T_{q-p+j+1}, & p > q. \end{cases}$$

In the case $s = 2$ the system (2.4.17) reduces to the form (2.4.9), and the matrices (2.4.11) and (2.4.18) are the same.

Comparing (2.3.5) and (2.4.16), and using the structure of solutions of the system (2.4.2) for the accompanying pencil $D(\lambda)$, obtain the following statement.

Lemma 2.4.1 *The integral family*

$$T_p = \frac{1}{2\pi i} \oint\limits_{\omega} \lambda^{p-1} F^{-1}(\lambda) d\lambda, \quad p = \overline{1, s}, \tag{2.4.19}$$

where ω is a closed contour separating the subset of spectrum $\sigma_0(F)$, presents one of the solutions of the system (2.4.17).

Lemma 2.4.2 *Let $T_1, \ldots, T_s$ be a nontrivial solution of the system (2.4.17). Then Δ is a projector of rank r of the matrix Θ, $1 \leq r \leq ns$, and at least r eigenvalues of the matrix Θ, taking into account the multiplicities, belong to the spectrum of the matrix polynomial $F(\lambda)$. Note that if $\lambda \in \sigma(\Theta)$, then either $\lambda \in \sigma_0(F)$, or $\lambda = 0$.*

This proposition can be proved through representation of matrices (2.4.18) in the form (2.4.5) with respect to the solutions of the system (2.4.2). The subset of spectrum $\sigma_0(F)$ of the matrix polynomial $F(\lambda)$ is determined, which coincides with $\sigma(J_0)$ and also with $\sigma(R^* AL)$, where L and R^* are multipliers of the skeleton expansion (2.4.7) of the corresponding solution of the system (2.4.2).

Using Lemma 2.4.2 and the above technique, we arrive at the following conclusion.

Theorem 2.4.1 *Let the equation (2.2.5) and the sets of matrices (2.2.6) be constructed for the operators*

$$M_f X = \sum_{p,q} \gamma_{pq} \, F_p X F_q^*, \qquad MX = \Delta X \Delta^*,$$

where

$$f(\lambda, \bar{\mu}) = \sum_{p,q} \gamma_{pq} \, f_p(\lambda) \overline{f_q(\mu)}, \quad F_p = \begin{cases} f_p(\Theta), & f_p(0) = 0, \\ \Delta f_p(\Theta), & f_p(0) \neq 0, \end{cases}$$

and the matrices Δ and Θ be determined in the form (2.4.18) with respect to the nontrivial solution of the system (2.4.17). Then for some *subset of the spectrum* $\sigma_0(F)$ *of the matrix polynomial* $F(\lambda)$*, consisting of* $r = \operatorname{rank}\Delta$ *eigenvalues, all statements of Lemmas 2.2.2–2.2.5 and Theorems 2.2.1–2.2.3 hold true.*

Thus, we have a general technique for construction of analogues of the Lyapunov equation and the corresponding theorems on eigenvalues location for linear, quadratic, and polynomial pencils of matrices. This technique is based on the usage of operators of the type M and M_f, formed by the solutions of the matrix algebraic system (2.4.17), e.g., integrals (2.4.19) or coefficients of the leading part of the Laurent expansion of the resolvent (2.3.6). The generalized results presented in the form of Theorem 2.4.1 belong to the class of Hermitian functions f with separable variables.

2.5 Right and Left Pairs of a Matrix Function

Let $F(\lambda)$ be a matrix function of dimensions $n \times n$, analytical in some domain Λ. We will introduce definitions generalizing the concepts of block eigenvalues and block eigenvectors of a matrix polynomial.

The matrices $U \in C^{m\times m}$ and $T \neq 0 \in C^{n\times m}$ form the *right pair* (U, T) of a matrix function $F(\lambda)$, if for some analytic matrix function $\Phi(\lambda)$ in the neighbourhood of points $\sigma(U)$ the identity

$$F(\lambda)T \equiv \Phi(\lambda)(\lambda I - U), \quad \lambda \in \Lambda. \tag{2.5.1}$$

holds true. The *left pairs* (U, T) of a matrix function $F(\lambda)$ are similarly determined by using the identity

$$TF(\lambda) \equiv (\lambda I - U)\Phi(\lambda), \quad \lambda \in \Lambda. \tag{2.5.2}$$

If a matrix function $F(\lambda)$ is presented in the form

$$F(\lambda) = A_0 + a_1(\lambda)A_1 + \cdots + a_s(\lambda)A_s, \tag{2.5.3}$$

where $a_j(\lambda)$ are scalar functions, A_j are constant matrices, then its right and left pairs (U, T) satisfy the corresponding equations

$$A_0T + A_1Ta_1(U) + \cdots + A_sTa_s(U) = 0, \tag{2.5.4}$$

$$TA_0 + a_1(U)TA_1 + \cdots + a_s(U)TA_s = 0. \tag{2.5.5}$$

This statement follows directly from (2.5.1), (2.5.2) and integral representation of analytic functions of the matrix $a_j(U)$. Conversely, if the matrices U and $T \neq 0$ satisfy the equation (2.5.4) ((2.5.5)), then (U, T) is the right (left) pair of the matrix function (2.5.3).

For the right and left pairs of the matrix function $F(\lambda)$, construct sequences of matrices E_k with the corresponding block structure:

$$E_k = \begin{bmatrix} T \\ TU \\ \vdots \\ TU^{k-1} \end{bmatrix}, \quad E_k = \left[T, UT, \ldots, U^{k-1}T\right], \quad k = 1, 2, \ldots . \tag{2.5.6}$$

For both sequences (2.5.6) the rank relations

$$r_1 < r_2 < \cdots < r_h = r_{h+1} = \cdots = r, \tag{2.5.7}$$

hold true, where $r_k = \operatorname{rank} E_k$, and h is the least value of the index k with $r_k = r_{k+1}$. The maximum $r = r_h$ of the rank sequence (2.5.7) is called *the observability (controllability) index* of the right (left) pair (U, T). The following estimates are true:

$$\operatorname{rank} T + h - 1 \leq r \leq m, \quad 1 \leq h \leq m_0, \tag{2.5.8}$$

where m_0 is the power of the minimal polynomial of the matrix U. In the case of $r = m$ the right (left) pair (U, T) is *observable (controllable)*. The observability and controllability of the pair (U, T) are equivalent to the conditions

$$\operatorname{rank} \begin{bmatrix} \lambda I - U \\ T \end{bmatrix} = m, \quad \operatorname{rank}[\lambda I - U, T] = m, \quad \lambda \in \sigma(U).$$

The observable (controllable) pairs of matrices (U, T) satisfying the condition (2.5.1) ((2.5.2)) will be called the right (left) eigenvalues of the matrix function $F(\lambda)$. For such pairs the inclusion $\sigma(U) \subseteq \sigma(F)$ holds true. The reverse inclusion $\sigma(F) \subseteq \sigma(U)$ holds true under the conditions

$$\operatorname{rank}[F(\lambda), \Phi(\lambda)] = n, \quad \lambda \in \sigma(F), \tag{2.5.9}$$

$$\operatorname{rank}\begin{bmatrix} F(\lambda) \\ \Phi(\lambda) \end{bmatrix} = n, \quad \lambda \in \sigma(F). \tag{2.5.10}$$

If $l < \infty$ is the number of points of the spectrum $\sigma(F)$, then under the condition (2.5.9) ((2.5.10)) the pair (U, T) has the maximum possible observability (controllability) index $r = l$.

Lemma 2.5.1 *Let (U, T) be the right (left) pair of the matrix function $F(\lambda)$ with the observability (controllability) index r. Then at least r points of the spectrum $\sigma(U)$ are eigenvalues of the matrix function $F(\lambda)$. Under the condition (2.5.9) ((2.5.10)) each point of the spectrum $\sigma(F)$ is an eigenvalue of matrix U.*

Proof. Let the identity (2.5.1) hold true. If r is the observability index of the pair (U, T), then there exists a nonsingular matrix $G \in C^{m\times m}$ transforming the matrices U and T to the form

$$GUG^{-1} = \begin{bmatrix} U_0 & 0 \\ U_2 & U_1 \end{bmatrix}, \qquad TG^{-1} = [T_0, 0],$$

where $U_0 \in C^{r\times r}$, $T_0 \in C^{n\times r}$, (U_0, T_0) is an observable pair. Taking this transform into account, in accordance with (2.5.1) obtain the relations

$$F(\lambda)T_0 \equiv \Phi_0(\lambda)(\lambda I - U_0), \qquad \Phi(\lambda) = [\Phi_0(\lambda), 0]\, G.$$

If u_0 is the right eigenvalue of the matrix U_0, corresponding to the eigenvalue $\lambda_0 \in \sigma(U_0)$, then in view of observability of the pair (U_0, T_0) the inequality $v_0 = T_0 u_0 \neq 0$ holds true. Therefore v_0 is the right eigenvalue of the matrix function $F(\lambda)$, corresponding to the eigenvalue $\lambda_0 \in \sigma(F)$. Using the described relations one can find that $\det F(\lambda) \equiv \varphi(\lambda) \det(\lambda I - U_0)$, where φ is some function. Hence, $\sigma(U_0)$ coincides with some subset of the spectrum $\sigma_0(F) \subseteq \sigma(U)$. If λ_0 is an eigenvalue of the matrix U_0 with the multiplicity n_0, then λ_0 is also an eigenvalue of the matrix function $F(\lambda)$ with the multiplicity $N_0 \geq n_0$. Under the condition (2.5.9) the converse is proved in a similar way.

The proof of the statements in the case of the left pair (U, T) of the matrix function $F(\lambda)$ follows from the relations

$$G^{-1}UG = \begin{bmatrix} U_0 & U_2 \\ 0 & U_1 \end{bmatrix}, \qquad G^{-1}T = \begin{bmatrix} T_0 \\ 0 \end{bmatrix},$$

$$T_0F(\lambda) \equiv (\lambda I - U_0)\Phi_0(\lambda), \quad \Phi(\lambda) = G\begin{bmatrix} \Phi_0(\lambda) \\ 0 \end{bmatrix}, \quad \lambda \in \Lambda,$$

where $U_0 \in C^{r\times r}$, $T_0 \in C^{r\times n}$, (U_0, T_0) is a controllable pair.

The selected subset of the spectrum $\sigma_0(F)$, corresponding to the right (left) pair (U,T), coincides with the observable (controllable) part of the spectrum $\sigma(U_0)$ of the matrix U. If the condition (2.5.9) ((2.5.10)) is satisfied, then the inclusion $\sigma(F) \subseteq \sigma(U)$ is proved by multiplication from the left (right) of the identity (2.5.1) ((2.5.2)) by the left (right) eigenvalues of the matrix function $F(\lambda)$.

The lemma is proved.

For the matrix polynomial (2.3.1) the relations determining the right and left pairs (U,T) have the form

$$A_0T + A_1TU + \cdots + A_sTU^s = 0, \tag{2.5.11}$$

$$TA_0 + UTA_1 + \cdots + U^sTA_s = 0. \tag{2.5.12}$$

In (2.5.1) and (2.5.2) $\Phi(\lambda)$ is determined by the corresponding expression:

$$\Phi(\lambda) = \sum_{i=1}^{s} \lambda^{i-1} \sum_{j=i}^{s} A_jTU^{j-i},$$

$$\Phi(\lambda) = \sum_{i=1}^{s} \lambda^{i-1} \sum_{j=i}^{s} U^{j-i}TA_j.$$

If in (2.5.11) ((2.5.12)) T is a matrix of full rank with respect to columns (rows), then the pair (U,T) is composed of the right (left) block eigenvalue and the eigenvector of the matrix polynomial $F(\lambda)$.

Lemma 2.5.2 (U,T) *is the right pair of the matrix polynomial (2.3.1) of the observability index r if and only if*

$$AE = CEU, \quad E = \begin{bmatrix} T \\ TU \\ \vdots \\ TU^{s-1} \end{bmatrix}, \quad \operatorname{rank} E = r. \tag{2.5.13}$$

Similarly, (U, T) is the left pair of the matrix polynomial $F(\lambda)$ with controllability index r if and only if

$$EA = UEB, \quad E = \left[T, UT, , \dots, U^{s-1}T\right], \quad \operatorname{rank} E = r. \quad (2.5.14)$$

Proof. The equivalence of the matrix equalities (2.5.11) and (2.5.13) ((2.5.12) and (2.5.14)) is the consequence of the structure of the block matrices (2.3.2). The fact that the rank of the matrix E coincides with the observability (controllability) index of the pair (U, T) is established by using the canonical form of regular pencil of matrices. (2.1.8) and (2.5.14) imply

$$E = [R, 0]\, P, \quad RJ = UR, \quad \operatorname{rank} E = \operatorname{rank}[E,\, UE].$$

Therefore columns of the matrix UE and, in particular, of the block U^sT are linearly expressed through columns of the matrix E. Similarly, in (2.5.13) rows of the matrix EU belong to the linear hull of rows E. Hence, for the matrix polynomial along with (2.5.8) the estimate $h \le s$ holds true.

The lemma is proved.

According to (2.3.2), $AS_1 = S_1A = S_3$ and $BS_1 = S_1C = S_2$. Therefore from (2.5.13) ((2.5.14)) follow the relations

$$AZ = BZU, \quad Z = S_1E \quad (ZA = UZC, \quad Z = ES_1), \quad (2.5.15)$$

that also determine the connection between the right (left) pairs of a matrix polynomial and its accompanying pencil.

Solutions of the system (2.4.17) can be used for finding the right and left pairs of a matrix polynomial. Indeed, the first block row (the first block column) of the matrix Z satisfying the system

$$AZB = BZA, \;\; Z = ZBZ \quad (AZC = CZA, \;\; Z = ZCZ), \quad (2.5.16)$$

composes the solution $T_1, \dots, T_s$ of (2.4.17). At the same time, the equalities (2.5.15) follow from (2.5.16) for $U = AZ$ ($U = ZA$).

Lemma 2.5.3 *If $T_1, \dots, T_s$ is a solution of the system (2.4.17),* *then the matrices*

$$T = [T_1, \ldots, T_s], \quad U = \|U_{pq}\|_1^s, \quad U_{pq} = \begin{cases} -\sum\limits_{i=0}^{p-1} A_i T_{q-p+i+1}, & p \leq q, \\ \sum\limits_{i=p}^{s} A_i T_{q-p+i+1}, & p > q, \end{cases}$$

form the right pair (U, T), *and the matrices*

$$T = \begin{bmatrix} T_1 \\ \vdots \\ T_s \end{bmatrix}, \quad U = \|U_{pq}\|_1^s, \quad U_{pq} = \begin{cases} -\sum\limits_{i=0}^{q-1} T_{p-q+i+1} A_i, & p \geq q, \\ \sum\limits_{i=q}^{s} T_{p-q+i+1} A_i, & p < q, \end{cases}$$

form the left pair (U, T) *of the matrix polynomial (2.3.1).*

The matrix system (2.4.17) is satisfied by the integrals (2.4.19). If the closed contour ω encloses the whole spectrum $\sigma(F)$, then in Lemma 2.6.3 (U, T) is the right (left) pair of the matrix polynomial $F(\lambda)$, for which the conditions (2.5.9) ((2.5.10)) and $\sigma(F) \subseteq \sigma(U)$ hold true. This statement follows from the canonical structure of the accompanying pencils $L(\lambda) = A - \lambda C$ and $D(\lambda) = A - \lambda B$ and block transforms of matrix expressions in the relations

$$\operatorname{rank}[L(\lambda), CE] = n\,s, \quad \operatorname{rank}\begin{bmatrix} D(\lambda) \\ EB \end{bmatrix} = n\,s, \quad \lambda \in \sigma(F),$$

reducing them to the corresponding form (2.5.9) and (2.5.10).

In construction of right and left pairs of a matrix polynomial one can only use linear equations of the systems (2.4.17) and (2.5.16). Thus, using (2.1.8), it is easy to show that if $AZB = BZA$, then for some matrix U the pair (U, E), where $E = (ZB)^k$, $k \geq \nu$, satisfies the relations (2.5.14). If $\operatorname{rank} Z = \operatorname{rank}(BZ)$, then as E one can also choose the matrix Z.

2.6 Theorems on Eigenvalues Location

Let (U, T) be the right (left) pair of the matrix function $F(\lambda)$ of the observability (controllability) index r and let a subset of the spectrum $\sigma_0(F) \subseteq \sigma(F)$ correspond to it, consisting of r eigenvalues (see Lemma 2.5.1). We study the location of the points $\sigma_0(F)$ in the complex plane with respect to the sets

$$\Lambda_f^+ = \{\lambda\colon f(\lambda,\bar{\lambda}) > 0\}, \quad \Lambda_f^- = \{\lambda\colon f(\lambda,\bar{\lambda}) < 0\},$$

$$\Lambda_f^0 = \{\lambda\colon f(\lambda,\bar{\lambda}) = 0\},$$

described by the Hermitian function

$$f(\lambda,\bar{\mu}) = \sum_{i,j} \gamma_{ij}\, f_i(\lambda)\overline{f_j(\mu)} \equiv z_\lambda \Gamma z_\mu^*.$$

Construct the linear matrix equation

$$\sum_{i,j} \gamma_{ij}\, F_i X F_j^* = EYE^*, \tag{2.6.1}$$

where the matrix coefficients are defined in terms of the right or left pair (U,T) of the matrix function $F(\lambda)$. To reduce the computation, in both cases we use the same notation. If (U,T) is the right pair, then

$$F_i = Ef_i(U), \qquad E = \begin{bmatrix} T \\ TU \\ \vdots \\ TU^{h-1} \end{bmatrix}.$$

In the case of the left pair (U,T) assume

$$F_i = f_i(U)E, \qquad E = \left[T, UT, \ldots, U^{h-1}T\right].$$

If $F(\lambda)$ is a matrix polynomial of order s, then along with (2.5.8) the inequality $h \le s$ holds true. Take the sets of Hermitian matrices

$$\mathcal{K} = \{X : EXE^* \ge 0\},$$

$$\mathcal{K}_{pq} = \{X : i_+(EXE^*) = p, \quad i_-(EXE^*) = q\},$$

where $i_\pm(\cdot)$ are the indices of inertia of a Hermitian matrix, equal to the quantities of its positive and negative eigenvalues. For the matrix Y in the equation (2.6.1) the following limitations will be used:

$$S_\lambda = EYE^* + E(\lambda I - U)(\lambda I - U)^* E^* \geq 0, \quad \operatorname{rank} S_\lambda \equiv r, \quad (2.6.2)$$

$$S_\lambda = EYE^* + (\lambda I - U)EE^*(\lambda I - U)^* \geq 0, \quad \operatorname{rank} S_\lambda \equiv r. \quad (2.6.3)$$

Theorem 2.6.1 *If the matrices $X \in \mathcal{K}$ and $Y \in \mathcal{K}$ satisfy the equation (2.6.1) and the conditions (2.6.2) ((2.6.3)), then a set of spectrum $\sigma_0(F)$, corresponding to the right (left) pair (U, T) of the matrix function $F(\lambda)$, is located in the domain Λ_f^+. If in addition the condition (2.5.9) ((2.5.10)) holds true, then $\sigma_0(F) = \sigma(F) \subset \Lambda_f^+$. Conversely, if $\sigma_0(F) \subset \Lambda_f^+$ and $f \in \mathcal{H}_0^r$, then for any matrix $Y \in \mathcal{K}$ the equation (2.6.1) has a solution $X \in \mathcal{K}$.*

Theorem 2.6.2 *If the matrices $X \in \mathcal{K}_{pq}$ and $Y \in \mathcal{K}_{r0}$ satisfy the equation (2.6.1), and $f \in \mathcal{H}_2^r$, then the equalities*

$$r_+ = p, \qquad r_- = q, \qquad r_0 = 0, \quad (2.6.4)$$

hold true, where r_+, r_-, and r_0 are the quantities of the points of $\sigma_0(F)$, belonging to Λ_f^+, Λ_f^- and Λ_f^0 respectively. Conversely, if for some p and q the equalities (2.6.4) hold true, then there exist matrices $X \in \mathcal{K}_{pq}$ and $Y \in \mathcal{K}_{r0}$ satisfying the equation (2.6.1).

Theorem 2.6.3 *If the matrices $X \in \mathcal{K}_{p0}$ and $Y \in \mathcal{K}_{00}$ satisfy the equation (2.6.1), then the estimate $r_0 \geq p$ holds true. In particular, with $p = r$ the inclusion $\sigma_0(F) \subset \Lambda_f^0$ is true. Conversely, if $r_0 \neq 0$, $Y \in \mathcal{K}_{00}$, $0 < p \leq \xi$, where ξ is the sum of geometric multiplicities of eigenvalues of the matrix U, belonging to the set $\sigma_0(F) \cap \Lambda_f^0$, then the equation (2.6.1) has a solution $X \in \mathcal{K}_{p0}$.*

Proof of Theorems 2.6.1–2.6.3. Let (U, T) be the right pair of the matrix function $F(\lambda)$ with the observability index r. Then from the proof of Lemma 2.5.1 we get

$$E = E_0 G_0, \quad E_0 = \begin{bmatrix} T_0 \\ T_0 U_0 \\ \vdots \\ T_0 U_0^{h-1} \end{bmatrix},$$

$$G_0 = [I, 0]\, G, \quad F_i = E_0 f_i(U_0) G_0,$$

where (U_0, T_0) is a right eigenvalue of the matrix function $F(\lambda)$, E_0 and G_0 are matrices of full rank with respect to their columns and rows respectively. Therefore the equation (2.6.1) is equivalent to the relation

$$\sum_{i,j} \gamma_{ij} f_i(U_0) X_0 f_j(U_0)^* = Y_0, \tag{2.6.5}$$

where $X_0 = G_0 X G_0^*$, $Y_0 = G_0 Y G_0^*$. The matrix (2.6.2) is representable in the form

$$S_\lambda = E_0 \left[(\lambda I - U_0) G_0 G_0^* (\lambda I - U_0)^* + Y_0\right] E_0^*.$$

Therefore the conditions (2.6.2) are equivalent to the controllability of the pair (U_0, Y_0).

Considering the case of the left pair (U, T) of the matrix function $F(\lambda)$ and using the proof of Lemma 2.5.1, obtain the relations

$$E = G_0 E_0, \quad E_0 = \left[T_0, U_0 T_0, \ldots, U_0^{h-1} T_0\right], \quad G_0 = G \begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$F_i = G_0 f_i(U_0) E_0, \quad X_0 = E_0 X E_0^*, \quad Y_0 = E_0 Y E_0^*,$$

where (U_0, T_0) is the left pair of eigenvalues of the matrix function $F(\lambda)$, E_0 and G_0 are matrices of full rank with respect to their columns and rows respectively. The equation (2.6.1) also reduces to (2.6.5), and the conditions (2.6.3) are equivalent to the controllability of the pair (U_0, Y_0).

In both cases described above the condition $X \in \mathcal{K}_{pq}$ means that $i_+(X_0) = p$ and $i_-(X_0) = q$. Similarly, the condition $Y \in \mathcal{K}_{pq}$ is equivalent to the equalities $i_+(Y_0) = p$, $i_-(Y_0) = q$.

Consequently, the statements of Theorems 2.6.1–2.6.3 follow from Lemma 2.5.1, the described technique, and Theorems 1.3.1, 1.5.1, 1.6.1, and 1.8.1 for the equation (2.6.5).

Theorems 2.6.1–2.6.3 are proved.

Remark 2.6.1 The conditions (2.5.9) and (2.6.2) of Theorem 2.6.1 are satisfied if

$$F(\lambda) F(\lambda)^* + \Phi(\lambda) Y \Phi(\lambda)^* > 0, \quad \lambda \in \Lambda. \tag{2.6.6}$$

The conditions (2.6.3) are the consequence of the matrix inequality

$$F(\lambda)F(\lambda)^* + \Theta(\lambda)Y\Theta(\lambda)^* > 0, \quad \lambda \in \Lambda, \tag{2.6.7}$$

where $\Theta(\lambda) = \left[I, \lambda I, \dots, \lambda^{h-1} I\right]$. All the relations (2.5.9), (2.5.10), (2.6.2), (2.6.3), (2.6.6), (2.6.7) in the respective statements of Theorem 2.6.1 must only hold true in some neighbourhood of the points $\sigma_0(F)$ for $\lambda \notin \Lambda_f^+$. If $Y > 0$, then the conditions (2.6.2), (2.6.3) and (2.6.7) hold true for any λ.

Remark 2.6.2 The limitations $f \in \mathcal{H}_0^r$ and $f \in \mathcal{H}_2^r$ in Theorems 2.6.1 and 2.6.2 hold true respectively for $i_+(\Gamma) = 1$ and $i_\pm(\Gamma) \le 1$. If

$$f(\lambda, \bar{\mu}) = f_1(\lambda)\overline{f_1(\lambda)} - f_2(\lambda)\overline{f_2(\lambda)} \in \mathcal{H}_2,$$

then the set of matrices $Y \in \mathcal{K}_{r0}$ in Theorem 2.6.2 can be enlarged, assuming $Y \in \mathcal{K}_{p0}$, $p \le r$ and using additional limitations on f and U (see Section 1.4). Thus, in the case of the left pair (U, T) in Theorem 2.6.2 instead of $Y \in \mathcal{K}_{r0}$ it is sufficient to require that

$$\widehat{Y} = EYE^* \ge 0, \quad \widetilde{Y} = \sum_{i=0}^{r-p} \varphi_i(U)\widehat{Y}\varphi_i(U)^* \ge 0,$$

$$\varphi_i(\lambda) = f_1(\lambda)^{r-p-i} f_2(\lambda)^i,\ p = \operatorname{rank}\widehat{Y} \le r = \operatorname{rank}\widetilde{Y},\ i = \overline{0, r-p}.$$

Note that the statements of Theorems 2.6.1–2.6.3, related to the application of the right (left) pairs of the matrix function $F(\lambda)$, hold true, if instead of the matrix E in the equation (2.6.1) and the expression for the sets $\mathcal{K}$ and $\mathcal{K}_{pq}$, one uses the product WE (EW), where W is any matrix for which $\operatorname{rank}(WE) = \operatorname{rank} E$ $(\operatorname{rank}(EW) = \operatorname{rank} E)$. This allows one to lower the order of the algebraic system to which the equation (2.6.1) is reduced. If the subset of the spectrum $\sigma_0(F)$ corresponding to the given pair (U, T) does not coincide with $\sigma(F)$, then at the further application of Theorems 2.6.1–2.6.3 one can use methods similar to procedures of deflation of block spectral characteristics of a matrix polynomial.

2.7 Sufficient Conditions of Spectrum Location

Let $\Xi(z) = A_0 + z_1 A_1 + \cdots + z_s A_s$ be a multi-parametric pencil of $n \times n$ matrices, satisfying the regularity condition

$$\det \Xi(z) \not\equiv 0, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_s \end{bmatrix} \in C^s. \tag{2.7.1}$$

The spectrum $\sigma(\Xi)$ of the pencil is defined as a locus of points z, with $\det \Xi(z) = 0$. Let us formulate a problem of localization of the spectrum $\sigma(\Xi)$, i.e. construction of vector sets $\mathcal{Z}$ containing all the points $\sigma(\Xi)$.

Consider the matrix equation

$$\sum_{i,j=0}^{s} \gamma_{ij}\, A_i X A_j^* = Y, \tag{2.7.2}$$

where γ_{ij} are scalar coefficients composing a Hermitian matrix Γ. In C^s determine a set of vectors

$$\mathcal{Z} = \{z\colon \operatorname{rank} \Delta(z) + \operatorname{sign} \Delta(z) \geq 2\}, \tag{2.7.3}$$

where $\Delta(z) = Z \Gamma Z^*$, $Z = [-z, I_s]$. The complement of this set $\mathcal{Z}_- = C^s \setminus \mathcal{Z}$ consists of those vectors z for which $\Delta(z) \leq 0$.

Theorem 2.7.1 *Let the Hermitian matrices X and Y satisfy the equation (2.7.2) and relations*

$$\left\| A_i X A_j^* + \Xi(z) C_{ij} \Xi(z)^* \right\|_{i,j=1}^{s} \geq 0, \quad z \in \mathcal{Z}_-, \tag{2.7.4}$$

$$Y + \Xi(z) S \Xi(z)^* > 0, \quad z \in \mathcal{Z}_-, \tag{2.7.5}$$

where C_{ij} are blocks of some matrix $C \geq 0$ and $S > 0$. Then each point of the spectrum $\sigma(\Xi)$ belongs to the set $\mathcal{Z}$.

Proof. Introduce the block matrices

$$A = [A_0, \ldots, A_s], \quad B = \begin{bmatrix} A_1 \\ \vdots \\ A_s \end{bmatrix}, \quad C = [A_1, \ldots, A_s].$$

The matrix equation (2.7.2) is represented in the form

$$A(\Gamma \otimes X)A^* = Y, \tag{2.7.6}$$

where $\otimes$ is a Kronecker product. Let $u^* \neq 0$ be a left eigenvector of the matrix $\Xi(z)$, corresponding to the point $z \in \sigma(\Xi)$. Then the relations

$$u^*A = u^*[A_0, C] = u^*C([-z, I_s] \otimes I_n) = u^*C(Z \otimes I_n)$$

are true. Here $u^*C \neq 0$. Otherwise the inequality $\operatorname{rank} A < n$ holds true, contradicting the condition (2.7.1).

Assume that $z \in \mathcal{Z}_-$. Multiplying (2.7.6) from the left (right) by u^* (u), taking into account (2.7.4), (2.7.5), and the properties of the Kronecker product, obtain the relation

$$u^*C(\Delta(z) \otimes X)C^*u = \operatorname{tr}\left(\Delta(z)W^T\right) = u^*Yu > 0,$$

where

$$W = UBXB^*U^* \geq 0, \qquad U = \begin{bmatrix} u^* & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u^* \end{bmatrix}.$$

In the case $Y \geq 0$ the inequalities $u^*Yu > 0$ and $u^*Y \neq 0$ are equivalent. Using the expansion of the nonnegative definite matrix $W^T = RR^* \geq 0$ and permuting the multipliers within the operation tr, arrive at the inequality

$$\operatorname{tr}(R^*\Delta(z)R) > 0.$$

From this, taking into consideration the law of inertia, it follows that the matrix $\Delta(z)$ cannot be negatively semidefinite, i.e. $z \in \mathcal{Z}$. This contradicts the assumption that $z \in \mathcal{Z}_-$. Hence, $\sigma(\Xi) \subset \mathcal{Z}$.

The theorem is proved.

Remark 2.7.1 The conditions (2.7.4) ((2.7.5)) of Theorem 2.7.1 are satisfied if $BXB^* \geq 0$ ($Y \geq CHC^*$, $H > 0$). In particular, for the conditions (2.7.4) ((2.7.5)) to be satisfied it is sufficient that X (Y)

be a nonnegative (positive) definite matrix. If $Y \geq 0$, then the limitation (2.7.5) is equivalent to an identity

$$\operatorname{rank}[\Xi(z), Y] \equiv n, \qquad z \in \mathcal{Z}_-,$$

which is an analogue of the conditions of controllability and stabilizability of linear systems in the Simon-Mitter form.

The vector sets (2.7.3) localizing the spectrum $\sigma(\Xi)$ in Theorem 2.7.1 are described in terms of rank and signature of the Hermitian matrix $\Delta(z)$ and defined by the values of only scalar coefficients of the equation (2.7.2). The condition $z \in \mathcal{Z}$ means that the matrix $\Delta(z)$ has at least one positive eigenvalue. For example, if the matrix Γ is presented in the form

$$\Gamma = \begin{bmatrix} \gamma & g^* \\ g & G \end{bmatrix}, \quad \gamma > 0, \quad G_0 = \gamma\, z_0\, z_0^* - G \geq 0, \quad z_0 = \frac{1}{\gamma}\, g,$$

then $\Delta(z) = \gamma\,(z - z_0)(z - z_0)^* - G_0$ and the set $\mathcal{Z}$ is located outside the s-dimensional sphere:

$$\mathcal{Z} \subset \{z\colon\ \|z - z_0\| > r\},$$

where $r = \sqrt{\gamma_0/\gamma}$, $\gamma_0 \geq 0$ is the minimum eigenvalue of the matrix G_0. In the description of the set (2.7.3) one can use the generalized law of inertia (see Chapter 4).

From Theorem 2.7.1 the technique of construction of domains in the complex plane follows, containing the spectrum of the matrix functions

$$F(\lambda) \stackrel{\Delta}{=} \Xi\,(z(\lambda)) = A_0 + z_1(\lambda)A_1 + \cdots + z_s(\lambda)A_s, \tag{2.7.7}$$

where $z(\lambda)$ is a given vector-function with components $z_i(\lambda)$, $i = \overline{1,s}$.

Theorem 2.7.2 *If Hermitian matrices X and Y satisfy the relations (2.7.2), (2.7.4), and (2.7.5), and in the conditions (2.7.4) and (2.7.5) $z = z(\lambda)$, $\lambda \notin \Lambda$, where $\Lambda = \{\lambda :\ z(\lambda) \in \mathcal{Z}\}$, and $\mathcal{Z}$ is a set of the form (2.7.3), then the spectrum of the matrix function (2.7.7) is located in the domain Λ.*

Corollary 2.7.1 *Let $F(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^s A_s$ be a matrix polynomial, and the conditions of Theorem 2.7.2 hold true. Then the spectrum $\sigma(F)$ is located in the domain*

$$\Lambda = \{\lambda \colon i_+(\Delta_\lambda) \geq 1\}, \tag{2.7.8}$$

where $\Delta_\lambda = \Gamma_0 - \lambda\Gamma_1 - \bar{\lambda}\Gamma_1^* + \lambda\bar{\lambda}\Gamma_2$,

$$\Gamma_0 = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1s} \\ \cdots & \cdots & \cdots \\ \gamma_{s1} & \cdots & \gamma_{ss} \end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix} \gamma_{01} & \cdots & \gamma_{0s} \\ \cdots & \cdots & \cdots \\ \gamma_{s-11} & \cdots & \gamma_{s-1s} \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} \gamma_{00} & \cdots & \gamma_{0s-1} \\ \cdots & \cdots & \cdots \\ \gamma_{s-10} & \cdots & \gamma_{s-1s-1} \end{bmatrix}.$$

The proof follows from the relations

$$\Delta(z) = S_\lambda \Delta_\lambda S_\lambda^*, \quad z = \begin{bmatrix} \lambda \\ \vdots \\ \lambda^s \end{bmatrix},$$

$$S_\lambda = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \lambda & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda^{s-1} & \lambda^{s-2} & \cdots & \lambda & 1 \end{bmatrix}.$$

If $\Delta_\lambda \leq 0$ ($\forall\lambda\colon \operatorname{Re}\lambda \geq 0$), then the domain (2.7.8) is located in the left half-plane. Similarly, if $\Delta_\lambda \leq 0$ ($\forall\, \lambda :\ |\lambda| \geq 1$), then the domain (2.7.8) is located inside a unit disk. These limitations on the matrix Γ will be used in construction of the algebraic conditions of stability of s-th order differential and difference systems.

Corollary 2.7.2 *Let the vector-function $z(\lambda)$ in (2.7.7) and the matrix Γ have the following structure:*

$$z(\lambda) = \begin{bmatrix} \lambda \\ w_\lambda \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \gamma & h^* \\ 0 & h & H \end{bmatrix},$$

$$\gamma < 0, \quad h \in C^{s-1}, \quad H = H^* < 0.$$

Then under the conditions of Theorem 2.7.2 the spectrum of the matrix function (2.7.7) is located in the domain

$$\Lambda = \left\{ \lambda \colon \lambda + \bar{\lambda} < \gamma - (h - w_\lambda)^* H^{-1} (h - w_\lambda) \right\}. \tag{2.7.9}$$

For the domain (2.7.9) to be located in the left half-plane it is sufficient to require that

$$\gamma \leq (h - w_\lambda)^* H^{-1} (h - w_\lambda) \quad (\forall\, \lambda : \operatorname{Re}\lambda \geq 0). \tag{2.7.10}$$

For example, if

$$w_\lambda = \begin{bmatrix} e^{-\lambda\tau_1} \\ \vdots \\ e^{-\lambda\tau_{s-1}} \end{bmatrix},$$

then the condition (2.7.10) is satisfied under the limitations

$$h = 0, \quad \gamma = \gamma_1 \leq \frac{1}{\gamma_2} + \ldots + \frac{1}{\gamma_s}, \quad H = \operatorname{diag}\{\gamma_2, \ldots, \gamma_s\}, \tag{2.7.11}$$

and also in the case

$$\sum_{i=1}^{s-1} (1 + |h_i|)^2 \leq \gamma\mu, \tag{2.7.12}$$

where μ is the maximum eigenvalue of the matrix H.

With the stated limitations on the matrix Γ Corollary 2.7.2 presents the conditions of absolute stability of the quasi-polynomial

$$F(\lambda) = A_0 + \lambda A_1 + e^{-\lambda\tau_1} A_2 + \cdots + e^{-\lambda\tau_{s-1}} A_s.$$

The limitation (2.7.10) is more general with respect to (2.7.11) and (2.7.12).

2.8 Notes and References

2.1 Construction and application of analogues of the Lyapunov equation for matrix functions is a new trend in systems theory. Such equations for matrices and linear matrix pencils were investigated in Bender [1], Stykel [1], Bulgakov [1] and others. The integral operator (2.1.3) proposed by Mazko [8, 11], with the use of the multiplicative derivative R_λ, underlies the construction of the generalized Lyapunov equation for matrix polynomials and functions. Some important properties of the matrix R_λ were found in Keldysh [1] and Gohberg, Sigal [1]. Representation of the operator (2.1.7) and its main properties have been obtained in Mazko [22, 25]. The Laurent expansion of the resolvent of a regular pencil of matrices was also used in Lewis [1] and Mazko [15]. More detailed information on the canonical form of a regular pencil of matrices of the form (2.1.8) can be found in Gantmacher [1].

2.2 Conditions of *regular factorization* of matrix and operator functions of the form (2.2.1) can be found in Markus, Matsaev [1]. Conclusion of the equality (2.2.2) follows from a similar formula in the case of resolvent (see Daletskii, Krein [1]). Lemmas 2.2.2–2.2.5 and Theorems 2.2.1–2.2.3 are proved in Mazko [8, 11, 12, 22, 25]. The conditions of controllability of linear systems not solved with respect to derivatives, are available, e.g., in Khasina [1] and Yamada, Luenberger [1].

2.3 The spectral properties of accompanying pencils of block matrices of the form (2.3.1) can be found in Lancaster [1], Markus [1], Khazanov [1]), and others. The statements of lemmas 2.2.2–2.2.5 and Theorems 2.2.1–2.2.3 with the use of the relations (2.3.4)–(2.3.8) are proved in Mazko [15, 25].

2.4 The known methods of splitting of a matrix spectrum are described in Valeev [1]. In Mazko [13, 14, 24] a technique of splitting of the spectrum of linear and polynomial pencils of matrices is proposed, which is based on the solution of the algebraic systems (2.4.2), (2.4.9), and (2.4.17). The main properties of the system (2.4.17) are given in the form of Lemmas 2.4.1 and 2.4.2, and Theorem 2.4.1 generalizes Lemmas 2.2.2–2.2.5 and Theorems 2.2.1–2.2.3 for arbitrary solutions of this system.

2.5 A block spectral problem for a matrix polynomial is studied by Khazanov [1]. The concepts of right and left pairs of a matrix function that are the solutions of the more general spectral problems (2.5.1) and (2.5.2) are introduced in Mazko [28], where their main properties are also determined; they are stated with the use of the concepts of observability and controllability in the form of Lemmas 2.5.1–2.5.3.

2.6 This Section is based on the results obtained in Mazko [28–30]. The general Theorems 2.6.1–2.6.3 present a universal technique of localization of the spectrum of matrix functions by using solutions of matrix equations and are the main results of Chapter 2. On the deflation of block spectral characteristics of a matrix polynomial see Kublanovskaya, Khazanov [1].

2.7 The results of Section 2.7 are published in Mazko [26]. Theorem 2.7.2 and its corollaries provide an effective technique of construction of domains in a complex plain, containing the spectrum of a given matrix function. Limitations of the type (2.7.11) were used in Zelentzovsky [1] and Korenevskii, Mazko [2] in construction of algebraic conditions of absolute stability of differential-difference delay systems by method of the Lyapunov–Krasovsky functionals (see also Boyd, Ghaoui, Feron, Balakhrishman [1]).

3

LINEAR DYNAMIC SYSTEMS. ANALYSIS OF SPECTRUM AND SOLUTIONS

3.0 Introduction

This chapter is devoted to the application of some results of the investigations described in Chapters 1 and 2 to the analysis of linear dynamic systems most often occurring in applications.

Proposed in Section 3.1 for a linear controllable object is a method of quadratic optimization by output feedback, using solutions of the generalized Lyapunov equation and ensuring the location of the system spectrum in a given domain. If the linear state feedback is sought, and the desirable domain of spectrum allocation is the left half-plane, then this method coincides with the known iteration algorithm of solving the Riccati equation and the optimal stabilization of the system.

In Sections 3.2 and 3.3, for linear descriptor systems and for second-order differential and difference systems, new criteria and sufficient conditions of asymptotic stability, spectrum location, and techniques of Lyapunov functions construction are formulated, based on solving the appropriate matrix equations.

In Section 3.4 algebraic conditions of stability of differential-difference and stochastic systems are formulated in terms of solutions of Sylvester–Lyapunov matrix equations.

In Section 3.5 the general technique of the study and numerical construction of solutions of higher order linear differential and difference systems is developed by using the concept of right pairs of matrix polynomials and functions.

3.1 Localization of Spectrum and Optimization of Linear Controllable Systems

At construction of real control systems such properties as their stability and optimality are put in the forefront. Many dynamic characteristics of linear controllable systems are described by using conditions put on the spectrum of a closed-loop system. Therefore the simultaneous solution of the problems of optimization and spectrum control (optimal modal control) is of great interest.

Attainment of a fixed set of eigenvalues of a closed system in a problem of optimal modal control limits the opportunity to meet other requirements (minimization of functional, physical realizability of control law, etc.). Prescribing the domain of the desirable location of a spectrum, one can overcome or reduce those restrictions and construct a relevant suboptimal control. Using the results of Chapter 1, we will enlarge the class of admissible domains of the spectrum location for a closed-loop system in the problem of quadratic optimization given incomplete information on the conditions of the object.

Consider a controllable object whose motion is described by the linear stationary system

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = x_0, \tag{3.1.1}$$

where $x \in R^n$ is a state vector, $u \in R^m$ is a control vector, $y \in R^l$ is a vector of measurable outputs, A, B and C are matrices of appropriate dimensions, and $\operatorname{rank} B = m$, $\operatorname{rank} C = l$. Determine the averaged quality criterion of the system

$$J = \int\limits_{\Delta} \rho(x_0) \int\limits_0^\infty (x^* Q x + u^* R u)\, dt\, dx_0, \tag{3.1.2}$$

where $\rho(x_0) > 0$ is a weight function determined on the set of admissible initial states $x_0 \in \Delta$, $Q = Q^* > 0$ and $R = R^* > 0$ are given matrices.

Let the desirable dynamic properties of the system be characterized by the location of its spectrum in the domain

$$\Lambda_f^+ = \{\lambda \colon f(\bar{\lambda}, \lambda) = \sum_{i,j} \gamma_{ij} \overline{f_i(\lambda)} f_j(\lambda) > 0\}, \tag{3.1.3}$$

wholly located in the left half-plane. It is required to construct a control in the form of a linear output feedback

$$u = -Ky, \tag{3.1.4}$$

which ensures the least value of the functional (3.1.2) and the location of the spectrum of the closed-loop system (3.1.1) in the domain (3.1.3). This problem can be formally represented in the form

$$J(K) = \operatorname{tr}\left[W(K)\Delta_0\right] \to \inf_{K\in\mathcal{K}}, \tag{3.1.5}$$

where

$$\Delta_0 = \int\limits_{\Delta} \rho(x_0)x_0\, x_0^*\, dx_0, \quad \mathcal{K} = \left\{\mathrm{K} \in \mathrm{C}^{\mathrm{m}\times\mathrm{l}} : \sigma(\mathrm{G}) \subset \Lambda_{\mathrm{f}}^{+}\right\},$$

$G = A - BKC$ is a closed-loop matrix, $W = W(K) > 0$ a solution of the Lyapunov equation

$$-G^*W - WG = Q + C^*K^*RKC. \tag{3.1.6}$$

Construct a solution of the equation (3.1.6) in the form

$$W = \sum_{i,j} \gamma_{ij} f_i(G)^* H f_j(G), \tag{3.1.7}$$

where H is a new unknown matrix. Substituting (3.1.7) into (3.1.6), obtain an equation with respect to H:

$$\sum_{i,j} \beta_{ij}\varphi_i(G)^* H \varphi_j(G) = Q + C^*K^*RKC, \tag{3.1.8}$$

where β_{ij} and φ_i are some coefficients and functions expressed through γ_{ij} and f_i. The domain Λ_φ^+, corresponding to the function

$$\varphi(\bar\lambda, \lambda) = \sum_{i,j} \beta_{ij}\overline{\varphi_i(\lambda)}\,\varphi_j(\lambda) = -(\bar\lambda + \lambda)f(\bar\lambda, \lambda)$$

consists of two disjoint subdomains — the domain (3.1.3) and the right half-plane. If H if a solution of the equation (3.1.8), then the inequality

$$X = -G^*H - HG > 0 \tag{3.1.9}$$

ensures the location of the closed-loop matrix spectrum in the domain (3.1.3). The matrix (3.1.9) is a solution of the equation

$$\sum_{i,j} \gamma_{ij}\, f_i(G)^* X f_j(G) = Q + C^* K^* R K C. \tag{3.1.10}$$

Thus, the original problem adds up to a mathematical programming problem. It is required to minimize the function (3.1.5) calculated by using the relations (3.1.7) and (3.1.8), with the limitations (3.1.9). The search of a suboptimal solution of the problem can be effected by gradient methods, using the known expression for gradient

$$\frac{dJ}{dK} = 2\,(RKC - B^* W)FC^*,$$

where W is a solution of the equation (3.1.6), in particular, the matrix (3.1.7), and F is a solution of the equation

$$-GF - FG^* = \Delta_0. \tag{3.1.11}$$

From the requirement for minimum of the function (3.1.5) the below relation follows:

$$K = R^{-1} B^* W F C^* (C F C^*)^{-1}. \tag{3.1.12}$$

The system of matrix relations (3.1.7), (3.1.8), (3.1.9), (3.1.11), and (3.1.12) represent the requirements for the minimum of functional and the location of the spectrum of closed-loop system in the domain (3.1.3). If the function f in (3.1.3) is representable in the form

$$f(\bar{\lambda}, \lambda) = -(\bar{\lambda} + \lambda)\psi(\bar{\lambda}, \lambda), \quad \psi(\bar{\lambda}, \lambda) = \sum_{i,j} \delta_{ij}\, \overline{\psi_i(\lambda)}\psi_j(\lambda), \tag{3.1.13}$$

then for the calculation of the matrix W one can use the expression

$$W = \sum_{i,j} \delta_{ij}\, \psi_i(G)^* X \psi_j(G), \tag{3.1.14}$$

where X is a solution of (3.1.10). In this case the equation (3.1.8) is not used.

Construct an iterative process according to the following rules:

1) select $K_0 \in \mathcal{K}$ and assume $s = 0$;

2) calculate the matrix $G_s = A - BK_sC$;

3) determine the matrices H_s and F_s from the equations

$$\sum_{i,j} \beta_{ij}\, \varphi_i(G_s)^* H_s \varphi_j(G_s) = Q + C^* K_s^* R K_s C,$$

$$G_s F_s + F_s G_s^* = -\Delta_0;$$

4) calculate the expressions

$$W_s = \sum_{i,j} \gamma_{ij}\, f_i(G_s)^* H_s f_j(G_s),$$

$$K_{s+1} = R^{-1} B^* W_s F_s C^* (C F_s C^*)^{-1};$$

5) increase s by one and revert to item 2).

The difference of this iterative process from the known quadratic optimization algorithms is the method of calculation of the matrix sequence W_s. Since at each step W_s is a solution of the Lyapunov equation (3.1.6), then the following inequalities hold true:

$$J(K_0) \geq J(K_1) \geq \ldots \geq J(K_s) \geq \ldots \quad .$$

Under the limitation (3.1.13) the matrices W_s can also be determined by using the formulae (3.1.10) and (3.1.14). In the case of the algebraic domains (3.1.3), the use of the matrix equations (3.1.8) or (3.1.10) instead of the Lyapunov equation (3.1.6) practically does not change the computational difficulties of the algorithm. At the same time, in the optimization process we have an opportunity to effectively control the belonging of the spectrum of the system to the domain (3.1.3) by using the inequality (3.1.9).

In individual cases under the conditions of controllability and observability of the system (3.1.1), the convergence of the matrix sequence K_s is proved. The initial approximation $K_0 \in \mathcal{K}$ can be determined by the known modal control techniques.

Note that in the case $C = I_n$, when all components of state vector are measurable, the sequence W_s reduces to the positive definite solution of the *Riccati equation*

$$A^*W + WA - WBR^{-1}B^*W + Q = 0.$$

Corresponding to the optimal control (3.1.4) for the system (3.1.1) is the limiting value of the coefficients

$$K = \lim_{s\to\infty} K_s = R^{-1}B^*W.$$

Example 3.1.1 Consider the system (3.1.1) and the functional (3.1.2) with parameters

$$A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \Delta_0 = I_3,$$

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad R = I_2.$$

As a domain Λ_f^+, take the exterior of a disk of radius r with the center in a point $(-r, 0)$, located in the left half-plane (see Section 1.4). In this case the function $f \in \mathcal{F}_1$ satisfies Theorem 1.3.1 and the conditions (3.1.13).

The functional (3.1.2) can be evaluated in the form

$$J(K) = \operatorname{tr} W, \quad W = G^TX + XG + r^{-1}G^TXG,$$

where X is a solution of the matrix equation

$$-G^{2T}X - XG^2 - 2G^TXG - r^{-1}G^{2T}XG - r^{-1}G^TXG^2 = Q + K^TK.$$

As an initial approximation the solution of the modal control problem

$$K_0 = \begin{bmatrix} 0,661 & -0,428 & 0,238 \\ -0,237 & 1,24 & 0,005 \end{bmatrix}$$

was used, for which the spectrum of the closed-loop system $\sigma(G_0) = \{-1{,}5;\ -1{,}2 \pm 1{,}5i\}$ and the functional $J(K_0) = 3,69$. The minimization was effected by the gradient method for two values of the radius r. For $r = r_0 = 0,4$ the optimal values of the following parameters were obtained:

$$K^{(0)} = \begin{bmatrix} 0,594 & -0,323 & 0,305 \\ -0,323 & 1,21 & -0,213 \end{bmatrix},$$

$$J(K^{(0)}) = 3,663, \quad G^{(0)} = A - BK^{(0)},$$

$$\sigma(G^{(0)}) = \{\lambda_1, \lambda_2, \lambda_3\}, \quad \lambda_1 = -1,43; \quad \lambda_{2,3} = -1,19 \pm 1,39i.$$

If $r = r_1 = 0,73$, then the values of absolute minimum cannot be obtained, since $\lambda_1 \notin \Lambda_f^+$. In this case, suboptimal values of the following parameters were obtained:

$$K^{(1)} = \begin{bmatrix} 0,622 & -0,353 & 0,279 \\ -0,28 & 1,23 & -0,134 \end{bmatrix},$$

$$J(K^{(1)}) = 3,666, \quad G^{(1)} = A - BK^{(1)},$$

$$\sigma(G^{(1)}) = \{\mu_1, \mu_2, \mu_3\}, \quad \mu_1 = -1,46, \quad \mu_{2,3} = -1,2 \pm 1,43i.$$

In the point $K^{(1)}$ the condition $X > 0$ is violated, since the eigenvalue μ_1 is located on the boundary of the domain Λ_f^+.

Example 3.1.2 Consider the set of equations that describe the perturbed motion of a rocket, taking into account the elastic oscillations of its airframe as a straight flexible nonuniform rod (see Fisher [1])

$$\ddot{z} = \frac{1}{\mu}\left[(F_1 - F_2)\varphi + F_3\psi + F_4\delta\right] + \sum_{j=1}^{\nu} d_j\eta_j,$$

$$\ddot{\varphi} + c_1\psi + c_2\delta + \sum_{j=1}^{\nu} e_j\eta_j = 0, \quad \psi = \varphi - \dot{z}/v_0,$$

$$\ddot{\eta}_j + 2\zeta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j = \xi_j\delta, \quad j = \overline{1,\nu},$$

where z is the shift of the center of gravity of the rocket in the direction perpendicular to the calculated trajectory, φ is the angle of pitch, ψ is the angle of attack, δ is the rotation angle of the engine, η_j is the j-th form of bending vibrations, F_1 is the tractive force, F_2 is the axial force acting upon the sides of the air flow, F_3 is the constituent of the force of the air flow, perpendicular to the longitudinal axis of the rocket, F_4 is the control power perpendicular to the longitudinal axis of the rocket, μ is the rocket mass, v_0 is the velocity of the rocket along the trajectory, d_j, c_j, e_j, ω_j, ζ_j, ξ_j are coefficients determined through the physical parameters of the rocket. Taking into account the three forms of the airframe's bending vibrations, we will reduce this system to the standard form (3.1.1), where $x = [\varphi, \dot{\varphi}, \psi, \eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, \eta_3, \dot{\eta}_3]^T, \quad u = \delta,$

$$A = \begin{bmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & 0 & a_4 & 0 & a_5 & 0 \\ a_6 & a_7 & a_8 & a_9 & 0 & a_{10} & 0 & a_{11} & 0 \\ 0 & 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{13} & a_{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{15} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{16} & a_{17} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{18} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{19} & a_{20} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b_1 \\ b_2 \\ 0 \\ b_3 \\ 0 \\ b_4 \\ 0 \\ b_5 \end{bmatrix}.$$

The values of the coefficients a_j and b_j are given in Table 3.1.1.

Table 3.1.1

j	a_j	j	a_j	j	a_j	j	a_j	j	b_j
1	1	6	-0,0458	11	7×10^{-4}	16	-169	1	-1,138
2	0,2165	7	1	12	1	17	-0,13	2	-0,0348
3	-0,0356	8	-0,0133	13	-29,81	18	1	3	29,56
4	-0,0299	9	4×10^{-4}	14	-0,0546	19	-334,3	4	47,25
5	-0,027	10	6×10^{-4}	15	1	20	-0,1828	5	16,4

The matrix A is unstable. Its spectrum consists of three real eigenvalues, including two positive ones, and three complex-conjugate pairs of eigenvalues located in the left half-plane and characterizing the respective forms of bending vibrations of the rocket airframe.

As an admissible domain for location of the spectrum of the closed system choose a domain limited by a cissoid of Diocles (see Section 1.4). This domain degenerates to the left half-plane for $a \to 0$.

This allows one to optimize, taking into account the spectrum location, by using the general procedure of computation for different values of the parameter a. An increase of the latter results in an increase of the stability factor and a decrease of the admissible frequency of the bending vibrations of the rocket airframe. For some values of $a > a_0$ the complex eigenvalues of the closed-loop systems may not belong to the domain Λ_f^+.

Assuming $a = 0,1$ and using the values of matrices of functional from (Fisher [1]), a stabilizing control was calculated according to the above-mentioned optimization algorithm given full information on the state vector ($C = I$). As a result, optimal control was obtained in the form of linear state feedback, which coincides with sufficient accuracy with the control obtained in the above-mentioned work by the method of solving the Riccati equation. The algorithm convergence was observed upon five iterations, and the inequality $X > 0$, where X was a solution of the equation

$$\frac{a}{2}(G^{2T}X + XG^2) - G^T(aX + G^TX + XG)G = Q + C^TK^TRKC,$$

equivalent to the location of the system spectrum in the domain Λ_f^+, held true for each iteration.

The main difficulties occurring during realization of the found control law are related to the determination of components of the state vector x. Sensors measure different linear combinations of those components. We will show the results of calculation of the suboptimal control in the form of a linear output feedback, assuming the presence of sensors of angular position, sensors of angular velocities, and accelerometers. The output signal equation $y = Cx$, where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4,368 \\ 0,02462 & 0 & -0,73673 \\ 0 & -0,04819 & -1,1663 \times 10^{-5} \\ 0,06918 & 0 & 53,7935 \\ 0 & -0,082347 & 0,041957 \\ -0,124168 & 0 & 161,21166 \\ 0 & -0,08976 & 0,08851 \end{bmatrix}^T.$$

Matrices of functional, selected in compliance with the requirement for the equilibration of bending vibration energies and the limitation on the lateral drift of the rocket, have the form

$$Q=\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad R=1, \quad \Delta_0=I_9, \quad Q_1=\begin{bmatrix} 0,1 & 0 & -0,1 \\ 0 & 0,05 & 0 \\ -0,1 & 0 & 0,5 \end{bmatrix},$$

$$Q_2 = \operatorname{diag}\left\{10^{-4}; 3\times 10^{-3}; 8\times 10^{-5}; 0,0137; 19\times 10^{-5}; 0,0621\right\}.$$

For the considered system a suboptimal control $u = -K_c y$ for $a = 0$ was obtained, proceeding from different initial values for the vector of multiplication factors K_0. At the same time, sufficiently good convergence of the proposed iteration algorithm was observed. The results of the calculations were the same as with $a = a_1 = 0,8$. This means that the spectrum of the closed-loop system is located in the given domain. If $a = a_2 = 0,9$, then after four iterations the inequality $X > 0$, i.e. the condition of belonging of the spectrum to the domain Λ_f^+, was violated.

In Table 3.1.2 you will find the obtained values of the multiplication factors $[k_1, k_2, k_3] = -K$, the spectrum of the closed-loop system $\sigma(G)$, and the functional $J(K)$, corresponding to suboptimal control for specified values of the parameter a. For $a = a_2$ (suboptimal control with limitation on the spectrum), frequencies of elastic vibrations of the rocket airframe are less, and the stability margin is more than in the case $a = a_1$.

In the plane of the first two multiplication factors k_1 and k_2 (for the fixed value of $k_3 = -0,0177$) the domains $\mathcal{K}_1$ and $\mathcal{K}_2$ are constructed, corresponding to the location of the spectrum of closed-loop system in the domain Λ_f^+ for $a = 0,8$ and $a = 0,9$ (Fig. 3.1.1). In these domains the movement of current values of the multiplication factors is represented, proceeding from different initial approximations $K_0^{(t)}$, $t = 1, \ldots, 4$.

Thus, the described algorithm, as against the known, allows one to control effectively, in the process of parameters optimization, the dynamic characteristics of the systems expressed in the form of a prescribed domain of the spectrum location.

Table 3.1.2

	Suboptimal control	Suboptimal control with limitation on spectrum
k_1 k_2 k_3	1,0401 1,5558 $-0,0177$	1,0450 1,6164 $-0,0179$
$\sigma(G)$	$-0,0528$ $-1,8661 \pm 18,1633i$ $-2,9326 \pm 14,1661i$ $-0,8486 \pm 5,2828i$ $-0,7098 \pm 0,4029i$	$-0,0530$ $-1,8617 \pm 18,0673i$ $-3,1098 \pm 14,1555i$ $-0,8828 \pm 5,2737i$ $-0,7441 \pm 0,3444i$
$J(K)$	15,7482	15,7510

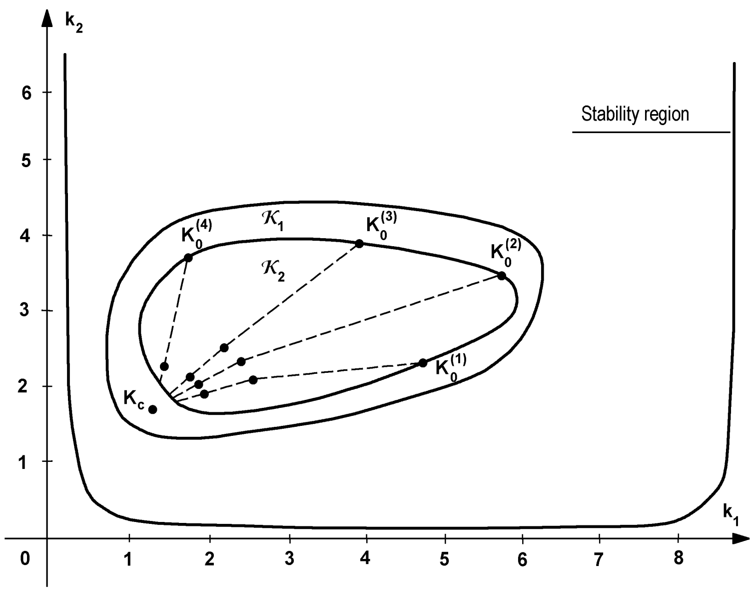


Fig. 3.1.1. Stability region and optimization of the parameters k_1 and k_2 in the domains $\mathcal{K}_1$ and $\mathcal{K}_2$.

3.2 Stability of Descriptor Continuous and Discrete Systems

The subject of inquiry in many applications is the sets of differential (difference) equations not solved with respect to derivatives (iterations). The construction of solutions of such systems and the analysis of their stability can be performed on the basis of the theory of canonical forms of matrix pencils, and by using different generalized inverse matrices as well.

In this Section the results of investigations are proposed, related to the development and application of the second Lyapunov's method for continuous and discrete-time systems of the form

$$B\dot{x}(t) = Ax(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.2.1}$$

$$Bx_{k+1} = Ax_k, \quad k = 0, 1, \ldots, \tag{3.2.2}$$

where A and B are $n \times n$ matrices of the regular pencil $L(\lambda) = A - \lambda B$, whose spectrum $\sigma(L)$ consists of l eigenvalues, taking into account the multiplicities, and x_0 is the vector of initial states. The stability conditions for the differential system (3.2.1) are determined by the location of the spectrum $\sigma(L)$ with respect to the imaginary axis. The number

$$\varepsilon = \min_{\lambda \in \sigma(L)} (-Re\lambda)$$

characterizes the spectral *stability factor* of the system (3.2.1). Similarly, the value

$$\varepsilon = \min_{\lambda \in \sigma(L)} (1 - |\lambda|)$$

determines the spectral stability factor of the system (3.2.2) with respect to the unit circle.

If B is a nonsingular matrix, then $l = n$. In this case the systems (3.2.1) and (3.2.2) reduce to the Cauchy form by the inversion of the matrix B. If B is singular, then the equality

$$l = n - \sum_{i=1}^{\tau} \nu_I = \operatorname{rank} B - \sum_{i=1}^{\tau} \nu_i + \tau$$

is true, where $\nu_1, \ldots, \nu_\tau$ are degrees of infinite elementary divisors of the matrix pencil $L(\lambda)$. This equality follows from the canonical form of a regular pencil of matrices. Let ν denote the maximum of the numbers $\nu_1, \ldots, \nu_\tau$. If B is a nonsingular matrix, then assume $\nu = 0$. The number l determines the dimension of some subspace $\mathcal{L}$ to which the initial states and trajectories of the systems (3.2.1) and (3.2.2) belong. The system (3.2.1), taking into account (2.1.8), adds up to the relations

$$\dot{y}_1(t) = Jy_1(t), \quad N\dot{y}_2(t) = y_2(t), \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = Q^{-1}x(t).$$

Since $N^\nu = 0$, then $y_2(t) \equiv 0$ and $x(t) \in \mathcal{L}$ for $t \geq 0$, where $\mathcal{L}$ is the linear span of the first l columns of the matrix Q. The zero solution of the system(3.2.1) is asymptotically stable if and only if the spectrum of the matrix J, coinciding with $\sigma(L)$, is located to the left of the imaginary axis.

Study the stability conditions and the spectral properties of the systems (3.2.1) and (3.2.2), using the matrix relations

$$\gamma_{00}BXB^* + \gamma_{10}AXB^* + \gamma_{01}BXA^* + \gamma_{11}AXA^* = Y \geq 0, \quad (3.2.3)$$

$$\operatorname{rank}[L(\lambda), Y] \equiv n, \quad \lambda \in C^1, \quad (3.2.4)$$

$$\operatorname{rank}(BXB^*) = l. \quad (3.2.5)$$

Let l_+, l_-, and l_0 denote the quantities of the points of the spectrum $\sigma(L)$, belonging to the respective sets

$$\Lambda_+ = \{\lambda : f(\lambda, \bar{\lambda}) > 0\},$$

$$\Lambda_- = \{\lambda : f(\lambda, \bar{\lambda}) < 0\},$$

$$\Lambda_0 = \{\lambda : f(\lambda, \bar{\lambda}) = 0\},$$

where $f(\lambda, \bar{\lambda}) = \gamma_{00} + \gamma_{10}\lambda + \gamma_{01}\bar{\lambda} + \gamma_{11}\lambda\bar{\lambda}$ is a prescribed Hermitian function. Λ_0 is some straight line or a circle with the center in the point $\gamma = -\gamma_{01}/\gamma_{11}$, separating the domains $\Lambda_\pm \subset C^1$.

Lemma 3.2.1 *If the Hermitian matrices X and Y satisfy the equation (3.2.3) under the condition (3.2.4), then the following relations hold true:*

$$l_+ \leq i_+(BXB^*), \quad l_- \leq i_-(BXB^*), \quad l_0 = 0. \quad (3.2.6)$$

All equalities in (3.2.6) hold true if and only if the condition (3.2.5) is true. Under the condition $l_0 = 0$ there exist Hermitian matrices X and Y satisfying the relations (3.2.3)–(3.2.5).

Proof. Assuming in (3.2.3)

$$PAQ = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \quad PBQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix},$$

$$X = Q \begin{bmatrix} X_1 & X_2 \\ X_2^* & X_3 \end{bmatrix} Q^*, \quad PYP^* = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^* & Y_3 \end{bmatrix},$$

where P and Q are nonsingular matrices, arrive at the equations

$$\gamma_{00} X_1 + \gamma_{10} J X_1 + \gamma_{01} X_1 J^* + \gamma_{11} J X_1 J^* = Y_1, \tag{3.2.7}$$

$$\gamma_{00} X_2 N^* + \gamma_{10} J X_2 N^* + \gamma_{01} X_2 + \gamma_{11} J X_2 = Y_2, \tag{3.2.8}$$

$$\gamma_{00} N X_3 N^* + \gamma_{10} X_3 N^* + \gamma_{01} N X_3 + \gamma_{11} X_3 = Y_3. \tag{3.2.9}$$

Taking into consideration (3.2.4), obtain

$$BXB^* = P^{-1} \begin{bmatrix} X_1 & X_2 N^* \\ N X_2^* & N X_3 N^* \end{bmatrix} P^{-1*}, \tag{3.2.10}$$

$$\operatorname{rank}[J - \lambda I, Y_1] \equiv l, \quad \lambda \in C^1. \tag{3.2.11}$$

The equivalence of the identities (3.2.4) and (3.2.11) follows from the inequality $Y \geq 0$. The identity (3.2.11) is the condition of controllability of the pair (J, Y_1) in the form of Simon–Mitter. It can be shown that the function f satisfies the conditions of the inertia theorem for equations of the type (3.2.7) (see Section 1.8). Hence the relations (3.2.6) hold true, and

$$l_+ = i_+(X_1) \leq i_+(BXB^*), \quad l_- = i_-(X_1) \leq i_-(BXB^*),$$

where X_1 is a nonsingular $l \times l$ matrix, satisfying the equation (3.2.7). The equalities are achieved if and only if the condition (3.2.5) holds true.

The lemma is proved.

Note that if the right-hand side of the equation (3.2.3) has the form

$$Y = BHB^*, \quad H > 0, \tag{3.2.12}$$

then the identity (3.2.4) holds true. In (3.2.8)–(3.2.10) the equalities $X_2N^* = 0$ and $NX_3N^* = 0$ hold true, as well as the condition (3.2.5) in each of the following cases:

1) $\nu \leq 1$;
2) $\gamma_{11} = 0$;
3) $\gamma_{11} \neq 0, \nu \leq 2, \gamma \notin \sigma(L)$.

If

$$f(\lambda, \bar{\mu}) \neq 0, \quad (\lambda, \mu) \in \sigma(L) \times \sigma(L), \tag{3.2.13}$$

then the equation (3.2.3) is solvable for any matrix (3.2.12) in each of the following cases:

1) $\nu \leq 1$;
2) $\gamma_{11} = 0, \nu \leq 2$;
3) $\gamma_{11} \neq 0, \gamma \notin \sigma(L)$;
4) $\gamma_{11} \neq 0, \gamma \in \sigma(L), \zeta(\gamma) = \xi(\gamma)$.

Here $\zeta(\gamma)(\xi(\gamma))$ is the algebraic (geometric) multiplicity of the spectrum point $\gamma \in \sigma(L)$. If $\gamma_{11} = 0$, then for any matrix (3.2.12) the equation (3.2.3) has a solution if and only if the conditions (3.2.13) and $\nu \leq 2$ hold true.

Lemma 3.2.2 *Let the equation (3.2.3) and the condition (3.2.4) be satisfied by Hermitian matrices of the form*

$$X = E\hat{X}E^*, \quad Y = BE\hat{Y}E^*B^*, \tag{3.2.14}$$

where $E \neq 0$ is any matrix determined by the relation

$$\operatorname{rank}[AE, BE] = \operatorname{rank}(BE). \tag{3.2.15}$$

Then the following equalities hold true:

$$l_+ = i_+(X), \quad l_- = i_-(X), \quad l_0 = 0. \tag{3.2.16}$$

Under the condition $l_0 = 0$ there exist matrices X, Y, and E for which the relations (3.2.3)–(3.2.5), (3.2.14)–(3.2.16) hold true.

Proof. The condition (3.2.15) means that for some matrix U the equality $AE = BEU$ holds true. Therefore the matrices E, X, and Y in (3.2.14)–(3.2.16) have the following structure:

$$E = Q\begin{bmatrix} R \\ 0 \end{bmatrix}, X = Q\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}Q^*, Y = P^{-1}\begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1*}, \quad (3.2.17)$$

where $X_1 = R\hat{X}R^*$, $Y_1 = R\hat{Y}R^*$, R is $l \times k$ matrix, satisfying the equality $JR = RU$. Substituting (3.2.17) into (3.2.3) and (3.2.4), arrive at the relations (3.2.7) and (3.2.11). From (3.2.11) it follows, in particular, that $\operatorname{rank} R = \operatorname{rank} E = l \leq k$. The equalities (3.2.16) follow from the relations (3.2.7), (3.2.11), (3.2.17), and the known inertia theorems. Under the condition $l_0 = 0$ one can select matrices $\hat{X}$ and $\hat{Y}$ so that the relations (3.2.3)–(3.2.5), (3.2.14)–(3.2.16) would hold true.

The lemma is proved.

If in (3.2.15) the matrix E has full rank with respect to its columns, then it is a block eigenvalue of the pencil $L(\lambda)$, $L(\lambda)$, corresponding to the block eigenvalue U. In this case $\sigma(U) \subseteq \sigma(F)$. The inverse inclusion $\sigma(F) \subseteq \sigma(U)$ holds under the condition

$$\operatorname{rank}[L(\lambda), BE] \equiv n, \quad \lambda \in C^1. \quad (3.2.18)$$

If the matrix Y has the structure (3.2.14), then the condition (3.2.18) is necessary for the identity (3.2.4) to hold true. In the case $\hat{Y} > 0$ the conditions (3.2.4) and (3.2.18) are equivalent.

The conditions for the existence of matrices X and Y that satisfy the equation (3.2.3) and have the given structure (3.2.14) depend on the spectrum $\sigma(L)$ only and are not connected with the properties of the infinite elementary divisors of the pencil $L(\lambda)$. If $\nu \leq 3$, then for the determination of the matrix E in Lemma 3.2.2 one can use the linear equation $AE = BS$ with respect to E and S instead of the condition (3.2.15). Lemma 3.2.2 remains valid in each of the following cases:

1) $\nu \leq 2$;
2) $\nu = 3, \gamma_{11} = 0$;
3) $\nu = 3, \gamma_{11} \neq 0, \gamma \notin \sigma(L)$.

From Lemma 3.2.2 the next propositions follow.

Lemma 3.2.3 *If the matrices (3.2.14) satisfy the equation*

$$-AXB^* - BXA^* = Y \geq 0 \tag{3.2.19}$$

under the conditions (3.2.4) and (3.2.15), then on the imaginary axis there are no points of the spectrum $\sigma(L)$, and of them exactly $i_+(X)$ and $i_-(X)$ eigenvalues are located respectively in the left and right half-planes.

Lemma 3.2.4. *If the matrices (3.2.14) satisfy the equation*

$$BXB^* - AXA^* = Y \geq 0 \tag{3.2.20}$$

under the conditions (3.2.4) and (3.2.15), then on the unit circle there are no points of the spectrum $\sigma(L)$, and of them exactly $i_+(X)$ and $i_-(X)$ eigenvalues are located respectively inside and outside the unit disk.

Theorem 3.2.1. *The differential system (3.2.1) is asymptotically stable if and only if there exist Hermitian matrices X and Y that satisfy the equation (3.2.19) and the relations*

$$BXB^* \geq 0, \quad \operatorname{rank}[L(\lambda), Y] \equiv n, \quad \mathrm{Re}\lambda \geq 0. \tag{3.2.21}$$

If the differential system (3.2.1) is asymptotically stable, then for any nonnegative definite matrix of the form $Y = BE\hat{Y}E^*B^* \geq 0$ the equation (3.2.19) under the condition (3.2.15) has a solution $X = E\hat{X}E^* \geq 0$.

Theorem 3.2.2 *The difference system (3.2.2) is asymptotically stable if and only if there exist Hermitian matrices X and Y that satisfy the equation (3.2.20) and the relations*

$$BXB^* \geq 0, \quad \operatorname{rank}[L(\lambda), Y] \equiv n, \quad |\lambda| \geq 1. \tag{3.2.22}$$

If the difference system (3.2.2) is asymptotically stable, then for any nonnegative definite matrix of the form $Y = BE\hat{Y}E^*B^* \geq 0$ the equation (3.2.20) under the condition (3.2.15) has a solution $X = E\hat{X}E^* \geq 0$.

In the construction of Lyapunov functions for the systems (3.2.1) and (3.2.2) one can use the solutions of the matrix equations

$$-2\alpha B^*XB - A^*XB - B^*XA = B^*YB, \tag{3.2.23}$$

$$\beta^2 B^* X B - A^* X A = B^* Y B, \tag{3.2.24}$$

where $Y = Y^* > 0$, $\alpha \geq 0$ and $0 < \beta \leq 1$ are real numbers. The expression for the Lyapunov quadratic functions is determined in the form

$$v(x) = x^* B^* X B x. \tag{3.2.25}$$

Theorem 3.2.3 *Let X be a solution of the equation (3.2.23) and $B^* X B \geq 0$. Then the zero solution of the system (3.2.1) is asymptotically stable with the stability factor $\varepsilon \geq \alpha$, and the function (3.2.25) and its derivatives along the nontrivial solution $x = x(t)$ satisfy the relations*

$$v(x) > 0, \quad \frac{dv(x)}{dt} = -x^*(B^* Y B + 2\alpha B^* X B)x < 0.$$

Theorem 3.2.4 *Let X be a solution of the equation (3.2.24) and $B^* X B \geq 0$. Then the zero solution of the system (3.2.2) is asymptotically stable with the stability factor $\varepsilon \geq 1 - \beta$, and the function (3.2.25) and its first difference along the nontrivial solution $x_k (k = 0, 1, \ldots)$ satisfy the relations*

$$v(x_k) > 0, \quad v(x_{k+1}) - v(x_k) = -{x_k}^* B^* \left(Y + (1 - \beta^2) X \right) B x_k < 0.$$

Lyapunov functions for the stable systems (3.2.1) and (3.2.2) can always be determined in the form (3.2.25), assuming, e.g., in (3.2.23) and (3.2.24)

$$X = Z^* \hat{X} Z \geq 0, \quad Y = Z^* \hat{Y} Z \geq 0, \quad \hat{Y} > 0,$$

where Z is a solution of the maximum rank l of the matrix system (3.2.16) in Chapter 2. The stability conditions for the systems (3.2.1) and (3.2.2) are described in terms of the matrices

$$X = E^* \hat{X} E \geq 0, \quad Y = E^* \hat{Y} E \geq 0, \quad \hat{Y} > 0,$$

where $E = (BZ)^s$, $s \geq \nu$, satisfying the equations (3.2.23) and (3.2.24). As Z one can select the solution of the linear equation $AZB = BZA$, in particular, $Z = (A - zB)^{-1}, z \notin \sigma(L)$.

3.3 Spectrum and Stability Analysis of Second-order Differential and Difference Systems

In problems of analysis and synthesis of controllable physical objects considerable attention is paid to methods of study of mathematical models described by the systems of second-order linear differential and difference equations

$$Ax(t) + B\dot{x}(t) + C\ddot{x}(t) = g(t), \quad t \geq 0, \tag{3.3.1}$$

$$Ax_t + Bx_{t+1} + Cx_{t+2} = g_t, \quad t = 0, 1, \ldots, \tag{3.3.2}$$

where $x \in R^n$ is the vector of generalized coordinates of the object, A, B and C are $n \times n$ matrices of dynamic coefficients, g is a vector-function depending on control parameters and external perturbations. Control parameters are usually determined in the form of dynamic state feedback or linear measurable output feedback. As a result, the closed-loop system (3.3.1) ((3.3.2)) is homogeneous, and its stability is described by the location of the spectrum of the quadratic pencil of matrices with respect to the imaginary axis (unit circle).

In those cases when the initial model is non-autonomous, the method of frozen coefficients is used, according to which the most characteristic points of time are selected on a prescribed motion interval, and the respective sets of equations with constant matrix coefficients A, B, and C are considered. The dynamics of the studied object, and its stability in particular, are judged by the solutions of stationary systems of the type (3.3.1) or (3.3.2).

An important role in the problem of motion stabilization is played by coefficient criteria of stability, formulated in terms of the matrices A, B, and C in the form of systems of algebraic equations and inequalities. In the construction of such criteria, different limitations on matrix coefficients are used, including such requirements as symmetry, non-singularity, positive definiteness, etc. Thus, for oscillating systems with friction the following limitations are peculiar:

$$A = A^*, \; B = H + K, \; H = H^* \geq 0, \; K = -K^*, \; C = C^*, \tag{3.3.3}$$

where A is the matrix of potential forces, C is the matrix of inertia, and the matrices H and K characterize respectively damping and gyroscopic forces.

The method of analysis of stability and spectrum of the systems (3.3.1) and (3.3.2) which is set forth below adds up to the construction and solving of matrix algebraic equations. For its substantiation (see Chapter 2) in some cases we use the only limitation — the condition of the regularity of the quadratic pencil of matrices $F(\lambda) = A + \lambda B + \lambda^2 C$.

The matrix equation of the left block spectral problem for the quadratic pencil $F(\lambda)$ has the form

$$TA + UTB + U^2TC = 0. \tag{3.3.4}$$

Let some solution (U, T) of this equation be known, for which $\operatorname{rank} E = r \neq 0$, where $E = [T, UT]$. According to Lemma 2.6.1, there exists a subset of the spectrum $\sigma_0(F)$, consisting of r eigenvalues of the matrix U. In particular, under the limitations

$$\operatorname{rank} \left[\frac{F(\lambda)}{TB + UTC + \lambda TC} \right] \equiv n, \quad \lambda \in C^1, \tag{3.3.5}$$

the pair (U, T) has maximum controllability index r and $\sigma_0(F) = \sigma(F) \subseteq \sigma(U)$.

Formulate the stability criteria for the systems (3.3.1) and (3.3.2), following from Theorem 2.6.1 for the respective matrix equations

$$-2\alpha X - UX - XU^* = Y, \tag{3.3.6}$$

$$\beta^2 X - UXU^* = Y, \tag{3.3.7}$$

where $\alpha \geq 0$, $0 < \beta \leq 1$ are real numbers characterizing the spectral stability factor.

Theorem 3.2.5 *Let (U, T) be a pair of matrices satisfying the relations (3.3.4) and (3.3.5). Then the zero solution of the homogeneous system (3.3.1) ((3.3.2)) is asymptotically stable with the stability factor $\varepsilon \geq \alpha$ ($\varepsilon \geq 1 - \beta$) if and only if for any prescribed matrix of the form $Y = E\hat{Y}E^*$, where $\hat{Y} > 0$, the equation (3.3.6) ((3.3.7)) has the solution $X = E\hat{X}E^* \geq 0$.*

For construction of matrices U and T satisfying Theorem 3.2.5 one can use the solutions of the algebraic system

$$\begin{gathered} AT_1B - BT_1A = CT_2A - AT_2C, \\ AT_1C - CT_1A = CT_2B - BT_2C, \\ T_1 = T_1BT_1 + T_1CT_2 + T_2CT_1, \\ T_2 = T_2CT_2 - T_1AT_1. \end{gathered} \tag{3.3.8}$$

In particular, if T_1 and T_2 are a solution of this system, then the matrices

$$U = \left[\begin{array}{c|c} -AT_1 & -AT_2 \\ \hline CT_2 & -AT_1 - BT_2 \end{array}\right], \quad T = \left[\begin{array}{c} BT_1 + CT_2 \\ \hline CT_1 \end{array}\right] \tag{3.3.9}$$

form the left pair of the quadratic pencil $F(\lambda)$, i.e. satisfy the equality (3.3.4). According to Lemma 2.4.2, the matrix

$$E = [T, UT] = \left[\begin{array}{c|c} BT_1 + CT_2 & -AT_1 \\ \hline CT_1 & CT_2 \end{array}\right] \tag{3.3.10}$$

is a projector of rank r of the matrix U, and at least r eigenvalues of the matrix U, taking into account the multiplicities, belong to the spectrum $\sigma(F)$ (if $\lambda \in \sigma(U)$, then either $\lambda \in \sigma(F)$, or $\lambda = 0$).

The system (3.3.8) consists of two linear homogeneous matrix equations and two matrix equations with quadratic nonlinearity. Its solutions can be found by methods of calculus mathematics. In addition, we can use partial solutions of the system (3.3.8) in integral form (see Lemma 2.4.1)

$$T_1 = \frac{1}{2\pi i} \oint\limits_{\omega} F(\lambda)^{-1} d\lambda, \quad T_2 = \frac{1}{2\pi i} \oint\limits_{\omega} \lambda F(\lambda)^{-1} d\lambda, \tag{3.3.11}$$

where ω is a closed contour separating some part of the spectrum $\sigma_0(F) \subseteq \sigma(U)$. In the particular case $\sigma_0(F) = \sigma(F)$ the conditions of stability of the system (3.3.1)((3.3.2)) are fully determined by the location of nonzero eigenvalues of the matrix U with respect to the imaginary axis (unit circle).

Note that for $T_1 = 0$ we have the reduction of the system (3.3.8) of the form

$$CT_2A = AT_2C, \quad CT_2B = BT_2C, \quad T_2 = T_2CT_2. \tag{3.3.12}$$

If C is a nonsingular matrix, then the system (3.3.12) has a solution $T_2 = C^{-1}$. In this case $E = I$, and for the mechanical system (3.3.1) under the conditions (3.3.3) the relation

$$-UX - XU^* = Y \geq 0,$$

holds true, where

$$U = \begin{bmatrix} 0 & -AC^{-1} \\ I & -BC^{-1} \end{bmatrix}, \quad X = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 2H \end{bmatrix}.$$

If, in addition, the condition of controllability of the pair of matrices (U, Y) holds true, which is equivalent to the identity

$$\operatorname{rank}[A + \lambda K + \lambda^2 C,\ \lambda H] \equiv n,$$

then, according to the inertia theorem, there are no eigenvalues of the quadratic pencil $F(\lambda)$ on the imaginary axis, and the quantity of eigenvalues with negative (positive) real part equals $i_+(A) + i_+(C)(i_-(A) + i_-(C))$. The inequalities $A > 0$ and $C > 0$ correspond to the case of asymptotic stability of the system (3.3.1), (3.3.3).

There is one more property of the system (3.3.8) which can also be used in the study of the spectrum of a quadratic pencil of matrices. Determine the skeleton expansion of rank r of the block matrix

$$Z = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline CT_2 & -AT_1 - BT_2 \end{array}\right] = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [R_1^*, R_2^*]\,.$$

If T_1 and T_2 are the solution of the system (3.3.8), then r eigenvalues of the matrix $\Sigma = R_2^*L_2 - R_1^*AL_1$ belong to the spectrum $\sigma(F)$. The order of the matrix Σ, equal to r, can be much lower than the order of the matrix U, which is very important in an eigenvalue problem. In particular, for $r = 1$ the number $\Sigma \in \sigma(F)$ is an eigenvalue of the quadratic pencil $F(\lambda)$.

Thus, problems of analysis of stability and movement stabilization for the second-order dynamic systems (3.3.1) and (3.3.2) are reduced to solving the matrix algebraic equation (3.3.4), in particular, the system (3.3.8) and the respective analogues of the Lyapunov equation (3.3.6) and (3.3.7) for a quadratic pencil of matrices.

At construction of sufficient conditions of stability and localization of eigenvalues of the quadratic pencil of matrices $F(\lambda)$ one can use the matrix equation

$$\begin{gathered}\gamma_{11}AXA^* + \gamma_{12}AXB^* + \gamma_{21}BXA^* + \\ +\gamma_{13}AXC^* + \gamma_{31}CXA^* + +\gamma_{22}BXB^* + \\ +\gamma_{23}BXC^* + \gamma_{32}CXB^* + \gamma_{33}CXC^* = Y,\end{gathered} \tag{3.3.13}$$

where X and Y are Hermitian $n \times n$ matrices subject to determination. Suppose that along with (3.3.13) the following relations hold true:

$$Y \geq [B, C]Q[B, C]^*, \quad \begin{bmatrix} B \\ C \end{bmatrix} X \begin{bmatrix} B \\ C \end{bmatrix}^* \geq 0, \tag{3.3.14}$$

where Q is some positive definite matrix. Then the spectrum $\sigma(F)$ is located in the domain (see Section 2.7)

$$\Lambda = \Lambda_1 \cup \Lambda_2, \tag{3.3.15}$$

where

$$\Lambda_1 = \{\lambda\colon\ f_1(\lambda, \bar{\lambda}) > 0\}, \quad \Lambda_2 = \{\lambda\colon\ f_2(\lambda, \bar{\lambda}) < 0\},$$

$$f_1(\lambda, \bar{\lambda}) = \mathrm{tr}\Delta_\lambda, \quad \Delta_\lambda = V_\lambda \Gamma V_\lambda^*, \quad f_2(\lambda, \bar{\lambda}) = \det \Delta_\lambda = z_\lambda^* \, \hat{\Gamma}^T z_\lambda,$$

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}, \quad V_\lambda = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \end{bmatrix}, \quad z_\lambda = \left[1, \ \lambda, \ \lambda^2\right],$$

$$\hat{\Gamma} = \begin{bmatrix} \gamma_{22}\gamma_{33} - \gamma_{32}\gamma_{23} & \gamma_{13}\gamma_{32} - \gamma_{12}\gamma_{33} & \gamma_{12}\gamma_{23} - \gamma_{22}\gamma_{13} \\ \gamma_{23}\gamma_{31} - \gamma_{21}\gamma_{33} & \gamma_{11}\gamma_{33} - \gamma_{13}\gamma_{31} & \gamma_{13}\gamma_{21} - \gamma_{11}\gamma_{23} \\ \gamma_{21}\gamma_{32} - \gamma_{22}\gamma_{31} & \gamma_{21}\gamma_{12} - \gamma_{11}\gamma_{32} & \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21} \end{bmatrix}.$$

Here $\hat{\Gamma}$ is an *adjoint matrix* for Γ, composed of algebraic adjuncts of its entries. The geometric properties of a domain Λ of the form

(3.3.15) are fully determined by the selection of entries of the matrix Γ.

We will give examples of the domains (3.3.15), for which $\Lambda_1 = \emptyset$.

1. $$\Gamma = \begin{bmatrix} -1 & \delta & \theta \\ \delta & -\delta^2 & \delta \\ \theta & \delta & -1 \end{bmatrix}, \quad 0 < \delta < 1, \quad -1 < \theta \leq 1 - 2\delta^2,$$

$$\Lambda = \{\lambda \colon \eta^2(c - \xi) > \xi(1 + d\xi + \xi^2)\},$$

$$\xi = \mathrm{Re}\lambda, \quad \eta = Im\lambda, \quad c = \frac{2\delta^2 + \theta - 1}{2\delta}, \quad d = \frac{2\delta^2 - \theta + 1}{2\delta}.$$

The domain Λ is located to the left of the imaginary axis, and for $\theta = 1 - 2\delta^2$ it degenerates to the open left half-plane. The relations (3.3.13) and (3.3.14) are the sufficient stability conditions of the differential system (3.3.1).

2. $$\Gamma = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2\rho \end{bmatrix}, \quad \Lambda = \{\lambda \colon |\lambda|^2 < \rho + \sqrt{\rho^2 + 2\rho}\}.$$

The domain Λ for $0 < \rho \leq 1/4$ is located inside the unit disk. The relations (3.3.13) and (3.3.14) are the sufficient conditions of stability of the difference system (3.3.2).

3. $$\Gamma = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1/2 & 0 \\ -1 & 0 & -4\alpha \end{bmatrix}, \quad \alpha \geq 0, \quad \Lambda = \{\lambda \colon |\mathrm{Re}\lambda| > \sqrt{\alpha}\}.$$

Under the conditions (3.3.13), (3.3.14) and $\alpha = 0$ the quadratic pencil $F(\lambda)$ does not have pure imaginary eigenvalues.

4. $$\Gamma = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1/2 & 0 \\ 1 & 0 & -4\beta \end{bmatrix}, \quad \beta \geq 0, \quad \Lambda = \{\lambda \colon |\mathrm{Im}\lambda| > \sqrt{\beta}\}.$$

Under the conditions (3.3.13), (3.3.14) and $\beta = 0$ all eigenvalues of the quadratic pencil $F(\lambda)$ are complex.

Note that if $Y > 0$, then it is always possible to select a matrix $Q > 0$ so that the limitation on the matrix Y of the form (3.3.14) would hold true.

3.4 Stability of Differential-difference and Stochastic Systems

This Section deals with construction and study of matrix equations playing the role of the generalized Lyapunov equation in stability problems for some classes of differential-difference and stochastic systems.

1. Consider the system of differential-difference delay equations

$$A_0\, x(t) + A_1 \frac{dx(t)}{dt} + \sum_{i=1}^{s-1} A_{i+1}\, x(t-\tau_i) = 0, \tag{3.4.1}$$

where $A_0, \ldots, A_s$ are constant $n \times n$ matrices, and $\tau_i \geq 0$ are parameters of permanent delays, $x(\theta) = x_0(\theta)$, $t_0 - \tau \leq \theta \leq t_0$, $0 \leq t_0 \leq t$, $\tau = \max\limits_i \tau_i$, $i = \overline{1, s-1}$. In the absence of the delays ($\tau = 0$) the system (3.4.1) is reduced to the form

$$Ax(t) + A_1 \frac{dx(t)}{dt} = 0, \quad A = A_0 + A_2 + \cdots + A_s. \tag{3.4.2}$$

The zero solution of the system (3.4.1) is said to be *stable by Lyapunov*, if for any $\varepsilon > 0$ there exists such $\delta = \delta(\varepsilon, t_0) > 0$, that $||x(t)|| < \varepsilon$ for $t > t_0$, as soon as $||x(\theta)|| < \delta$ for $t_0 - \tau \leq \theta \leq t_0$. The zero solution of the system (3.4.1) is said to be *asymptotically stable* if it is stable by Lyapunov and $||x(t)|| \to 0$ while $t \to \infty$. The problem of *absolute stability* for the system (3.4.1) lies in the construction of (algebraic) conditions imposed on the matrix coefficients, under which the zero solution is asymptotically stable for any constant values of the delays $\tau_i \geq 0, i = \overline{1, s-1}$.

Lemma 3.4.1 *For the asymptotic stability of the system (3.4.1) it is necessary and sufficient that all eigenvalues of the matrix quasi-polynomial*

$$F(\lambda) = A_0 + \lambda A_1 + e^{-\lambda\tau_1} A_2 + \ldots + e^{-\lambda\tau_{s-1}} A_s$$

have negative real parts.

From Theorem 2.7.2 and Lemma 3.4.1 the next proposition follows:

Theorem 3.4.1 *If the Hermitian matrices X, Y, Q, and G satisfy the relations*

$$A_0 X A_1^* + A_1 X A_0^* + C(G \otimes X)C^* = Y, \tag{3.4.3}$$

$$BXB^* \geq 0, \quad Y \geq CQC^*, \tag{3.4.4}$$

$$\gamma \leq g_\lambda^* H^{-1} g_\lambda, \quad \forall \lambda : \mathrm{Re}\lambda \geq 0, \tag{3.4.5}$$

where

$$B^* = [A_1^*, \ldots, A_s^*], \quad C = [A_1, \ldots, A_s], \quad G = \begin{bmatrix} \gamma & h^* \\ h & H \end{bmatrix},$$

$$Q > 0, \quad H < 0, \quad g_\lambda = h - [e^{-\tau_1 \lambda}, \ldots, e^{-\tau_{s-1}\lambda}]^T,$$

then the zero solution of the system (3.4.1) is asymptotically stable.

In the case of a diagonal matrix G we have sufficient conditions of absolute stability of the system (3.4.1).

Theorem 3.4.2 *If the Hermitian matrices X and Y satisfy the relations (3.4.4) and the equation*

$$A_0 X A_1^* + A_1 X A_0^* + \sum_{i=1}^{s} \gamma_i \, A_i X A_i^* = Y, \tag{3.4.6}$$

where $\gamma_1 = 1/\gamma_2 + \ldots + 1/\gamma_s$, $\gamma_i < 0$, $i = \overline{1, s}$, then the system (3.4.1) is absolutely stable.

Remark 3.4.1 Statements analogous to Theorems 3.4.1 and 3.4.2 can be formulated in terms of solutions of the adjoint matrix equations

$$A_0^* Z A_1 + A_1^* Z A_0 + B^*(G^T \otimes Z)B = S, \tag{3.4.7}$$

$$A_0^* Z A_1 + A_1^* Z A_0 + \sum_{i=1}^{s} \gamma_i \, A_i^* Z A_i = S. \tag{3.4.8}$$

Here instead of (3.4.4) one should use the inequalities $C^* Z C \geq 0$ and $S \geq B^* Q B$. Operators of the left-hand sides of the equations (3.4.7) and (3.4.8) are adjoint to the respective operators of the equations (3.4.3) and (3.4.6) (see Section 6.1). If the equation (3.4.6) ((3.4.8)) is satisfied by the positive definite matrices X and Y (Z and S), then the inequality $-2\varepsilon < \gamma_1 < 0$ must hold true, where ε is the spectral stability factor of the pencil of matrices $A_0 + \lambda A_1$.

We will show that the solutions of the matrix equations (3.4.6) and (3.4.8) can be used at construction of quadratic functionals of the form

$$v = x^*(t)X_0x(t) + \sum_{i=1}^{s-1}\int\limits_{t-\tau_i}^{t} x^*(\tau)X_ix(\tau)d\tau, \tag{3.4.9}$$

that satisfy the Lyapunov–Krasovsky theorem on asymptotic stability of the system (3.4.1). In the case $A_1 = I$ some methods of selection of the weight matrices $X_i \geq 0$ are known that ensure the conditions of absolute stability of the system (3.4.1).

The following statement yields general estimates for those matrices and for the derivative of the functional (3.4.9) with respect to solutions of the system (3.4.1).

Lemma 3.4.2 *Let $X_0 = A_1^*ZA_1$ and the following system of inequalities*

$$A^*ZA_1 + A_1^*ZA \geq Z_0, \quad X_1 \geq Z_1, \ldots, X_{s-1} \geq Z_{s-1}, \tag{3.4.10}$$

$$\Delta = \begin{bmatrix} Z_0 & X_1 + A_1^*ZA_2 & \cdots & X_{s-1} + A_1^*ZA_s \\ X_1 + A_2^*ZA_1 & Z_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ X_{s-1} + A_s^*ZA_1 & 0 & \cdots & Z_{s-1} \end{bmatrix} \geq 0, \tag{3.4.11}$$

hold true, where A is a matrix of the system (3.4.2), $Z_i \geq 0$ are some nonnegative definite matrices. Then the derivative of the functional (3.4.9), in view of the system (3.4.1), satisfies the estimate

$$\left.\frac{dv}{dt}\right|_{(55)} \leq -x^*(t)S_0x(t), \quad t \geq t_0. \tag{3.4.12}$$

where $S_0 = A^*ZA_1 + A_1^*ZA - Z_0 \geq 0$.

Proof. Use the known formula of differentiation of integral with respect to parameter

$$\frac{d}{dt}\int\limits_{p(t)}^{q(t)} f(\tau,t)d\tau = \int\limits_{p(t)}^{q(t)} \frac{\partial}{\partial t} f(\tau,t)d\tau + f(q,t)\frac{dq}{dt} - f(p,t)\frac{dp}{dt}.$$

As a result, obtain the expression for the derivative of functional in view of the system (3.4.1)

$$\left.\frac{dv}{dt}\right|_{(55)} = -y^*Wy, \quad y = \begin{bmatrix} x(t) \\ x(t-\tau_1) \\ \vdots \\ x(t-\tau_{s-1}) \end{bmatrix},$$

$$W = \begin{bmatrix} A_0^*ZA_1 + A_1^*ZA_0 - \sum\limits_{i=1}^{s-1} X_i & A_1^*ZA_2 & \cdots & A_1^*ZA_s \\ A_2^*ZA_1 & X_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_s^*ZA_1 & 0 & \cdots & X_{s-1} \end{bmatrix}.$$

Using block transformations of the matrix W, obtain the relations

$$\left.\frac{dv}{dt}\right|_{(55)} = -z^*\Omega z, \quad z = \begin{bmatrix} x(t) \\ x(t-\tau_1) - x(t) \\ \vdots \\ x(t-\tau_{s-1}) - x(t) \end{bmatrix},$$

$$\Omega = \begin{bmatrix} A^*ZA_1 + A_1^*ZA & X_1 + A_1^*ZA_2 & \ldots & X_{s-1} + A_1^*ZA_s \\ X_1 + A_2^*ZA_1 & X_1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ X_{s-1} + A_s^*ZA_1 & 0 & \ldots & X_{s-1} \end{bmatrix}.$$

Under the conditions (3.4.10) and (3.4.11) obtain $\Omega \geq \Theta \geq 0$, where $\Theta = \Omega - \Delta$ is a block-diagonal matrix. From this the estimate (3.4.12) follows.

The lemma is proved.

Assume

$$X_{i-1} = Z_{i-1} = -\gamma_i\, A_i^*ZA_i, \quad Z_0 = \sum_{i=2}^{s} D_i^*ZD_i,$$

where $D_i = (-\gamma_i)^{-1/2}A_1 + (-\gamma_i)^{1/2}A_i$, $\gamma_i < 0$, $i = \overline{2,s}$. Then the conditions of Lemma 3.5.2 hold true if

$$A^*ZA_1 + A_1^*ZA - \sum_{i=2}^{s} D_i^*ZD_i = S, \tag{3.4.13}$$

$$\begin{bmatrix} A_1^* \\ A_i^* \end{bmatrix} Z\,[A_1, A_i] \geq 0, \quad i = \overline{2,s}, \tag{3.4.14}$$

where A is a matrix of the system (3.4.2), $S \geq 0$. It is easy to see that the equations (3.4.8) and (3.4.13) are equivalent, and the inequalities (3.4.14) are the consequence of the relation $C^*ZC \geq 0$.

If

$$X_{i-1} = Z_{i-1} = -\frac{1}{\gamma_i}X^{-1}, \quad \gamma_i < 0, \quad i = \overline{2,s},$$

$$Z_0 = A_1^*Z\left(\sum_{i=2}^{s} D_iXD_i^*\right)ZA_1, \quad X^{-1} = A_1^*ZA_1,$$

then for the conditions (3.4.10) – (3.4.11) of Lemma 3.5.2 to hold true it is sufficient that the matrices $X > 0$ and $Y \geq 0$ satisfy the equation

$$AXA_1^* + A_1XA^* - \sum_{i=2}^{s} D_iXD_i^* = Y. \tag{3.4.15}$$

In the relation (3.4.12) $S_0 = A_1^*ZYZA_1$. The matrix equations (3.4.6) and (3.4.15) are equivalent.

2. Consider the system of Ito's stochastic differential equations

$$dx(t) = Ax(t)\,dt + \sum_{i=1}^{s} B_ix(t)\,dw_i(t), \tag{3.4.16}$$

where A, B_i are constant $n \times n$ matrices, w_i are components of the standard Wiener process, $x(t_0) = x_0$, $t \geq t_0$. At the study of the conditions of *mean-square stability* of the system (3.4.16), the Lyapunov function of the form

$$v(x) = x^*(t)Xx(t) \tag{3.4.17}$$

is used, where X is a positive definite matrix subject to determination. The average of distribution of the derivative of the function (3.4.17) in view of the system (3.4.16) is represented in the form

$$M\left\{\frac{dv}{dt}\right\} = x^*(t)\left(A^*X + XA + \sum_{i=1}^{s} B_i^* X B_i\right)x(t).$$

We formulate known algebraic criterion of the asymptotic mean-square stability of system (3.4.16) following from the second Lyapunov's method.

Theorem 3.4.3 *If for some positive definite matrix Y the matrix equation*

$$-A^*X - XA - \sum_{i=1}^{s} B_i^* X B_i = Y \tag{3.4.18}$$

has a positive definite solution X, then the zero solution of the system (3.4.16) is asymptotically mean-square stable. The inverse statement holds true also.

Rewrite the matrix equation (3.4.18) in the form

$$LX - PX = Y, \tag{3.4.19}$$

where

$$LX = -A^*X - XA, \quad PX = \sum_{i=1}^{s} B_i^* X B_i.$$

Under the conditions of Theorem 3.4.3 the operator L is *positively invertible*, and the operator P is *positive* with respect to the cone $\mathcal{K}$ of nonnegative definite matrices, i.e. $P\mathcal{K} \subseteq \mathcal{K} \subseteq L\mathcal{K}$. At the study of the equation (3.4.19) one can use *majorants of the operator P*, satisfying the condition $(\hat{P} - P)\mathcal{K} \subseteq \mathcal{K}$. If the matrix equation

$$LZ - \hat{P}Z = S, \tag{3.4.20}$$

where $S > 0$, and $\hat{P}$ is a majorant of the operator P, has a solution $Z > 0$, then for some matrix $Y > 0$ the equation (3.4.19) also has a positive definite solution $X > 0$. As the majorant $\hat{P}$ in (3.4.20) the following linear operators can serve:

$$\hat{P}X = r_0(X)Q_0, \quad r_0(X) = \mathrm{tr}X, \quad Q_0 = \sum_{i=1}^{s} B_i^* B_i, \tag{3.4.21}$$

$$\hat{P}X = \sum_{i=1}^{s} r_i(X)Q_i, \quad r_i(X) = \mathrm{tr}(E_i^* X E_i), \quad Q_i = F_i F_i^*, \tag{3.4.22}$$

where E_i and F_i are components of skeleton expansions $B_i = E_i F_i^*$, $i = \overline{1, s}$. Thus, if the relations

$$\mathrm{tr}H_0 < 1, \quad -A^* H_0 - H_0 A = Q_0, \tag{3.4.23}$$

hold true, then for any matrix $S > 0$ the equation (3.4.20) with the operator (3.4.21) has a positive definite solution $Z > 0$. If the system of relations

$$\det(I_i - \Sigma_i) > 0, \quad -A^* H_i - H_i A = Q_i, \quad i = \overline{1, s}, \tag{3.4.24}$$

is true, where Σ_i are successive principal submatrices of dimension $i \times i$ of the nonnegative matrix

$$\Sigma = \begin{bmatrix} r_1(H_1) & r_1(H_2) & \dots & r_1(H_s) \\ r_2(H_1) & r_2(H_2) & \dots & r_2(H_s) \\ \cdots & \cdots & \cdots & \cdots \\ r_s(H_1) & r_s(H_2) & \dots & r_s(H_s) \end{bmatrix},$$

then the equation (3.4.20) with the operator (3.4.22) is also solvable in the form $Z > 0$ for any matrix $S > 0$ (see Section 6.2).

Each of the systems of relations (3.4.23) and (3.4.24) plays the role of sufficient conditions of mean-square stability of the system (3.4.16). The spectrum of the matrix A must be located to the left of the imaginary axis. The stability conditions similar to the relations (3.4.24) can be obtained, proceeding from the equation (3.4.18), when the matrix coefficients B_i have the unit rank.

Note that above stability analysis technique of the system (3.4.16), based on the construction and solution of matrix equations of the type (3.4.19), can be extended to more general classes of differential-difference stochastic systems. The main properties of linear equations of the form (3.4.19) are described in Section 6.2.

3.5 Representation of Solutions of Linear Dynamic Systems

We will describe the technique for construction of the solutions of linear dynamic systems, which is based on application of the right pairs of matrix functions. First, consider the first-order differential system

$$Az(t) - B\dot{z}(t) = y(t), \quad z(0) = z_0, \tag{3.5.1}$$

where $L(\lambda) = A - \lambda B$ is a regular pencil of matrices. If the matrix B is nonsingular, then this system can be reduced to the normal Cauchy form. In the general case the solution $z(t)$ has two constituents corresponding to the finite and infinite elementary divisors of the pencil $L(\lambda)$. We will show that these constituents can be described in terms of solutions of the algebraic systems

$$AE = BEU, \quad BH = AHV. \tag{3.5.2}$$

Determine the solution and the right part of the system (3.5.1) in the form

$$z(t) = Eu(t) + Hv(t), \quad y(t) = -BEp(t) + AHq(t), \tag{3.5.3}$$

where u, v, p, and q are vector-functions. Substitution of these expressions into (3.5.1), taking into consideration (3.5.2), gives

$$-BE\left[\dot{u}(t) - Uu(t) - p(t)\right] + AH\left[v(t) - V\dot{v}(t) - q(t)\right] = 0.$$

Hence, if for some ν the condition

$$AHV^{\nu}q^{(\nu)}(t) \equiv 0, \tag{3.5.4}$$

holds true, then the system (3.5.1) is solvable in the form (3.5.3), where

$$u(t) = e^{tU}u_0 + \int_0^t e^{(t-\tau)U}p(\tau)d\tau, \quad v(t) = \sum_{i=0}^{\nu-1} V^i q^{(i)}(t), \tag{3.5.5}$$

$q^{(i)}(t)$ is an i-th order derivative of the vector-function $q(t)$.

Consider the s-th order differential system

$$A_0 x(t) + A_1 x^{(1)}(t) + \ldots + A_s x^{(s)}(t) = g(t), \tag{3.5.6}$$

where

$$x^{(i)}(0) = x_0^{(i)}, \quad i = \overline{0, s-1}, \quad \det F(\lambda) \not\equiv 0, \quad F(\lambda) = \sum_{i=0}^{s} \lambda^i A_i.$$

The right pairs of the matrix polynomials $F(\lambda)$ and $\lambda^s F(1/\lambda)$ are determined by the relations

$$\sum_{i=0}^{s} A_i T U^i = 0, \quad \sum_{i=0}^{s} A_i K V^{s-i} = 0. \tag{3.5.7}$$

At the same time the following identities hold true:

$$F(\lambda)T \equiv \Phi(\lambda)(\lambda I - U), \quad F(\lambda)K \equiv \Psi(\lambda)(I - \lambda V), \tag{3.5.8}$$

where

$$\Phi(\lambda) = \sum_{i=0}^{s-1} \lambda^i \Phi_i, \quad \Psi(\lambda) = \sum_{i=0}^{s-1} \lambda^i \Psi_i,$$

$$\Phi_i = \sum_{j=i+1}^{s} A_j T U^{j-i-1}, \quad \Psi_i = \sum_{j=0}^{i} A_j K V^{i-j}, \quad i = \overline{0, s-1}.$$

Theorem 3.5.1 *Let the pairs of matrices (U, T) and (V, K) satisfy the equalities (3.5.7), and the following relations be true:*

$$\det(I - \lambda V) \neq 0, \quad \lambda \in \sigma(F), \tag{3.5.9}$$

$$x_0 = T u_0 + K v_0, \quad \sum_{j=i}^{s} A_j \, x_0^{(j-i)} = \Phi_{i-1} \, u_0 - \Psi_{i-1} V \, v_0, \tag{3.5.10}$$

$$g(t) = \Phi_0 p(t) + \Psi_0 q(t), \quad \Phi_i p(t) + \Psi_i q(t) \equiv 0, \tag{3.5.11}$$

where u_0, v_0, $p(t)$, and $q(t)$ are some vectors, $i = \overline{1, s-1}$. Then the system (84) is solvable in the form

$$x(t) = T u(t) + K v(t), \tag{3.5.12}$$

where $u(t)$ and $v(t)$ are the vector-functions determined in (3.5.5).

Conversely, if $x(t)$ is a solution of the system (3.5.6), then there exist matrices U, T, V, and K, for which all the relations (3.5.7) – (3.5.12) hold true.

Proof. Rewrite the system (3.5.6) and the inequalities (3.5.7) in the compact form (3.5.1) and (3.5.2), assuming

$$A = \begin{bmatrix} 0 & \dots & 0 & -A_0 \\ I & \dots & 0 & -A_1 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & -A_{s-1} \end{bmatrix}, \quad B = \begin{bmatrix} I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & I & 0 \\ 0 & \dots & 0 & A_s \end{bmatrix},$$

$$z(t) = \begin{bmatrix} A_1 x(t) + \ldots + A_s x^{(s-1)}(t) \\ \vdots \\ A_{s-1} x(t) + A_s x^{(1)}(t) \\ x(t) \end{bmatrix}, \quad y(t) = -\begin{bmatrix} g(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$E = \begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_{s-2} \\ T \end{bmatrix}, \quad H = \begin{bmatrix} -\Psi_0 V \\ \vdots \\ -\Psi_{s-2} V \\ K \end{bmatrix},$$

$$BE = \begin{bmatrix} \Phi_0 \\ \vdots \\ \Phi_{s-2} \\ \Phi_{s-1} \end{bmatrix}, \quad AH = -\begin{bmatrix} \Psi_0 \\ \vdots \\ \Psi_{s-2} \\ \Psi_{s-1} \end{bmatrix}.$$

According to (3.5.2)–(3.5.5), we have representation of the solution $z(t)$ of the system (3.5.1) of the form (3.5.3) with the respective limitations on the initial vector z_0 and the right hand part $y(t)$.

Go to the canonical form of the regular pencil

$$P(A - \lambda B)Q = \begin{bmatrix} J - \lambda I & 0 \\ 0 & I - \lambda N \end{bmatrix}. \tag{3.5.13}$$

According to (3.5.2) and (3.5.13), obtain the relations

$$[E, H] = Q \begin{bmatrix} R & S \\ 0 & G \end{bmatrix}, \quad [BE, AH] = P^{-1} \begin{bmatrix} R & JS \\ 0 & G \end{bmatrix},$$

$$JR = RU, \quad NG = GV, \quad S = JSV.$$

Since $N^\nu = 0$, then $GV^\nu = 0$, where ν is the maximum degree of infinite elementary divisors of the pencil $L(\lambda)$. If, in addition, the condition (3.5.9) holds true, then $S = 0$, and in (3.5.4) $HV^\nu = 0$. In particular, if the matrix V is nilpotent, then the condition (3.5.9) holds true for any λ.

Consequently, under the conditions (3.5.7)–(3.5.11) the expression (3.5.3) is a solution of the system (3.5.1). Taking into account the block structure of the vector $z(t)$, obtain the solution of the form (3.5.12) of the initial system (3.5.6). The remaining equalities for the block components of the vector $z(t)$ follow from (3.5.5), (3.5.7) – (3.5.12).

To prove the inverse proposition we can assume

$$E = -\frac{1}{2\pi i}\oint\limits_{\sigma}(A - \lambda B)^{-1}d\lambda = Q\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}P,$$

$$H = -\frac{1}{2\pi i}\oint\limits_{\omega}(B - \lambda A)^{-1}d\lambda = Q\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}P,$$

$$U = AE = P^{-1}\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}P, \quad V = BH = P^{-1}\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}P,$$

where σ and ω are closed contours enclosing respectively the spectrum $\sigma(L)$ and the point 0. Apparently, $V^\nu = 0$ and the condition (3.5.9) holds true, and the matrices BE and AH (EB and HA) are orthogonal projectors, and $BE + AH = I$ ($EB + HA = I$). Any vectors $z(t)$ and $y(t)$ are therefore representable in the form (3.5.3). In particular, we can assume $u(t) = Bz(t)$, $v(t) = Az(t)$, $p(t) = -y(t)$, $q(t) = y(t)$.

The theorem is proved.

The described technique applies to wider classes of systems. Suppose that the matrix function $F(\lambda)$ describes the differential (difference) system

$$F(D)\,x(t) = g(t), \quad t \geq t_0, \tag{3.5.14}$$

where D is a differentiation (shift) operator in the case of continuous (discrete) time t, and the pairs of matrices (U, T) and (V, K) are determined, satisfying the identities (3.5.8), where $\Phi(\lambda)$ and $\Psi(\lambda)$ are some matrix functions. Then, with some limitations, the vector functions $x(t)$ and $g(t)$ in the system (3.5.14) can be determined in the form

$$x(t) = Tu(t) + Kv(t), \quad g(t) = \Phi(D)p(t) + \Psi(D)q(t). \tag{3.5.15}$$

Indeed, substitution of these expressions into (3.5.14) gives

$$\Phi(D)[Du(t) - Uu(t) - p(t)] + \Psi(D)[v(t) - VDv(t) - q(t)] = 0.$$

If for some ν the identity

$$\Psi(D)V^\nu D^\nu q(t) \equiv 0, \tag{3.5.16}$$

is true, in particular, V is a nilpotent matrix, then the differential system (3.5.14) is solvable in the form (3.5.15), where $u(t)$ and $v(t)$ are determined in (3.5.5).

Similarly, solutions of the difference system (3.5.14) in the case of the shift operator D for $t = 0, 1, \ldots$ are also determined by the relations (3.5.8), (3.5.15), and (3.5.16), where

$$u(t+1) = U^{t+1}u_0 + \sum_{i=0}^{t} U^{t-i}p(i), \quad v(t) = \sum_{i=0}^{\nu-1} V^i q(t+i). \tag{3.5.17}$$

3.6 Notes and References

3.1 There are a lot of publications devoted to the techniques of control of spectral properties of linear systems (see Andreev [1], Kozhinskaya, Vornovitsky [1], Kirichenko [1], Porter, Crossley [1], Simon, Mitter [1], and others).

Averaged quality criteria of the type (3.1.2) were used in Kirichenko [1]. The system of matrix relations (3.1.7)–(3.1.9), (3.1.11), and (3.1.12) expresses the necessary conditions for minimum of a functional, under which the spectrum of a closed-loop system is located in the given domain (3.1.3). This system and the optimization algorithm following from it are obtained by Mazko [3, 6]. There the results of Athans, Levine [1] were used, in particular, an expression for the gradient of an averaged functional. The known algorithms of measurable output optimization of systems coincide with the described computation scheme if the domain (3.1.3) is the left half-plane (see Anderson, Moor [1], Athans, Levine [1], Söderström [1], and others).

The illustrative example of a control system is taken from Maki, Van de Vegte [1]. The example of a rocket control system with consideration for elastic vibrations of the airframe is taken from Fisher [1].

3.2 Some known techniques of construction and analysis of solutions of descriptor systems of the form 3.2.1 are described in Boyarintsev [1], Gantmacher [1], Rutkas [1], and others. Matrix equations of the Lyapunov equation type for such systems is studied in Bender [1], Stykel [1], and others. Lemmas 3.2.1–3.2.4 and Theorems 3.2.1 and 3.2.2 are proved by Mazko [27]. Stability criteria in the form of (3.2.19)–(3.2.22) are given by Mazko [26].

3.3 Systems of the form (3.3.1) and (3.3.2) occur at simulation of transport, electromechanical, space, and other objects (see, e.g., Krein, Langer [1], Lazaryan, Dlugach, Korotenko [1], Rabinovich [1], Kilchevsky [1], and others). In Leang Shieh, Mohamad Mehio, Rani Dib [1] a number of sufficient conditions for stability of the system (3.3.1) were obtained within additional limitations on the matrix coefficients. In Wimmer [2] in the study of a system of the form (3.3.1) the inertia theorem was used. Theorem 3.3.1 and the described techniques of analysis of the spectrum of a quadratic pencil of matrices follow from the more general results of Chapter 2.

3.4 The statement of Lemma 3.4.1 is available in Rezvan [1]. The study of absolute stability of delay systems with the use of quadratic functionals of the type (3.4.9) became possible owing to Krasovsky [1]. In the case $A_1 = I$ the techniques of selection of weight matrices of functionals are known, ensuring the conditions of absolute stability of the system (3.4.1) (see Korenevskii [1], Zelentzovsky [1], Resvan [1], Skorodinskii [1], and others). Theorems 3.4.1 and 3.4.2 generalize the conditions of absolute stability of the system (3.4.1), formulated in Korenevskii, Mazko [2].

Algebraic conditions of mean-square stability of Ito's stochastic systems with the use of matrix equations are described in Korenevskii

[1], Korenevskii, Mazko [1], Valeev, Karelova, Gorelov [1], Boyd, Ghaoui, Feron, Balakrishman [1], and others. Theorem 3.4.3 yields such conditions in terms of solutions of the Silvester equation (3.4.18) (see Korenevskii [1]).

3.5 The general technique of construction and study of linear differential and difference systems solutions, based on the application of generalized spectral problems for matrix polynomials and functions, is proposed in Mazko [27-29].

4

MATRIX EQUATIONS AND LAW OF INERTIA

4.0 Introduction

In this chapter the techniques of the study of matrix equations of the general type are described. First of all, we are interested in solvability conditions, inertial properties of Hermitian solutions, and methods of construction of matrix equation solutions. Furthermore, for the systematization of the obtained data it seems important to study the spectral and analytic properties of operators in the space of matrices, which determine the considered classes of matrix equations.

Using the operation of matrix semi-inversion, in Section 4.1 you will find the estimates for the rank of a matrix satisfying the linear matrix equation of the general form. As a corollary, formulae for the calculation of block matrix rank are given.

In Section 4.2 symmetric matrix equations are considered whose solutions are Hermitian or symmetric matrices. A number of relations are given, presenting the generalized principle of inertia and the technique of computation of the rank, signature, and indices of inertia of solutions of such equations and various matrix expressions.

In Section 4.3 the classes of matrix equations allowing transformation of their coefficients to the easy-to-study canonical form are determined, and the solvability conditions for such equations are formulated. In particular, if the matrix coefficients of an equation are simultaneously reducible to triangular form through similarity transformation, then the conditions of one-valued solvability of this equation are only determined by the diagonal entries (eigenvalues) of the transformed matrix coefficients.

In Section 4.4 transformation systems for the class of symmetric matrix equations are proposed, and general theorems of inertia of Hermitian solutions are formulated. The main properties of the respective classes of operators are determined.

In Section 4.5 the classes of matrix equations are studied whose coefficients form so-called collectives. The concept of the property of collective of order α is determined, and theorems of its distribution that follow from the main results of Section 4.4 are formulated.

In Section 4.6 you will find a review of basic numerical and analytic methods of construction of solutions of linear matrix equations. For special classes of equations some additional facts are formulated. In particular, a spectral criterion of solvability of the Sylvester binomial equation is found by applying the method of transformations. Using the integral representation of a solution of matrix and more general equations, a criterion of asymptotic stability of a class of differential systems described by the operators of these equations is formulated.

4.1 Estimate of the Rank of a Matrix Solution

Consider the linear matrix equation

$$\sum_{i=1}^{k}\sum_{j=1}^{s} c_{ij}A_iXB_j = Y, \tag{4.1.1}$$

where A_i, B_j, and Y are given matrices of dimensions $p \times n$, $m \times q$, and $p \times q$ respectively, c_{ij} the scalar coefficients composing the $k \times s$ matrix C. Let the equation (4.1.1) be solvable and X be one of its solutions of dimension $n \times m$. The equation (4.1.1) is equivalent to the system

$$AZB = Y, \quad C \otimes X = Z, \tag{4.1.2}$$

where $\otimes$ is the *Kronecker product*, A, B, and Z are block matrices of the form

$$A = [A_1, \dots, A_k], \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_s \end{bmatrix}, \quad Z = \begin{bmatrix} c_{11}X & \dots & c_{1s}X \\ \cdots & \cdots & \cdots \\ c_{k1}X & \dots & c_{ks}X \end{bmatrix}.$$

Along with (4.1.1) and (4.1.2) consider the relations

$$W = Z - ZBY^{-}AZ, \quad YY^{-}Y = Y, \tag{4.1.3}$$

where Y^{-} is an arbitrary *semi-inverse* $q \times p$ matrix for Y. According to (4.1.2) and (4.1.3), the matrix W is a solution of the homogeneous equation

$$AWB = 0. \tag{4.1.4}$$

If Y is a matrix of full rank by column (row), then $WB = 0$ ($AW = 0$).

Theorem 4.1.1. *For the matrix system (4.1.2), (4.1.3) the following equality holds true:*

$$\operatorname{rank} C \operatorname{rank} X = \operatorname{rank} Y + \operatorname{rank} W. \tag{4.1.5}$$

Proof. Let P, Q, and D be square nonsingular matrices of order p, q, and δ respectively, such that

$$PYQ = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \quad \operatorname{rank} Y = \delta \neq 0.$$

The semi-inversion operation has the following properties:

$$(PYQ)^{-} = Q^{-1}Y^{-}P^{-1}, \quad \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}^{-} = \begin{bmatrix} D^{-1} & U \\ V & S \end{bmatrix},$$

where U, V, and S are arbitrary blocks of appropriate dimensions. Using these relations, represent the expression (4.1.3) in the form

$$W = Z - ZR_0(L_0ZR_0)^{-1}L_0Z - ZR_1\Delta L_1Z, \tag{4.1.6}$$

where

$$L_0 = [I_\delta, DU]\,PA, \quad L_1 = [0, I_{q-\delta}]\,PA, \quad \Delta = S - VDU,$$

$$R_0 = BQ\begin{bmatrix} I_\delta \\ VD \end{bmatrix}, \quad R_1 = BQ\begin{bmatrix} 0 \\ I_{p-\delta} \end{bmatrix},$$

I_δ is a unit matrix of order δ. Here the equalities

$$L_0ZR_0 = D, \quad L_1ZR_1 = 0, \quad L_0ZR_1 = 0, \quad L_1ZR_0 = 0,$$

$$L_0 W = 0, \quad W R_0 = 0, \quad L_1 W = L_1 Z, \quad W R_1 = Z R_1$$

hold true. Let L_2 and R_2 be arbitrary matrices such that

$$\operatorname{rank} L_3 = k\,n, \;\; L_3 = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \end{bmatrix}, \;\; \operatorname{rank} R_3 = s\,m, \;\; R_3 = [R_0, R_1, R_2]\,.$$

Then the block matrices L_4 and R_4 of the form

$$L_4 = \begin{bmatrix} L_0 \\ L_1 \\ L_2 - L_2 Z R_0 D^{-1} L_0 - 0.5 L_2 Z R_1 \Delta L_1 \end{bmatrix},$$

$$R_4 = \left[R_0, R_1, R_2 - R_0 D^{-1} L_0 Z R_2 - \frac{1}{2} R_1 \Delta L_1 Z R_2\right],$$

have full rank equaling respectively to kn and sm. Calculating and comparing the products of matrices

$$L_3 W R_3 = \left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & L_1 Z R_2 \\ 0 & L_2 Z R_1 & L_2 W R_2 \end{array}\right],$$

$$L_4 Z R_4 = \left[\begin{array}{c|cc} D & 0 & 0 \\ \hline 0 & 0 & L_1 Z R_2 \\ 0 & L_2 Z R_1 & L_2 W R_2 \end{array}\right],$$

we see that the equality $\operatorname{rank} Z = \operatorname{rank} W + \delta$ holds true. This equality can be reduced to the form (4.1.5), because $\operatorname{rank}(C \otimes X) = \operatorname{rank} C \operatorname{rank} X$. If $Y = 0$, then Y^- is an arbitrary matrix, and the equality (4.1.5) follows from

$$\begin{bmatrix} A \\ L_5 \end{bmatrix} W [B, R_5] = \begin{bmatrix} A \\ L_5 - \frac{1}{2} L_5 Z B Y^- A \end{bmatrix} Z \left[B, R_5 - \frac{1}{2} B Y^- A Z R_5\right].$$

Here the blocks L_5 and R_5 are selected so that the left and right multipliers are matrices of full rank.

The theorem is proved.

Remark 4.1.1 If we require that the semi-inverse matrix Y^- satisfies the second condition in the Penrose semi-inversion system $((Y^-)^- = Y)$, then in (4.1.6) $\Delta = 0$, and the proof of Theorem 4.1.1 is simplified.

Corollary 4.1.1 *For any solution of the equation (4.1.1) the following inequalities hold true*

$$\operatorname{rank} C \, \operatorname{rank} X \geq \operatorname{rank} Y, \tag{4.1.7}$$

$$\operatorname{rank} C \, \operatorname{rank} X \leq \operatorname{rank} Y - \operatorname{rank} A - \operatorname{rank} B + k\,n + s\,m, \tag{4.1.8}$$

$$\operatorname{rank} C \, \operatorname{rank} X \leq \operatorname{rank} Y + \operatorname{rank}(C^- \otimes X^- - BX^-A). \tag{4.1.9}$$

The inequality (4.1.7) follows directly from the equality (4.1.5). The inequality (4.1.8) is a consequence of the relations (4.1.3)–(4.1.5), and also of the Sylvester–Frobenius inequalities for the rank of a matrix product. The inequality (4.1.9) follows from (4.1.5) and the equality $W = Z(Z^- - BY^-A)Z$, where Z^- is a semi-inverse matrix for Z, in particular, $Z^- = C^- \otimes X^-$. The equality in (4.1.7) is achieved if and only if the expression BY^-A is a semi-inverse matrix for Z.

Corollary 4.1.2 *If $BY^-A = C^- \otimes X^-$, then the solution X of the equation (4.1.1) has the minimum rank.*

Corollary 4.1.3 *If the matrices X_1, X_2, X_3, and X_4 have the dimensions $p \times q$, $p \times \tau$, $t \times q$, and $t \times \tau$ respectively, then the following equalities hold true:*

$$\begin{aligned} \operatorname{rank}\,[X_1, X_2] &= \operatorname{rank} X_1 + \operatorname{rank} L, \\ \operatorname{rank}\begin{bmatrix} X_1 \\ X_3 \end{bmatrix} &= \operatorname{rank} X_1 + \operatorname{rank} R, \\ \operatorname{rank}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} &= \operatorname{rank} X_1 + \operatorname{rank} L + \operatorname{rank} R + \operatorname{rank} G, \end{aligned} \tag{4.1.10}$$

where

$$L = X_2 - X_1 X_1^- X_2, \quad R = X_3 - X_3 X_1^- X_1,$$

$$G = (I_t - RR^-)T(I_\tau - L^-L), \quad T = X_4 - X_3 X_1^- X_2.$$

For the proof of the equalities (4.1.10) in Theorem 4.1.1 assume

$$A = [I_p, 0], \quad B = \begin{bmatrix} I_q \\ 0 \end{bmatrix}, \quad C = 1, \quad Z = X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}.$$

At the same time the following relations hold true:

$$Y = X_1, \quad W = \begin{bmatrix} 0 & L \\ R & T \end{bmatrix}, \quad L_6 W R_6 = \begin{bmatrix} 0 & 0 & L \\ 0 & G & 0 \\ R & 0 & 0 \end{bmatrix}, \qquad (4.1.11)$$

where the multipliers L_6 and R_6 have full rank and the following structure:

$$L_6 = \begin{bmatrix} I_p & 0 \\ (RR^- - I_t)TL^- & I_t - RR^- \\ -\frac{1}{2}RR^-TL^- & RR^- \end{bmatrix}, \quad \operatorname{rank} L_6 = p + t,$$

$$R_6 = \begin{bmatrix} I_q & R^-T(L^-L - I_q) & -\frac{1}{2}R^-TL^-L \\ 0 & I_\tau - L^-L & L^-L \end{bmatrix}, \quad \operatorname{rank} R_6 = q + \tau.$$

Therefore the equalities (4.1.10) follow from the formulae (4.1.5) and (4.1.11).

Corollary 4.1.4 *The following criterion takes place:*

$$\operatorname{rank} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \operatorname{rank} X_1 \quad \Longleftrightarrow \quad L = 0, \ R = 0, \ T = 0.$$

If X_1 is a square nonsingular block, then $X_1^- = X_1^{-1}$, $L = 0$, $R = 0$. In this case the statement of Corollary 4.1.4 is known.

4.2 Inertia of Hermitian Solutions

Inertia of a Hermitian matrix $X = X^*$ is composed by the triple of numbers

$$i(X) = \{i_+(X), i_-(X), i_0(X)\}$$

determined by the quantity of its positive (i_+), negative (i_-), and zero (i_0) eigenvalues, taking into account the multiplicities. If the Hermitian matrices X and Y are connected by the relation

$$AXA^* = Y, \qquad (4.2.1)$$

where A is a square nonsingular matrix, then $i(X) = i(Y)$. It means that inertia is invariant with respect to Sylvester's *congruent transformation* (*law of inertia*). If A is a rectangular matrix of full rank by column, then the invariants of the transformation (4.2.1) are the indices of inertia $i_\pm$, as well is the *rank* and the *signature*

$$\operatorname{rank} X = i_+(X) + i_-(X), \quad \operatorname{sign} X = i_+(X) - i_-(X).$$

Apparently, the rank and the signature determine all the three components of inertia.

Study the connection between inertias of the matrices X and Y satisfying the equation

$$\sum_{i,j=1}^{k} c_{ij} A_i X A_j^* = Y, \tag{4.2.2}$$

in particular, the relation (4.2.1) without any limitations on the matrix coefficients A and A_i. Consider the relations (4.1.2)–(4.1.6) under the conjugation conditions:

$$B = A^*,\ C = C^*,\ X = X^*,\ Y = Y^*,\ Z = Z^*,\ Y^- = Y^{-*}. \tag{4.2.3}$$

Repeating the proof of Theorem 4.1.1, in consideration of the equalities $R_i = L_i^*$ and the law of inertia, obtain the following statement.

Theorem 4.2.1 *For the matrix system (4.1.2), (4.1.3), and (4.2.3) the following equalities hold true:*

$$\begin{gathered} \operatorname{rank} C \, \operatorname{rank} X = \operatorname{rank} Y + \operatorname{rank} W, \\ \operatorname{sign} C \, \operatorname{sign} X = \operatorname{sign} Y + \operatorname{sign} W, \\ i_+(C)\, i_+(X) + i_-(C)\, i_-(X) = i_+(Y) + i_+(W), \\ i_-(C)\, i_+(X) + i_+(C)\, i_-(X) = i_-(Y) + i_-(W), \\ i_0(C)\, i_0(X) - n\, i_0(C) - k\, i_0(X) = \operatorname{rank} Y - i_0(W). \end{gathered} \tag{4.2.4}$$

Corollary 4.2.1 *For any Hermitian matrix solution X of the equation (4.2.2) the following inequalities hold true:*

$$i_+(Y) \le i_+(C)i_+(X) + i_-(C)i_-(X) \le i_+(Y) + i_+(C^- \otimes X^- - A^* Y^- A),$$

$$i_-(Y) \le i_-(C)i_+(X)+i_+(C)i_-(X) \le i_-(Y)+i_-(C^-\otimes X^- - A^*Y^-A).$$

Corollary 4.2.2 *Let* $X = X^*$ *be a solution of the equation (4.2.2) under the condition* $\operatorname{sign} C = 0$*. Then the inequality*

$$\operatorname{rank} C \operatorname{rank} X \ge \operatorname{rank} Y + |\operatorname{sign} Y| = 2\max\{i_+(Y), i_-(Y)\}$$

holds true. In particular, if $Y > 0$ *or* $Y < 0$*, then under the conditions* $\operatorname{rank} C = 2$ *and* $\operatorname{sign} C = 0$ *the solution* X *is a nonsingular matrix.*

We will formulate corollaries of Theorem 4.2.1 for the relation (4.2.1), assuming

$$C = 1, \quad Z = X, \quad W = X - XA^*Y^-AX,$$

where A is a rectangular matrix of any rank. According to (4.2.4), obtain the inequalities

$$i_+(X) \ge i_+(Y), \quad i_-(X) \ge i_-(Y).$$

Corollary 4.2.3 *The equalities*

$$i_+(X) = i_+(Y), \quad i_-(X) = i_-(Y) \tag{4.2.5}$$

hold true if and only if the expression A^*Y^-A *is a semi-inverse matrix for* X*.*

If A is a square nonsingular matrix, then $W = 0$ and the equalities (4.2.5) representing Sylvester's law of inertia hold true.

Corollary 4.2.4 *The equalities* $i_+(X) = i_+(Y) + z$ *and* $i_-(X) = i_-(Y)$ *hold true if and only if* W *is a nonnegative definite matrix of rank* z*. Similarly, the equalities* $i_+(X) = i_+(Y)$ *and* $i_-(X) = i_-(Y) + z$ *are equivalent to the relations* $W \le 0$*,* $\operatorname{rank} W = z$*.*

These statements can be used in calculation of indices of inertia of a Hermitian matrix. No limitations on its minors, similar to the conditions of Jacobi's theorem, are required. The search of inertia of the matrix X adds up to the application of criteria of sign definiteness of the matrices Y and W. Thus, if the $p \times n$ matrix A is selected so that

$Y > 0$ and $W \leq 0$, then $i_+(X) = p$ and $i_-(X) = \operatorname{rank} W$. Similarly, under the conditions $Y < 0$, $W \geq 0$ we have $i_+(X) = \operatorname{rank} W$ and $i_-(X) = p$. In the case $p = 1$ the following statement is true.

Corollary 4.2.5 *If the Hermitian form $\alpha(z) = z^* X z > 0$ is positive for some vector z, then the relations $i_+(X) = 1$ and $\alpha(z) X \leq X z z^* X$ are equivalent.*

Corollary 4.2.6 *Let a matrix X be represented in the block form*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad X_1 = X_1^*, \quad X_2 = X_3^*, \quad X_4 = X_4^*.$$

Then along with (4.1.10) the following equality holds true:

$$\operatorname{sign} X = \operatorname{sign} X_1 + \operatorname{sign} G. \tag{4.2.6}$$

The proof of Corollary 4.2.6 follows from (4.1.11), (4.2.4) and the fact that the signatures of the matrix W and the block G in (4.1.11) coincide. The equalities (4.1.10) and (4.2.6) are equivalent to the relations

$$i_\pm(X) = i_\pm(X_1) + i_\pm(G) + \operatorname{rank} L,$$

$$i_0(X) = i_0(X_1) + i_0(G) - 2 \operatorname{rank} L.$$

In particular, we have the following criteria:

$$i_+(X) = i_+(X_1) \iff X_2 = X_1 X_1^- X_2,\ X_4 \leq X_3 X_1^- X_2;$$

$$i_-(X) = i_-(X_1) \iff X_2 = X_1 X_1^- X_2,\ X_4 \geq X_3 X_1^- X_2;$$

$$X \geq 0 \iff X_1 \geq 0,\ X_4 \geq X_3 X_1^- X_2,\ X_2 = X_1 X_1^- X_2;$$

$$X > 0 \iff X_1 > 0,\ X_4 > X_3 X_1^{-1} X_2.$$

Similar results can be obtained, considering the block X_4 of X.

4.3 Transformations and Solvability Conditions of Matrix Equations

In the study of matrix equations an important role is played by systems of *transformations* that reduce them to a simpler form. In particular, we are interested in the possibility of reduction of the equation (4.1.1) to a similar equation with triangular matrix coefficients; the solvability conditions of this equation are well studied. Consider two equations of the form (4.1.1):

$$MX \triangleq \sum_{i=1}^{k} \sum_{j=1}^{s} c_{ij}\, A_i X B_j = Y, \tag{4.3.1}$$

$$\hat{M}\hat{X} \triangleq \sum_{i=1}^{\hat{k}} \sum_{j=1}^{\hat{s}} d_{ij}\, L_i \hat{X} R_j = \hat{Y}. \tag{4.3.2}$$

The linear operator $M(\hat{M})$ acts from the space of matrices of dimensions $n \times m$ $(\hat{n} \times \hat{m})$ to the space of matrices of dimensions $p \times q$ $(\hat{p} \times \hat{q})$. According to (4.1.2), we have the representations

$$\begin{gathered} MX = A(C \otimes X)B, \quad A = [A_1, \ldots, A_k], \\ B = \begin{bmatrix} B_1 \\ \vdots \\ B_s \end{bmatrix}, \quad C = \begin{bmatrix} c_{11} & \ldots & c_{1s} \\ \cdots & \cdots & \cdots \\ c_{k1} & \ldots & c_{ks} \end{bmatrix}, \\ \hat{M}\hat{X} = L(D \otimes \hat{X})R, \quad L = \left[L_1, \ldots, L_{\hat{k}}\right], \\ R = \begin{bmatrix} R_1 \\ \vdots \\ R_{\hat{s}} \end{bmatrix}, \quad D = \begin{bmatrix} d_{11} & \ldots & d_{1\hat{s}} \\ \cdots & \cdots & \cdots \\ d_{\hat{k}1} & \ldots & d_{\hat{k}\hat{s}} \end{bmatrix}. \end{gathered}$$

Determine the connection between the parameters and the structure of solutions of the equations (4.3.1) and (4.3.2), using the matrix system of transformations

$$\begin{gathered} P_1 A P^{(2)} = P_3 L P^{(4)}, \quad Q^{(1)} B Q_2 = Q^{(3)} R Q_4, \\ C = S_1 G S_2, \quad D = S_3 G S_4, \\ P_1 Y Q_2 = P_3 \hat{Y} Q_4, \end{gathered} \tag{4.3.3}$$

where $P^{(2)} = S_1 \otimes P_2$, $P^{(4)} = S_3 \otimes P_4$, $Q^{(1)} = S_2 \otimes Q_1$, $Q^{(3)} = S_4 \otimes Q_3$, P_t, Q_t, S_t, and G are some matrices of relevant dimensions. The solutions X and $\hat{X}$ will be constructed in the form

$$X(H) = P_2 H Q_1, \quad \hat{X}(H) = P_4 H Q_3, \tag{4.3.4}$$

where H is an unknown matrix. Here we use rank limitations on the transformation matrices, in particular

$$\operatorname{rank} P_1 = p, \quad \operatorname{rank} Q_2 = q, \tag{4.3.5}$$

$$\operatorname{rank} P_2 = n, \quad \operatorname{rank} Q_1 = m, \tag{4.3.6}$$

$$\operatorname{rank} P_3 = \hat{p}, \quad \operatorname{rank} Q_4 = \hat{q}, \tag{4.3.7}$$

$$\operatorname{rank} P_4 = \hat{n}, \quad \operatorname{rank} Q_3 = \hat{m}, \tag{4.3.8}$$

$$\operatorname{rank} S_1 = k, \quad \operatorname{rank} S_2 = s, \tag{4.3.9}$$

$$\operatorname{rank} S_3 = \hat{k}, \quad \operatorname{rank} S_4 = \hat{s}. \tag{4.3.10}$$

Theorem 4.3.1 *Let the equations (4.3.1) and (4.3.2) be connected by the system (4.3.3). Then, if the conditions (4.3.5) hold true and the equation (4.3.2) is solvable in the form $\hat{X} = \hat{X}(H)$, then $X = X(H)$ is a solution of the equation (4.3.1). If the conditions (4.3.7) are true and the equation (4.3.1) is solvable in the form $X = X(H)$, then $\hat{X} = \hat{X}(H)$ is a solution of the equation (4.3.2).*

Proof. Use the prescribed structure of the solutions (4.3.4) and calculate the Kronecker products

$$C \otimes X = P^{(2)} F Q^{(1)}, \quad D \otimes \hat{X} = P^{(4)} F Q^{(3)},$$

where $F = G \otimes H$. If one of the matrices (4.3.4) is a solution of the respective equation (4.3.1) or (4.3.2), then, according to (4.3.1)–(4.3.4), the equalities

$$P_1 Y Q_2 = P_1 (MX) Q_2 = P_1 A P^{(2)} F Q^{(1)} B Q_2 =$$

$$= P_3 L P^{(4)} F Q^{(3)} R Q_4 = P_3 (\hat{M} \hat{X}) Q_4 = P_3 \hat{Y} Q_4$$

hold true. Thus, if $\hat{X}$ is a solution of the equation (4.3.2) and the conditions (4.3.5) hold true, then $P_1^- P_1 = I_p$, $Q_2 Q_2^- = I_q$, and the matrix X satisfies the equation (4.3.1). Similarly, if X is a solution of the equation (4.3.1) under the conditions (4.3.7), then $\hat{X}$ is a solution of the equation (4.3.2). In this case $P_3^- P_3 = I_{\hat{p}}$ and $Q_4 Q_4^- = I_{\hat{q}}$,

The theorem is proved.

The connection between the right-hand sides of the equations (4.3.1) and (4.3.2) in the system (4.3.3) can be represented in the form

$$Y = P_1^- P_3 \hat{Y} Q_4 Q_2^- + Y_0, \quad \hat{Y} = P_3^- P_1 Y Q_2 Q_4^- + \hat{Y}_0,$$

where Y_0 and $\hat{Y}_0$ are arbitrary matrices such that $P_1 Y_0 Q_2 = 0$ and $P_3 \hat{Y}_0 Q_4 = 0$. Under the conditions (4.3.5) ((4.3.7)) of Theorem 4.3.1 it is necessary that $Y_0 = 0$ ($\hat{Y}_0 = 0$). A similar connection exists between the solutions with the structure (4.3.4). Excluding the matrix H, obtain

$$X = P_2 P_4^- \hat{X} Q_3^- Q_1 + X_0, \quad \hat{X} = P_4 P_2^- X Q_1^- Q_3 + \hat{X}_0,$$

where $X_0 = P_2 H_0 Q_1$, $\hat{X}_0 = P_4 \hat{H}_0 Q_3$, H_0 and $\hat{H}_0$ are arbitrary matrices such that $P_4 H_0 Q_3 = 0$ and $P_2 \hat{H}_0 Q_1 = 0$. In particular, for $H_0 = 0$ and $\hat{H}_0 = 0$ we have the following statements.

Corollary 4.3.1 *For the matrix X to be a solution of the equation (4.3.1), under the conditions (4.3.5) and (4.3.6) it is sufficient, and under the conditions (4.3.6) and (4.3.7) it is necessary that the matrix $\hat{X} = P_4 P_2^- X Q_1^- Q_3$ must satisfy the equation (4.3.2).*

Corollary 4.3.2 *For the equation (4.3.1) to be solvable in the form $X = P_2 P_4^- \hat{X} Q_3^- Q_1$, under the conditions (4.3.5) and (4.3.8) it is sufficient, and under the conditions (4.3.7) and (4.3.8) it is necessary that the matrix $\hat{X}$ must satisfy the equation (4.3.2).*

Consider the following variants of the system (4.3.3):

$$P_1 A P^{(2)} = L, \ Q^{(1)} B Q_2 = R, \ C = S_1 D S_2, \ P_1 Y Q_2 = \hat{Y}; \quad (4.3.11)$$

$$A P^{(2)} = P_3 L, \ Q^{(1)} B = R Q_4, \ C = S_1 D S_2, \ Y = P_3 \hat{Y} Q_4; \quad (4.3.12)$$

$$P_1 A = L P^{(4)}, \ B Q_2 = Q^{(3)} R, \ S_3 C S_4 = D, \ P_1 Y Q_2 = \hat{Y}; \quad (4.3.13)$$

$$A = P_3 L P^{(4)}, \ B = Q^{(3)} R Q_4, \ S_3 C S_4 = D, \ Y = P_3 \hat{Y} Q_4. \quad (4.3.14)$$

If one succeeds in constructing the system (4.3.11) or (4.3.12), then the solution of the equation (4.3.1) can be determined, according to (4.3.4), in the form $X = X(\hat{X})$, where $\hat{X}$ is a solution of the equation

(4.3.2). Similarly, using the systems (4.3.13) and (4.3.14), we have $\hat{X} = \hat{X}(X)$.

Note that each of the limitations (4.3.5)–(4.3.10) allows us to simplify the system (4.3.3). Thus, if the equalities (4.3.7), (4.3.8), and (4.3.10) hold true, then, proceeding from (4.3.3), we can construct a new transformation system of the form (4.3.11) by semi-inversion of matrices of full rank.

Study the solvability conditions of the equations (4.3.1) and (4.3.2), assuming that all matrices L_i and R_i in the system (4.3.3) simultaneously have quasitriangular structure:

$$L_i = \begin{bmatrix} L_{11}^{(i)} & \dots & 0 \\ \vdots & \ddots & \vdots \\ L_{\alpha 1}^{(i)} & \dots & L_{\alpha\alpha}^{(i)} \end{bmatrix}, \quad i = \overline{1, \hat{k}};$$
$$R_j = \begin{bmatrix} R_{11}^{(j)} & \dots & R_{1\beta}^{(j)} \\ \vdots & \ddots & \vdots \\ 0 & \dots & R_{\beta\beta}^{(j)} \end{bmatrix}, \quad j = \overline{1, \hat{s}}. \tag{4.3.15}$$

Denote the dimensions of the diagonal blocks $L_{tt}^{(i)}$ and $R_{\tau\tau}^{(j)}$ respectively by $l_{t1} \times l_{t2}$ and $r_{\tau 1} \times r_{\tau 2}$ ($t = \overline{1, \alpha}$, $\tau = \overline{1, \beta}$). Using the block form of the matrices

$$\hat{X} = \begin{bmatrix} \hat{X}_{11} & \dots & \hat{X}_{1\beta} \\ \dots & \dots & \dots \\ \hat{X}_{\alpha 1} & \dots & \hat{X}_{\alpha\beta} \end{bmatrix}, \quad \hat{Y} = \begin{bmatrix} \hat{Y}_{11} & \dots & \hat{Y}_{1\beta} \\ \dots & \dots & \dots \\ \hat{Y}_{\alpha 1} & \dots & \hat{Y}_{\alpha\beta} \end{bmatrix},$$

represent the equation (4.3.2) in the form of a system $\alpha\beta$ of matrix equations with respect to $\hat{X}_{t\tau}$:

$$K_{11}\hat{X}_{11} = \hat{Y}_{11}, \quad K_{t\tau}\hat{X}_{t\tau} + N_{t\tau}\hat{X} = \hat{Y}_{t\tau}, \quad t + \tau > 2, \tag{4.3.16}$$

where $\hat{X}_{t\tau}$ and $\hat{Y}_{t\tau}$ are blocks of the dimensions $l_{t2} \times r_{\tau 1}$, and $l_{t1} \times r_{\tau 2}$

respectively, $K_{t\tau}$¡ and $N_{t\tau}$ are linear operators determined by

$$K_{t\tau}\hat{X}_{t\tau} = \sum_{i=1}^{\hat{k}} \sum_{j=1}^{\hat{s}} d_{ij}\, L_{tt}^{(i)} \hat{X}_{t\tau} R_{\tau\tau}^{(j)},$$

$$N_{t\tau}\hat{X} = \sum_{\xi=1}^{t} \sum_{\zeta=1}^{\tau} \sum_{i=1}^{\hat{k}} \sum_{j=1}^{\hat{s}} d_{ij}\, L_{t\xi}^{(i)} \hat{X}_{\xi\zeta} R_{\zeta\tau}^{(j)}, \quad \xi + \zeta < t + \tau. \tag{4.3.17}$$

Let all diagonal blocks of the matrices (4.3.15) be square: $l_{t1} = l_{t2}$, $r_{\tau 1} = r_{\tau 2}$. The operators $N_{t\tau}$ only act on the blocks $\hat{X}_{\xi\zeta}$ of the matrix $\hat{X}$ for $\xi \leq t$, $\zeta \leq \tau$, and $\xi + \zeta \neq t + \tau$. This can be used in a recurrent procedure of exclusion of unknown systems (4.3.16), providing the proof of the following proposition.

Lemma 4.3.1 *The matrix equation (4.3.2) with quasitriangular coefficients (4.3.15) has a unique solution for any right-hand side if and only if all operators $K_{t\tau}$ are invertible.*

Represent the operator of the equation (4.3.2) in the form

$$\hat{M} = K + N, \tag{4.3.18}$$

where

$$K\hat{X} = \begin{bmatrix} K_{11}\hat{X}_{11} & \dots & K_{1\beta}\hat{X}_{1\beta} \\ \cdots & \cdots & \cdots \\ K_{\alpha 1}\hat{X}_{\alpha 1} & \dots & K_{\alpha\beta}\hat{X}_{\alpha\beta} \end{bmatrix},$$

$$N\hat{X} = \begin{bmatrix} 0 & N_{12}\hat{X} & \dots & N_{1\beta}\hat{X} \\ N_{21}\hat{X} & N_{22}\hat{X} & \dots & N_{2\beta}\hat{X} \\ \cdots & \cdots & \cdots & \cdots \\ N_{\alpha 1}\hat{X} & N_{\alpha 2}\hat{X} & \dots & N_{\alpha\beta}\hat{X} \end{bmatrix}.$$

According to Lemma 4.3.1, the conditions of invertibility of operators $\hat{M}$ and K coincide. In addition, the unique solution of the equation (4.3.2) has the form

$$\hat{X} = \hat{Y}_1 + N_1\hat{Y}_1 + \ldots + N_1^{\nu-1}\hat{Y}_1, \tag{4.3.19}$$

where

$$\nu = \alpha + \beta - 1, \quad N_1 = -K^{-1}N,$$

$$\hat{Y}_1 = K^{-1}\hat{Y} = \begin{bmatrix} K_{11}^{-1}\hat{Y}_{11} & \dots & K_{1\beta}^{-1}\hat{Y}_{1\beta} \\ \cdots & \cdots & \cdots \\ K_{\alpha 1}^{-1}\hat{Y}_{\alpha 1} & \dots & K_{\alpha\beta}^{-1}\hat{Y}_{\alpha\beta} \end{bmatrix}.$$

Indeed, the operators N and N_1 are nilpotent. Their indices of nilpotency coincide and do not exceed ν,

From (4.3.15) and (4.3.18) the following statement follows.

Lemma 4.3.2 *Under the conditions* (4.3.15) *the spectrum of the operator* $\hat{M}$ *consists of the eigenvalues of the operators* $K_{t\tau}$*:*

$$\sigma(\hat{M}) = \sigma(K) = \bigcup_{t,\tau} \sigma(K_{t\tau}). \tag{4.3.20}$$

Let the quasitriangular matrices (4.3.15) be triangular, i.e. $l_{t1} = l_{t2} = r_{\tau 1} = r_{\tau 2} = 1$. Then the actions of the operators K and K^{-1} give the *Schur products*:

$$K\hat{X} = \Omega \odot \hat{X}, \quad K^{-1}\hat{Y} = \Delta \odot \hat{Y}, \tag{4.3.21}$$

where

$$\Omega = \Sigma_l D \Sigma_r^T = \begin{bmatrix} \omega_{11} & \dots & \omega_{1\beta} \\ \cdots & \cdots & \cdots \\ \omega_{\alpha 1} & \dots & \omega_{\alpha\beta} \end{bmatrix},$$

$$\Delta = \begin{bmatrix} \dfrac{1}{\omega_{11}} & \dots & \dfrac{1}{\omega_{1\beta}} \\ \cdots & \cdots & \cdots \\ \dfrac{1}{\omega_{\alpha 1}} & \dots & \dfrac{1}{\omega_{\alpha\beta}} \end{bmatrix},$$

$$\Sigma_l = \begin{bmatrix} L_{11}^{(1)} & \dots & L_{11}^{(\hat{k})} \\ \cdots & \cdots & \cdots \\ L_{\alpha\alpha}^{(1)} & \dots & L_{\alpha\alpha}^{(\hat{k})} \end{bmatrix}, \quad \Sigma_r = \begin{bmatrix} R_{11}^{(1)} & \dots & R_{11}^{(\hat{s})} \\ \cdots & \cdots & \cdots \\ R_{\beta\beta}^{(1)} & \dots & R_{\beta\beta}^{(\hat{s})} \end{bmatrix}.$$

In this case the spectrum (4.3.20) is formed by elements of the matrix Ω, and the inequalities

$$w_{t\tau} = \sum_{i=1}^{\hat{k}} \sum_{j=1}^{\hat{s}} d_{ij}\, L_{tt}^{(i)} R_{\tau\tau}^{(j)} \neq 0, \quad t = \overline{1,\alpha}, \quad \tau = \overline{1,\beta}, \tag{4.3.22}$$

represent the criterion of one-valued solvability of the equation (4.3.2).

Solvability conditions and the solution of the original equation (4.3.1) can be obtained by using (4.3.19), (4.3.22), and the corollaries of Theorem 4.3.1 for different variants of the transformation system (4.3.3), in particular, (4.3.11)–(4.3.14).

4.4 Inertial Properties of Transformable Equations

Consider the class of matrix equations

$$MX = Y, \quad MX \triangleq \sum_{i,j=1}^{k} c_{ij}\, A_i X A_j^* \equiv A(C \otimes X)A^*, \qquad (4.4.1)$$

where C, X, and Y are Hermitian matrices of order k, n, and p respectively. The notation of a block matrix $A = [A_1, \dots, A_k]$ will be also used as a family of $p \times n$ matrices A_i, $i = 1, \dots, k$.

Study the properties of the operator M and the inertia of Hermitian solutions of the equation (4.4.1), using transformation systems of the type (4.3.11)–(4.3.14):

$$P_1 A P^{(2)} = L,\ C = S_1 D S_1^*,\ P_1 Y P_1^* = \hat{Y},\ X = P_2 \hat{X} P_2^*; \quad (4.4.2)$$

$$A P^{(2)} = P_3 L,\ C = S_1 D S_1^*,\ Y = P_3 \hat{Y} P_3^*,\ X = P_2 \hat{X} P_2^*; \quad (4.4.3)$$

$$P_1 A = L P^{(4)},\ S_3 C S_3^* = D,\ P_1 Y P_1^* = \hat{Y},\ P_4 X P_4^* = \hat{X}; \quad (4.4.4)$$

$$A = P_3 L P^{(4)},\ S_3 C S_3^* = D,\ Y = P_3 \hat{Y} P_3^*,\ P_4 X P_4^* = \hat{X}. \quad (4.4.5)$$

Suppose that there exists one of the transformation systems (4.4.2)–(4.4.5), giving a family of triangular matrices L:

$$L_i = \begin{bmatrix} l_{11}^{(i)} & 0 & \dots & 0 \\ l_{21}^{(i)} & l_{22}^{(i)} & \dots & 0 \\ \dots & \dots & \dots & \\ l_{\alpha 1}^{(i)} & l_{\alpha 2}^{(i)} & \dots & l_{\alpha\alpha}^{(i)} \end{bmatrix}, \quad i = 1, \dots, \hat{k}. \qquad (4.4.6)$$

At the same time $P^{(2)} = S_1 \otimes P_2$, $P^{(4)} = S_3 \otimes P_4$, and $P_1, \dots, P_4$ are some matrices of full rank α.

For each of the systems (4.4.2)–(4.4.5) determine a family of matrices

$$\mathcal{X} = \bigcup_{t,\tau=0}^{\alpha} \mathcal{X}_{t\tau}, \quad \mathcal{X}_{t\tau} = \{X : i_+(\hat{X}) = t,\, i_-(\hat{X}) = \tau\},$$

$$\mathcal{Y} = \bigcup_{t,\tau=0}^{\alpha} \mathcal{Y}_{t\tau}, \quad \mathcal{Y}_{t\tau} = \{Y : i_+(\hat{Y}) = t,\, i_-(\hat{Y}) = \tau\}.$$

Thus, if the systems (4.4.2) or (4.4.3) are used, then $\mathcal{X}$ is a set of Hermitian matrices representable in the form $X = P_2 \hat{X} P_2^*$. If $X \in \mathcal{X}_{t\tau}$, then $i_+(X) = t$ and $i_-(X) = \tau$, because P_2 is a matrix of full rank by column. For the systems (4.4.4) or (4.4.5) the relation $X \in \mathcal{X}_{t\tau}$ means that $\hat{X} = P_4 X P_4^*$ is a Hermitian matrix with indices of inertia $i_+(\hat{X}) = t \le i_+(X)$ and $i_-(\hat{X}) = \tau \le i_-(X)$. Similarly, the sets $\mathcal{Y}$ and $\mathcal{Y}_{t\tau}$ are described by using the matrices P_1 and P_3. Note that if $\alpha = n$ ($\alpha = p$), then in each case (4.4.2)–(4.4.5) the set $\mathcal{X}(\mathcal{Y})$ consists of all Hermitian $\alpha \times \alpha$ matrices, and $\mathcal{X}_{\alpha 0}$ ($\mathcal{Y}_{\alpha 0}$) is a subset of positive definite matrices. The sets $\mathcal{X}_{00}$ and $\mathcal{Y}_{00}$ are subspaces. In particular, for the systems (4.4.3) and (4.4.5), as well as (4.4.2) and (4.4.4) in the case $\alpha = p$, the subspace $\mathcal{Y}_{00}$ is zero.

Lemma 4.4.1 *The equality $\mathcal{Y} = M\mathcal{X} + \mathcal{Y}_{00}$ holds true if and only if*

$$\omega_{t\tau} = \sum_{i,j=1}^{\hat{k}} d_{ij}\, l_{tt}^{(i)}\, \overline{l_{\tau\tau}^{(j)}} \ne 0, \quad t, \tau = \overline{1, \alpha}. \tag{4.4.7}$$

Proof. Consider the relations

$$\hat{M}\hat{X} = \hat{Y}, \quad \hat{M}\hat{X} \triangleq \sum_{i,j=1}^{\hat{k}} d_{ij}\, L_i \hat{X} L_j^* \equiv L(D \otimes \hat{X})L^*, \tag{4.4.8}$$

where L is the family of triangular matrices (4.4.6). The spectrum of the operator $\hat{M}$ consists of α^2 numbers $\omega_{t\tau}$, and the inequalities (4.4.7) are equivalent to its invertibility. The operators M and $\hat{M}$ are connected by one of the relations

$$P_1\,(MX)\,P_1^* = \hat{M}\hat{X}, \quad MX = P_3\,(\hat{M}\hat{X})\,P_3^*. \tag{4.4.9}$$

The first (second) of them holds true for the systems (4.4.2) and (4.4.4) ((4.4.3) and (4.4.5)). Therefore $M\mathcal{X}+\mathcal{Y}_{00} \subseteq \mathcal{Y}$, in particular, $M\mathcal{X} \subseteq \mathcal{Y}$. The inverse inclusion $\mathcal{Y} \subseteq M\mathcal{X} + \mathcal{Y}_{00}$ means that each matrix $Y \in \mathcal{Y}$ is representable in the form

$$Y = MX + Y_0, \quad X \in \mathcal{X}, \quad Y_0 \in \mathcal{Y}_{00}, \tag{4.4.10}$$

and, owing to rank limitations on $P_1, \ldots, P_4$, equivalent to the solvability of the equation (4.4.8) for any right-hand side, i.e. to the inequalities (4.4.7).

The lemma is proved.

Lemma 4.4.2 *The operators $\hat{M}$ and $\hat{M}^{-1}$ are representable in the form*

$$\hat{M}\hat{X} = \sum_{t,\tau=1}^{\alpha} \sum_{i,j=1}^{t,\tau} \gamma_{t\tau}^{ij}\, E_{ti}\hat{X}E_{\tau j}^{*}, \tag{4.4.11}$$

$$\hat{M}^{-1}\hat{Y} = \sum_{t,\tau=1}^{\alpha} \sum_{i,j=1}^{t,\tau} \theta_{t\tau}^{ij}\, E_{ti}\hat{Y}E_{\tau j}^{*}, \tag{4.4.12}$$

where $\gamma_{t\tau}^{ij}$ and $\theta_{t\tau}^{ij}$ are scalar coefficients, E_{pq} is a matrix with a single nonzero element equal to 1 *and located at the intersection of the p-th row and the q-th column.*

Proof. Substituting the expansions of the triangular matrix coefficients

$$L_\xi = \sum_{t=1}^{\alpha} \sum_{i=1}^{t} l_{ti}^{(\xi)} E_{ti}, \quad \xi = 1, \ldots, \hat{k},$$

into (4.4.8), arrive at the expression (4.4.11) for the operator $\hat{M}$. At the same time

$$\gamma_{t\tau}^{ij} = \sum_{\xi,\zeta=1}^{\hat{k}} d_{\xi\zeta}\, l_{ti}^{(\xi)}\, \overline{l_{\tau j}^{(\zeta)}}, \quad i \le t, \quad j \le \tau. \tag{4.4.13}$$

For the inverse operator $\hat{M}^{-1}$ existing under the conditions (4.4.7) one can construct a similar expression (4.4.12). Indeed, all its matrix coefficients formed as a result of the use of the formulae (4.3.17)–(4.3.19) are sums or products of left triangular matrices and therefore also have triangular structure. For the calculation of scalar coefficients of the expansion (4.4.12) we have the system of linear recurrent relations

$$\gamma_{t\tau}^{t\tau}\,\theta_{t\tau}^{t\tau} = 1, \quad \sum_{\xi=i}^{t}\sum_{\zeta=j}^{\tau}\gamma_{\xi\zeta}^{ij}\,\theta_{t\tau}^{\xi\zeta} = 0, \tag{4.4.14}$$

$$i \le t, \quad j \le \tau, \quad i + j < t + \tau$$

following from the identity $\hat{M}^{-1}\hat{M}\hat{X} \equiv \hat{X}$. The inequalities (4.4.7) are the criterion of one-valued solvability of the system (4.4.14) for prescribed values of the coefficients of (4.4.13).

The lemma is proved.

From the scalar coefficients of the expansions (4.4.11) and (4.4.12) construct the block matrices

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \dots & \Gamma_{1\alpha} \\ \cdots & \cdots & \cdots \\ \Gamma_{\alpha 1} & \dots & \Gamma_{\alpha\alpha} \end{bmatrix}, \quad \Gamma_{t\tau} = \begin{bmatrix} \gamma_{t\tau}^{11} & \dots & \gamma_{t\tau}^{1\tau} \\ \cdots & \cdots & \cdots \\ \gamma_{t\tau}^{t1} & \dots & \gamma_{t\tau}^{t\tau} \end{bmatrix}, \tag{4.4.15}$$

$$\Theta = \begin{bmatrix} \Theta_{11} & \dots & \Theta_{1\alpha} \\ \cdots & \cdots & \cdots \\ \Theta_{\alpha 1} & \dots & \Theta_{\alpha\alpha} \end{bmatrix}, \quad \Theta_{t\tau} = \begin{bmatrix} \theta_{t\tau}^{11} & \dots & \theta_{t\tau}^{1\tau} \\ \cdots & \cdots & \cdots \\ \theta_{t\tau}^{t1} & \dots & \theta_{t\tau}^{t\tau} \end{bmatrix}, \tag{4.4.16}$$

Since $\gamma_{t\tau}^{t\tau} = \omega_{t\tau}$ and $\theta_{t\tau}^{t\tau} = 1/\omega_{t\tau}$, then in (4.4.15) and (4.4.16) one can find principal submatrices consisting of eigenvalues of the operators (4.4.11) and (4.4.12), of the form

$$\Omega = \begin{bmatrix} \omega_{11} & \cdots & \omega_{1\alpha} \\ \cdots & \cdots & \cdots \\ \omega_{\alpha 1} & \cdots & \omega_{\alpha\alpha} \end{bmatrix}, \quad \Delta = \begin{bmatrix} \dfrac{1}{\omega_{11}} & \cdots & \dfrac{1}{\omega_{1\alpha}} \\ \cdots & \cdots & \cdots \\ \dfrac{1}{\omega_{\alpha 1}} & \cdots & \dfrac{1}{\omega_{\alpha\alpha}} \end{bmatrix}. \tag{4.4.17}$$

Lemma 4.4.3 *Let the matrix* Γ *have exactly one positive eigenvalue:*

$$i_+(\Gamma) = 1. \tag{4.4.18}$$

Then the conditions (4.4.7) and the matrix inequality

$$\Theta \ge 0 \tag{4.4.19}$$

hold true if and only if all diagonal elements of the matrix Ω *are positive:*

$$\omega_{11} > 0, \quad \omega_{22} > 0, \ \ldots \ , \omega_{\alpha\alpha} > 0. \tag{4.4.20}$$

Proof. The matrix Θ is determined under the conditions (4.4.7), and Δ is its principal submatrix. Therefore from (4.4.19) it follows that $\Delta \geq 0$ and the inequalities (4.4.20) are true.

Let the relations (4.4.18) and (4.4.20) hold true. Then all elements of the matrix Ω are nonzero. Indeed, the indices of inertia $i_\pm$ of any principal submatrix do not exceed the respective indices of inertia of the whole matrix. In particular, the following inequalities hold true:

$$i_\pm(\Omega) \leq i_\pm(\Gamma), \quad i_\pm(\Delta) \leq i_\pm(\Theta). \tag{4.4.21}$$

If $\omega_{t\tau} = 0$ and $t \neq \tau$, then Ω contains a second-order positive definite principal submatrix located at the intersection of rows and columns with the numbers t and τ. This, in accordance with (4.4.21), contradicts the assumption (4.4.18).

Prove the inequality (4.4.19). Under the condition (4.4.18) the entries of the matrix (4.4.15) are expanded in the form

$$\gamma_{t\tau}^{ij} = u_{ti}^{(0)}\, \overline{u_{\tau j}^{(0)}} - \sum_s u_{ti}^{(s)}\, \overline{u_{\tau j}^{(s)}}, \quad i \leq t,\ j \leq \tau. \tag{4.4.22}$$

Substituting these expressions into (4.4.11), obtain

$$\hat{M} = \hat{M}_+ - \hat{M}_-, \quad \hat{M}_+\hat{X} = U_0\hat{X}U_0^*, \quad \hat{M}_-\hat{X} = \sum_s U_s\hat{X}U_s^*,$$

where U_s are left triangular matrices with elements $u_{ti}^{(s)}$ for $i \leq t$. From (4.4.20) and (4.4.22) it follows that all diagonal elements of the matrix U_0 are nonzero and the operator $\hat{M}_+$ is invertible. The eigenvalues of the operator $W = \hat{M}_-\hat{M}_+^{-1}$, taking into consideration (4.4.20), (4.4.22) and the Cauchy inequality, satisfy the conditions

$$w_{t\tau} = \sum_s \frac{u_{tt}^{(s)}\overline{u_{\tau\tau}^{(s)}}}{u_{tt}^{(0)}\overline{u_{\tau\tau}^{(0)}}}, \quad |w_{t\tau}|^2 \leq w_{tt}w_{\tau\tau} < 1.$$

Therefore for the operator $\hat{M}^{-1}$ it is possible to construct the expression

$$\hat{M}^{-1}\hat{Y} = U_0^{-1}\left(\sum_{s=0}^{\infty} W^s\hat{Y}\right)U_0^{-1*} = \sum_s V_s\hat{Y}V_s^*,$$

where V_s are left triangular matrices with the elements $v_{ti}^{(s)}$ for $i \leq t$. Reduce this expression to the form (4.4.12):

$$\hat{M}^{-1}\hat{Y} = \sum_{t,\tau=1}^{\alpha}\sum_{i,j=1}^{t,\tau} \breve{\theta}_{t\tau}^{ij}\, E_{ti}\hat{Y}E_{\tau j}^*, \quad \breve{\theta}_{t\tau}^{ij} = \sum_s v_{ti}^{(s)}\overline{v_{\tau j}^{(s)}}.$$

Here the coefficients $\breve{\theta}_{t\tau}^{ij}$ $(i \leq t, j \leq \tau)$ form the matrix $\breve{\Theta} = \breve{\Theta}^* \geq 0$. However, the coefficients of the expansion (4.4.12) are uniquely determined. Hence, $\Theta = \breve{\Theta} \geq 0$.

The lemma is proved.

Lemma 4.4.4 *If the equality*

$$i_+(\Omega) = 1, \tag{4.4.23}$$

holds true, then the conditions (4.4.7) and the matrix inequality

$$\Delta \geq 0 \tag{4.4.24}$$

are equivalent to the scalar inequalities (4.4.20).

This statement holds true for any Hermitian matrices of the form (4.4.17) and follows from Lemma 4.4.3 if and only if the matrices (4.4.6) are diagonal. In this case all the elements of the matrices (4.4.15) and (4.4.16), that do not belong to the respective submatrices (4.4.17), are zero and in (4.4.21) equality is obtained.

Lemma 4.4.5 *Let Δ and H be Hermitian matrices of the same dimensions. Then the inequality $\Delta \odot H \geq 0$ holds true for all $H \geq 0$ if and only if $\Delta \geq 0$. The strict inequality $\Delta \odot H > 0$ holds true for all $H > 0$ if and only if $\Delta \geq 0$ and all diagonal elements of the matrix Δ are positive.*

Proof. According to the theorem of the Schur product, from $\Delta \geq 0$, $H \geq 0$ $(\Delta > 0, H > 0)$ it follows that $\Delta \odot H \geq 0$ $(\Delta \odot H > 0)$.

Let $E_\varepsilon = E + \varepsilon I$, where $\varepsilon > 0$ is a small parameter, I is a unit matrix, and all elements of the matrix E are equal to 1. Apparently $E \odot \Delta = \Delta$ and $E_\varepsilon > 0$ for any $\varepsilon > 0$. If the inequality $\Delta \odot H \geq 0$ holds true for any matrix $H > 0$, then $\Delta \odot E_\varepsilon = = \Delta + \varepsilon\Delta \odot I \geq 0$, and for $\varepsilon \to 0$ we have $\Delta \geq 0$.

Let $H > 0$, $\Delta \geq 0$, and $\Delta \odot I > 0$. The latter means that all diagonal elements of the matrix Δ are positive. For a sufficiently small $\varepsilon > 0$ the inequality $H_\varepsilon = H - \varepsilon I > 0$ is true, and hence $\Delta \odot H = \Delta \odot H_\varepsilon + \Delta \odot I > 0$ is also true.

The lemma is proved.

Theorem 4.4.1 *If the inequalities*

$$\omega_{11} \neq 0, \quad \omega_{22} \neq 0, \ \dots \ , \omega_{\alpha\alpha} \neq 0, \tag{4.4.25}$$

hold true, then there exist matrices $X \in \mathcal{X}_{t\tau}$ and $Y \in \mathcal{Y}_{\alpha 0}$ satisfying the equation (4.4.1). At the same time t and τ coincide with the quantity of positive and negative numbers (4.4.25):

$$t = \sum_{s=1}^{\alpha} i_+(\omega_{ss}), \quad \tau = \sum_{s=1}^{\alpha} i_-(\omega_{ss}), \quad t + \tau = \alpha. \tag{4.4.26}$$

If there exist matrices $X \in \mathcal{X}_{t\tau}$ and $Y \in \mathcal{Y}_{\alpha 0}$ satisfying the conditions (4.4.10) under the limitations

$$i_+(\Gamma) \leq 1, \quad i_-(\Gamma) \leq 1, \tag{4.4.27}$$

then the relations (4.4.25) and (4.4.26) hold true.

Theorem 4.4.2 *If the inequalities (4.4.20) hold true, then there exist matrices $X \in \mathcal{X}_{\alpha 0}$ and $Y \in \mathcal{Y}_{\alpha 0}$ satisfying the equation (4.4.1). If there exist matrices $X \in \mathcal{X}_{\alpha 0}$ and $Y \in \mathcal{Y}_{\alpha 0}$ satisfying the conditions (4.4.10) under the limitation (4.4.18), then the inequalities (4.4.20) hold true.*

Theorem 4.4.3 *If the inequalities (4.4.7) and (4.4.19) hold true, then each matrix $Y \in \mathcal{Y}_{\alpha 0}$ is representable in the form (4.4.10) for $X \in \mathcal{X}_{\alpha 0}$, i.e.*

$$\mathcal{Y}_{\alpha 0} \subseteq M\mathcal{X}_{\alpha 0} + \mathcal{Y}_{00}. \tag{4.4.28}$$

The inequalities (4.4.7), (4.4.20), and (4.4.24) are the consequence of the inclusion (4.4.28).

Proof of Theorems 4.4.1–4.4.3. If

$$i_+(\hat{X}) = t, \quad i_-(\hat{X}) = \tau, \tag{4.4.29}$$

where $\hat{X}$ is a solution of the equation (4.4.8) for $\hat{Y} > 0$, then according to (4.4.9) there exist matrices $X \in \mathcal{X}_{t\tau}$ and $Y \in \mathcal{Y}_{\alpha 0}$ satisfying the equation (4.4.1). In particular, $X \in \mathcal{X}_{\alpha 0}$ and $\hat{X} > 0$. Conversely, if for some $X \in \mathcal{X}_{t\tau}$ and $Y \in \mathcal{Y}_{\alpha 0}$ the relations (4.4.10) hold true, in particular (4.4.1), then there exists a matrix $\hat{X}$ with the indices of inertia (4.4.29) such that $\hat{M}\hat{X} > 0$.

Let X_s and Y_s be sequential principal submatrices of order s of the respective matrices $\hat{X}$ and $\hat{Y}$ in the equation (4.4.8), $s = 1, \ldots, \alpha$. Taking into account the formula (4.4.11), obtain the relations

$$Y_s = \sum_{t,\tau=1}^{s} \sum_{i,j=1}^{t,\tau} \gamma_{t\tau}^{ij} E_{ti} X_s E_{\tau j}^*, \quad s = 1, \ldots, \alpha.$$

Here the matrices E_{ti} have the dimensions $s \times s$. According to Theorem 4.2.1,

$$i_+(Y_s) \le i_+(\Gamma_s \otimes X_s) \le i_+(\Gamma \otimes X_s) = i_+(\Gamma) i_+(X_s) + i_-(\Gamma) i_-(X_s),$$

where Γ_s is the principal submatrix of the matrix Γ, consisting of the blocks $\Gamma_{t\tau}$, $t \le s$, $\tau \le s$. The equalities $i_+(Y_s) = s$ hold true in all statements of Theorems 4.4.1–4.4.3 and in Theorems 4.4.2 and 4.4.3 $i_+(X_s) = s$. Under the conditions (4.4.27) we have the relations $s \le i_+(X_s) + i_-(X_s) = \operatorname{rank} X_s$. Therefore we only consider such matrices $\hat{X}$ that satisfy the equation (4.4.8) for $\hat{Y} > 0$ and whose all sequential principal minors are all nonzero.

Represent the submatrices X_s and Y_s in the form

$$\begin{aligned} X_s &= \begin{bmatrix} X_{s-1} & u_s^* \\ u_s & x_s \end{bmatrix} = \Psi_s \begin{bmatrix} X_{s-1} & 0 \\ 0 & \kappa_s \end{bmatrix} \Psi_s^*, \\ Y_s &= \begin{bmatrix} Y_{s-1} & v_s^* \\ v_s & y_s \end{bmatrix} = \Phi_s W_s \Phi_s^* + H_s, \end{aligned} \tag{4.4.30}$$

where

$$\Psi_s = \begin{bmatrix} I_{s-1} & 0 \\ u_s X_{s-1}^{-1} & 1 \end{bmatrix}, \ W_s = \Gamma_s \otimes \begin{bmatrix} X_{s-1} & 0 \\ 0 & 0 \end{bmatrix}, \ H_s = \begin{bmatrix} 0 & 0 \\ 0 & \kappa_s\, \omega_{ss} \end{bmatrix},$$

$$\Phi_s = [E_{11}\Psi_s, E_{21}\Psi_s, \ldots, E_{s1}\Psi_s, \ldots, E_{ss}\Psi_s], \quad \kappa_s = x_s - u_s X_{s-1}^{-1} u_s^*.$$

Note that all entries of the matrix Y_s, except y_s, do not depend on x_s. If $X_{s-1} > 0$, then $X_s > 0$ for $\kappa_s > 0$. Similarly, if $Y_{s-1} > 0$ and $y_s > v_s Y_{s-1}^{-1} v_s^*$, then $Y_s > 0$.

Under the conditions (4.4.25) we can select the entries x_s of the matrices X_s successively so that the inequalities $\kappa_s \omega_{ss} > 0$, $Y_s > 0$ and the equalities

$$i_\pm(X_1) = i_\pm(\omega_{11}), \ i_\pm(X_s) = i_\pm(X_{s-1}) + i_\pm(\omega_{ss}), \ s = \overline{2,\alpha}. \quad (4.4.31)$$

hold true. This means that there exists a matrix $\hat{X}$ with indices of inertia (4.4.29) such that $\hat{M}\hat{X} > 0$. In particular, $\hat{X} > 0$ under the conditions (4.4.20).

To prove the inverse statements of Theorems 4.4.1 and 4.4.2, also use the relations (4.4.30). Let $\hat{Y} = \hat{M}\hat{X} > 0$. Then under the conditions (4.4.27) the relations

$$s = i_+(Y_s) > i_+(W_s) = i_+(\Gamma_s)\, i_+(X_{s-1}) + i_-(\Gamma_s)\, i_-(X_{s-1}) \quad (4.4.32)$$

hold true, and taking into consideration the monotonicity of the numbers $i_+(\cdot)$ we have the inequalities $H_s \geq 0$ and $H_s \neq 0$ which mean that $\kappa_s \omega_{ss} > 0$. The signs of the numbers κ_s and ω_{ss} coincide, and the equalities (4.4.26), (4.4.29), and (4.4.31) hold true. If $\hat{X} > 0$, then the relations (4.4.32) are also true under the condition (4.4.18). In this case we similarly arrive at the inequalities (4.4.20).

Let us move on to the proof of Theorem 4.4.3 The inclusion (4.4.28) means that for any matrix $\hat{Y} > 0$ the equation (4.4.8) has a solution $\hat{X} > 0$. The inequalities (4.4.7) are equivalent to the invertibility of the operator (4.4.11). Using the spectral expansion of entries of the matrix Θ

$$\theta_{t\tau}^{ij} = \sum_{s=1}^{r} \sigma_s\, g_{ti}^{(s)}\, \overline{g_{\tau j}^{(s)}}, \quad \sigma_s \in \sigma(\Theta), \quad r = \operatorname{rank} \Theta,$$

transform the formula of the inverse operator (4.4.12) to the form

$$\hat{M}^{-1}\hat{Y} = \sum_{s=1}^{r} \sigma_s\, G_s \hat{Y} G_s^*, \tag{4.4.33}$$

where

$$G_s = \begin{bmatrix} g_{11}^{(s)} & 0 & \dots & 0 \\ g_{21}^{(s)} & g_{22}^{(s)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ g_{\alpha 1}^{(s)} & g_{\alpha 2}^{(s)} & \dots & g_{\alpha\alpha}^{(s)} \end{bmatrix}, \quad \mathrm{tr}(G_s G_q^*) = \delta_{sq} = \begin{cases} 1 & s = q \\ 0 & s \neq q \end{cases}.$$

The matrix inequality (4.4.19) is equivalent to the scalar inequalities $\sigma_s > 0$, $s = 1, \dots, r$. For any matrix $\hat{Y} \geq 0$ the equation (4.4.8) has a solution $\hat{X} = \hat{M}^{-1}\hat{Y} \geq 0$. Moreover, if $\hat{Y} > 0$, then for a sufficiently small $\varepsilon > 0$ the inequalities $\hat{X} \geq \varepsilon \hat{X}_0 > 0$ and $\hat{Y} > \varepsilon \hat{Y}_0$ hold true, where $\hat{X}_0 > 0$ and $\hat{Y}_0 = \hat{M}\hat{X}_0 > 0$ are some matrices existing under the conditions (4.4.20).

Show that the inequalities (4.4.7), (4.4.20), and (4.4.24) are the consequence of the inclusion (4.4.28). If the equation (4.4.8) has a solution for $\hat{Y} > 0$, then it is also solvable for any right-hand side $\hat{Y}$. Indeed, an arbitrary matrix $\hat{Y}$ and the respective solution $\hat{X}$ of the equation (4.4.8) can be represented in the form of linear combinations of Hermitian positive definite matrices (see the proof of Lemma 1.3.3). Therefore under the condition (4.4.28) the inequalities (4.4.7) hold true, and the operator $\hat{M}$ is invertible.

For any vectors $a = [a_1, \dots, a_\alpha]^T$ and $b = [b_1, \dots, b_\alpha]^T$, according to (4.4.33), the relations

$$b^* \hat{M}^{-1}(aa^*)b = \sum_{s=1}^{r} \sigma_s \, |\mathrm{tr}(G_s a b^*)|^2 \geq 0$$

hold true. In particular, if $b_t = \varepsilon^t$ and $a_t = \bar{c}_t/\varepsilon^t$, then for $\varepsilon \to 0$ we have $c^*\Delta c \geq 0$, where $c = [c_1, \dots, c_\alpha]^T$ is an arbitrary vector. Hence (4.4.28) implies (4.4.20) and (4.4.24).

Theorems 4.4.1–4.4.3 are proved.

Remark 4.4.1 In Lemmas 4.4.3, 4.4.4 and Theorems 4.4.1–4.4.3 instead of the conditions (4.4.18), (4.4.23), (4.4.27), similar limitations on the indices of inertia of the matrices C and D can be used.

The matrix (4.4.15) is representable in the form

$$\Gamma = ZDZ^*, \quad Z = \begin{bmatrix} Z_1 \\ \vdots \\ Z_\alpha \end{bmatrix}, \; Z_t = \begin{bmatrix} l_{t1}^{(1)} & \dots & l_{t1}^{(\hat{k})} \\ \cdots & \cdots & \cdots \\ l_{tt}^{(1)} & \dots & l_{tt}^{(\hat{k})} \end{bmatrix}, \; t = \overline{1, \alpha}.$$

Therefore the inequalities $i_\pm(\Gamma) \leq i_\pm(D)$ hold true. If the transformation systems (4.4.4) and (4.4.5) are used, then $i_\pm(D) \leq i_\pm(C)$. For the systems (4.4.2) and (4.4.3) the opposite inequalities hold true.

Remark 4.4.2 If all matrices (4.4.6) are diagonal, then the operators (4.4.11) and (4.4.12) reduce to Schur products of the form (4.3.21). In this case the limitations (4.4.27) and (4.4.18) used in Lemma 4.4.3 and Theorems 4.4.1 and 4.4.2 are equivalent to the relations $i_\pm(\Omega) \leq 1$ and $i_+(\Omega) = 1$ respectively, and in Theorem 4.4.3 following from Lemma 4.4.5 the inclusion (4.4.28) is equivalent to the system of inequalities (4.4.7) and (4.4.24). If in the expansion (4.4.12) $\theta_{t\tau}^{ij} = 0$ for $(t, i) \notin \sigma$ or $(\tau, j) \notin \sigma$, where

$$\sigma = \{(t, i)/ \max(t - 1, 1) \leq i \leq t \leq \alpha\},$$

then the matrices G_s in (4.4.33) have the left two-diagonal form, and the inclusion (4.4.28) is equivalent to the system of inequalities (4.4.7) and (4.4.19) (see Section 6.1).

4.5 Distribution of Property of a Matrix Collective

Let a family of $n \times m$ matrices A_λ be given, where $\lambda \in \Lambda$ is some set of indices. As A_λ we can take a scalar function uniquely determined on a scalar or vector set of parameters Λ. If Λ is a finite (countable) set, then we consider a finite (countable) set of matrices A_λ,

Determine the classes of matrix families, using the transformations

$$P_1 A_\lambda P_2 = L_\lambda \quad (\lambda \in \Lambda), \tag{4.5.1}$$

$$P_1 A_\lambda = L_\lambda P_4 \quad (\lambda \in \Lambda), \tag{4.5.2}$$

$$A_\lambda P_2 = P_3 L_\lambda \quad (\lambda \in \Lambda), \tag{4.5.3}$$

$$A_\lambda = P_3 L_\lambda P_4 \quad (\lambda \in \Lambda), \tag{4.5.4}$$

where $P_1, \ldots, P_4$ are matrices of full rank α, independent of λ, $L_\lambda \in C^{\alpha\times\alpha}$. The family A_λ is called a *collective of order* α, if there exist matrices $P_1 \in C^{\alpha\times n}$ and $P_2 \in C^{m\times\alpha}$ of full rank α such that all square matrices L_λ of the form (4.5.1) have triangular form of the same type (lower or upper one). In particular, if all matrices L_λ are diagonal, then A_λ is an ideal collective of order α. Similarly, by using the relations (4.5.2), (4.5.3), and (4.5.4), the *left*, the *right*, and the *neutral* collective of order α are respectively determined. All square matrices L_λ of order α also have triangular form and, owing to the rank limitations on the transformation matrices $P_3 \in C^{n\times\alpha}$ and $P_4 \in C^{\alpha\times m}$, are representable in the form (4.5.1). Vectors l_λ of order α, composed of diagonal elements (eigenvalues) of the triangular matrices L_λ, form the *property of the collective* A_λ,

Definitions of collectives by using the transformations (4.5.1)–(4.5.4) for $\alpha = n = m$ are equivalent. If $P_2 = P_1^{-1}$, then the collective A_λ of order n represents a family of matrices simultaneously reducible to triangular form by using the similarity transformation (4.5.1). In this case the property vectors l_λ consist of eigenvalues of the respective matrices A_λ.

See examples of matrix families that are collectives.

1. A family of analytic functions of the matrix $A_f = f(A)$. This collective is ideal if the matrix A has a simple structure. Using the Jordan form of the matrix A, one can construct property vectors l_f with the values of functions f on the spectrum of the matrix A serving as their components.

2. Families of pairwise commutative and *quasi-commutative* matrices:

$$A_k(A_iA_j - A_jA_i) = (A_iA_j - A_jA_i)A_k,\ \forall i, j, k.$$

Their properties with respect to the similarity transformation are the vectors composed of the eigenvalues of each matrix.

3. The analytic matrix function

$$A_\lambda = \sum_k f_k(\lambda)\, A_k, \quad \lambda \in C^1,$$

where f_k are scalar functions, and A_k is a given collective. In particular, the regular pencil of $n \times n$ matrices

$$A_\lambda = A - \lambda B, \quad \det A_\lambda \not\equiv 0,$$

is a collective of order n. Its property vectors can be constructed, proceeding from the canonical Kronecker form.

Consider the matrix equation (4.4.1) and assume that the family of matrix coefficients A is a collective of order $\alpha \leq \min\{n, p\}$. Then we can construct a transformation system (4.4.2) leading to the equation (4.4.8) with triangular coefficients (4.4.6). The scalar coefficients can be left unchanged, assuming $C = D$. If the collective A is left, right, or neutral, we will use the respective transformation systems (4.4.4), (4.4.3), or (4.4.5). The matrix Ω composed of eigenvalues ω_{ij} of the operator $\hat{M}$ is representable in the form

$$\Omega = \Sigma\, C\, \Sigma^*, \quad \Sigma = [l_1, \ldots, l_k],$$

where $l_t \in C^\alpha$ are property vectors of the collective A. Theorems 4.4.1–4.4.3 give the general technique of study and estimation of property elements of the collective A in terms of inertia of Hermitian solutions of the equation (4.4.8). See the corollaries of these theorems in the case $\alpha = n = p$.

Theorem 4.5.1 *The matrix inequality*

$$\sum_{i,j=1}^{k} c_{ij}\, A_i X A_j^* > 0 \tag{4.5.5}$$

is solvable if and only if $\omega_{ii} \neq 0$, $i = \overline{1,n}$. There exists a solution $X = X^*$ satisfying the relations

$$i_+(X) = \sum_{s=1}^{n} i_+(\omega_{ss}), \quad i_-(X) = \sum_{s=1}^{n} i_-(\omega_{ss}), \quad i_0(X) = 0. \tag{4.5.6}$$

If $X = X^*$ is an arbitrary solution of the matrix inequality (4.5.5) under the limitations $i_\pm(C) \leq 1$, then the relations (4.5.6) hold true.

Theorem 4.5.2 *If $\omega_{ii} > 0$, $i = \overline{1,n}$, then there exists a positive definite solution $X > 0$ of the matrix inequality (4.5.5). The converse proposition is true under the limitation $i_+(C) = 1$.*

Theorem 4.5.3 *The inequalities*

$$\left\| \frac{1}{\omega_{ij}} \right\|_1^n \geq 0, \quad \omega_{ij} \neq 0, \quad i,j = \overline{1,n}$$

are necessary and the relations

$$i_+(C) = 1, \quad \omega_{ii} > 0, \quad i = \overline{1,n}$$

are sufficient for the equation (4.4.1) to have a positive definite solution $X > 0$ for any positive definite right-hand side $Y > 0$.

These statements can be strengthened in the case of an ideal collective A (see Remark 4.4.2).

Example 4.5.1 Consider the equation (4.4.1) with the operator

$$MX = A_1 X A_2^* + A_2 X A_1^* + c\, A_3 X A_3^*,$$

$$A_1 = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & c \end{bmatrix},$$

where $c \neq -2$ and a are real parameters. The matrices A_i are not simultaneously reducible to triangular form through a similarity transformation, but form a neutral collective A of order 2:

$$PA_1 = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \quad PA_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad PA_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix},$$

$$\Omega = \begin{bmatrix} c+2 & 3 \\ 3 & 4 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1/(c+2) & 1/3 \\ 1/3 & 1/4 \end{bmatrix}.$$

The operators M and $\hat{M}$ are connected by the relation $\hat{M}X = P(MX)P^*$, and the matrices (4.4.15) and (4.4.16) have the form

$$\Gamma = \begin{bmatrix} c+2 & a & 3 \\ a & 0 & 2a \\ 3 & 2a & 4 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} 1/(c+2) & -a/(3c+6) & 1/3 \\ -a/(3c+6) & a^2/(3c+6) & -a/6 \\ 1/3 & -a/6 & 1/4 \end{bmatrix}.$$

The condition $c \neq -2$ is equivalent to the invertibility of the operator M and to the solvability of the inequality (4.5.5) as well.

Consider the following cases.

a) $a = 0$. In this case the collective A is ideal, and in Theorems 4.4.1, 4.4.2, 4.5.1, and 4.5.2 one can assume

$$X = \begin{bmatrix} c+2 & 0 \\ 0 & 4 \end{bmatrix}, \quad MX = \begin{bmatrix} 16 & 0 \\ 0 & (c+2)^2 \end{bmatrix} > 0.$$

The conditions $i_\pm(\Omega) \leq 1$, $i_+(\Omega) = 1$ and $\Delta \geq 0$ reduce to the respective inequalities $c \leq 1/4$, $c \leq 1/4$, and $-2 < c \leq 1/4$. If $c \leq 1/4$, then for any solution X of the inequality (4.5.5) $i_+(X) = i_+(c+2)+1$ and $i_-(X) = i_-(c+2)$. The inequality (4.5.5) has a solution $X > 0$ for $c > -2$. For any matrix $Y > 0$ the equation (4.4.1) has a solution $X > 0$ if and only if $-2 < c \leq 1/4$.

b) $a \neq 0$. The relations $i_\pm(\Gamma) \leq 1$, $i_+(\Gamma) = 1$ and $\Theta \geq 0$ are equivalent to the corresponding conditions $c = 0$, $c \leq 0$, and $-2 < c \leq 0$. In Theorems 4.4.1, 4.4.2, 4.5.1, and 4.5.2 one can assume

$$X = \begin{bmatrix} c+2 & 0 \\ 0 & \alpha \end{bmatrix}, \quad MX = \begin{bmatrix} 4\alpha & a(c+2) \\ a(c+2) & (c+2)^2 \end{bmatrix} > 0,$$

where $\alpha > a^2/4$. Here $i_+(X) = i_+(c+2)+1$, $i_-(X) = i_-(c+2)$. In particular, with $c = 0$ the arbitrary solution of the inequality (4.5.5) is positive definite. If $X > 0$ is a solution of (4.5.5) with $c \leq 0$, then it is necessary that $c > -2$. The equation (4.4.1) for any matrix $Y > 0$ has a solution $X > 0$ if and only if $-2 < c \leq 0$.

4.6 Construction of Solutions of Matrix Equations

1. Methods of reduction. The matrix equation (4.1.1) is equivalent to the system of linear algebraic equations

$$G\,x = y, \tag{4.6.1}$$

where

$$G = \sum_{i,j} c_{ij}\, A_i \otimes B_j^T, \quad x = [x_{1*}, \dots, x_{n*}]^T, \quad y = [y_{1*}, \dots, y_{p*}]^T.$$

In this case the vectors x and y are composed from elements of the matrices X and Y ordered by rows. It is possible to construct systems similar to (4.6.1) by using other methods of ordering the elements X and Y.

The criterion of compatibility of the system (4.6.1) and hence of the matrix equation (4.1.1) is the equality $\operatorname{rank}[G, y] = \operatorname{rank} G$. For any matrix $Y \in C^{p\times q}$ the equation (4.1.1) has a solution $X \in C^{n\times m}$ if and only if $\operatorname{rank} G = p\,q$. Here $x = G^- y$. In the case $p\,q = n\,m$, $\det G \neq 0$ we have the unique solution $x = G^{-1}y$.

2. Methods of transformations. The essence of transformation techniques is presented by Theorem 4.3.1 and its corollaries. For the matrix equation (4.3.1) construct the system (4.3.3) transforming it to a simpler form (4.3.2). If the matrix coefficients of the equation (4.3.2) have quasitriangular, in particular, triangular structure (4.3.15), then under the conditions of Lemma 4.3.1 the solution of the equation (4.3.2) is constructed in the form (4.3.19), and the solution of the original equation (4.3.1) is determined by using the relations (4.3.4).

For the class of matrix equations (4.4.1) one can use the transformation systems (4.4.2)–(4.4.5), and while constructing their solutions, use the formulae (4.4.12) and (4.4.14).

As an example take the binomial *Sylvester equation*

$$A_1XB_2 - A_2XB_1 = Y. \tag{4.6.2}$$

The operator of the left-hand side of this equation is representable in the form

$$MX = \frac{1}{w}\,[(a_1A_1 + a_2A_2)X(b_1B_1 + b_2B_2) -$$
$$-(b_1A_1 + b_2A_2)X(a_1B_1 + a_2B_2)],$$

where $w = a_1b_2 - a_2b_1 \neq 0$. In particular, obtain the representation

$$MX = \frac{1}{\lambda - \mu}\,[A(\lambda)XB(\mu) - A(\mu)XB(\lambda)], \quad \lambda \neq \mu, \tag{4.6.3}$$

where $A(\lambda) = A_1 - \lambda A_2$ and $B(\lambda) = B_1 - \lambda B_2$ are pencils of matrices of dimensions $p \times n$ and $m \times q$ respectively.

At the study of solvability conditions and at construction of algorithms of the solution of the equation (4.6.2) one can use the equivalent transformations of the matrix pencils $A(\lambda)$ and $B(\lambda)$ to the canonical form:

$$P_1 A(\lambda) P_2 = \hat{A}_1 - \lambda \hat{A}_2, \quad Q_1 B(\lambda) Q_2 = \hat{B}_1 - \lambda \hat{B}_2,$$

where P_1, P_2, Q_1, and Q_2 are square nonsingular matrices. As a result, instead of (4.6.2) it is necessary to solve a simpler equation

$$\hat{A}_1 \hat{X} \hat{B}_2 - \hat{A}_2 \hat{X} \hat{B}_1 = \hat{Y}, \tag{4.6.4}$$

where $X = P_2 \hat{X} Q_1$, $\hat{Y} = P_1 Y Q_2$,

The *canonical form* of an arbitrary *singular pencil* of matrices has the following structure:

$$\begin{bmatrix} J - \lambda I & 0 & 0 & 0 & 0 \\ 0 & I - \lambda N & 0 & 0 & 0 \\ 0 & 0 & U(\lambda) & 0 & 0 \\ 0 & 0 & 0 & V(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.6.5}$$

where the first two diagonal blocks form a regular kernel corresponding to the finite and infinite elementary divisors of the pencil, and the diagonal blocks of the block-triangular matrices $U(\lambda)$ and $V(\lambda)$ have the form

$$U_i(\lambda) = [0, I_{k_i}] + \lambda\, [I_{k_i}, 0], \quad V_j(\lambda) = [0, I_{s_j}]^T + \lambda\, [I_{s_j}, 0]^T.$$

Here I_r is a unit matrix of order r, 0 is a zero vector of appropriate dimensions and the numbers k_i, $i = \overline{1, t}$ ($s_j, j = \overline{1, \tau}$) coincide with nonzero *minimal indices* for the columns (rows) of the pencil. The dimensions of the nonzero diagonal block $0 \in C^{\xi \times \eta}$ in (4.6.5) are determined by the number $\eta(\xi)$ of its zero minimal indices for the columns (rows).

Denote the full set of minimal indices for the columns (rows) of a pencil $A(\lambda)$ by $\Delta(A)$ ($\Delta(A^T)$) and determine the set of points $\sigma_p(A) = \{\lambda : \operatorname{rank} A(\lambda) < p\}$, generalizing the concept of spectrum.

Theorem 4.6.1 *For any matrix $Y \in C^{p\times q}$ the equation (4.6.2) has a solution $X \in C^{n\times m}$ if and only if the following relations hold true:*

$$(\operatorname{rank} A_2 - p)(\operatorname{rank} B_2 - q) = 0, \tag{4.6.6}$$

$$\operatorname{rank}[A_1, A_2] = p, \quad \operatorname{rank}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = q, \tag{4.6.7}$$

$$\sigma_p(A) \cap \sigma_q(B) = \emptyset, \tag{4.6.8}$$

$$k \in \Delta(A), \quad s \in \Delta(B) \quad \Longrightarrow \quad k \le s, \tag{4.6.9}$$

$$k \in \Delta(A^T), \quad s \in \Delta(B^T) \quad \Longrightarrow \quad k \ge s. \tag{4.6.10}$$

The solution X is unique if $p\,q = n\,m$.

This statement is determined through reduction of the matrix pencils $A(\lambda)$ and $B(\lambda)$ to the canonical form (4.6.5) and the usage of the relations (4.6.3) and (4.6.4). The matrix equation (4.6.4) breaks out into independent matrix equations, the analysis of whose solvability leads to the relations (4.6.6)–(4.6.10). In particular, the inequalities $k \le s$ in (4.6.9) are equivalent to the solvability for any right-hand side of matrix equations of the type

$$[I_k, 0]\tilde{X}[0, I_s] - [0, I_k]\tilde{X}[I_s, 0] = \tilde{Y}$$

occurring in the presence of nonzero minimal indices for the rows of the pencils $A(\lambda)$ and $B(\lambda)$. The equalities (4.6.7) express the absence of zero minimal indices for the rows $A(\lambda)$ and the columns $B(\lambda)$.

Note that the criterion of one-valued solvability of the equation (4.6.2) (i.e. the invertibility of the operator M) is the fulfillment of one of the following requirements: a) the matrix pencils $A(\lambda)$ and $B(\lambda)$ are regular and satisfy the conditions (4.6.6)–(4.6.8); b) the matrix pencils $A(\lambda)$ and $B(\lambda)$ do not have regular kernels and minimal indices for rows, and all elements of the set $\Delta(A) \cup \Delta(B)$ are nonzero and coincide; c) the matrix pencils $A(\lambda)$ and $B(\lambda)$ do not have regular kernels and minimal indices for columns, and all elements of the set $\Delta(A^T) \cup \Delta(B^T)$ are nonzero and coincide. In each of the cases a), b), and c) the equality $pq = nm$ holds true. In the case of regular matrix pencils $A(\lambda)$ and $B(\lambda)$ the conditions

(4.6.7), (4.6.9), and (4.6.10) always hold true, and the relation (4.6.8) is a limitation on the spectra $\sigma(A) \cap \sigma(B) = \emptyset$.

The described technique of analysis and construction of solutions of the equation (4.6.2) is the generalization of techniques of the transformation type, based on the reduction of matrices to the Schur form by using orthogonal transformations.

3. Method of series. Consider the class of matrix equations

$$X - WX = Y, \quad \rho(W) < 1, \tag{4.6.11}$$

where $\rho(W)$ is the spectral radius of the operator $W : C^{n\times m} \to C^{n\times m}$. Solutions of such equations are represented in the form of a convergent series

$$X = \lim_{s\to\infty} X_s, \quad X_s = Y + WY + \ldots + W^{s-1}Y, \quad s = 1, 2, \ldots .$$

Direct computation of partial sums X_s for large n and m is ineffective. For faster construction of a solution X with given accuracy, one can use the following recurrent relations:

$$X_{s_0} = Y, \quad X_{s_{k+1}} = X_{s_k} + W^{s_k} X_{s_k}, \quad W^{s_{k+1}} = (W^{s_k})^2,$$

where $s_k = 2^k$, $k = 0, 1, \ldots$.

The method of series is preferable to that of reduction when it is required to save computer memory and computation time. The main weak point of the method of series is related to the limitation on the spectral radius of the operator W. For example, in the case of the Lyapunov equation for discrete systems in (4.6.11) assume $WX = AXA^*$, where the matrix A must be convergent, i.e. $\rho(A) < 1$.

4. Method of matrix sign-function. If the real parts of all eigenvalues of the matrices $A \in C^{n\times n}$ and $B \in C^{m\times m}$ are negative, then to solve the equation

$$-AX - XB = Y \tag{4.6.12}$$

one can use the method of *matrix sign-function*, which is based on the computation of sequences of matrices

$$A_{k+1} = \frac{1}{2}\,(A_k + A_k^{-1}), \quad B_{k+1} = \frac{1}{2}\,(B_k + B_k^{-1}),$$

$$Y_{k+1} = \frac{1}{2}\,(Y_k + A_k^{-1} Y_k B_k^{-1}),$$

where $A_0 = A$, $B_0 = B$, $Y_0 = Y$. Here the relations

$$-A_k X - X B_k = Y_k, \quad k = 0, 1, \dots\,,$$

$$X = \frac{1}{2} \lim_{k\to\infty} Y_k, \quad \mathrm{sgn} A = \lim_{k\to\infty} A_k, \quad \mathrm{sgn} B = \lim_{k\to\infty} B_k.$$

A matrix sign-function $\mathrm{sgn} A$ is determined under the condition of *dichotomy of the spectrum* $\sigma(A)$, i.e. the absence of eigenvalues on the imaginary axis. And

$$(\mathrm{sgn} A)\, x = \begin{cases} x, & x \in \mathcal{A}_+ \\ -x, & x \in \mathcal{A}_- \end{cases},$$

where $\mathcal{A}_+(\mathcal{A}_-)$ is an invariant subspace of the matrix A, corresponding to a part of spectrum in the right (left) half-plane. In our case, $\mathrm{sgn} A = -I_n$ and $\mathrm{sgn} B = -I_m$,

It should be noted that the speed of convergence of the method of matrix sign-function essentially depends on the closeness of the spectra of the matrices A and B to the imaginary axis.

5. Integral methods. In theoretical studies integral representations of solutions of matrix equations are used. Thus, in the theory of controllable and observable systems an important role is played by the integral of the form

$$X = \int_0^\infty e^{At} Y e^{Bt} dt, \tag{4.6.13}$$

which is a solution of the matrix equation (4.6.12). The matrices A and B must be stable. The solution of a more general class of equations

$$MX = Y, \tag{4.6.14}$$

where M is a linear operator with all its eigenvalues having positive real parts, is also representable in the integral form

$$X = \int_0^\infty Z(t) dt, \quad Z(t) = e^{-Mt} Y. \tag{4.6.15}$$

Here $Z(t)$ is a solution of the following Cauchy problem:

$$\dot{Z}(t) + MZ(t) = 0, \quad Z(0) = Y. \tag{4.6.16}$$

If M is an operator of the matrix equation (4.6.12), then the integral (4.6.15) is reducible to the form (4.6.13).

Let the operator M preserve the set of Hermitian matrices and the system (4.6.16) be *positive* with respect to the cone of nonnegative definite matrices $\mathcal{K}$, i.e. $Y = Y^* \geq 0 \implies Z(t) = Z(t)^* \geq 0$, $\forall t > 0$. This property of the system is equivalent to the positivity of the evolutional operator e^{-Mt} with respect to $\mathcal{K}$.

Theorem 4.6.2 *The positive system (4.6.16) is asymptotically stable if and only if for any matrix $Y = Y^* > 0$ the equation (4.6.14) has a unique solution $X = X^* > 0$.*

The statement of the necessity of this criterion follows from the integral representation (4.6.15) of the solution of the equation (4.6.14). The statement of sufficiency can be proved by using the generalized Frobenius theorem on spectral radius of a positive operator. Indeed, the spectral radius of the positive operator $f(M) = M^{-1}e^{-Mt}$ is a point of its spectrum, i.e.

$$\left| \frac{e^{-\lambda t}}{\lambda} \right| \leq \frac{e^{-at}}{a}, \quad \forall \lambda \in \sigma(M),$$

where $a > 0$ is the minimum real point of $\sigma(M)$. For this inequality to hold true for large values of $t > 0$ it is necessary that the spectrum of the operator M be located in the half-plane $\mathrm{Re}\lambda \geq a$ (asymptotic stability of the system (4.6.16) with a reserve a).

Note that for the class of operators

$$M = L - P, \quad LX = -A^*X - XA, \quad PX = \sum_{k=1}^{s} B_k^* X B_k, \tag{4.6.17}$$

the differential system (4.6.16) is positive. This fact is proved on the basis of the relations

$$e^{-Mt} = W(t) + t^3 R(t) = \lim_{k \to \infty} \left[W\left(\frac{t}{k}\right) \right]^k, \tag{4.6.18}$$

where $W(t) = \frac{1}{2}(e^{-Lt}e^{Pt} + e^{Pt}e^{-Lt})$, and $R(t)$ is an entire operator-function. In this case, the positivity of the operators

$$e^{-Lt}X = e^{A^*t}Xe^{At}, \quad e^{Pt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} P^k$$

and the closedness of the cone of positive operators implies the positivity of the operator e^{-Mt}. The relations (4.6.18) hold true for a wide class of bounded operators L and P.

The solution of the system (4.6.16) with the operator (4.6.17) can be considered as a second-moment matrix for the stochastic system (3.4.16). The mean-square asymptotic stability of this system is equivalent to the asymptotic stability of the system (4.6.16) and, according to Theorem 4.6.2, to the existence of a positive definite solution of the equation (4.6.14) for any positive definite right-hand side.

The positivity of the system (4.6.16) and the statements similar to Theorem 4.6.2 will be proved in Chapter 5 for more general classes of operators of the form

$$M = L - P, \quad P\mathcal{K} \subseteq \mathcal{K} \subseteq L\mathcal{K}, \quad \mathcal{K} \subseteq e^{Lt}\mathcal{K}, \quad \forall t \geq 0,$$

acting in partially ordered space with a cone $\mathcal{K}$.

If $\mathcal{K}$ is a cone of nonnegative matrices, then the positivity of the system (4.6.16), determined in the form $\mathcal{K} \subseteq e^{Mt}\mathcal{K}$, $\forall t \geq 0$, comes to the fact that all off-diagonal elements of the matrix G of the operator M are not positive. In this case the asymptotic stability of the system (4.6.16) is equivalent to the nonnegativity of the matrix G^{-1}, and also to the positivity of all principal leading minors of the matrix G.

4.7 Notes and References

4.1 Representation of linear matrix equations by using the Kronecker product can be found in Lancaster [1], Ikramov [1], and others. Theorem 4.1.1 and its Corollaries 4.1.1–4.1.4 are proved in Mazko [13, 16, 17, 19]. The usage of the semi-inversion operation during computation of the rank of a block matrix is also described in Korsukov

[1]. Theory of generalized inverse matrices is given in Zuhair Nashed [1].

4.2 Theorem 4.2.1 and its Corollaries 4.2.1–4.2.6 are given in Mazko [13, 16, 17, 19]. The statement of Corollary 4.2.5 is proved in Mazko, Kharitonov [1].

4.3 The method of transformations to binomial matrix equations was used in Ikramov [1]. The generalization of this method, formulated in the form of Theorem 4.3.1 and its corollaries, was obtained in Mazko [20, 21, 23].

4.4 In Hill [1] and Schneider [1] see the theory of inertia of a family of matrices simultaneously reducible to triangular form through similarity transformation. Lemmas 4.4.1–4.4.5 and Theorems 4.4.1–4.4.3 developing the inertia theory for the transformation systems (4.4.2)–(4.4.5) of the matrix equation (4.4.1) were obtained by Mazko [19–21, 23].

4.5 Matrix collectives generalizing the families of matrices simultaneously reducible to triangular form were introduced by Mazko [19]. The main properties of matrix equations constructed with the use of collections are formulated as Theorems 4.5.1–4.5.3.

4.6 The known methods of solving matrix equations are described in Lancaster [1], Ikramov [1], Afanasiev, Kolmanovskii, Nosov [1], Larin [1], Vetter [1], Vicente, Maite [1], Ran, Reurings [1]), and others. The estimation of solutions of operator equations on the basis of the Cayley transformation is available in Gavrilyuk, Makarov [1].

Note that the method of series was successfully used for the solution of the Lyapunov equation of large dimension at calculation of control systems of aerospace equipment of the USA and USSR (see, e.g., Afanasiev, Kolmanovskii, Nosov [1] and Larin [2]).

Spectral conditions of solvability of the equation (4.6.2) under some limitations on matrix coefficients are described in Ikramov [1], Korenevskii, Mazko [1] and Lewis, Mertzios [1]. Theorem 4.6.1 gives the criterion of solvability of this equation without any limitations. Concepts of minimal indices of a singular pencil of matrices and its canonical form are available in Gantmacher [1].

Theorem 4.6.2 follows from the integral representation of the solution of the equation (4.6.14) and the theorem of the spectral radius of a monotone operator (see Krein, Rutman [1] and Glazman, Lyubich [1]). The conditions of positiveness of linear differential systems with respect to a cone of nonnegative vectors follow from Fiedler, Pták [1] and Krasnoselskii, Lifshits, Sobolev [1]).

5

STABILITY OF DYNAMIC SYSTEMS IN PARTIALLY ORDERED SPACE

5.0 Introduction

This chapter is devoted to the study of dynamic systems in a partially ordered Banach space. The ordering relationship in a phase space is determined by some normal reproducing cone. In the considered stability problems for differential systems a given cone can be both constant and changing in time.

In Section 5.1 the main concepts of the theory of convex cones are given, and some classes of operators with respect to given cones of normalized space are described.

In Section 5.2 classes of positive and monotone dynamic systems with respect to given cones of phase space are defined. Conditions of positivity and monotonicity of linear and nonlinear systems are formulated by using the elements of a conjugate cone. Examples of differential and difference systems with the above properties are given.

In Section 5.3 stability conditions for linear positive systems are described in terms of positive and positively invertible operators. Autonomous and positively reducible systems are described, as well as systems with functionally commutative operators.

Classes of nonlinear systems that have properties like monotonicity with respect to two cones are considered in Section 5.4. Under additional limitations with respect to given cones, stability conditions for the considered systems are successfully formulated.

In Section 5.5 stability conditions for given families of differential systems (robust stability) are formulated that are determined in space by given cones and the corresponding operator inequalities.

In Section 5.6 the known technique of comparison of systems in a partially ordered space is developed, applied to the study of stability of complex nonlinear systems.

In Section 5.7 a generalized comparison principle of systems in a partially ordered space with a variable cone is given, as well as conditions of robust stability of a family of nonlinear systems.

5.1 Properties of Operators with Respect to Cones

A convex closed set $\mathcal{K}$ of a real normalized space $\mathcal{E}$ is called a *cone* if the following conditions hold true:

1) $X \in \mathcal{K},\ \alpha \geq 0 \implies \alpha X \in \mathcal{K}$;
2) $X \in \mathcal{K},\ Y \in \mathcal{K} \implies X + Y \in \mathcal{K}$;
3) $X \in \mathcal{K},\ -X \in \mathcal{K} \implies X = 0$.

The concept of a *wedge* is defined by the conditions 1) and 2) only. A wedge $\mathcal{K}$ is a cone if its *blade* $\mathcal{K} \cap -\mathcal{K}$ consists of a zero element.

A space containing the wedge, a cone in particular, is *partially ordered*. The inequality $X \geq Y (X > Y)$ means that $X - Y \in \mathcal{K}$ $(X - Y \in \mathcal{K}^0)$, where $\mathcal{K}^0$ is a set of interior points of $\mathcal{K}$. For a *solid wedge* $\mathcal{K}^0 \neq \emptyset$. If several cones are taken in a space, then, when using inequalities between the elements, we will indicate their generating cone. For example, $X \overset{\mathcal{K}}{\geq} Y$, $X \overset{\mathcal{K}}{>} Y$, etc.

Hereinafter we will use the properties of normal and reproducing cones. A cone $\mathcal{K}$ is *normal* if the norm in $\mathcal{E}$ is semi-monotone, i.e. from $0 \leq X \leq Y$ it follows that $\|X\| \leq \nu\|Y\|$, where ν is a universal constant independent of X and Y. Minimum of such numbers ν is *a normality constant of* $\mathcal{K}$. A wedge, in particular a cone $\mathcal{K}$ in a space $\mathcal{E}$ is *reproducing* if $\mathcal{E} = \mathcal{K} - \mathcal{K}$, i.e. any element $X \in \mathcal{E}$ is representable in the form $X = X_+ - X_-$, where $X_+ \geq 0$ and $X_- \geq 0$.

Let in the Banach spaces $\mathcal{E}$ and $\mathcal{E}_0$ the cones $\mathcal{K} \subset \mathcal{E}$ and $\mathcal{K}_0 \subset \mathcal{E}_0$ be taken, determining the corresponding ordering relationship between the elements. Let $M : \mathcal{E}_0 \to \mathcal{E}$ be a linear operator acting from $\mathcal{E}_0$ into $\mathcal{E}$. An operator M is called *monotone* if any $X, Y \in \mathcal{E}_0$ from $X \overset{\mathcal{K}_0}{\geq} Y$ implies $MX \overset{\mathcal{K}}{\geq} MY$.

The property of monotonicity of a linear operator is equivalent to its *positivity*: $X \overset{\mathcal{K}_0}{\geq} 0 \Longrightarrow MX \overset{\mathcal{K}}{\geq} 0$. A set of linear positive operators is a cone. The inequality between the operators $M \leq L$ means that the operator $L - M$ is positive. A positive operator $\hat{M}$ is called a *majorant* (*minorant*) of a positive operator M, if $M \leq \hat{M}(M \geq \hat{M})$. A positive operator M is called *extremal* if it cannot be represented in the form of a sum of linear independent minorants. All minorants of the extremal operator M have the form αM, $0 \leq \alpha \leq 1$. An operator M is called *strictly positive* (*strongly positive*) if $MX \overset{\mathcal{K}}{>} MY$ for $X \overset{\mathcal{K}_0}{>} Y$ ($X \overset{\mathcal{K}_0}{\geq} Y, X \neq Y$). Strongly positive (extremal) operators are inner (extreme) points of a solid cone of linear positive operators.

Note that an operator M in a Hilbert space is positive if and only if the conjugate operator M^* is positive. Similarly, the properties of strict and strong positivity must be true or not true simultaneously for the operators M and M^*. For the positive operator M to be strictly positive with respect to solid cones it is necessary and sufficient that for some $X_0 \overset{\mathcal{K}_0}{\geq} 0$ the inequality $MX_0 \overset{\mathcal{K}}{>} 0$ be true. Indeed, for any $X \overset{\mathcal{K}_0}{>} 0$ there exists $\varepsilon > 0$ such that $X \overset{\mathcal{K}_0}{\geq} \varepsilon X_0$, and hence $MX \overset{\mathcal{K}}{\geq} MX_0 \overset{\mathcal{K}}{>} 0$.

An operator M is called *positively invertible* if for any $Y \overset{\mathcal{K}}{\geq} 0$ the equation

$$MX = Y \tag{5.1.1}$$

has a solution $X \overset{\mathcal{K}_0}{\geq} 0$. If the cone $\mathcal{K}$ in the space $\mathcal{E}$ is reproducing, then the positively invertible operator is invertible.

Classes of positive, strictly positive, strongly positive, and positively invertible operators can be described in the form of the corresponding inclusions

$$M\mathcal{K}_0 \subseteq \mathcal{K}, \quad M\mathcal{K}_0^0 \subseteq \mathcal{K}^0, \quad M\mathcal{K}_0\backslash\{0\} \subseteq \mathcal{K}^0, \quad \mathcal{K} \subseteq M\mathcal{K}_0,$$

where $\mathcal{K}_0^0(\mathcal{K}^0)$ is a set of inner points of the cone $\mathcal{K}_0(\mathcal{K})$.

In the further study we will use the classes of positive and positively invertible operators with respect to normal and reproducing cones. Such operators, as in the case of matrices (Chapters 1-4), play an important role in problems of spectrum localization and stability analysis of dynamic systems.

We eliminate a class of linear operators representable in the form

$$M = L - P, \quad P\mathcal{K}_0 \subseteq \mathcal{K} \subseteq L\mathcal{K}_0. \tag{5.1.2}$$

If the cone $\mathcal{K}$ is normal and reproducing, then the criterion of positive invertibility of the operator (5.1.2) is the inequality

$$\rho(T) < 1, \tag{5.1.3}$$

where $\rho(T)$ is a spectral radius of the operator pencil $T(\lambda) = P - \lambda L$ (see Section 6.2). If the cone $\mathcal{K}$ is normal and solid, then the inequality (5.1.3) holds true if and only if for some $Y \overset{\mathcal{K}_0}{>} 0$ the equation (5.1.1) has a solution $X \overset{\mathcal{K}}{\geq} 0$.

In the study of the conditions of positive invertibility of operators one can use operator inequalities. For instance, if a cone $\mathcal{K}$ is normal and reproducing, and for a prescribed operator M the two-sided estimate $M_1 \leq M \leq M_2$ is true, then the positive invertibility of the operators M_1 and M_2 implies the positive invertibility of the operator M, and $M_2^{-1} \leq M^{-1} \leq M_1^{-1}$. In the case of a normal solid cone $\mathcal{K}$ the positive invertibility of the operator M under the conditions $M \leq L$ and $L^{-1} \geq 0$ is equivalent to the existence of $X \overset{\mathcal{K}_0}{\geq} 0$ and $Y \overset{\mathcal{K}}{>} 0$ in the equation (5.1.1).

The properties of monotonicity and positivity are similarly determined for the nonlinear operators $F : \mathcal{E}_0 \to \mathcal{E}$. Generally, the property of monotonicity of a nonlinear operator

$$X \overset{\mathcal{K}_0}{\geq} Y \implies F(X) \overset{\mathcal{K}}{\geq} F(Y) \tag{5.1.4}$$

does not add up to its positivity. The classes of monotone and positive operators, defined by given cones in $\mathcal{E}_0$ and $\mathcal{E}$, will be denoted respectively by $\mathcal{M}$ and $\mathcal{M}_0$. Furthermore, define classes of operators $\mathcal{M}_1^+$, $\mathcal{M}_2^+$, $\mathcal{M}_1^-$, and $\mathcal{M}_2^-$ that have the property (5.1.4) under the additional requirements $X \in \mathcal{K}_0$, $Y \in \mathcal{K}_0$, $Y \in -\mathcal{K}_0$ and $X \in -\mathcal{K}_0$ respectively. Obviously, $\mathcal{M} \subseteq \mathcal{M}_1^\pm \subseteq \mathcal{M}_2^\pm$. Operators of the class $\mathcal{M}_2^+$ $(\mathcal{M}_2^-)$ are *monotone in the cone* $\mathcal{K}_0$ $(-\mathcal{K}_0)$. If $F(X) \in \mathcal{K}$ for any $X \in \mathcal{E}_0$, then the operator F is *everywhere positive.*

5.2 Positive and Monotone Systems

Numerous real systems posses the properties of positivity and monotonicity. These properties should be taken into account and used in analysis and synthesis problems, in particular during the study of stability conditions and spectral characteristics, in numerical procedures of construction of solutions and appropriate control laws, etc.

In a Banach space $\mathcal{E}$ containing cones $\mathcal{K}_0$ and $\mathcal{K}$ consider an abstract dynamic system with continuous or discrete time $t \geq \theta$. Let $\Omega(t,t_0) : \mathcal{E} \to \mathcal{E}$ be some operator uniquely determining the transition of the system from the state X_0 into the state

$$X(t) = \Omega(t,t_0)X_0, \quad \Omega(t_0,t_0) = E, \quad t \geq t_0 \geq \theta. \tag{5.2.1}$$

where E is an identity operator. For the given points of time $t > t_0$ define the properties of (t,t_0)*–positivity* and (t,t_0)*–monotonicity* of the system in the form of the respective conditions

$$X(t_0) = X_0 \overset{\mathcal{K}_0}{\geq} 0 \implies X(t) \overset{\mathcal{K}}{\geq} 0,$$

$$X_1(t_0) = X_{10} \overset{\mathcal{K}_0}{\geq} X_2(t_0) = X_{20} \implies X_1(t) \overset{\mathcal{K}}{\geq} X_2(t),$$

where $X_k(t) = \Omega(t,t_0)X_{k0}$, $k = 1,2$. A system is called *positive* (*monotone*) if it is (t,t_0)–positive ((t,t_0)–monotone) for any $t > t_0 \geq \theta$.

The properties of positivity (monotonicity) of a system is equivalent to the positivity (monotonicity) of the operator $\Omega(t,t_0)$ for $t > t_0 \geq \theta$. In other words, to the operator $\Omega(t,t_0)$ of the class $\mathcal{M}_0$ ($\mathcal{M}$) with respect to cones $\mathcal{K}_0$ and $\mathcal{K}$ a *system of the class* $\mathcal{M}_0$ ($\mathcal{M}$) corresponds (see Section 5.1). Similarly, to the operator $\Omega(t,t_0)$ of the class $\mathcal{M}_k^{\pm}$ a *system of the class* $\mathcal{M}_k^{\pm}$ corresponds, $k = 1,2$. A cone $\mathcal{K}$ is an *invariant set* of a system positive with respect to $\mathcal{K}$ and of a corresponding positive operator $\Omega(t,t_0)$ (the case $\mathcal{K}_0 = \mathcal{K}$). A system of the class $\mathcal{M}_2^+$ ($\mathcal{M}_2^-$) is *monotone in* $\mathcal{K}_0$ ($-\mathcal{K}_0$). If $\Omega(t,t_0)$ is an operator of the class $\mathcal{M}_0$ ($\mathcal{M}$) with respect to one of the embedded cones $\mathcal{K}_0 \subseteq \mathcal{K}$, then the respective system belongs to the class $\mathcal{M}_0$ ($\mathcal{M}$) with respect to $\mathcal{K}_0$ and $\mathcal{K}$. A similar statement is true for systems of the classes $\mathcal{M}_1^{\pm}$ and $\mathcal{M}_2^{\pm}$.

Properties of positivity and monotonicity of control systems are similarly determined with respect to given cones in state and control spaces. If an operator $\Phi(t) : \mathcal{E}_0 \to \mathcal{E}$ determines the functioning of a system with input $U(t)$ and output $X(t) = \Phi(t)U(t)$, then its positivity (monotonicity) with respect to the cones $\mathcal{K}_0 \subset \mathcal{E}_0$ and $\mathcal{K} \subset \mathcal{E}$ is equivalent to the positivity (monotonicity) of the given control system. The operator $\Phi(t)$ can be prescribed explicitly or in the form of solutions of differential, difference, integro-differential and other types of systems.

We will show the properties of some classes of positive and monotone dynamic systems described by differential and difference equations, in the case $\mathcal{K}_0 = \mathcal{K}$, and provide a number of typical examples.

1. Consider a linear differential system

$$\dot{X}(t) + M(t)X(t) = G(t), \quad t \geq \theta, \tag{5.2.2}$$

where $M(t)$ is a linear bounded operator acting in a partially ordered Banach space $\mathcal{E}$ with a normal reproducing cone $\mathcal{K}$, and $G(t) \in \mathcal{E}$ is a prescribed function. Each initial condition $X(t_0) = X_0 \in \mathcal{E}$ for $t \geq t_0 \geq \theta$ determines the unique solution

$$X(t) = W(t, t_0)X_0 + \int_{t_0}^{t} W(t, s)G(s)ds, \tag{5.2.3}$$

where $W(t, s) = W(t, t_0)[W(s, t_0)]^{-1}$ is an *evolutionary operator* which is the unique solution of the Cauchy problem

$$\dot{W}(t) + M(t)W(t) = 0, \quad W(s) = E, \quad t \geq s, \tag{5.2.4}$$

where E is an identity operator. The linear operator $W(t, t_0)$ is developed as series

$$W(t, t_0) = E - \int_{t_0}^{t} M(t_1)\, dt_1 + \int_{t_0}^{t} M(t_2) \int_{t_0}^{t_2} M(t_1)\, dt_1 dt_2 - \cdots, \tag{5.2.5}$$

uniformly converging in operator norm.

According to (5.2.3), $X_0 \geq 0$ implies $X(t) \geq 0$ for $t > t_0$, if

$$W(t, t_0) \geq 0, \quad \int_{t_0}^{t} W(t, s)G(s)ds \geq 0. \tag{5.2.6}$$

Here the first inequality means the monotonicity of the operator with respect to the cone $\mathcal{K}$,, and the second one means the belonging of the function value to the given cone. The converse proposition is easily proved with consideration for the closedness of the cone $\mathcal{K}$.

Consequently, the system (5.2.2) is positive if and only if the relations (5.2.6) hold true for $t > t_0 \geq \theta$. Using (5.2.3) and (5.2.6), one can easily prove the equivalence of the following statements:

(a) the system (5.2.2) is (t, θ)–positive for any function $G(t) \geq 0$;
(b) the operator $W(t, s)$ is positive for $t \geq s \geq \theta$;
(c) the system (5.2.2) is positive;
(d) for any function $Z(t)$ satisfying the relations

$$\dot{Z}(t) + M(t)Z(t) \geq 0, \quad Z(\theta) = Z_\theta,$$

$Z_\theta \geq 0$ implies $Z(t) \geq 0$ for $t > \theta$.

If $G(t) \geq 0$, in particular, $G(t) \equiv 0$, then each of the statements (a)–(d) is equivalent to the positivity of the system (5.2.2). If $M(t) = M$ is a constant operator, then $W(t, s) = e^{-M(t-s)}$, and the positivity conditions for $W(t, s)$ and $W(t, \theta)$ for $t \geq s \geq \theta$ coincide.

Note the properties of an evolutionary operator, following from (5.2.4) and (5.2.5). For $t \geq s \geq \tau$ the relations

$$W(t, t) = E, \quad [W(t, s)]^{-1} = W(s, t), \quad W(t, \tau) = W(t, s)W(s, \tau)$$

hold true. If $M = M_1 + M_2$, then $W(t, s) = W_{M_1}(t, s)W_{M_3}(t, s)$, where $W_{M_1}(t, s)$ and $W_{M_3}(t, s)$ are evolutionary operators of systems of the type (5.2.2) with the respective operators $M_1(t)$ and $M_3(t) = W_{M_1}(s, t)M_2(t)W_{M_1}(t, s)$. The operator $W(t, s)$ is positive if such are the operators $W_{M_1}(t, s)$ and $W_{M_3}(t, s)$.

Lemma 5.2.1 *For an evolutionary operator $W(t, s)$ to be positive under $t \geq s \geq \theta$, it is necessary and sufficient that an exponential operator $e^{-M(t)h}$ be positive under $t \geq 0$ and $h \geq 0$.*

Proof. Use the procedure of representation of an operator $W(t, s)$ in the form of a so-called *multiplicative integral.* Splitting the segment $[s, t]$ by points $t_{kn} = s + kh_n$, where $h_n = (t - s)/n$, $k = 0, \ldots, n$, for large n we have

$$W(t, s) = W\left(t_{nn}, t_{n-1n}\right) W\left(t_{n-1n}, t_{n-2n}\right) \ \ldots \ W\left(t_{1n}, t_{0n}\right),$$

$$W\left(t_{kn}, t_{k-1n}\right) = e^{-M(\theta_{kn})h_n} + o(h_n), \quad k = 1, \ldots, n,$$

where $\theta_{kn} \in [t_{k-1n}, t_{kn}]$ are some intermediate points. Therefore

$$W(t,s) = \lim_{n\to\infty}\left[e^{-M(\theta_{nn})h_n} \quad \ldots \quad e^{-M(\theta_{1n})h_n}\right].$$

If $e^{-M(t)h} \geq 0$ for any $t \geq 0$ and $h \geq 0$, then the operator $W(t,s)$ is a limit of some sequence of positive operators and, due to the closedness of the cone of linear positive operators, must be positive.

The converse proposition is similarly proved on the basis of the relations

$$W\left(t, t - \frac{h}{n}\right) = e^{-M(\theta_n)\,\frac{h}{n}} + o\left(\frac{1}{n}\right),$$

$$e^{-M(t)h} = \lim_{n\to\infty}\left[W\left(t, t - \frac{h}{n}\right)\right]^n,$$

where $\theta_n \in [t - h/n, t]$, $n = 1, 2, \ldots$.

The lemma is proved.

Lemma 5.2.2 *If $M(t) = M_1(t) + M_2(t)$ and the operators $W_{M_1}(t,s)$ and $W_{M_2}(t,s)$ are positive for $t \geq s \geq \theta$, then the operator $W_M(t,s)$ is also positive for $t \geq s \geq \theta$.*

Proof. Use Lemma 5.2.1 to represent the exponential operator in the form

$$e^{-(M_1+M_2)h} \equiv R(h) - h^3 \sum_{k=3}^{\infty} \frac{(-h)^{k-3}}{k!}\, S_k,$$

where

$$R(h) = \frac{1}{2}\left(e^{-M_1 h} e^{-M_2 h} + e^{-M_2 h} e^{-M_1 h}\right),$$

$$S_k = (M_1 + M_2)^k - \frac{1}{2}\sum_{i=0}^{k} C_k^i \left(M_1^i M_2^{k-i} + M_2^i M_1^{k-i}\right).$$

For simplicity here the dependence of M_1 and M_2 from t is not indicated. Assuming $h = \tau/k$, obtain the relation

$$e^{-(M_1+M_2)\tau} = \lim_{k\to\infty}\left[R\left(\frac{\tau}{k}\right)\right]^k.$$

Taking into account Lemma 5.2.1 and the assumptions, hence follows the positivity of the operator $W_M(t,s)$ for $t \geq s \geq \theta$.

The lemma is proved.

The positivity of a system can be used for estimation of its solutions. If the functions $X_1(t)$ and $X_2(t)$ satisfy the inequalities

$$\dot{X}_1(t) + M(t)X_1(t) \leq G_1(t), \quad \dot{X}_2(t) + M(t)X_2(t) \geq G_2(t),$$

then under the conditions (5.2.6) and $Z_0 \geq 0$ the relations

$$X_2(t) - X_1(t) \geq W(t,t_0)Z_0 + \int_{t_0}^{t} W(t,s)G(s)ds \geq 0$$

hold true, where $Z_0 = X_2(t_0) - X_1(t_0)$, $G(t) = G_2(t) - G_1(t)$. Hence follows the next proposition.

Lemma 5.2.3 *Let $X(t)$ be a solution of the positive system (5.2.2), and let functions $X_1(t)$ and $X_1(t)$ satisfy the inequalities*

$$\dot{X}_1(t) + M(t)X_1(t) \leq \alpha_1 G(t), \quad \dot{X}_2(t) + M(t)X_2(t) \geq \alpha_2 G(t),$$

where $\alpha_1 \leq 1 \leq \alpha_2$. Then $X_{10} \leq X_0 \leq X_{20}$ implies

$$X_1(t) \leq X(t) \leq X_2(t) \;\; for \;\; t > t_0.$$

If $\alpha_1 = 0$, then in this statement the lower estimate $X_1(t)$ of the solution of the system (5.2.2) does not depend on its right-hand side $G(t)$. In the case $\alpha_1 = \alpha_2 = 1$ Lemma 5.2.3 holds true under the condition of positivity of the operator $W(t,s)$, $t \geq s \geq t_0$.

Let us give examples of linear positive systems with respect to cones of nonnegative vectors and nonnegative definite matrices.

Example 5.2.1 Consider the differential system

$$\dot{x} + A(t)x = g(t), \quad x \in R^n, \tag{5.2.7}$$

where $A(t)$ is a continuous matrix function of dimension $n \times n$. It is known that the positivity of an evolutionary operator of the system (5.2.7) with respect to the cone of nonnegative vectors

$$\mathcal{K} = \{x \in R^n : x_i \geq 0,\; i = 1, \dots, n\}$$

is equivalent to the off-diagonal non-positivity of the entries of the matrix $A(t)$. Consequently, the system (5.2.7) under the conditions

$$a_{ij}(t) \leq 0, \quad i \neq j, \quad g(t) \geq 0, \quad t \geq \theta,$$

is positive. For its solutions one can construct double-sided estimates, using Lemma 5.2.3.

Example 5.2.2 Consider the matrix differential Lyapunov equation

$$\dot{X} + A(t)X + XA^T(t) = Y(t), \quad X \in R^{n\times n}, \tag{5.2.8}$$

where $A(t)$ and $Y(t)$ are given matrix functions. In this case the operator of the system (5.2.2) and its evolutionary operator in consideration of (5.2.5) are determined in the form

$$M(t)X = A(t)X + XA^T(t), \quad W(t,s)X = W_A(t,s)XW_A^T(t,s),$$

where $W_A(t,s)$ is an evolutionary operator (*matrizant*) of the system (5.2.7). Obviously, the operator $W(t,s)$ is positive with respect to the cone of symmetric nonnegative definite matrices

$$\mathcal{K} = \{X \in R^{n\times n} \colon X = X^T \geq 0\}.$$

Therefore if $Y(t) = Y^T(t) \geq 0$ for $t \geq \theta$, then the matrix differential equation (5.2.8) is a positive system with respect to the prescribed cone $\mathcal{K}$.

2. Generalize the differential system (5.2.2) in the form

$$\dot{X} + M(t)X = G(X,t), \quad t \geq \theta, \tag{5.2.9}$$

where $G(X,t)$ is a nonlinear operator ensuring the existence and uniqueness of the solution $X(t)$ for $t \geq t_0 \geq \theta$, $X(t_0) = X_0$.

Let $\Omega(t,t_0)$ be a *shift operator* along the paths of the system (5.2.9), determining the transition from the state $X(t_0)$ into the state $X(t) = \Omega(t,t_0)X(t_0)$ for $t > t_0$. Then the property of positivity (monotonicity) of the system (5.2.9) with respect to the cone $\mathcal{K}$ is equivalent to the positivity (monotonicity) of the operator $\Omega(t,t_0)$ for any $t > t_0 \geq \theta$.

The solutions of the systems (5.2.9) satisfy the integral equation

$$X(t) = W(t,t_0)X_0 + \int_{t_0}^{t} W(t,s)G(X(s),s)\,ds, \tag{5.2.10}$$

where $W(t,s)$ is an evolutionary operator of the linear system (5.2.2). From (5.2.10) it follows, in particular, that the system (5.2.9) is positive if the operator $W(t,t_0)$ is positive and the operator-function $W(t,s)G(X,t)$ is positive for any $t \geq s \geq t_0$.

Formulate the condition of positivity and monotonicity for the system (5.2.9), using the conjugate cone of linear functionals

$$\mathcal{K}^* = \{\varphi \in \mathcal{E}^* : \varphi(X) \geq 0, \forall X \in \mathcal{K}\}.$$

Let $\mathcal{F}_0$ and $\mathcal{F}$ denote the families of continuous operator-functions $F(X,t)$ satisfying the following conditions for $t \geq \theta$:

$$X \geq 0,\ \varphi(X) = 0 \implies \varphi\left(F(X,t)\right) \geq 0, \tag{5.2.11}$$

$$X \geq Y,\ \varphi(X-Y) = 0 \implies \varphi\left(F(X,t) - F(Y,t)\right) \geq 0, \tag{5.2.12}$$

where $\varphi \in \mathcal{K}^*$. We will also determine the families of the operator-functions $\mathcal{F}_1^+$, $\mathcal{F}_2^+$, $\mathcal{F}_1^-$, and $\mathcal{F}_2^-$ which have the property (5.2.12) under the additional requirements $X \geq 0$, $Y \geq 0$, $Y \leq 0$, and $X \leq 0$ respectively. Obviously, $\mathcal{F} \subseteq \mathcal{F}_1^{\pm} \subseteq \mathcal{F}_2^{\pm}$.

Lemma 5.2.4 *If the cone $\mathcal{K}$ is solid, the evolutionary operator of the system (5.2.2) is positive, and $G \in \mathcal{F}_0$ ($G \in \mathcal{F}$), then the system (5.2.9) is positive (monotone). If the system (5.2.9) is positive (monotone), then $F \in \mathcal{F}_0$ ($F \in \mathcal{F}$), where $F(X,t) = G(X,t) - M(t)X$.*

Proof. Consider an auxiliary system

$$\dot{Z} = F(Z,t) + \varepsilon Q, \quad t \geq \theta,$$

where $\varepsilon > 0$, $Q > 0$ is an internal element $\mathcal{K}$. Let $Z(t)$ be its solution satisfying the conditions $Z(t_0) = Z_0 \geq 0$, $Z(\tau) = Z_\tau \in \partial\mathcal{K}$ be a point of the boundary of $\mathcal{K}$ for some $\tau \geq t_0 \geq \theta$. Then $\varphi(Z_\tau) = 0$ and $\varphi(Q) > 0$ for some $\varphi \in \mathcal{K}^*$ and $\varphi \neq 0$.

The positivity of the exponential operator $e^{-M(t)h}$ and the relation (see the proof of Lemma 5.2.1)

$$M(t)Z = \lim_{h \to 0+} \frac{1}{h}\left(Z - e^{-M(t)h}Z\right)$$

imply the inequality $\varphi\left(M(\tau)Z_\tau\right) \leq 0$. If, in addition, $G \in \mathcal{F}_0$, then $F \in \mathcal{F}_0$ and under the conditions of continuity for some $\delta > 0$ obtain the relations

$$\varphi\left(\dot{Z}(\tau)\right) = \varphi\left(F(Z_\tau, \tau)\right) + \varepsilon\varphi(Q) > 0,$$

$$\int_\tau^{\tau+\delta} \varphi\left(\dot{Z}(t)\right)dt = \varphi\left(Z(\tau+\delta)\right) > 0.$$

Consequently, the trajectory $Z(t)$ cannot exceed the bounds of the cone $\mathcal{K}$ at the time $t = \tau$, i.e. $Z(t) \geq 0$ for $\tau \leq t \leq \tau + \delta$. Otherwise, for some $\varphi \in \mathcal{K}^*$ and $\delta > 0$ the relations $\varphi\left(Z(\tau+\delta)\right) < 0$ and $\varphi\left(Z(\tau)\right) = 0$ must hold true which leads to contradiction. Due to the closedness of the cone, under $\varepsilon \to 0$ obtain $Z(t) \to X(t) \geq 0$ for any $Z_0 = X_0 \geq 0$ and $t \geq t_0$, i.e. the system (5.2.9) is positive.

The fact that the condition $F \in \mathcal{F}_0$ is necessary for the positive system (5.2.9) follows from the relations

$$\varphi\left(X(t_0+\delta)\right) = \delta\varphi\left(F(X(\xi),\xi)\right), \ \ \varphi(X_0) = 0,$$

where $X(t_0) = X_0 \in \partial\mathcal{K}$, $\varphi \in \mathcal{K}^*$, $t_0 < \xi < t_0 + \delta$, for sufficiently small values $\delta > 0$.

The formulated necessary and sufficient conditions of the monotonicity of the system (5.2.9) are proved analogously.

The lemma is proved.

From Lemma 5.2.4 it follows, in particular, that in the case of a solid cone $\mathcal{K}$ the positivity (monotonicity) of the differential systems $\dot{X} = F_1(X,t)$ and $\dot{X} = F_2(X,t)$ implies the positivity (monotonicity) of the differential system

$$\dot{X} = \alpha F_1(X,t) + \beta F_2(X,t), \quad \alpha \geq 0, \ \beta \geq 0.$$

It can be proved that the positive system $\dot{X} = F(X,t)$ with respect to the solid cone $\mathcal{K}$ $(-\mathcal{K})$ belongs to the classes $\mathcal{M}_1^+$ and $\mathcal{M}_2^+$ ($\mathcal{M}_1^-$ and $\mathcal{M}_2^-$) if and only if respectively $F \in \mathcal{F}_1^+$ and $F \in \mathcal{F}_2^+$ ($F \in \mathcal{F}_1^-$ and $F \in \mathcal{F}_2^-$).

A technique of construction of double-sided estimates with respect to a cone for the solutions of the class of monotone systems follows from the next proposition below.

Lemma 5.2.5 *Let $X(t)$ be a solution of the system (5.2.9), the operator $W(t,s)$ be positive with respect to the solid cone $\mathcal{K}$, $G \in \mathcal{F}$, and the functions $X_1(t)$ and $X_2(t)$ satisfy the relations*

$$\dot{X}_1 + M(t)X_1 \leq G(X_1,t), \quad \dot{X}_2 + M(t)X_2 \geq G(X_2,t), \quad t \geq t_0.$$

Then $X_{10} \leq X_0 \leq X_{20}$ implies $X_1(t) \leq X(t) \leq X_2(t)$ for $t \geq t_0$.

This proposition is proved by using the method of the proof of Lemma 5.2.4.

Example 5.2.3 Consider the nonlinear system

$$\dot{x} + A(t)x = g(x,t), \quad x \in R^n, \tag{5.2.13}$$

where $A(t)$ is a matrix with non-positive off-diagonal entries. Let $\mathcal{K} \subset R^n$ be a cone of nonnegative vectors. The system (5.2.13) is positive with respect to $\mathcal{K}$ and $-\mathcal{K}$ if the vector-function $g(x,t)$ satisfies the respective conditions

$$x \geq 0,\ x_i = 0 \implies g_i(x,t) \geq 0, \tag{5.2.14}$$

$$x \leq 0,\ x_i = 0 \implies g_i(x,t) \leq 0, \tag{5.2.15}$$

where $i = \overline{1,n}$, $t \geq \theta$. The system (5.2.13) is monotone if $g(x,t)$ is a *quasimonotone nondecreasing* with respect to x (Wazewski's condition):

$$x \geq y,\ x_i = y_i \implies g_i(x,t) \geq g_i(y,t), \tag{5.2.16}$$

where $i = \overline{1,n}$, $t \geq \theta$. The system (5.2.13) under the condition (5.2.14) ((5.2.15)) belongs to the classes $\mathcal{F}_1^+$ and $\mathcal{F}_2^+$ ($\mathcal{F}_1^-$ and $\mathcal{F}_2^-$) if (5.2.16) holds true under the auxiliary requirements $x \geq 0$ and $y \geq 0$ ($y \leq 0$ and $x \leq 0$). In particular, the system (5.2.13) is monotone in $\mathcal{K}$ if (5.2.14) and (5.2.16) hold true for $x \geq y \geq 0$.

Example 5.2.4 Consider the nonlinear control system with a dynamic feedback

$$\dot{x} = f(x,u,t), \quad \dot{u} = g(x,u,t), \tag{5.2.17}$$

where $x \in R^n$, $u \in R^1$, $t \geq \theta$, and assign the circular *Minkovsky cone*

$$\mathcal{K} = \left\{ z : \; z = \begin{bmatrix} x \\ u \end{bmatrix}, \; \|x\| \leq u \right\}. \tag{5.2.18}$$

This cone is normal, solid, and *self-conjugate*. The property of self-conjugacy of a cone means that $l^T z \geq 0, \; \forall z \in \mathcal{K} \iff l \in \mathcal{K}$. The criterion of the positivity of the system (5.2.17) can be presented, by using Lemma 5.2.4, in the form

$$x^T f(x,u,t) \leq u\, g(x,u,t), \quad u = \|x\|, \quad x \in R^n.$$

In the case of a linear system assume

$$f(x,u,t) = A(t)x + b(t)u, \quad g(x,u,t) = c^T(t)x + d(t)u,$$

and foregoing inequality can be presented as

$$z^T \left(M^T(t)\Delta + \Delta M(t) \right) z = z^T \left(M^T(t)\Delta + \Delta M(t) - \gamma(t)\Delta \right) z \geq 0,$$

where

$$M(t) = \begin{bmatrix} A(t) & b(t) \\ c^T(t) & d(t) \end{bmatrix}, \; \Delta = \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}, \; z = \begin{bmatrix} x \\ u \end{bmatrix}, \; u = \|x\|,$$

$\gamma(t)$ is any bounded function. Therefore each of the conditions

$$\frac{1}{2}\lambda_{max}\left(A_S(t)\right) + \|b(t) - c(t)\| \leq d(t),$$

$$M^T(t)\Delta + \Delta M(t) - \gamma(t)\Delta = \begin{bmatrix} \gamma(t)I - A_S(t) & c(t) - b(t) \\ c^T(t) - b^T(t) & 2d(t) - \gamma(t) \end{bmatrix} \geq 0,$$

where $A_S(t) = A(t) + A^T(t)$, and $\lambda_{max}(\cdot)$ is the maximal eigenvalue of a symmetric matrix, provides this system the properties of positivity and monotonicity with respect to the cone (5.2.18).

Example 5.2.5 Consider the matrix differential equation

$$\dot{X} + \mathcal{A}(X,t)X + X\mathcal{A}^T(X,t) = \mathcal{P}(X,t), \quad X = X^T \in R^{n\times n}, \tag{5.2.19}$$

where $\mathcal{A}$ and $\mathcal{P}$ are given operators. Let the operator $\mathcal{P}$ preserve the cone of symmetric nonnegative definite matrices $\mathcal{K} \subset R^{n\times n}$, and the equation (5.2.19) have a continuous solution $X(t)$ for $t \geq t_0$, $X(t_0) = X_0 \geq 0$. Then $\mathcal{K}$ is an invariant set of this equation. Indeed, for any functional $\varphi \in \mathcal{K}^*$ represented in the form $\varphi(X) = \mathrm{tr}(SX)$, where $S = HH^T \geq 0$, the equality $\varphi(X) = 0$ for $X \geq 0$ means that $XH = 0$, and, taking into consideration the permutation of matrices within the operation tr, obtain the relations

$$\varphi\left(\mathcal{A}(X,t)X + X\mathcal{A}^T(X,t)\right) = 0, \quad \varphi(\mathcal{P}(X,t)) \geq 0.$$

Basing on Lemma 5.2.4, we assert that the equation (5.2.19) is positive with respect to $\mathcal{K}$.

Let's reduce particular cases of the matrix equation (5.2.19).

The generalized Riccati equation of the form

$$\dot{X} + A(t)X + XA^T(t) = X\,\mathcal{R}(X,t)\,X$$

under the above mentioned assumptions and the condition of monotonicity of the operator $\mathcal{R}(X,t)$, is monotone with respect to the cone $\mathcal{K}$.

The linear matrix equation

$$\dot{X} + A(t)X + XA^T(t) = \sum_k B_k(t)XB_k^T(t)$$

is known as a second-moment equation for the stochastic Ito's system

$$dx(t) + A(t)x(t)\,dt = \sum_k B_k(t)x(t)\,dw_k(t),$$

where w_k are components of the standard Wiener process. The mean-square asymptotic stability of the solution $x \equiv 0$ of the given system is equivalent to the asymptotic stability by Lyapunov of the solution

$X \equiv 0$ of the second-moment matrix equation. To the linear operator $M(t) = L(t) - P(t)$, where

$$L(t)X = A(t)X + XA^T(t), \quad P(t)X = \sum_k B_k(t)XB_k^T(t),$$

an evolutionary operator $W(t,s)$ corresponds which is positive with respect to the cone $\mathcal{K}$ (see Example 5.2.2 and Lemma 5.2.2). Therefore the second-moment equation is positive and monotone with respect to $\mathcal{K}$.

3. Consider the discrete system

$$X_{k+1} = M_k X_k + G(X_k, k), \quad k = 0, 1, \dots , \tag{5.2.20}$$

where $M_k : \mathcal{E} \to \mathcal{E}$ are linear operators, and $G(X,k)$ is a nonlinear operator-function. Let $\mathcal{E}$ be a Banach space partially ordered by the cone $\mathcal{K}$. If $X_0 \in \mathcal{K}$ entails $X_k \in \mathcal{K}$ for any $k = 0, 1, \dots$, then the system (5.2.20) is positive.

Each solution of the system satisfies the relation

$$X_{k+1} = W_{k0}X_0 + \sum_{s=0}^{k} W_{ks+1}G(X_s, s),$$

where $W_{kk+1} = E$, $W_{ks} = M_k \cdots M_s$, $k \geq s$. Therefore the system (5.2.20) is positive with respect to $\mathcal{K}$ if the operators W_{k0} and $W_{ks+1}G(X,s)$ are positive for $k \geq s \geq 0$. In the general case, these conditions are not necessary for the positivity of the system (5.2.20). If $G(X,k) \equiv 0$, then the positivity of the system (5.2.20) is equivalent to the positivity of all operators W_{k0}, $k \geq 0$.

Example 5.2.6 Consider a discrete control system with a dynamic feedback

$$x_{k+1} = Ax_k + bu_k, \quad u_{k+1} = c^T x_k + du_k, \quad k = 0, 1, \dots , \tag{5.2.21}$$

where x_k is a state vector, u_k is a control. Rewrite it in the form

$$z_{k+1} = Mz_k, \quad M = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}, \quad z_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix},$$

and formulate the conditions of positivity with respect to the Minkovsky cone $\mathcal{K} \subset R^{n+1}$ of the form (5.2.18).

The positivity of the system (5.2.21) is equivalent to the inclusion $M\mathcal{K} \subseteq \mathcal{K}$ and, taking into account the self-conjugacy of the cone $\mathcal{K}$, adds up to the inequality $l^T Mz \geq 0$ which must hold true for any $l, z \in \mathcal{K}$. Using the Cauchy inequality, obtain the sufficient condition of the positivity of the system (5.2.21)

$$\sqrt{\lambda_{max}(A^T A)} + \|b\| + \|c\| \leq d.$$

The belonging of a vector to the cone $\mathcal{K}$ can be described in terms of nonnegative definite matrices:

$$z = \begin{bmatrix} x \\ u \end{bmatrix} \in \mathcal{K} \iff u \geq 0,\ u^2 I \geq x\,x^T \iff S(z) = \begin{bmatrix} uI & x \\ x^T & u \end{bmatrix} \geq 0.$$

Therefore the positivity of the system (5.2.21) is equivalent to the condition

$$S(z) \geq 0 \quad \Longrightarrow \quad S(Mz) = \begin{bmatrix} (c^T x + du)I & Ax + bu \\ x^T A^T + ub^T & c^T x + du \end{bmatrix} \geq 0.$$

For any vectors $z \in \mathcal{K}$ and $g \in R^{n+1}$ the inequality $g^T S(Mz) g = l_g^T z \geq 0$ must hold true, where

$$l_g = \begin{bmatrix} g^T S(h_1) g \\ \vdots \\ g^T S(h_n) g \\ g^T S(h) g \end{bmatrix}, \quad h_1 = \begin{bmatrix} a_1 \\ c_1 \end{bmatrix}, \ldots, h_n = \begin{bmatrix} a_n \\ c_n \end{bmatrix}, \ h = \begin{bmatrix} b \\ d \end{bmatrix},$$

a_i are columns of the matrix A, and c_i are components of the vector c.

The condition $l_g \in \mathcal{K}$ equivalent to the inequality $S(l_g) \geq 0$ holds true if

$$S = \begin{bmatrix} S(h) & \cdots & 0 & S(h_1) \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & S(h) & S(h_n) \\ S(h_1) & \cdots & S(h_n) & S(h) \end{bmatrix} \geq 0,$$

where

$$S(h) = \begin{bmatrix} d\,I & b \\ b^T & d \end{bmatrix}, \quad S(h_i) = \begin{bmatrix} c_i\,I & a_i \\ a_i^T & c_i \end{bmatrix}, \quad i = 1, \dots, n.$$

If $h > 0$ is an inner point of $\mathcal{K}$, then the positivity conditions for the system (5.2.21) have the form

$$S(h) > 0, \quad S(h) \geq \sum_i S(h_i) S^{-1}(h) S(h_i).$$

The condition $h > 0$ means that $d > \sqrt{b^T b}$. If $d = \sqrt{b^T b} > 0$, then for the positivity of the system it is necessary that the relation $dc^T = b^T A$ holds true.

We rewrite the condition $l_g \in \mathcal{K}$ as

$$g^T S(h)\, g = w^T h \geq 0, \quad (g^T S(h)\, g)^2 - \sum_{i=1}^{n} (g^T S(h_i)\, g)^2 = w^T S w \geq 0,$$

where

$$g = \begin{bmatrix} y \\ v \end{bmatrix}, \quad w = \Phi(g) = \begin{bmatrix} 2vy \\ y^T y + v^2 \end{bmatrix},$$

$$S = hh^T - \sum_{i=1}^{n} h_i h_i^T = M \Delta M^T, \quad \Delta = \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}.$$

It is to be noted that the nonlinear transformation $\Phi : R^{n+1} \to R^{n+1}$ preserves the cone $\mathcal{K}$. Moreover, $\Phi(\mathcal{K}) = \mathcal{K}$ and $w^T S w \geq 0$ for any $w \in \mathcal{K}$ if and only if there is $\gamma \geq 0$ such that $S \geq \gamma \Delta$ (see Loewy, Schneider [1]).

Hence, necessary and sufficient for positivity of the system (5.2.21) with respect to the cone $\mathcal{K}$ is the existence of $\gamma \geq 0$ such that

$$M \Delta M^T \geq \gamma \Delta, \quad \|b\| \leq d. \tag{5.2.22}$$

We generalize this result for the ellipsoidal cone

$$\mathcal{K}(Q) = \{z \in R^{n+1} :\ z^T Q z \geq 0,\ \ z^T q \geq 0\}, \tag{5.2.23}$$

where Q is a symmetric matrix with inertia $i(Q) = \{1, n, 0\}$, and q is a unit eigenvector of Q corresponding to its positive eigenvalue.

Necessary and sufficient for positivity of the system (5.2.21) with respect to the cone $\mathcal{K}(Q)$ is the existence of $\gamma \geq 0$ such that

$$M^T Q M \geq \gamma Q, \quad q^T M q \geq 0, \quad q^T M Q^{-1} M^T q \geq 0. \tag{5.2.24}$$

If $Q = \Delta$, then the inequalities (5.2.24) reduce to the form (5.2.22).

5.3 Stability of Linear Positive Systems

The stability problem for the class of nonstationary system (5.2.2) without additional limitations is quite complex and has not yet been fully solved for applications. Among the well-known directions of stability analysis of such systems one can mark out the Lyapunov function method, theory of characteristic measures, method of comparison systems, theory of reducible and periodic systems, etc.

We will show that, under some additional conditions, the estimate of the asymptotic stability of the system (5.2.2) can add up to the solution of linear equations of the type (5.1.1) with positively invertible operator. One such condition is the existence of a cone in a phase space, with respect to which a given or some auxiliary system has the positivity property.

The stability analysis of the system (5.2.2) for any bounded function $G(t)$ adds up to the study of the conditions of stability of the zero solution of the homogeneous system

$$\dot{X} + M(t)X = 0. \tag{5.3.1}$$

The system (5.3.1) is *stable*, if each of its solutions

$$X(t) = W(t, t_0)X_0, \quad t \geq t_0, \quad X_0 = X(t_0),$$

is bounded. At the same time we have *asymptotic stability*, if $\|X(t)\| \to 0$ for $t \to \infty$. The properties of stability and asymptotic stability of the system (5.3.1) are equivalent to the respective conditions

$$\sup_{t \geq t_0} \|W(t, t_0)\| < \infty; \quad \|W(t, t_0)Y\| \to 0,\ Y \in \mathcal{E},\ t \to \infty.$$

The heterogeneous system (5.2.2) is stable (asymptotically stable) if and only if the corresponding homogeneous system (5.3.1) is stable (asymptotically stable). The system (5.3.1) is *exponentially stable* if each of its solutions with $\|X_0\| \leq \delta$ satisfies the estimate

$$\|X(t)\| \leq \beta e^{-\gamma(t-t_0)}\|X_0\|, \quad t \geq t_0, \tag{5.3.2}$$

where γ and β are positive constants independent of the selection of a solution. The exponential stability of the system (5.3.1) implies its asymptotic stability. For autonomous and periodic systems, the properties of exponential and asymptotic stability are equivalent.

Let a cone $\mathcal{K}$ be given in the phase space of the system (5.3.1). If the system (5.3.1) is asymptotically stable, then for any solution $X(t)$ the following relation is true:

$$\int_{t_0}^{\infty} M(t)X(t)dt = Y, \tag{5.3.3}$$

where $X(t_0) = Y$. If $Y \geq 0$ and the system (5.3.1) is positive then $X(t) \geq 0$ for $t \geq t_0$.

Consider a class of stationary systems

$$\dot{X} + MX = 0, \tag{5.3.4}$$

where M is a linear bounded operator, $\sigma(M) \neq \emptyset$. The evolutionary operator of the system (5.3.4) in consideration of (5.2.5) has the form

$$W(t,s) = \sum_{k=0}^{\infty} \frac{(s-t)^k}{k!} M^k = e^{-M(t-s)},$$

and (5.3.3) reduces to the form (5.1.1), where

$$X = \int_{t_0}^{\infty} X(t)\, dt, \quad X(t) = e^{-M(t-t_0)}Y.$$

The positivity of the system (5.3.4) is equivalent to the positivity of the exponential operator e^{-Mt} for $t \geq 0$. Therefore it is possible to use the theory of one-parameter positive semigroups.

Determine the bound of the increase of the operator exponent in the form

$$\gamma_M = \lim_{t\to\infty} \frac{1}{t} \ln \|e^{-Mt}\| < \infty.$$

From the theorem on mapping of the spectrum of bounded operators it follows that $\gamma_M = -\alpha_M$, where

$$\alpha_M = \inf\{\operatorname{Re}\lambda\colon\ \lambda \in \sigma(M)\}.$$

The spectral radius of a positive operator is a point of its spectrum. Therefore for the positive system (5.3.4) $\alpha_M \in \sigma(M)$.

Lemma 5.3.1 *If the system (5.3.4) is positive, then the operator $M+\gamma E$ is positively invertible if and only if $\gamma > \gamma_M$. If the operator $M+\gamma E$ is positively invertible for each $\gamma \geq \gamma_0$, then the system (5.3.4) is positive, and $\gamma_0 > \gamma_M$.*

Proof. If the system (5.3.4) is positive, then for any $\gamma > \gamma_M$ the relation

$$(M+\gamma E)^{-1} = \int_0^\infty e^{-\gamma t} e^{-Mt}\,dt \geq 0$$

holds true. Conversely, if the operator $M+\gamma E$ is positively invertible for any $\gamma \geq \gamma_0$, where γ_0 is some real number, then

$$e^{-Mt} = \lim_{k\to\infty}\left[\frac{k}{t}\left(M+\frac{k}{t}E\right)^{-1}\right]^k \geq 0, \quad t > 0.$$

We will show that for the positive system (5.3.4) the operator $M+\gamma E$ is not positively invertible for $\gamma \leq \gamma_M$. Suppose that for some numbers γ_1 and γ_2 such that $\gamma_1 < \gamma_M < \gamma_2$ the operators $M_1 = M+\gamma_1 E$ and $M_2 = M+\gamma_2 E$ are positively invertible. Then from the relation

$$M_1 \leq M+\gamma_M E \leq M_2$$

and the theorem on two-sided estimate of positively invertible operators it follows that the operator $M+\gamma_M E$ must also be positively invertible. However, this is contrary to the condition $\alpha_M = -\gamma_M \in \sigma(M)$.

Hence, under the condition of positivity of the system (5.3.4) the operator $M+\gamma E$ is positively invertible for $\gamma > \gamma_M$ only.

The lemma is proved.

Remark 5.3.1 From the theorems on mapping of the spectrum and on the spectral radius of a positive operator it follows that under the conditions $e^{-Mt} \geq 0$ and $(M-\alpha E)^{-1} \geq 0$ there exist such points of spectrum α_* and β_* of the operator M that for any $\lambda \in \sigma(M)$ the inequalities

$$e^{-\mathrm{Re}\lambda\, t} \leq e^{-\alpha_* t}, \quad \frac{e^{-\mathrm{Re}\lambda\, t}}{\mid \lambda - \alpha \mid} \leq \frac{e^{-\beta_* t}}{\beta_* - \alpha},$$

hold true. The right-hand sides of these inequalities take on real positive values for $t \geq 0$. From the first inequality (for sufficiently small $t < 2\pi/\rho(M)$) it follows that α_* is a real point of the spectrum, such that $\mathrm{Re}\,\lambda \geq \alpha_*, \forall \lambda \in \sigma(M)$. For the second inequality to hold true for any arbitrary large t and $\forall \lambda \in \sigma(M)$ it is necessary to assume $\beta_* = \alpha_* = \alpha_M > \alpha$.

Lemma 5.3.2 *If the operator $M - \alpha E$ is positively invertible for any $\alpha \leq \alpha_0$, then the spectrum of the operator M is located on the half-plane* $\mathrm{Re}\,\lambda > \alpha_0$.

Proof. From the invertibility of the operator $M - \alpha E$ for $\alpha \leq \alpha_0$ it follows that the operator M does not have real points of spectrum in the interval $(-\infty, \alpha_0]$. The spectral radius of the positive inverse operator $(M - \alpha E)^{-1}$ equals $1/(\alpha_* - \alpha)$, where α_* is a real point of the spectrum $\sigma(M)$ such that

$$|\lambda - \alpha| \geq \alpha_* - \alpha > 0, \quad \forall \lambda \in \sigma(M).$$

Here $\alpha_* > \alpha_0 \geq \alpha$ and α_* do not depend on α.

If $\mathrm{Re}\,\lambda \leq \alpha_0$ then it is possible to select such a value of α that the opposite inequality $|\lambda - \alpha| < \alpha_* - \alpha$ would be true. Consequently, $\mathrm{Re}\,\lambda > \alpha_0$ for $\lambda \in \sigma(M)$. Here α_* coincides with α_M.

The lemma is proved.

If $\alpha_M > 0$, then for any solution of the system (5.3.4) the estimate (5.3.2) holds true, where $0 < \gamma < \alpha_M$, i.e. the system (5.3.4) is exponentially stable. Conversely, if the system (5.3.4) is exponentially stable and positive then from the inequality (5.3.2) for the partial solution

$$X(t) = e^{-\alpha_M (t-t_0)} V, \quad V \neq 0,$$

it follows that $\alpha_M > 0$. Using Lemmas 5.3.1 and 5.3.2 respectively for $\gamma_0 = 0$ and $\alpha_0 = 0$, obtain the following result.

Theorem 5.3.1 *The positive system (5.3.4) is exponentially stable if and only if the operator M is positively invertible. If the operator $M + \gamma E$ is positively invertible for any $\gamma \geq 0$, then the system (5.3.4) is positive and exponentially stable.*

Note that the exponential stability of the system (5.3.4) follows from the positive invertibility of the two operators M and $M + \gamma_0 E$, where $\gamma_0 > 0$ is a sufficiently large number. Indeed, each operator $M + \gamma E$ for $\gamma \in [0, \gamma_0]$ must be positively invertible, and $|\lambda + \gamma_0| \geq \alpha_M + \gamma_0 > 0$, $\forall \lambda \in \sigma(M)$ (see the proofs of Lemmas 5.3.1 and 5.3.2). Hence for sufficiently large γ_0 the inequality $\alpha_M > 0$ follows for which the system (5.3.4) is exponentially stable.

Corollaries of Theorem 5.3.1 are the known criteria of asymptotic stability in the mean square of stochastic Ito's systems (see Section 3.4).

Example 5.3.1 Consider the system (5.3.4) in R^{n+1} with $(n+1) \times (n+1)$ matrix M and the ellipsoidal cone $\mathcal{K}(Q) \subset R^{n+1}$ of the form (5.2.23). The system is positive with respect to $\mathcal{K}(Q)$ if and only if there is $\alpha \in R^1$ such that

$$M^T Q + QM + \alpha Q \leq 0. \tag{5.3.5}$$

The inverse matrix M^{-1} preserves the cone $\mathcal{K}(Q)$ if and only if there is $\beta > 0$ such that (see Example 5.2.6)

$$M^T QM \leq \beta Q, \quad q^T M^{-1} q \geq 0, \quad q^T (M^T QM)^{-1} q \geq 0. \tag{5.3.6}$$

Hence Theorem 5.3.1 implies that the system (5.3.4) is exponentially stable and positive with respect to $\mathcal{K}(Q)$ if there are the matrix $Q = Q^T$ with the inertia $i(Q) = \{1, n, 0\}$ and the constants $\alpha \in R^1$ and $\beta > 0$ satisfying the inequalities (5.3.5) and (5.3.6).

Let us move on to the study of non-autonomous systems. The system (5.3.1) will be called *positively reducible*, if there exists a Lyapunov transformation $X(t) = Q(t)H(t)$ resulting in a positive autonomous system

$$\dot{H} + M_0 H = 0, \tag{5.3.7}$$

where M_0 is some constant operator. In this definition $Q(t)$ is a uniformly bounded differentiable operator which has a uniformly bounded inverse $Q^{-1}(t)$ and satisfies the operator differential equation

$$\dot{Q} + M(t)Q - QM_0 = 0.$$

The stability conditions of the systems (5.3.1) and (5.3.7) are equivalent.

Theorem 5.3.2 *The positive reducible system (5.3.1) is exponentially stable if and only if the operator M_0 is positively invertible.*

An important subclass of reducible systems of the form (5.3.1) includes *ω-periodic systems* for which

$$M(t+\omega) = M(t), \quad W(t+\omega) = W(t)W(\omega), \quad t \geq \theta,$$

where $W(t) = W(t,\theta)$. If the spectrum of a *monodromy operator* $W(\omega)$ does not surround zero, then the system (5.3.1) is reducible. The operator of the Lyapunov transformation has the form

$$Q(t) = W(t)e^{M_0 t}, \quad M_0 = -\frac{1}{\omega}\ln W(\omega).$$

Hence, the positive reducible ω–periodic system (5.3.1) is exponentially stable if and only if the operator M_0 is positively invertible.

Consider a subclass of systems (5.3.1) described by a *functionally commutative operator $M(t)$*, i.e.

$$M(t)M(\tau) = M(\tau)M(t), \quad \forall t, \tau \geq \theta. \tag{5.3.8}$$

In this case the evolutionary operator is determined by

$$W(t,s) = e^{-N(t,s)}, \quad N(t,s) = \int_s^t M(\tau)d\tau, \quad t \geq s. \tag{5.3.9}$$

Suppose that there exists a limiting bounded operator

$$M_0 = \lim_{t\to\infty} \frac{1}{\varphi(t)} \int_{t_0}^t M(\tau)d\tau, \tag{5.3.10}$$

where $\varphi(t) > 0$ is some function such that $\varphi(t) \to \infty$ while $t \to \infty$.

Theorem 5.3.3 *Let the conditions (5.3.8) be satisfied and the system (5.3.7) with the operator (5.3.10) be positive. Then the positive* *invertibility of the operator (5.3.10) implies the asymptotic stability of the system (5.3.1).*

Proof. From (5.3.8)–(5.3.10) follow the relations

$$M(t)N(t,\tau) = N(t,\tau)M(t), \quad M_0 N(t,t_0) = N(t,t_0)M_0,$$

$$M_0\Delta(t,t_0) = \Delta(t,t_0)M_0, \quad \Delta(t,t_0) = \frac{1}{\varphi(t)}N(t,t_0) - M_0,$$

and $\Delta(t,t_0) \to 0$ while $t \to \infty$. Therefore an arbitrary solution of the system (5.3.1) in consideration of (5.3.9) can be represented in the form

$$X(t) = e^{-\varphi(t)[M_0+\Delta(t,t_0)]}X_0 = e^{-\varphi(t)M_0}e^{-\varphi(t)\Delta(t,t_0)}X_0.$$

Let $\alpha_0 = \inf\{\mathrm{Re}\,\lambda : \lambda \in \sigma(M_0)\}$. Then, according to Lemma 5.3.2, $\alpha_0 > 0$ and for any positive number $\varepsilon < \alpha_0/2$ there exists t_1 such that $\|\Delta(t,t_0)\| < \varepsilon$ for $t > t_1$. The following estimate holds true:

$$\|X(t)\| \le \beta e^{-\varphi(t)(\alpha_0-\varepsilon)}e^{\varphi(t)\varepsilon}\|X_0\| = \beta e^{-\varphi(t)(\alpha_0-2\varepsilon)}\|X_0\|,$$

where $\beta > 0$ is some constant. Since $\varphi(t) \to \infty$ and $\alpha_0 > 2\varepsilon$, then $\|X(t)\| \to 0$ while $t \to \infty$. Consequently, the system (5.3.1) is asymptotically stable.

The theorem is proved.

Example 5.3.2 Consider a matrix system (5.3.1), assuming

$$M(t) = \begin{bmatrix} a(t) & -b(t) \\ -b(t) & a(t) \end{bmatrix},$$

where $a(t)$ and $b(t)$ are prescribed functions. Obviously, the matrix $M(t)$ satisfies the condition of functional commutativity (5.3.8).

Suppose that

$$\varphi(t) = \int_{t_0}^{t} b(s)ds \to \infty, \quad \frac{1}{\varphi(t)}\int_{t_0}^{t} a(s)ds \to \alpha, \quad t \to \infty.$$

Then the limiting matrix (5.3.10) has the form

$$M_0 = \begin{bmatrix} \alpha & -1 \\ -1 & \alpha \end{bmatrix}.$$

The autonomous system (5.3.7) with the matrix M_0 is positive with respect to a cone of nonnegative vectors (see Section 5.2). The condition of positive invertibility of the matrix M_0 adds up to the inequality $\alpha > 1$. According to Theorem 5.3.3, the initial system (5.3.1) is asymptotically stable.

5.4 Stability of Nonlinear Monotone Systems

Consider the autonomous differential system

$$\dot{x} + Ax = g(x), \quad x(t_0) = x_0 \in R^n, \tag{5.4.1}$$

where $A \in R^{n\times n}$ is a matrix with nonpositive off-diagonal elements, $g(x)$ a continuous vector-function satisfying Wazewski's conditions (5.2.16). Let $f(0) = 0$ and $x = 0$ be an isolated equilibrium state of the system (5.4.1). Suppose that a solution of the Cauchy problem (5.4.1) exists and is locally unique for any $x_0 \in R^n$.

It is known that under the above mentioned assumptions the system (5.4.1) has the property of monotonicity with respect to the cone of nonnegative vectors $\mathcal{K} \subset R^n$, and the following statement is true (see Martynyuk, Obolenskii [1]).

Theorem 5.4.1 *The solution $x \equiv 0$ of the system (5.4.1) is asymptotically stable in the cone $\mathcal{K}$ if and only if for some vector $x > 0$ the inequality $g(x) < Ax$ is true.*

The solution $x \equiv 0$ of the system (5.4.1) has the property of *stability in a cone* $\mathcal{K}$ if for any $\varepsilon > 0$ and $t_0 \geq \theta$ it is possible to indicate such $\delta > 0$ that

$$\|x_0\| \leq \delta,\ x_0 \in \mathcal{K} \quad \Longrightarrow \quad \|x(t)\| \leq \varepsilon,\ x(t) \in \mathcal{K},\ t > t_0.$$

If, in addition, for some $\delta_0 > 0$ $\|x_0\| \leq \delta_0$ implies $\|x(t)\| \to 0$ while $t \to \infty$, then the solution $x \equiv 0$ is asymptotically stable in $\mathcal{K}$.

The statement of Theorem 5.4.1 applies to some more general classes of systems and cones in the space R^n. Limitations of the type (5.2.12) are provided to the considered systems by their monotonicity with respect to the given cones.

We set out the generalized stability analysis technique for linear and nonlinear dynamic systems with their states $X(t) \in \mathcal{E}$ determined as (5.2.1). Suppose that $X(t) \equiv 0$ is an equilibrium state, i.e. $\Omega(t, t_0)0 \equiv 0$.

Let in a space $\mathcal{E}$ two cones $\mathcal{K}$ and $\mathcal{K}_0 \subseteq \mathcal{K}$ be given. The cone $\mathcal{K}$ must be normal, and the cone $\mathcal{K}_0$, reproducing.

Note that the normality of the cone $\mathcal{K}$ is equivalent to the condition

$$U \overset{\mathcal{K}}{\leq} X \overset{\mathcal{K}}{\leq} V \quad \Longrightarrow \quad \|X\| \leq \nu_1\|U\| + \nu_2\|V\|, \tag{5.4.2}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are universal constants. Indeed, this condition for $U = 0$ coincides with the definition of normality of the cone. If the cone is normal, then from $0 \overset{\mathcal{K}}{\leq} X - U \overset{\mathcal{K}}{\leq} V - U$, it follows that

$$\|X\| - \|U\| \leq \|X - U\| \leq \nu\|V - U\| \leq \nu\|V\| + \nu\|U\|$$

and in (5.4.2), in particular, one can assume $\nu_1 = \nu + 1$ and $\nu_2 = \nu$, where ν is a normality constant of $\mathcal{K}$.

The state $X \equiv 0$ of the system (5.2.1) will be called *stable from* $\mathcal{K}_0$ *into* $\mathcal{K}$, if for any $\varepsilon > 0$ and $t_0 \geq \theta$ it is possible to point out such $\delta > 0$ that $X_0 \in \mathcal{S}_\delta(\mathcal{K}_0)$ would imply $X(t) \in \mathcal{S}_\varepsilon(\mathcal{K})$ for $t > t_0$, where

$$\mathcal{S}_\varepsilon(\mathcal{K}) = \{X \in \mathcal{K} : \ \|X\| \leq \varepsilon\}.$$

If for some $\delta_0 > 0$ from $X_0 \in \mathcal{S}_{\delta_0}(\mathcal{K}_0)$ follows $\|X(t)\| \to 0$ while $t \to \infty$, then the state $X \equiv 0$ is asymptotically stable from $\mathcal{K}_0$ into $\mathcal{K}$.

If the system (5.2.1) is positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$, and its state $X \equiv 0$ is stable (asymptotically stable) by Lyapunov, it is stable (asymptotically stable) from $\mathcal{K}_0$ into $\mathcal{K}$.

Lemma 5.4.1 *If the following conditions are satisfied*

$$X_0 \overset{\mathcal{K}_0}{\geq} 0 \implies X(t) \overset{\mathcal{K}}{\geq} 0, \ \dot{X}(t) \overset{\mathcal{K}}{\leq} 0, \tag{5.4.3}$$

$$X_0 = X_+ - X_-, \ X_\pm \overset{\mathcal{K}_0}{\geq} 0 \implies -PX_-(t) \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} QX_+(t), \tag{5.4.4}$$

where $X_\pm(t) = \Omega(t,t_0)X_\pm$, $t \geq t_0$, P and Q are positive linear operators with respect to the cone $\mathcal{K}$, then the equilibrium state $X \equiv 0$ of the system (5.2.1) is stable by Lyapunov.

Proof. a) $X_0 \in \mathcal{K}_0$. According to the Lagrange theorem,

$$X(t) - X(t_0) = \dot{X}(\xi)(t - t_0), \quad \xi \in (t, t_0), \quad t > t_0.$$

Taking into account (5.4.3), we obtain the inequalities $0 \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} X_0$, hence follows $\|X(t)\| \leq \nu \|X_0\|$, where $\nu > 0$ is the normality constant of $\mathcal{K}$. Therefore for any $\varepsilon > 0$ from $\|X_0\| \leq \delta = \varepsilon/\nu$ follows $\|X(t)\| \leq \varepsilon$.

b) $X_0 \in \mathcal{E}$. Since the cone $\mathcal{K}_0$ is reproducing, then $X_0 = X_+ - X_-$, where $X_\pm \in \mathcal{K}_0$. There exists a universal constant $\gamma > 0$ such that $\|X_\pm\| \leq \gamma \|X_0\|$ (the property of *unflattenedness* of a cone).

Let $\varepsilon > 0$. Select $\delta_\pm$ in accordance with item a) so that $\|X_\pm\| \leq \delta_\pm$ implies $\|X_+(t)\| \leq \varepsilon/(2\nu_2 q)$ and $\|X_-(t)\| \leq \varepsilon/(2\nu_1 p)$, where $p = \|P\|$, $q = \|Q\|$. For this one can assume $\delta_+ = \varepsilon/(2\nu\nu_2 q)$ and $\delta_- = \varepsilon/(2\nu\nu_1 p)$. Here we use the fact that positive linear operators with respect to a normal reproducing cone are limited by norm. If $\|X_0\| \leq \delta$, where $\delta = min\{\delta_+, \delta_-\}/\gamma$, then, taking into consideration (5.4.2) and (5.4.4), obtain the inequality

$$\|X(t)\| \leq \nu_1 p \|X_-(t)\| + \nu_2 q \|X_+(t)\| \leq \varepsilon, \quad t \geq t_0.$$

Consequently, the zero state of the system is stable by Lyapunov.

The lemma is proved.

The condition (5.4.3) of Lemma 5.4.1 provides the property of stability from $\mathcal{K}_0$ into $\mathcal{K}$ of the zero state of the system (5.2.1). For the class of linear positive systems (5.3.1) the condition (5.4.4) follows, e.g., from the operator inequalities $P \geq E$ and $Q \geq E$ (with respect to $\mathcal{K}$), and the conditions (5.4.3) can be rewritten in terms of the evolutionary operator $W(t, t_0)$.

Theorem 5.4.2 *The state $X \equiv 0$ of the system of classes $\mathcal{M}_1^\pm$ is stable by Lyapunov if*

$$X_0 \in \pm\mathcal{K}_0 \implies \dot{X}(t) \in \mp\mathcal{K}, \quad t > t_0.$$

Proof. Use the properties of the operator $\Omega(t,t_0)$ belonging to the classes $\mathcal{M}_1^{\pm}$. From the condition $\Omega(t,t_0)0 \equiv 0$ it follows that $X(t) \in \pm\mathcal{K}$ for any $X_0 \in \pm\mathcal{K}_0$. If $X_0 \in \mathcal{K}_0$, then $0 \overset{\mathcal{K}}{\le} X(t) \overset{\mathcal{K}}{\le} X_0$ and hence $\|X(t)\| \le \nu\|X_0\|$ (see the proof of Lemma 5.4.1). This estimate also holds true in the case $X_0 \in -\mathcal{K}_0$, because in this case $0 \overset{\mathcal{K}}{\le} -X(t) \overset{\mathcal{K}}{\le} -X_0$. In a general case $X_0 = X_+ - X_- \in \mathcal{E}$ and, taking into account the properties of the operator $\Omega(t,t_0)$, we have the inequalities

$$\Omega(t,t_0)(-X_-) \overset{\mathcal{K}}{\le} X(t) \overset{\mathcal{K}}{\le} \Omega(t,t_0)X_+,$$

where $X_\pm \overset{\mathcal{K}_0}{\ge} 0$. Hence, taking into consideration (5.4.2) and the unflattenedness of the reproducing cone $\mathcal{K}_0$, obtain the estimate

$$\|X(t)\| \le \gamma\nu\,(\nu_1+\nu_2)\|X_0\|$$

guaranteeing the stability of the state $X \equiv 0$.

The theorem is proved.

Theorem 5.4.2 can be used for construction of stability conditions of the class of monotone differential systems (5.2.9) expressed in terms of operators $M(t)$ and $G(X,t)$. The conditions of Theorem 5.4.2 for the system (5.2.9) are the inequalities

$$G(X,t) \overset{\mathcal{K}}{\le} M(t)X, \quad G(X,t) \overset{\mathcal{K}}{\ge} M(t)X$$

that hold true over its solutions with initial values from $\mathcal{K}_0$ and $-\mathcal{K}_0$ respectively.

Under the conditions of Theorem 5.4.2 the state $X \equiv 0$ of the system (5.2.1) has the properties of stability from $\mathcal{K}_0$ into $\mathcal{K}$ and from $-\mathcal{K}_0$ into $-\mathcal{K}$. These properties ensure the stability by Lyapunov of the state $X \equiv 0$ of this system. The inverse statement is also true, because under the condition $\Omega(t,t_0)0 \equiv 0$ a system of the class $\mathcal{M}_1^+$ ($\mathcal{M}_1^-$) must be positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$ ($-\mathcal{K}_0$ and $-\mathcal{K}$).

Theorem 5.4.3 *The state $X \equiv 0$ of the system (5.2.1) belonging to the classes $\mathcal{M}_1^{\pm}$ is stable (asymptotically stable) by Lyapunov if* *and only if it is stable (asymptotically stable) from $\mathcal{K}_0$ into $\mathcal{K}$ and from $-\mathcal{K}_0$ into $-\mathcal{K}$.*

Note that for the class of linear systems, the properties of stability from $\mathcal{K}_0$ into $\mathcal{K}$ and from $-\mathcal{K}_0$ into $-\mathcal{K}$ of the zero state are equivalent.

For the study of stability and asymptotic properties of the system (5.2.1) one can use different estimates of the states $X(t)$ or the operator $\Omega(t, t_0)$ with respect to the cones $\mathcal{K}_0$ and $\mathcal{K}$. For example, if for any $X_0 \in \mathcal{E}$ the following relations are true

$$-\Psi(t, t_0)|X_0| \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} \Phi(t, t_0)|X_0|, \quad t \geq t_0, \tag{5.4.5}$$

where $X_0 = X_+ - X_-$, $|X_0| = X_+ + X_-$, $X_\pm \in \mathcal{K}_0$, $\Psi(t, t_0)$ and $\Phi(t, t_0)$ are uniformly bounded linear operators, then the stability of the zero state $X \equiv 0$ follows from the estimate

$$\|X(t)\| \leq 2\gamma(\nu_1\rho_1 + \nu_2\rho_2)\|X_0\|,$$

where $\rho_1 = \sup \|\Psi(t, t_0)\|$, $\rho_2 = \sup \|\Phi(t, t_0)\|$, which is proved by using (5.4.2), (5.4.5), and assumptions related to the cones $\mathcal{K}$ and $\mathcal{K}_0$. The similar proposition holds true under the condition

$$-\Psi(t, t_0)X_+ - \Phi(t, t_0)X_- \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} \Phi(t, t_0)X_+ + \Psi(t, t_0)X_-, \tag{5.4.6}$$

which, in the case of a linear operator $\Omega(t, t_0)$, is equivalent to the double-sided estimate

$$-\Psi(t, t_0) \leq \Omega(t, t_0) \leq \Phi(t, t_0), \quad t \geq t_0.$$

Note that under the positivity conditions the operators $\Psi(t, t_0)$ and $\Phi(t, t_0)$ in (5.4.5) and (5.4.6) must be bounded by norm.

Formulate a corollary of Lemma 5.4.1 for the linear system (5.3.1) in terms of an evolutionary operator $W(t, t_0)$.

Theorem 5.4.4 *If the conditions*

$$W(t, t_0)\mathcal{K}_0 \subseteq \mathcal{K}, \quad M(t)W(t, t_0)\mathcal{K}_0 \subseteq \mathcal{K}, \quad t > t_0, \tag{5.4.7}$$

are true, then the linear system (5.3.1) is stable.

Note that under the conditions (5.4.7) in the case $M(t) \equiv M$ and $\mathcal{K} \subseteq M\mathcal{K}_0$ the following systems of inclusions

$$\mathcal{K}_0 \subseteq M\mathcal{K}_0, \quad e^{-Mt}\mathcal{K}_0 \subseteq \mathcal{K}_0, \quad t > 0, \tag{5.4.8}$$

$$\mathcal{K} \subseteq M\mathcal{K}, \quad e^{-Mt}\mathcal{K} \subseteq \mathcal{K}, \quad t > 0, \tag{5.4.9}$$

hold true, and each of them, according to Theorem 5.3.1, provides the exponential stability to the stationary system (5.3.4). If $M\mathcal{K}_0 \subseteq \mathcal{K}$, in particular, $\mathcal{K} = M\mathcal{K}_0$, then the conditions (5.4.7) follow from (5.4.8) or (5.4.9).

5.5 Robust Stability of a Family of Systems

In applied research there occurs a problem of stability of a given family of dynamic systems described by differential or difference equations with indefinite parameters (*robust stability* problem). Kharitonov's theorem and its analogues present a solution of this problem for some classes of autonomous systems. For example, if the admissible region for the coefficients of characteristic polynomial of a differential system is a right parallelepiped

$$\{a \in R^n : \underline{a} \leq a \leq \overline{a}\},$$

then its robust asymptotic stability is equivalent to the asymptotic stability of some set of systems corresponding to the extreme values of the coefficients $\underline{a}$ and $\overline{a}$.

Consider a family of dynamic systems which is determined by the relations

$$\dot{X} + M(t)X = G(X,t), \quad t \geq \theta, \tag{5.5.1}$$

$$\underline{M}(t) \leq M(t) \leq \overline{M}(t), \tag{5.5.2}$$

$$\underline{G}(X,t) \overset{\mathcal{K}}{\leq} G(X,t) \overset{\mathcal{K}}{\leq} \overline{G}(X,t), \tag{5.5.3}$$

where

$$\underline{G}(X,t) = -M_1(t)X + G_1(t), \quad \overline{G}(X,t) = -M_2(t)X + G_2(t),$$

$\underline{M}(t), \overline{M}(t), M_1(t)$, and $M_2(t)$ are linear operators, $G_1(t)$ and $G_2(t)$ are given functions. Assume that the operators $\underline{G}(X,t)$ and $\overline{G}(X,t)$

are bounded, and the inequalities between the elements of the phase space $\mathcal{E}$ and the operator inequalities are determined with respect to the normal cone $\mathcal{K} \subset \mathcal{E}$. In each initial point of time $t_0 \geq \theta$ the inequalities in $\mathcal{E}$ are determined with respect to the reproducing cone $\mathcal{K}_0 \subset \mathcal{K}$.

The two-sided estimate (5.5.3) for the operator $G(X,t)$ in a general case must hold true in each point of the phase space $X \in \mathcal{E}$, where a solution of the system (5.5.1) is determined. If the positive solutions $X(t) \overset{\mathcal{K}}{\geq} 0$ are determined, then one can suppose that the inequalities (5.5.3) hold true for $X \in \mathcal{K}$.

In the family (5.5.1)–(5.5.3) take two systems

$$\dot{X}_1 + \left[\overline{M}(t) + M_1(t)\right] X_1 = G_1(t), \tag{5.5.4}$$

$$\dot{X}_2 + \left[\underline{M}(t) + M_2(t)\right] X_2 = G_2(t). \tag{5.5.5}$$

Lemma 5.5.1 *Let the evolutionary operator of the system (5.5.4) be positive, and the inequalities (5.5.3) hold true for $X \in \mathcal{K}$. Then the positive solutions $X(t) \overset{\mathcal{K}}{\geq} 0$ of each system of the family (5.5.1)–(5.5.3) are bounded by the respective solutions of the linear systems (5.5.4) and (5.5.5), i. e.*

$$X_{10} \overset{\mathcal{K}_0}{\leq} X_0 \overset{\mathcal{K}_0}{\leq} X_{20} \implies X_1(t) \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} X_2(t), \quad t > t_0.$$

Besides, the positivity of the system (5.5.4) implies the positivity of each system of the given family.

Proof. Subtracting (5.5.4) from (5.5.1) and (5.5.1) from (5.5.5) in consideration of (5.5.2) and (5.5.3) obtain the inequalities

$$\dot{H}_1(t) + \left[\overline{M}(t) + M_1(t)\right] H_1(t) \overset{\mathcal{K}}{\geq} \left[\overline{M}(t) - M(t)\right] X(t),$$

$$\dot{H}_1(t) + \left[M(t) + M_1(t)\right] H_1(t) \overset{\mathcal{K}}{\geq} \left[\overline{M}(t) - M(t)\right] X_1(t),$$

$$\dot{H}_2(t) + \left[M(t) + M_2(t)\right] H_2(t) \overset{\mathcal{K}}{\geq} \left[M(t) - \underline{M}(t)\right] X_2(t),$$

$$\dot{H}_2(t) + \left[\underline{M}(t) + M_2(t)\right] H_2(t) \overset{\mathcal{K}}{\geq} \left[M(t) - \underline{M}(t)\right] X(t),$$

where $H_1(t) = X(t) - X_1(t)$, $H_2(t) = X_2(t) - X(t)$. Here the relations

$$\overline{M}(t) + M_1(t) \geq M(t) + M_1(t) \geq M(t) + M_2(t) \geq \underline{M}(t) + M_2(t)$$

hold true. If the system (5.5.4) is positive, then its evolutionary operator $W_{\overline{M}+M_1}(t,s)$ should be positive. The positivity of the operator $W_{\overline{M}+M_1}(t,s)$ implies the positivity of the operators $W_{M+M_1}(t,s), W_{M+M_2}(t,s)$, and $W_{\underline{M}+M_2}(t,s)$ (see Section 5.2).

If $X(t) \overset{\mathcal{K}}{\geq} 0$ or $X_1(t) \overset{\mathcal{K}}{\geq} 0$, then $H_{10} \overset{\mathcal{K}_0}{\geq} 0$ implies $H_1(t) \overset{\mathcal{K}}{\geq} 0$, i.e. $X_1(t) \overset{\mathcal{K}}{\leq} X(t)$ for $t > t_0$. Similarly, if $X(t) \overset{\mathcal{K}}{\geq} 0$ or $X_2(t) \overset{\mathcal{K}}{\geq} 0$, then $H_{20} \overset{\mathcal{K}_0}{\geq} 0$ implies $H_2(t) \overset{\mathcal{K}}{\geq} 0$, i.e. $X(t) \overset{\mathcal{K}}{\leq} X_2(t)$ for $t > t_0$. Therefore the positiveness of the system (5.5.4) implies the positiveness of each system of the family (5.5.1)–(5.5.3). Since $X(t) \overset{\mathcal{K}}{\geq} 0$, the inequalities (5.5.3) above are only used for $X \in \mathcal{K}$.

The lemma is proved.

Lemma 5.5.1 can be used for construction of the robust stability conditions for a family of differential systems of the form (5.5.1), i.e. asymptotic stability of each system of the given family. As a consequence of Lemma 5.5.1, formulate the conditions of robust stability of the family of linear systems

$$\dot{X} + M(t)X = 0, \quad \underline{M}(t) \leq M(t) \leq \overline{M}(t). \tag{5.5.6}$$

In this case the systems (5.5.4) and (5.5.5) have the form

$$\dot{X}_1 + \overline{M}(t)X_1 = 0, \tag{5.5.7}$$

$$\dot{X}_2 + \underline{M}(t)X_2 = 0. \tag{5.5.8}$$

If the initial conditions of the systems (5.5.6) and (5.5.8) satisfy the inequalities $0 \overset{\mathcal{K}_0}{\leq} X_0 \overset{\mathcal{K}_0}{\leq} X_{20}$, then the positivity of the evolutionary operator of the system (5.5.7) implies $0 \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} X_2(t)$ for $t > t_0$. If, in addition, the system (5.5.8) is asymptotically stable, then, taking into consideration the normality of the cone $\mathcal{K}$, obtain $\|X(t)\| \to 0$ while $t \to \infty$. Solutions $X(t)$ have this property for any initial $X_0 \in \mathcal{E}$ if the cone $\mathcal{K}_0$ is reproducing. Here

$$X(t) = X_+(t) - X_-(t), \quad X_\pm(t) \overset{\mathcal{K}}{\geq} 0, \quad t \geq t_0,$$

where $X_{\pm}(t)$ are some functions taking values in $\mathcal{K}$ and being the solutions of the system (5.5.6) (due to the uniqueness of the solution of the Cauchy problem), for which the above reasoning is true.

Thus, obtain the following proposition.

Theorem 5.5.1 *If the system (5.5.8) is stable (asymptotically stable) and the system (5.5.7) is positive, then each system of the family (5.5.6) is stable (asymptotically stable) and positive.*

Note that for the family of stationary systems (5.5.6), the positive invertibility of the operators $\underline{M}$ and $\overline{M}$ implies the positive invertibility of each operator M from the segment $\underline{M} \leq M \leq \overline{M}$. From Theorems 5.3.1 and 5.5.1, in particular, it follows that for the positive invertibility of the operator M it is sufficient that the operator $e^{-\overline{M}t}$ be positive for $t \geq 0$ and the spectrum of the operator $\underline{M}$ be located in the half-plane $\operatorname{Re}\lambda > 0$.

Example 5.5.1 Consider the family of linear systems

$$\dot{x} + A(t)x = 0, \quad \underline{a}_{ij} \leq a_{ij}(t) \leq \left\{ \begin{array}{cc} 0, & i \neq j \\ \overline{a}_{ij}(t), & i = j \end{array} \right., \quad t \geq \theta,$$

where $\underline{a}_{ij}$ are entries of an off-diagonal non-positive matrix with positive principal leading minors, and $\overline{a}_{ij}(t)$ are given continuous functions. The system $\dot{x}_1 + \overline{A}(t)x_1 = 0$ with the diagonal matrix $\overline{A}(t)$ is positive with respect to the cone of nonnegative vectors $\mathcal{K} \subset R^n$, and the system $\dot{x}_2 + \underline{A}x_2 = 0$ is asymptotically stable. Therefore each system of this family is asymptotically stable and positive with respect to $\mathcal{K}$. Here $A^{-1}(t) \geq 0$ for any matrix $A(t)$ from the interval $\underline{A} \leq A(t) \leq \overline{A}(t)$.

Generalize the described technique of analysis of robust stability for a family of nonlinear systems

$$\dot{X} = F(X,t), \quad F(0,t) \equiv 0, \quad t \geq \theta, \tag{5.5.9}$$

$$\underline{F}(X,t) \overset{\mathcal{K}}{\leq} F(X,t) \overset{\mathcal{K}}{\leq} \overline{F}(X,t), \quad X \in \mathcal{E}, \quad t \geq \theta. \tag{5.5.10}$$

Suppose that the right-hand sides of the systems are continuous and satisfy the conditions of existence and uniqueness of solutions for $t \geq t_0 \geq \theta$.

Let $\overline{\mathcal{F}}$ denote a family of operator-functions $F(X,t)$ providing such correspondence between the solutions of the system (5.5.9) and the differential inequality $\dot{Z} \overset{\mathcal{K}}{\leq} F(Z,t)$ that for any $t_0 \geq \theta$ from $Z_0 \overset{\mathcal{K}_0}{\leq} X_0$ it follows that $Z(t) \overset{\mathcal{K}}{\leq} X(t)$ for $t > t_0$. This property of solutions under the additional requirement $X_0 \overset{\mathcal{K}_0}{\geq} 0$ ($Z_0 \overset{\mathcal{K}_0}{\geq} 0$) determines some family of operator-functions $F \in \overline{\mathcal{F}}_1$ ($F \in \overline{\mathcal{F}}_2$), and $\overline{\mathcal{F}} \subseteq \overline{\mathcal{F}}_1 \subseteq \overline{\mathcal{F}}_2$.

Classes $\underline{\mathcal{F}}$, $\underline{\mathcal{F}}_1$ and $\underline{\mathcal{F}}_2$ are determined in a similar way, by substituting all the used inequality signs by the opposite ones. Here $\underline{\mathcal{F}} \subseteq \underline{\mathcal{F}}_1 \subseteq \underline{\mathcal{F}}_2$.

Obviously, under the condition $F \in \overline{\mathcal{F}} \cup \underline{\mathcal{F}}$ the system (5.5.9) is monotone with respect to $\mathcal{K}_0$ and $\mathcal{K}$. If $F \in \overline{\mathcal{F}}_2$ and $F(0,t) \overset{\mathcal{K}}{\geq} 0$ ($F \in \underline{\mathcal{F}}_2$ and $F(0,t) \overset{\mathcal{K}}{\leq} 0$), then the system (5.5.9) must be positive with respect to $\mathcal{K}_0$ and $\mathcal{K}$ ($-\mathcal{K}_0$ and $-\mathcal{K}$). In the given case $F(0,t) \equiv 0$ and for $F \in \overline{\mathcal{F}}_2$ ($F \in \underline{\mathcal{F}}_2$) the system (5.5.9) is monotone in $\mathcal{K}_0$ ($-\mathcal{K}_0$).

In the family (5.5.9), (5.5.10) take two systems

$$\dot{\underline{X}} = \underline{F}(\underline{X}, t), \quad \underline{F}(0,t) \equiv 0, \tag{5.5.11}$$

$$\dot{\overline{X}} = \overline{F}(\overline{X}, t), \quad \overline{F}(0,t) \equiv 0. \tag{5.5.12}$$

If $\underline{F} \in \underline{\mathcal{F}}$ and $\overline{F} \in \overline{\mathcal{F}}$, then the solutions of each system of the given family are bounded by the corresponding solutions of the systems (5.5.11) and (5.5.12), i.e.

$$\underline{X}_0 \overset{\mathcal{K}_0}{\leq} X_0 \overset{\mathcal{K}_0}{\leq} \overline{X}_0 \implies \underline{X}(t) \overset{\mathcal{K}}{\leq} X(t) \overset{\mathcal{K}}{\leq} \overline{X}(t), \ t > t_0. \tag{5.5.13}$$

Theorem 5.5.2 *If $\underline{F} \in \underline{\mathcal{F}}_1$, $\overline{F} \in \overline{\mathcal{F}}_1$ and the zero solutions of the systems (5.5.11) and (5.5.12) are stable (asymptotically stable) respectively from $-\mathcal{K}_0$ into $-\mathcal{K}$ and from $\mathcal{K}_0$ into $\mathcal{K}$, then the zero solution of each system of the family (5.5.9), (5.5.10) is stable (asymptotically stable) by Lyapunov.*

Proof. Since the cone $\mathcal{K}_0$ is reproducing, then $X_0 = X_+ - X_-$, where $X_\pm \in \mathcal{K}_0$. Let $\underline{X}(t)$ and $\overline{X}(t)$ be solutions of the respective systems (5.5.11) and (5.5.12) for the initial conditions $\underline{X}(t_0) = -X_-$ and $\overline{X}(t_0) = X_+$. If the zero solution of the system (5.5.11) ((5.5.12)) is stable from $-\mathcal{K}_0$ into $-\mathcal{K}$ (from $\mathcal{K}_0$ into $\mathcal{K}$), then for any $\varepsilon > 0$ one can select $\delta_- > 0$ ($\delta_+ > 0$) so that from $\|X_-\| < \delta_-$ ($\|X_+\| < \delta_+$) it follows that $\|\underline{X}(t)\| \leq \varepsilon/(2\nu_1)$ ($\|\overline{X}(t)\| \leq \varepsilon/(2\nu_2)$) for $t > t_0$. Here $\underline{X}(t) \in -\mathcal{K}$ ($\overline{X}(t) \in \mathcal{K}$).

Assuming $\|X_0\| \leq \delta$, where $\delta = \min\{\delta_-, \delta_+\}/\gamma$, $\gamma > 0$ is the unflattenedness constant of the cone $\mathcal{K}_0$, taking into account (5.4.2) and (5.5.13), obtain the inequality

$$\|X(t)\| \leq \nu_1\|\underline{X}(t)\| + \nu_2\|\overline{X}(t)\| \leq \varepsilon, \quad t > t_0,$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are universal constants. This means that the zero state of the system (5.5.9) is stable, and $\|X(t)\| \to 0$, if $\|\underline{X}(t)\| \to 0$ and $\|\overline{X}(t)\| \to 0$ while $t \to \infty$.

The theorem is proved.

Using Theorem 5.5.2, it is required to construct the estimate (5.5.10) and determine the belonging of operator-functions to the classes $\underline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_1$, in particular, $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$. Belonging to the classes $\underline{\mathcal{F}}$ and $\overline{\mathcal{F}}$ determined by using the cone of nonnegative vectors $\mathcal{K} \subset R^n$ are functions satisfying Wazewski's conditions (5.2.16). Operator-functions of the class $\mathcal{F}$ that satisfy the conditions (5.2.12) have the general property of quasi-monotonicity with respect to the cone $\mathcal{K} \subset \mathcal{E}$.

Lemma 5.5.2 *If the conditions (5.2.12) hold true with respect to a solid cone $\mathcal{K}$, then $F \in \overline{\mathcal{F}} \cap \underline{\mathcal{F}}$.*

Proof. Use the fact that $X \in \mathcal{K}$ is only in the case when $\varphi(X) \geq 0$ for any $\varphi \in \mathcal{K}^*$. Here $\varphi(X) > 0$, if $X \overset{\mathcal{K}}{>} 0$ and $\varphi \neq 0$.

Let $F \in \mathcal{F}$ and functions $Y(t)$ and $Z(t)$ for $t \geq t_0$ satisfy the relations

$$\dot{Y} = F(Y, t) + \varepsilon Q, \quad \dot{Z} \overset{\mathcal{K}}{\leq} F(Z, t), \quad Z_0 \overset{\mathcal{K}_0}{\leq} Y_0,$$

where $\varepsilon > 0$, $Q \overset{\mathcal{K}}{>} 0$, and in the point of time $\tau \geq t_0$ the function $Y(t) - Z(t)$ exceeds the bound of $\mathcal{K}$. Then for some $\varphi \in \mathcal{K}^*$ and $\delta > 0$ the relations

$$Z(\tau) \overset{\mathcal{K}}{\leq} Y(\tau), \;\; \varphi\left(Z(\tau)\right) = \varphi\left(Y(\tau)\right), \;\; \varphi\left(Z(t)\right) > \varphi\left(Y(t)\right),$$

where $\tau < t \leq \tau + \delta$, hold true. Taking into consideration the assumptions, obtain the inequalities

$$\dot{Y}(\tau) - \dot{Z}(\tau) \overset{\mathcal{K}}{\geq} F(Y(\tau), \tau) - F(Z(\tau), \tau) + \varepsilon Q,$$

$$\varphi\left(\dot{Y}(\tau) - \dot{Z}(\tau)\right) \geq \varepsilon\varphi(Q) > 0.$$

Therefore for some $\delta > 0$ the inequality

$$\int\limits_{\tau}^{\tau+\delta} \varphi\left(\dot{Y}(t) - \dot{Z}(t)\right) dt = \varphi\left(Y(\tau+\delta)\right) - \varphi\left(Z(\tau+\delta)\right) > 0$$

is contrary to the assumption.

Consequently, $Z(t) \overset{\mathcal{K}}{\leq} Y(t)$, $t \geq t_0$, and while $\varepsilon \to 0$ obtain $Z(t) \overset{\mathcal{K}}{\leq} X(t)$, where $X(t)$ is a solution of the system (5.5.9), i.e. $F \in \overline{\mathcal{F}}$. The inequality $Z_0 \overset{\mathcal{K}_0}{\leq} X_0$ can be considered with respect to an arbitrary cone $\mathcal{K}_0 \subseteq \mathcal{K}$. In a similar way it can be proved that under the conditions (5.2.12) $F \in \underline{\mathcal{F}}$.

The lemma is proved.

It can be shown that under the condition of positiveness of the system (5.5.9) with respect to $\mathcal{K}_0$ and $\mathcal{K}$, from $F \in \mathcal{F}_1^+$ ($F \in \mathcal{F}_2^+$) it follows that $F \in \overline{\mathcal{F}}_1$ ($F \in \overline{\mathcal{F}}_2$). Similarly, if the system (5.5.9) is positive with respect to $-\mathcal{K}_0$ and $-\mathcal{K}$, then from $F \in \mathcal{F}_1^-$ ($F \in \mathcal{F}_2^-$) it follows that $F \in \underline{\mathcal{F}}_1$ ($F \in \underline{\mathcal{F}}_2$).

5.6 Differential Comparison Systems

In applied and theoretical research the methods of comparison of systems are used, that are based on mapping of the state space of the initial (complex) system into the state spaces of auxiliary (studied) systems. In problems of stability analysis, it is expedient to take as comparison systems the classes of positive and monotone systems with respect to appropriate cones, as well as nonlinear systems satisfying the conditions of theorems of Chaplygin and Wazewski type. In this case the above results obtained for such systems may be of use.

In a Banach space $\mathcal{X}$ consider the differential system

$$\dot{x} = f(x,t), \quad x \in \mathcal{X}, \quad t \geq \theta, \tag{5.6.1}$$

Let $\mathcal{E}$ be a Banach space partially ordered by a normal cone $\mathcal{K}$. In $\mathcal{E}$ construct the classes of differential systems

$$\dot{X} = F(X,t), \quad X \in \mathcal{E}, \quad t \geq \theta, \tag{5.6.2}$$

acting as comparison systems for the initial system (5.6.1). Suppose that the systems (5.6.1) and (5.6.2) have unique continuous solutions for $t \geq t_0$ and the considered initial conditions $x(t_0) = x_0$ and $X(t_0) = X_0$. The inequalities in $\mathcal{E}$ between the values of functions in each initial point of time $t_0 \geq \theta$ will be determined with respect to some reproducing cone $\mathcal{K}_0 \subseteq \mathcal{K}$.

Let $\overline{\mathcal{M}}$, $\overline{\mathcal{M}}_k$, $\underline{\mathcal{M}}$, and $\underline{\mathcal{M}}_k$ denote the classes of systems (5.6.2) describing the respective families of operator-functions $\overline{\mathcal{F}}$, $\overline{\mathcal{F}}_k$, $\underline{\mathcal{F}}$, and $\underline{\mathcal{F}}_k$, $k = 1, 2$ (see Section 5.5). For example, for the solution $X(t)$ of a system of the class $\overline{\mathcal{M}}$ obtain the estimate $X(t) \overset{\mathcal{K}}{\geq} Z(t)$, if only $X_0 \overset{\mathcal{K}_0}{\geq} Z_0$ and the function $Z(t)$ satisfy the relations $Z(t_0) = Z_0$ and $\dot{Z} \overset{\mathcal{K}}{\leq} F(Z,t)$ for $t > t_0$. This estimate must only be true for positive solutions of the positive systems from the classes $\overline{\mathcal{M}}_1$ and $\overline{\mathcal{M}}_2$ with respect to $\mathcal{K}_0$ and $\mathcal{K}$.

Let $V(x,t)$ be an operator continuously mapping some neighbourhood $\mathcal{D}$ of the point $x = 0 \in \mathcal{X}$ for $t \geq t_0$ into the space $\mathcal{E}$. If the expression $V(x,t)$ and its generalized derivative along the trajectories of the system (5.6.1) for $x \in \mathcal{D}$ and $t \geq \theta$ satisfy the relation

$$D_t V(x,t)|_{(5.6.1)} \overset{\mathcal{K}}{\leq} F(V(x,t),t), \tag{5.6.3}$$

then the system (5.6.2) of the class $\overline{\mathcal{M}}$ is an *upper comparison system*, i.e.

$$V(x_0,t_0) \overset{\mathcal{K}_0}{\leq} X_0 \implies V(x(t),t) \overset{\mathcal{K}}{\leq} X(t), \quad t > t_0. \tag{5.6.4}$$

In (5.6.3) the derivative along the trajectories of the system (5.6.1) can be determined in the form

$$D_t V(x,t)|_{(5.6.1)} = \limsup_{h \to 0+} \frac{1}{h} \left[V(x + hf(x,t), t+h) - V(x,t) \right].$$

The positive system (5.6.2) of the class $\overline{\mathcal{M}}_1$ with respect to $\mathcal{K}_0$ and $\mathcal{K}$ under the condition (5.6.3) is also an upper comparison system for the system (5.6.1). If the operator $V(x,t)$ is everywhere positive, then we have an upper comparison system of the class $\overline{\mathcal{M}}_2$.

Similarly over the classes of systems $\underline{\mathcal{M}}$, $\underline{\mathcal{M}}_1$ and $\underline{\mathcal{M}}_2$ *lower comparison systems* (5.6.2) are determined for the system (5.6.1) by substituting all the cone inequalities in (5.6.3) and (5.6.4) by the opposite ones.

If we require that in (5.6.3) the following equality

$$D_t V(x,t)|_{(5.6.1)} = F(V(x,t),t), \tag{5.6.5}$$

must hold true, then from definition of monotonicity of the system (5.6.2) obtain

$$X_{10} \overset{\mathcal{K}_0}{\leq} V(x_0,t_0) \overset{\mathcal{K}_0}{\leq} X_{20} \Longrightarrow X_1(t) \overset{\mathcal{K}}{\leq} V(x(t),t) \overset{\mathcal{K}}{\leq} X_2(t), \tag{5.6.6}$$

where $X_1(t)$ and $X_2(t)$ are some solutions of the system (5.6.2) for $t \geq t_0$ with the initial conditions $X_1(t_0) = X_{10}$ and $X_2(t_0) = X_{20}$. It means that the relation (5.6.5) determines the class of monotone systems (5.6.2) acting simultaneously as the lower and upper comparison systems for the system (5.6.1).

The estimates (5.6.4) and (5.6.6) can be used for the comparison of dynamic properties of the systems (5.6.1) and (5.6.2), and for the construction of the attraction domain in the phase space of the system (5.6.1). For example, if the operator V is selected so that the inequality $V(x,t) \overset{\mathcal{K}}{\leq} 0$ is only possible for $x = 0$, then under the conditions (5.6.4) $X(t) \to 0$ implies $x(t) \to 0$, $t \to \infty$.

In the space $\mathcal{E}$ consider two systems

$$\dot{X}_1 = F_1(X_1,t), \quad X_1 \in \mathcal{E}, \quad t \geq \theta, \tag{5.6.7}$$

$$\dot{X}_2 = F_2(X_2,t), \quad X_2 \in \mathcal{E}, \quad t \geq \theta, \tag{5.6.8}$$

of the class $\underline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ respectively. If for $x \in \mathcal{D}$ and $t \geq \theta$

$$F_1(V(x,t),t) \overset{\mathcal{K}}{\leq} D_t V(x,t)|_{(5.6.1)} \overset{\mathcal{K}}{\leq} F_2(V(x,t),t), \tag{5.6.9}$$

then the solution of the initial system (5.6.1) satisfies the estimate (5.6.6), where $X_1(t)$ and $X_2(t)$ are the solutions of the corresponding systems (5.6.7) and (5.6.8).

We formulate the main statement on stability of solutions of the system (5.6.1), using the comparison systems (5.6.2), (5.6.7), and (5.6.8). Suppose that the identities $f(0,t) \equiv 0$ and $F(0,t) \equiv F_1(0,t) \equiv F_2(0,t) \equiv 0$ are true and the operator V has the additional properties

$$V(0,t) \equiv 0, \quad \|V(x,t)\| \geq v(x) > 0, \quad x \neq 0, \quad t \geq \theta, \qquad (5.6.10)$$

where $v(x) \geq 0$ is a continuous function such that $v(0) = 0$ and $v(x) \leq v(y) \Longrightarrow \|x\| \leq \|y\|$.

Theorem 5.6.1 *Let the operator V satisfy the relations (5.6.9) and (5.6.10), $F_1 \in \underline{\mathcal{F}}_1$ and $F_2 \in \overline{\mathcal{F}}_1$. Then the solution $x \equiv 0$ of the system (5.6.1) is stable (asymptotically stable) by Lyapunov, if the solution $X_1 \equiv 0$ of the system (5.6.7) is stable (asymptotically stable) from $-\mathcal{K}_0$ into $-\mathcal{K}$ and the solution $X_2 \equiv 0$ of the system (5.6.8) is stable (asymptotically stable) from $\mathcal{K}_0$ into $\mathcal{K}$.*

Proof. Since the cone $\mathcal{K}_0$ is reproducing and has the nonflattenedness property, then

$$-X_-^0 \overset{\mathcal{K}_0}{\leq} V(x_0,t_0) = X_+^0 - X_-^0 \overset{\mathcal{K}_0}{\leq} X_+^0, \quad \|X_\pm^0\| \leq \gamma \|V(x_0,t_0)\|,$$

where $X_\pm^0 \in \mathcal{K}_0$, $\gamma > 0$ is a universal constant.

Let $X_1(t)$ and $X_2(t)$ be solutions of the systems (5.6.7) and (5.6.8) with the initial conditions $X_1(t_0) = -X_-^0$ and $X_2(t_0) = X_+^0$. Since $F_1 \in \underline{\mathcal{F}}_1$ and $F_2 \in \overline{\mathcal{F}}_1$, then $X_1(t) \in -\mathcal{K}$ and $X_2(t) \in \mathcal{K}$ for $t \geq t_0$. Taking into consideration (5.6.6) and the normality of the cone $\mathcal{K}$, obtain the inequality

$$\|V(x(t),t)\| \leq \nu_1 \|X_1(t)\| + \nu_2 \|X_2(t)\|, \quad t \geq t_0,$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are universal constants.

From the continuity of the function $V(x,t)$ and the conditions (5.6.10) it follows that for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that $\|x(t)\| \leq \varepsilon$, as soon as $\|V(x(t),t)\| \leq \delta_0$. Use the properties of stability from $-\mathcal{K}_0$ into $-\mathcal{K}$ and from $\mathcal{K}_0$ into $\mathcal{K}$ of zero solutions of the systems (5.6.7) and (5.6.8) respectively. Select $\delta_1 > 0$ and $\delta_2 > 0$ so that from $\|X_-^0\| \le \delta_1$ and $\|X_+^0\| \le \delta_2$ the respective inequalities

$$\|X_1(t)\| \le \delta_0/(2\nu_1), \quad \|X_2(t)\| \le \delta_0/(2\nu_2), \quad t \ge t_0$$

follow.

Finally, select $\delta > 0$ so that $\|x_0\| \le \delta$ implies the inequality $\|V(x_0, t_0)\| \le \min\{\delta_1, \delta_2\}/\gamma$. Then, taking into account the above reasoning, obtain the inequality $\|x(t)\| \le \varepsilon$ for $t > t_0$, i.e. the zero solution of the system (5.6.1) is stable by Lyapunov. $\|x(t)\| \to 0$, if $\|X_1(t)\| \to 0$ and $\|X_2(t)\| \to 0$ while $t \to \infty$.

The theorem is proved.

The stability of the zero solution of the system (5.6.1) can be analyzed, basing on construction of upper comparison systems only, under additional limitations on the operator V.

Theorem 5.6.2 *Let the operator V satisfy the relations (5.6.3) and (5.6.10), and $F \in \overline{\mathcal{F}}_2$ and $V(x,t) \overset{\mathcal{K}}{\ge} 0$ for $x \in \mathcal{D}$ and $t \ge \theta$. Then the solution $x \equiv 0$ of the system (5.6.1) is stable (asymptoticall stable) by Lyapunov, if the solution $X \equiv 0$ of the system (5.6.2) is stable (asymptotically stable) from $\mathcal{K}_0$ into $\mathcal{K}$.*

The proof of this proposition is similar to the proof of Theorem 5.6.1.

Remark 5.6.1 The upper comparison systems in Theorems 5.5.2, 5.6.1, and 5.6.2 must be positive. Therefore the conditions of stability (asymptotic stability) from $\mathcal{K}_0$ into $\mathcal{K}$ of the zero solutions of these systems can be replaced by the requirement of their stability (asymptotic stability) by Lyapunov. Similarly, the conditions of stability (asymptotic stability) from $-\mathcal{K}_0$ into $-\mathcal{K}$ of the zero solutions of the lower comparison systems in Theorems 5.5.2 and 5.6.1 can be replaced by the requirement of their stability (asymptotic stability) by Lyapunov. At the construction of positive or monotone upper comparison systems (5.6.2) the operator V can be selected from the class of everywhere positive operators. The statement of Theorem 5.6.2 holds true if instead of the condition $V(x,t) \overset{\mathcal{K}}{\ge} 0$ we require that for some $\varphi_0 \in \mathcal{K}^*$ the milder restriction

$$\varphi_0(V(x,t)) > 0, \quad x \ne 0 \in \mathcal{D}, \quad t \ge \theta.$$

must be true. Theorem 5.6.2 is also true if instead of the requirement $F \in \overline{\mathcal{F}}_2$ we use the monotonicity of the system (5.6.2) in the cone $\mathcal{K}_0$. At that the relation (5.6.5) must hold true.

Example 5.6.1 Consider the differential system

$$\dot{x} = g(V(x), t) \odot x, \quad x \in R^n, \tag{5.6.11}$$

where g is some vector-function, $V(x) = x \odot x \overset{\mathcal{K}}{\geq} 0$, $\mathcal{K} \subset R^n$ is a cone of nonnegative vectors, and $\odot$ is an elementwise Schur product. The derivative of $V(x)$ along trajectories of the system (5.6.11) is equal to $2g(V(x), t) \odot V(x)$, and we have the comparison system

$$\dot{X} = 2\,g(X, t) \odot X, \quad X \in R^n. \tag{5.6.12}$$

The system (5.6.12) is positive with respect to the cone $\mathcal{K}$. If the vector-function g satisfies Wazewski's condition (5.2.16), then the system (5.6.12) is monotone in $\mathcal{K}$ and can be used in Theorem 5.6.2. For the solutions of the systems (5.6.11) and (5.6.12) the estimate

$$|x_0| \overset{\mathcal{K}}{\leq} \sqrt{X_0} \implies |x(t)| \overset{\mathcal{K}}{\leq} \sqrt{X(t)}, \quad t > t_0,$$

holds true, where the operations of module and root of vector are executed elementwise. The stability (asymptotic stability) in $\mathcal{K}$ of the solution $X \equiv 0$ of the system (5.6.12) implies the stability (asymptotic stability) by Lyapunov of the solution $x \equiv 0$ of the system (5.6.11).

Example 5.6.2 Consider the differential system

$$\dot{x} = \mathcal{A}(V(x), t)\, x, \quad x \in R^n, \quad t \geq \theta, \tag{5.6.13}$$

where $\mathcal{A}$ is some operator, and $V(x) = x\,x^T \geq 0$. Calculating the derivative of $V(x)$ along trajectories of the system (5.6.13), based on the relation (5.6.5) obtain the matrix equation

$$\dot{X} = \mathcal{A}(X, t)\, X + X\, \mathcal{A}^T(X, t), \quad X = X^T \in R^{n \times n}.$$

Let $\mathcal{A}(X, t) = -A(t) + X\mathcal{B}(X, t)$. Then this equation reduces to the form

$$\dot{X} + A(t)X + XA^T(t) = X\,\mathcal{R}(X, t)\, X, \tag{5.6.14}$$

where $\mathcal{R}(X,t) = \mathcal{B}(X,t) + \mathcal{B}^T(X,t)$. If the operator $\mathcal{R}(X,t)$ is monotone with respect to the cone of symmetric nonnegative definite matrices $\mathcal{K}$, then the equation (5.6.14) can be used as an upper comparison system in Theorem 5.6.2 for the initial system (5.6.1). Each solution $x(t)$ of the system (5.6.13) satisfies the estimate

$$0 \le x_0\, x_0^T \le X_0 \implies 0 \le x(t)\, x^T(t) \le X(t),$$

where $X(t) \ge 0$ is a solution of the equation (5.6.14), $t \ge t_0$. The stability (asymptotic stability) in $\mathcal{K}$ of the solution $X \equiv 0$ of the equation (5.6.14) implies the stability (asymptotic stability) by Lyapunov of the solution $x \equiv 0$ of the system (5.6.13).

Studying the system (5.6.2) one can use the two-sided estimates of its right-hand side of the form (5.5.3) or (5.5.10) and the respective comparison systems constructed in the same space $\mathcal{E}$. For example, if the estimate (5.5.10) is obtained, then the system (5.5.11) ((5.5.12)) for $\underline{F} \in \underline{\mathcal{F}}_1$ ($\overline{F} \in \overline{\mathcal{F}}_1$) is the lower (upper) comparison system for the system (5.6.2). The transformation of the phase space is not applied. The statement of Theorem 5.5.2 for these systems can be obtained as a corollary of Theorem 5.6.1, assuming $V(X,t) \equiv X$.

The lower and upper comparison systems

$$\dot{X}_1 = F_1(X_1, t), \quad \dot{X}_2 = F_2(X_2, t)$$

can be constructed in different partially ordered spaces $\mathcal{E}_1$ and $\mathcal{E}_2$. The properties of the respective operators $V_1(x,t)$ and $V_2(x,t)$, as well as the ordering relationship determined by the selected cones in the relation

$$V_1(x(t),t) \overset{\mathcal{K}_1}{\ge} X_1(t), \quad V_2(x(t),t) \overset{\mathcal{K}_2}{\le} X_2(t), \quad t \ge t_0,$$

should be coordinated with the purpose of the study of certain characteristics of the initial system (5.6.1). For example, it can be required that the conditions $V_1(x,t) \overset{\mathcal{K}_1}{\ge} 0$ and $V_2(x,t) \overset{\mathcal{K}_2}{\le} 0$ must hold true simultaneously for $x = 0$ only. In this case it should be expected that $x(t) \to 0$ as $t \to \infty$ if $X_1(t) \to 0$ and $X_2(t) \to 0$, where $X_1(t)(X_2(t))$ is a solution of the lower (upper) comparison system.

If the operators $V_1(x,t)$ and $V_2(x,t)$ coincide then this property follows from the lemma on two policemen in a partially ordered space. Note that if the requirement of the uniqueness of solutions of lower and upper comparison systems does not hold true, then instead of $X_1(t)$ and $X_2(t)$ in the above reasoning one should consider respectively the minimal and maximal solutions of the given systems with respect to the selected cones $\mathcal{K}_0$ and $\mathcal{K}$.

5.7 Dynamics of Systems with Respect to Variable Cone

The described methods of research of dynamic properties of differential and difference systems are based on the estimation and comparison of their states with respect to normal reproducing cones $\mathcal{K}_0$ and $\mathcal{K}$. The similar and more general results are established by using the sets $\mathcal{K}_t \subset \mathcal{E}$, $t \geq t_0 \geq \theta$, changed in accordance with the given law. As $\mathcal{K}_t$ for each t one can use, e.g., some cone, polyhedron, etc.

$\mathcal{K}_t$ is called an *invariant set* of the system (5.2.1), if for any $t_0 \geq \theta$ the inclusion $\Omega(t,t_0)\mathcal{K}_0 \subseteq \mathcal{K}_t$ holds true, where $\mathcal{K}_{t_0} = \mathcal{K}_0$, $t \geq t_0$. If the systems(5.2.1) have an invariant cone $\mathcal{K}_t$, then it is *positive* with respect to $\mathcal{K}_t$. The positivity of systems means that $X(t) \overset{\mathcal{K}_t}{\geq} 0$ for $t \geq t_0$ as soon as $X_0 \overset{\mathcal{K}_0}{\geq} 0$ and $t_0 \geq \theta$. The system (5.2.1) is called *monotone* with respect to the cone $\mathcal{K}_t$, if for any $t_0 \geq \theta$

$$X_{10} \overset{\mathcal{K}_0}{\leq} X_{20} \implies X_1(t) \overset{\mathcal{K}_t}{\leq} X_2(t), \quad t > t_0, \tag{5.7.1}$$

where $X_k(t) = \Omega(t,t_0)X_{k0}$, $k = 1,2$. For the classes of positive and monotone systems, as well as for the systems that have the property (5.7.1) under the additional requirements $X_{20} \in \mathcal{K}_0$, $X_{10} \in \mathcal{K}_0$, $X_{10} \in -\mathcal{K}_0$ and $X_{20} \in -\mathcal{K}_0$, as before, use the respective notation $\mathcal{M}_0$, $\mathcal{M}$, $\mathcal{M}_1^+$, $\mathcal{M}_2^+$, $\mathcal{M}_1^-$ and $\mathcal{M}_2^-$. A system of the class $\mathcal{M}_2^+$ ($\mathcal{M}_2^-$) is *monotone in the cone* $\mathcal{K}_t$ ($-\mathcal{K}_t$).

The state $X \equiv 0$ of the system (5.2.1) is *stable in* $\mathcal{K}_t$, if for any $\varepsilon > 0$ and $t_0 \geq \theta$ it is possible to find such $\delta > 0$ that $X_0 \in \mathcal{S}_\delta(\mathcal{K}_0)$ implies $X(t) \in \mathcal{S}_\varepsilon(\mathcal{K}_t)$ at $t > t_0$, where

$$\mathcal{S}_\varepsilon(\mathcal{K}_t) = \{X \in \mathcal{K}_t \colon \|X\| \leq \varepsilon\}.$$

If for some $\delta_0 > 0$ from $X_0 \in \mathcal{S}_{\delta_0}(\mathcal{K}_0)$ follows $\|X(t)\| \to 0$ while $t \to \infty$, then the state $X \equiv 0$ of the system is asymptotically stable in $\mathcal{K}_t$.

If the system (5.2.1) is positive with respect to $\mathcal{K}_t$ and its state $X \equiv 0$ is stable (asymptotically stable) by Lyapunov, then it is stable (asymptotically stable) in $\mathcal{K}_t$.

For the differential system (5.6.2) and the given cone $\mathcal{K}_t$ define the classes of operator-functions $F \in \mathcal{F}_0$ and $F \in \mathcal{F}$ satisfying the respective conditions

$$X \overset{\mathcal{K}_t}{\geq} 0,\ \varphi(X) = 0 \implies \varphi\left(F(X,t)\right) \geq 0, \tag{5.7.2}$$

$$X \overset{\mathcal{K}_t}{\geq} Y,\ \varphi(X-Y) = 0 \implies \varphi\left(F(X,t) - F(Y,t)\right) \geq 0, \tag{5.7.3}$$

where $\varphi \in \mathcal{K}_t^*$, $t \geq \theta$. We will also define the families of operator-functions $\mathcal{F}_1^+$, $\mathcal{F}_2^+$, $\mathcal{F}_1^-$ and $\mathcal{F}_2^-$ that have the property (5.7.3) under the additional requirements $X \in \mathcal{K}_t$, $Y \in \mathcal{K}_t$, $Y \in -\mathcal{K}_t$, and $X \in -\mathcal{K}_t$ respectively. The families $\mathcal{F}_0$, $\mathcal{F}$, $\mathcal{F}_1^{\pm}$ and $\mathcal{F}_2^{\pm}$ are wedges, and $\mathcal{F} \subseteq \mathcal{F}_1^{\pm} \subseteq \mathcal{F}_2^{\pm}$.

In problems of robust stability and comparison of system we will use the families of operator-functions $\overline{\mathcal{F}}$ and $\underline{\mathcal{F}}$ determined by the respective conditions

$$\dot{X} = F(X,t),\ \ \dot{Z} \overset{\mathcal{K}_t}{\leq} F(Z,t),\ \ X_0 \overset{\mathcal{K}_0}{\geq} Z_0, \implies X(t) \overset{\mathcal{K}_t}{\geq} Z(t), \tag{5.7.4}$$

$$\dot{X} = F(X,t),\ \ \dot{Z} \overset{\mathcal{K}_t}{\geq} F(Z,t),\ \ X_0 \overset{\mathcal{K}_0}{\leq} Z_0, \implies X(t) \overset{\mathcal{K}_t}{\leq} Z(t). \tag{5.7.5}$$

These conditions must hold true for $t \geq t_0$ and any initial point of time $t_0 \geq \theta$.

The condition (5.7.4) under the additional requirement $X_0 \in \mathcal{K}_0$ ($Z_0 \in \mathcal{K}_0$) determines some family of operator-functions $F \in \overline{\mathcal{F}}_1$ ($F \in \overline{\mathcal{F}}_2$), and $\overline{\mathcal{F}} \subseteq \overline{\mathcal{F}}_1 \subseteq \overline{\mathcal{F}}_2$. Similarly, the family $\underline{\mathcal{F}}_1$ ($\underline{\mathcal{F}}_2$) is determined by the condition (5.7.5) for $X_0 \in -\mathcal{K}_0$ ($Z_0 \in -\mathcal{K}_0$), and $\underline{\mathcal{F}} \subseteq \underline{\mathcal{F}}_1 \subseteq \underline{\mathcal{F}}_2$. The families $\overline{\mathcal{F}}$, $\overline{\mathcal{F}}_k$, $\underline{\mathcal{F}}$ and $\underline{\mathcal{F}}_k$ describe the respective classes of systems $\overline{\mathcal{M}}$, $\overline{\mathcal{M}}_k$, $\underline{\mathcal{M}}$, and $\underline{\mathcal{M}}_k$ of the form (5.6.2) ($k = 1, 2$).

Formulate analogues of Lemmas 5.2.4 and 5.5.2, using the variable cone $\mathcal{K}_t$.

Lemma 5.7.1 *Let $\mathcal{K}_t$ be a solid cone with the property*

$$\mathcal{K}_t \supseteq \mathcal{K}_\tau, \quad t > \tau \geq \theta. \tag{5.7.6}$$

Then the system (5.6.2) is positive (monotone) with respect to $\mathcal{K}_t$, if $F \in \mathcal{F}_0$ ($F \in \mathcal{F}$).

Lemma 5.7.2 *Under the conditions of Lemma 5.7.1 the inclusion $\mathcal{F} \subseteq \overline{\mathcal{F}} \cap \underline{\mathcal{F}}$ holds true.*

The positive system (5.6.2) with respect to the solid cone $\mathcal{K}_t$ ($-\mathcal{K}_t$) belongs to the class $\mathcal{M}_k^+$ ($\mathcal{M}_k^-$) if $F \in \mathcal{F}_k^+$ ($F \in \mathcal{F}_k^-$), $k = 1, 2$. If the right-hand side of the system (5.6.2) is representable in the form

$$F(X,t) = \alpha F_1(X,t) + \beta F_2(X,t), \quad \alpha \geq 0, \;\; \beta \geq 0,$$

and $F_1, F_2 \in \mathcal{F}_0$ ($F_1, F_2 \in \mathcal{F}$), then under the conditions of Lemma 5.7.1 this system is positive (monotone) with respect to $\mathcal{K}_t$.

Let $\mathcal{K}_t$ be a normal cone with a bounded normality constant $\nu_t \leq \nu < \infty$, and for any $t_0 \geq \theta$ the cone $\mathcal{K}_0$ is reproducing. We will also suppose that the cone has the following property:

$$\mathcal{K}_t \subseteq \mathcal{K}_\tau \quad \text{or} \quad \mathcal{K}_t \supseteq \mathcal{K}_\tau, \quad \forall t, \tau \geq \theta. \tag{5.7.7}$$

Show that at the study of the conditions of stability in the cone and the stability by Lyapunov of the zero state of the system (5.2.1) one can use the following properties of derivatives:

$$X_0 \overset{\mathcal{K}_0}{\geq} 0 \implies \dot{X}(t) \overset{\mathcal{K}_t}{\leq} 0, \quad t > t_0, \tag{5.7.8}$$

$$X_0 \overset{\mathcal{K}_0}{\leq} 0 \implies \dot{X}(t) \overset{\mathcal{K}_t}{\geq} 0, \quad t > t_0. \tag{5.7.9}$$

Lemma 5.7.3 *The state $X \equiv 0$ of the positive system (5.2.1) with respect to $\mathcal{K}_t$ ($-\mathcal{K}_t$), having the property (5.7.8) ((5.7.9)), is stable in $\mathcal{K}_t$ ($-\mathcal{K}_t$).*

Now we formulate analogues of Theorems 5.4.2, 5.4.3, 5.5.2, 5.6.1, and 5.6.2.

Theorem 5.7.1 *The state $X \equiv 0$ of the system (5.2.1) belonging to the classes $\mathcal{M}_1^+$ and $\mathcal{M}_1^-$ is stable (asymptotically stable) by*

Lyapunov if and only if it is stable (asymptotically stable) in $\mathcal{K}_t$ *and* $-\mathcal{K}_t$.

Note that under the conditions of Theorem 5.7.1 the system (5.2.1) must be positive with respect to the cones $\pm\mathcal{K}_t$. This follows from the assumption $\Omega(t, t_0)\, 0 \equiv 0$ and the belonging of the operator $\Omega(t, t_0)$ to the classes $\mathcal{M}_1^{\pm}$.

Corollary 5.7.1 *If the system (5.2.1) belongs to the classes* $\mathcal{M}_1^+$ *and* $\mathcal{M}_1^-$*, in particular, if it is monotone with respect to* $\mathcal{K}_t$*, then under the conditions (5.7.8) and (5.7.9) its state* $X \equiv 0$ *is stable by Lyapunov.*

Corollary 5.7.2 *If the evolutionary operator* $W(t, t_0)$ *of the differential system (5.3.1) satisfies the relations*

$$W(t, t_0)\, \mathcal{K}_0 \subseteq \mathcal{K}_t, \quad M(t)\, W(t, t_0)\, \mathcal{K}_0 \subseteq \mathcal{K}_t, \quad t > t_0,$$

then this system is stable by Lyapunov.

Theorem 5.7.2 *If* $\underline{F} \in \underline{\mathcal{F}}_1$, $\overline{F} \in \overline{\mathcal{F}}_1$ *and the zero solutions of the systems (5.5.11) and (5.5.12) are stable (asymptotically stable) respectively in* $-\mathcal{K}_t$ *and* $\mathcal{K}_t$*, then the zero solution of each system (5.5.9), for which the following relations*

$$\underline{F}(X, t) \overset{\mathcal{K}_t}{\leq} F(X, t) \overset{\mathcal{K}_t}{\leq} \overline{F}(X, t), \quad X \in \mathcal{E},\ t \geq \theta,$$

hold true, is stable (asymptotically stable) by Lyapunov.

Theorem 5.7.3 *Let* $F_1 \in \underline{\mathcal{F}}_1$, $F_2 \in \overline{\mathcal{F}}_1$ *and let the operator* $V(x, t)$ *for* $x \in \mathcal{D}$ *and* $t \geq \theta$ *satisfy the relations (5.6.10) and*

$$F_1(V(x, t), t) \overset{\mathcal{K}_t}{\leq} D_t V(x, t)|_{(5.6.1)} \overset{\mathcal{K}_t}{\leq} F_2(V(x, t), t).$$

Then the zero solution of the system (5.6.1) is stable (asymptotically stable) by Lyapunov if the zero solutions of the systems (5.6.7) and (5.6.8) are stable (asymptotically stable) respectively in $-\mathcal{K}_t$ *and* $\mathcal{K}_t$.

Theorem 5.7.4 *Let* $F \in \overline{\mathcal{F}}_2$ *and the operator* $V(x, t)$ *for* $x \in \mathcal{D}$ *and* $t \geq \theta$ *satisfy the relations (5.6.10) and*

$$D_t V(x, t)|_{(5.6.1)} \overset{\mathcal{K}_t}{\leq} F(V(x, t), t), \quad V(x, t) \overset{\mathcal{K}_t}{\geq} 0.$$

Then the zero solution of the system (5.6.1) is stable (asymptotically stable) by Lyapunov if the zero solution of the system (5.6.2) is stable (asymptotically stable) in $\mathcal{K}_t$.

The proof of the above propositions is made in the same way as in the case of a constant cone in consideration of the supplementary assumptions with respect to the cone $\mathcal{K}_t$.

Note that in Theorems 5.7.2 – 5.7.4 the property (5.7.7) of the cone $\mathcal{K}_t$ is not used, and the conditions of stability (asymptotic stability) in $-\mathcal{K}_t$ and $\mathcal{K}_t$ of the zero solutions of respectively the lower and upper comparison systems should be replaced by the requirement of stability (asymptotic stability) by Lyapunov of these solutions.

In the phase space $\mathcal{E}$ consider the sets

$$\mathcal{K}_t = \{X:\, R(t)X \in \mathcal{K}\}, \quad \hat{\mathcal{K}}_t = \{R(t)X:\, X \in \mathcal{K}\} = R(t)\,\mathcal{K},$$

where $\mathcal{K}$ is a given cone, $R(t)$ a linear operator, $t \geq \theta$. These sets are wedges, and $\ker R(t) = \{X:\, R(t)X = 0\}$ coincides with the blade of the wedge $\mathcal{K}_t$.

Suppose that $\ker R(t) \equiv \{0\}$ and the cone $\mathcal{K}$ is normal. Then $\mathcal{K}_t$ is a normal cone if the following inequalities

$$r_-(t)\,\|X\| \leq \|R(t)X\| \leq r_+(t)\,\|X\|, \quad X \in \mathcal{K}_t, \tag{5.7.10}$$

where $r_\pm(t) > 0$ are some functions independent of X, hold true. In addition, its normality constant must not exceed $\nu r_+(t)/r_-(t)$, where ν is a normality constant of $\mathcal{K}$. If similar inequalities hold true for $X \in \mathcal{K}$, then the cone $\hat{\mathcal{K}}_t$ is normal. In the case of Euclidean norm in R^n, the inequalities (5.7.10) hold true if for instance $r_+^2(t)(r_-^2(t))$ is maximal (minimal) eigenvalue of the matrix $R^T(t)R(t) > 0$.

Let there exist a time derivative $\dot{R}(t)$ and an inverse operator $R^{-1}(t)$ for $t \geq \theta$. Assuming $Y(t) = R(t)X(t)$, transform the system (5.6.2) to the form

$$\dot{Y} + N(t)Y = G(Y,t), \quad t \geq \theta, \tag{5.7.11}$$

where $N(t) = -\dot{R}(t)R^{-1}(t)$, $G(Y,t) = R(t)F\left(R^{-1}(t)Y,t\right)$. Then the positivity conditions of the system (5.6.2) with respect to $\mathcal{K}_t$ are equivalent to the positivity with respect to the constant cone $\mathcal{K}$ of the system (5.7.11). The system (5.7.11) is positive with respect to $\mathcal{K}$, if such is the linear system $\dot{Z} + N(t)Z = 0$ and the inclusion $R(t)F(R^{-1}(t)\mathcal{K}, t) \subseteq \mathcal{K}$ holds true. The condition $\mathcal{K}_\tau \subseteq \mathcal{K}_t$ is equivalent to the inclusion $W_N(t,\tau)\mathcal{K} \subseteq \mathcal{K}$, where $W_N(t,\tau) = R(t)R^{-1}(\tau)$, $t \geq \tau$.

Similarly, the positivity of the system (5.6.2) with respect to $\hat{\mathcal{K}}_t$ is equivalent to positivity with respect to the constant cone $\mathcal{K}$ of some system obtained from (5.6.2) with the use of the transform $X(t) = R(t)Y(t)$.

The above reasoning can be useful in the study of the positivity conditions with respect to variable cones and the stability of the solutions of linear and nonlinear systems with the use of Lemmas 5.7.1 – 5.7.3 and Theorems 5.7.1 – 5.7.4.

Example 5.7.1 Consider the linear system (5.3.1) and the set $\mathcal{K}_t = \{X : R(t)X \in \mathcal{K}\}$ assuming

$$M(t) = \begin{bmatrix} A(t) & b(t) \\ c^T(t) & d(t) \end{bmatrix}, \quad \mathcal{K} = \left\{ X : X = \begin{bmatrix} x \\ u \end{bmatrix}, \, \|x\| \leq u \right\},$$

where $\mathcal{K}$ is a circular cone. If $\det R(t) \neq 0$, then $\mathcal{K}_t$ is an ellipsoidal cone that can be represented as

$$\mathcal{K}_t = \{X \in R^{n+1} : \, X^T Q(t) X \geq 0, \;\; X^T Q(t)p(t) \geq 0\},$$

where the symmetric matrix $Q(t)$ with the inertia $i(Q(t)) \equiv \{1, n, 0\}$ and the vector $p(t)$ satisfy the conditions

$$Q(t) = R^T(t)\Delta R(t), \quad \Delta = \begin{bmatrix} -I & 0 \\ 0 & 1 \end{bmatrix}, \quad p^T(t)Q(t)p(t) = \omega(t) > 0.$$

Note, that (see Section 4.2)

$$S(t) = Q(t) - \frac{1}{\omega(t)} Q(t)p(t)p^T(t)Q^T(t) \leq 0, \quad i(S(t)) \equiv \{0, n, 1\},$$

and $p(t)$ may be an eigenvector of $Q(t)$ corresponding to unique positive eigenvalue.

The transformed system (5.7.11) has the form

$$\dot{Y} + L(t)Y = 0, \quad L(t) = R(t)M(t)R^{-1}(t) - \dot{R}(t)R^{-1}(t).$$

It is positive with respect to $\mathcal{K}$ if and only if there is $\gamma(t) \in R^1$ such that

$$L^T(t)\Delta + \Delta L(t) + \gamma(t)\Delta \leq 0.$$

We present the criterion for positivity of the system (5.3.1) with respect to $\mathcal{K}_t$ as the matrix inequality

$$\dot{Q} \geq M^T(t)Q + QM(t) + \gamma(t)Q,$$

where $\gamma(t)$ is some continuous function. Various sufficient conditions for positivity of (5.3.1) with respect to $\mathcal{K}_t$ can be constructed (see Example 5.2.4). For example, if

$$R(t) = \begin{bmatrix} I & 0 \\ 0 & r(t) \end{bmatrix}, \quad 0 < r(t) \leq r_0,$$

then one of such conditions has the form

$$\frac{1}{2}\lambda_{min}\left(A(t) + A^T(t)\right) + \frac{\dot{r}(t)}{r(t)} \ \geq \ d(t) + \left\| \frac{b(t)}{r(t)} - r(t)c(t) \right\|.$$

In that case (5.7.6) hold true if $r(t)$ is a nondecreasing function and in (5.7.10) one can put

$$r_-(t) = \min\{r(t), 1\}, \quad r_+(t) = \min\{\sqrt{2}\, r(t), \max\{r(t), 1\}\}.$$

In particular, the normality constant of $\mathcal{K}_t$ does not exceed $\sqrt{2}$ if $r_0 = 1/\sqrt{2}$.

5.8 Notes and References

5.1 The main concepts and facts from the theory of operators in a partially ordered space are described in Kantorovich, Vulih, Pinsker [1], Krein, Rutman [1], Krasnoselskii, Lifschits, Sobolev [1], Glazman, Lyubich [1], and others. Classes of operators of the form (5.1.2) were studied in Schneider [1], Mazko [18] and Korenevskii, Mazko [1]).

5.2 Positivity of the systems in control problems were used in Krasnoselskii, Lifschits, Sobolev [1], Berman, Neumann, Stern [1], Angeli, Sontag [1], Farina, Rinaldi [1], Bru, Coll, Romero, Sánchez [1], and others. In the works of Hirsch, Smith [1], Farina, Rinaldi [1], Mazko [31–37] the classes of positive and monotone dynamic systems were determined in terms of their solutions with respect to a given cone.

In the proof of Lemma 5.2.1 the known procedure of representation of an evolutionary operator of a system in the form of a multiplicative integral was used (see Daletsky, Krein [1]).

The positivity conditions for an evolutionary operator of the system (5.2.7) with respect to a cone of nonnegative vectors are determined in Krasnoselskii, Lifschits, Sobolev [1]. The conditions of monotonicity of the shift operator applied along the trajectories of the system (5.2.13) with respect to a cone of nonnegative vectors are found in Krasnoselskii [1].

Some properties of the Minkovsky cone are described in Glazman, Lyubich [1]. The condition of invariance of this cone for linear differential system in the form of a matrix inequality is obtained in Stern, Wolkowicz [1].

5.3 The known methods of analysis of stability of the systems of the form (5.3.1) can be found in Barbashin [1], Demidovich [1], Grujich, Martynyuk, Ribbens–Pavella [1], Kuntsevich, Lychak [1], Lakshmikantham, Leela, Martynyuk [1], Matrosov, Anapolskii, Vassilyev [1], Martynyuk [1], and others.

The proofs of theorems of Krein–Bonsall–Karlin on the spectral radius of a positive operator and the theorem on a two-sided estimate of positively invertible operators can be found in Krasnoselskii, Lifshits, Sobolev [1].

The theory of one-parameter positive semigroups is described in Clement, Heijmans, Angenent, C. van Duijn, B. de Pagter [1].

The statements of Theorems 5.3.1 – 5.3.3 are proved in Mazko [30–32]. The first proposition of Theorem 5.3.1 in the case of a solid cone can be also proved on the basis of the results of Milshtein [1]. The demonstrative example to Theorem 5.3.3 is taken from Demidovich [1].

5.4 The stability conditions in a cone of nonlinear Wazewski equations of the type of Theorem 5.4.1 reduce to the solvability of some systems of algebraic inequalities determined by a given cone in the space R^n and the right-hand sides of these equations (see, e.g., Martynyuk, Obolenskii [1,2]). The described approach to the stability analysis of positive and monotone dynamic systems with respect to two cones is obtained in Mazko [33–35].

5.5 The stability conditions of some systems of polynomials are described in Zhabko, Kharitonov [1], Soh, Berger, Dabke [1]. The stability conditions of families of linear and nonlinear systems determined by prescribed cones are formulated in Mazko [32–36].

5.6 The methods for construction of comparison systems satisfying the conditions of theorems of Chaplygin and Wazewski type are described in Matrosov, Anapolskii, Vassilyev [1], Lakshmikantham, Leela, Martynyuk [1], Postnikov, Sabaev [1], Lakshmikantham, Leela [1], Martynyuk [1, 2], and others.

The problem of comparison of the systems (5.6.1) and (5.6.2) gets complicated without the requirement for the uniqueness of solutions. In Matrosov, Anapolskii, Vassilyev [1] and Lakshmikantham, Leela, Martynyuk [1] the comparison systems of the form (5.6.2) in a partially ordered space R^n are considered, using the concepts of a maximal and minimal solutions, as well as generalized Dini derivatives of a vector-function $V(x,t)$ on the solutions of the initial system (5.6.1). Here $V(x,t)$ must be locally the Lipschitz function of x, and the vector-function $F(X,t)$ must be quasimonotone nondecreasing with respect to X in the cone $\mathcal{K} \subset R^n$. This limitation on F provides the monotonicity property to the system (5.6.2) and its belonging to the classes $\mathcal{F}_{\pm}$, and in the case of a cone of nonnegative vectors it adds up to the Wazewski's conditions (5.2.16).

The lemma on two policemen in a partially ordered space is formulated in Krasnoselskii, Lifshits, Sobolev [1].

5.7 The study of positive and monotone dynamic systems with respect to a variable cone and some its applications in stability problems are described in Mazko [36, 37]. In Example 5.7.1, we use some results by Aliluyko and Mazko [1]. Comparison technique is developed for a set of dynamical systems (see Aliluyko and Mazko [2]).

6

APPENDIX

6.1 Representations of Linear Operators in Matrix Space

During the study and the use of matrix equations of the general form an important role is played by different representations of linear operators $M\colon C^{n\times m} \to C^{p\times q}$, in particular,

$$MX = \sum_{i=1}^{k}\sum_{j=1}^{s} c_{ij} A_i X B_j, \tag{6.1.1}$$

$$MX = \sum_{t=1}^{n}\sum_{\tau=1}^{m} x_{t\tau} H_{t\tau}, \tag{6.1.2}$$

$$MX = \sum_{t=1}^{\xi}\sum_{\tau=1}^{\zeta} (V_{t\tau}, X)\, U_{t\tau}, \tag{6.1.3}$$

$$MX = \begin{bmatrix} (G_{11}, X) & \dots & (G_{1q}, X) \\ \cdots & \cdots & \cdots \\ (G_{p1}, X) & \dots & (G_{pq}, X) \end{bmatrix}, \tag{6.1.4}$$

where $(P, Q) \triangleq \operatorname{tr}(P^* Q)$ is a *scalar product* of the matrices P and Q. The properties of the operator (6.1.1) are characterized by the matrix families A, B and the matrix of weighting coefficients C. The operators (6.1.2) and (6.1.4) define the block matrices

$$H = \begin{bmatrix} H_{11} & \dots & H_{1m} \\ \cdots & \cdots & \cdots \\ H_{n1} & \dots & H_{nm} \end{bmatrix}, \quad G = \begin{bmatrix} G_{11} & \dots & G_{1q} \\ \cdots & \cdots & \cdots \\ G_{p1} & \dots & G_{pq} \end{bmatrix}.$$

The expression (6.1.3) is a product of operators of the type (6.1.2) and (6.1.4).

If the operator M is represented in the form (6.1.2) or (6.1.4), then the construction of solvability conditions of the equation $MX = Y$ adds up to the determination of linearly independent blocks of the respective matrices H or G. Thus, in the case $nm = pq$, the operator (6.1.2) ((6.1.4)) is invertible if and only if all the blocks $H_{t\tau}$ $(G_{t\tau})$ are linearly independent.

Let the operator M be given in a standard form (6.1.1). Take the rows and columns of matrix coefficients

$$A_i = [a^i_{*1}, \dots, a^i_{*n}] = \begin{bmatrix} a^i_{1*} \\ \vdots \\ a^i_{p*} \end{bmatrix}, \quad B_j = [b^j_{*1}, \dots, b^j_{*q}] = \begin{bmatrix} b^j_{1*} \\ \vdots \\ b^j_{m*} \end{bmatrix}.$$

Then in (6.1.2) and (6.1.4) one can assume

$$H = \begin{bmatrix} a^1_{*1} & \dots & a^k_{*1} \\ \dots & \dots & \dots \\ a^1_{*n} & \dots & a^k_{*n} \end{bmatrix} C \begin{bmatrix} b^1_{1*} & \dots & b^1_{m*} \\ \dots & \dots & \dots \\ b^s_{1*} & \dots & b^s_{m*} \end{bmatrix},$$

$$G^* = \begin{bmatrix} b^1_{*1} & \dots & b^s_{*1} \\ \dots & \dots & \dots \\ b^1_{*q} & \dots & b^s_{*q} \end{bmatrix} C^T \begin{bmatrix} a^1_{1*} & \dots & a^1_{p*} \\ \dots & \dots & \dots \\ a^k_{1*} & \dots & a^k_{p*} \end{bmatrix}.$$

Here the inequalities

$$\operatorname{rank} H \le \operatorname{rank} C, \quad \operatorname{rank} G \le \operatorname{rank} C,$$

hold true. The equalities here are achieved if and only if the sets A and B consist of linearly independent matrices. Conversely, proceeding from the representations (6.1.2) and (6.1.4) of the operator M, it is possible to construct expressions of the type (6.1.1). Parameters A, B, and C are determined ambiguously.

For the determination of a *conjugate operator* $M^*\colon C^{p\times q} \to C^{n\times m}$ we use the relation $(MX, Y) = (X, M^*Y)$. If the operator M is given in the form (6.1.1), then, taking into consideration the permutability of matrices within the operation tr we have

$$M^*Y = \sum_{i=1}^{k} \sum_{j=1}^{s} \overline{c}_{ij}\, A_i^* Y B_j^*.$$

Consider the class of operators $M : \mathcal{H}_n \to \mathcal{H}_p$ preserving the property of self-conjugacy of matrices. This property is peculiar to operators described by the relations (6.1.1) – (6.1.4) under the conditions $A = B^*$, $C = C^*$, $H = H^*$, $V = V^*$, $U = U^*$, $G = G^*$. The inequalities

$$i_\pm(H) \le i_\pm(C), \quad i_\pm(G) \le i_\pm(C),$$

hold true which can be used for strengthening of Theorems 4.4.1 – 4.4.3 and 4.5.1 – 4.5.3. Thus, in Theorems 4.5.1 – 4.5.3 instead of the relations $i_\pm(C) \le 1$ and $i_+(C) = 1$ one can use similar limitations on the indices of inertia of the matrices H and G.

It can be determined that the linear independence of the matrices $A_1, \dots, A_k$ is equivalent to the linear independence of the operator family $A_i X A_j^*$, $i, j = \overline{1,k}$. Corresponding to an arbitrary basis in the space $C^{p\times n}$ is some representation of the given operator M, in particular

$$MX = \sum_{i,j=1}^{k} c_{ij}\, A_i X A_j^* \equiv \sum_{t,\tau=1}^{n} x_{t\tau} H_{t\tau}, \tag{6.1.5}$$

where

$$H_{t\tau} = ||h_{t\tau}^{ij}||_{i,j=1}^{p}, \quad h_{t\tau}^{ij} = [a_{it}^{(1)}, \dots, a_{it}^{(k)}]\, C\, [a_{j\tau}^{(1)}, \dots, a_{j\tau}^{(k)}]^*.$$

Here the relations $M\mathcal{H}_n \subseteq \mathcal{H}_p$, $C = C^*$ and $H = H^*$ are equivalent. If the matrices $A_1, \dots, A_k$ are linearly independent, then $i_\pm(C) = i_\pm(H)$.

Let $\mathcal{K}_n \subset \mathcal{H}_n$ ($\mathcal{K}_n^0 \subset \mathcal{K}_n$) be a set of nonnegative (positive) definite matrices of order n. The set $\mathcal{K}_n$ is a normal solid cone in the space $\mathcal{H}_n$ (see Section 5.1). Choose the classes of positive, strictly positive, strongly positive, and positively invertible operators of the form (6.1.5), using the respective inclusions

$$M\mathcal{K}_n \subset \mathcal{K}_p, \quad M\mathcal{K}_n^0 \subset \mathcal{K}_p^0, \quad M\mathcal{K}_n\backslash\{0\} \subset \mathcal{K}_p^0, \quad \mathcal{K}_p \subset M\mathcal{K}_n.$$

Note that the operator M is positive if and only if the conjugate operator M^* is positive. Similarly, the properties of strict and strong positivity must be true or not true for the operators M and M^*. For the positive operator M to be strictly positive it is necessary and sufficient that for some matrix $X_0 \geq 0$ the inequality $MX_0 > 0$ holds true.

A strictly positive operator can be noninvertible. This fact is confirmed by the following example:

$$MX = X + AXA^*, \quad A = \begin{bmatrix} a & 0 \\ 0 & -1/\bar{a} \end{bmatrix}, \quad a \neq 0.$$

Moreover, the linear operator $MX = (\operatorname{tr} X)L$, where $L > 0$, is strictly positive but noninvertible.

Lemma 6.1.1 *If $AA^* \geq BB^*$, then $B = AC$, where $A \in C^{n\times k}$, $B \in \in C^{n\times s}$, $C \in C^{k\times s}$. Here $CC^* \leq I_k$, if* $\operatorname{rank} A = k$. *Conversely, the inequality $AA^* \geq BB^*$ is a consequence of the relations $B = AC$ and $CC^* \leq I_k$.*

Using Lemma 6.1.1, it can be found that in a cone of positive operators, operators of the type AXA^* and AX^TA^* are extremal. Taking into consideration Lemmas 4.4.1 – 4.4.5, formulate the properties of the Schur operator.

Lemma 6.1.2 *Let $MX = \Omega \odot X$ $(\Omega \in \mathcal{H}_n)$ be the Schur operator, then:*

1) M is invertible $\iff \omega_{ij} \neq 0 \quad (\forall i, j)$;

2) M is positive $\iff \Omega \geq 0$;

3) M is strictly positive $\iff \Omega \geq 0,\ \omega_{ii} > 0 \quad (\forall i)$;

4) M is not strongly positive for $n > 1$;

5) M is positively invertible if $i_+(\Omega) = 1,\ \omega_{ii} > 0\ (\forall i)$;

6) M is positively invertible $\iff \left\| \dfrac{1}{\omega_{ij}} \right\|_1^n \geq 0,\ \omega_{ij} \neq 0\ (\forall i, j)$.

Using the spectral expansion of the matrix H, obtain the representation of the operator (6.1.5) with orthonormalized matrix coefficients:

$$MX = \sum_{s=1}^{r} \sigma_s \, D_s X D_s^*, \tag{6.1.6}$$

where $\sigma_1, \ldots, \sigma_r$ are nonzero eigenvalues of the matrix H,

$$h_{t\tau}^{ij} = \sum_{s=1}^{r} \sigma_s \, d_{it}^{(s)} \, \overline{d_{j\tau}^{(s)}}, \quad D_s = ||d_{it}^{(s)}||_{i,t=1}^{p,n}, \quad (D_s, D_q) = \begin{cases} 1, & s = q, \\ 0, & s \neq q. \end{cases}$$

The class of positive operators of the type (6.1.6) can be defined in terms of real matrices. Taking real and imaginary parts of the matrices $X = S + iK$ and $D_s = R_s + iG_s$, obtain the following criterion. The operator (6.1.6) is positive if and only if

$$\tilde{M}\tilde{X} = \sum_{s=1}^{r} \sigma_s \, \tilde{D}_s \tilde{X} \tilde{D}_s^T \geq 0, \quad \forall \tilde{X} \geq 0,$$

where

$$\tilde{X} = \begin{bmatrix} S & K \\ -K & S \end{bmatrix}, \quad \tilde{D}_s = \begin{bmatrix} R_s & G_s \\ -G_s & R_s \end{bmatrix},$$

$$S^T = S, \quad K^T = -K, \quad s = 1, \ldots, r.$$

This criterion follows from the equivalence of the matrix inequalities $X \geq 0$ and $\tilde{X} \geq 0$.

If $H \geq 0$, in particular $C \geq 0$, then the operator (6.1.5) is positive. However, the operator (6.1.5) can be positive even if $i_-(H) \neq 0$ or $i_-(C) \neq 0$. The simplest example of such an operator is the transposition operator

$$X^T = \sum_{t,\tau=1}^{n} x_{t\tau} \Delta_{\tau t}, \ \Delta = \begin{bmatrix} \Delta_{11} & \ldots & \Delta_{n1} \\ \cdots & \cdots & \cdots \\ \Delta_{1n} & \ldots & \Delta_{nn} \end{bmatrix}, \ i_{\pm}(\Delta) = \frac{n(n \pm 1)}{2},$$

where each block $\Delta_{\tau t}$ has a single nonzero (τ, t)-entry equal to 1.

It can be proved that if one of the following conditions holds true

$$\operatorname{rank}[A_1 x, \ldots, A_k x] = k, \quad \operatorname{rank} \begin{bmatrix} z^* a_{*1}^1 & \ldots & z^* a_{*1}^k \\ \cdots & \cdots & \cdots \\ z^* a_{*n}^1 & \ldots & z^* a_{*n}^k \end{bmatrix} = k,$$

where $x \in C^n$, $z \in C^p$ are some vectors, the inequality $C \geq 0$ is equivalent to the positivity of the operator M. Similarly, if for some $x \in C^n$ or $z \in C^p$ the corresponding condition

$$\operatorname{rank}[D_1 x, \ldots, D_r x] = r, \quad \operatorname{rank}\begin{bmatrix} z^* d_{*1}^1 & \ldots & z^* d_{*1}^r \\ \cdots & \cdots & \cdots \\ z^* d_{*n}^1 & \ldots & z^* d_{*n}^r \end{bmatrix} = r,$$

holds true, then the positivity criterion of the operator M is the inequality $H \geq 0$.

If for any matrix $X \geq 0$ of rank 1 the inequality $MX \geq 0$ holds true, then the operator M is positive. This proposition follows from the linearity of the operator M and the spectral expansion of non-negative definite matrices. Therefore the positivity of the operator (6.1.5) can be determined in the form

$$F_z \triangleq ||z^* H_{t\tau} z||_1^n \geq 0, \quad \forall z \in C^p. \tag{6.1.7}$$

The conditions of strict (strong) positivity are equivalent to the relations $F_z \geq 0$, $F_z \neq 0$, $\forall z \neq 0$ ($F_z > 0, \forall z \neq 0$).

If the following expansion

$$H_{t\tau} = U_t U_\tau^* + V_\tau V_t^*, \quad t, \tau = 1, \ldots, n\,, \tag{6.1.8}$$

holds true, then for any $z \in C^p$

$$F_z = ||z^* U_t U_\tau^* z||_1^n + ||z^T \overline{V}_t V_\tau^T \overline{z}||_1^n \geq 0.$$

Conversely, under the conditions (6.1.7) the blocks $H_{t\tau}$ are representable in the form (6.1.8). The last proposition can be proved based on the relations

$$F_z = L_z L_z^*, \quad f_{t\tau} = z^* H_{t\tau} z \equiv \sum_s l_{ts} \overline{l}_{\tau s}\,, \quad h_{t\tau}^{ij} = \frac{\partial^2 f_{t\tau}}{\partial \overline{z}_i \partial z_j}\,,$$

$$z^T = [z_1, \ldots, z_p], \quad t, \tau = 1, \ldots, n, \quad i, j = 1, \ldots, p,$$

where l_{ts} are some functions of z and $\overline{z}$, comprising the matrix L_z. In particular, for each vector $z \neq 0$, as L_z one can select a (unique) Hermitian nonnegative definite matrix satisfying the equality $F_z = L_z^2$ and representable in the form of a polynomial of F_z.

Lemma 6.1.3 *The operator (6.1.5) is positive if and only if blocks of the matrix H are representable in the form (6.1.8).*

Using the relations (6.1.6) – (6.1.8), one can obtain different algebraic conditions of positivity of the operator M. For example, consider the inequalities (6.1.7) and calculate the principal minors of the matrix F_z, corresponding to the given sets of numbers of rows and columns t:

$$\mu_t(z) = w_z^* \, \Phi_t \, w_z, \quad t = \{t_1, \dots, t_\nu\},$$

$$1 \le t_1 < \dots < t_\nu \le n, \quad 1 \le \nu \le n.$$

Here w_z is a vector of order $C^\nu_{p+\nu-1}$, consisting of the products $z_{j_1} \dots z_{j_\nu}$, and Φ_t is a matrix determined by the expressions

$$\Phi_t = ||\phi_t^{ij}||, \quad \phi_t^{ij} = \sum_{\xi,\eta} \det \begin{bmatrix} h_{t_1 t_1}^{\xi_1 \eta_1} & \dots & h_{t_1 t_\nu}^{\xi_1 \eta_\nu} \\ \dots & \dots & \dots \\ h_{t_\nu t_1}^{\xi_\nu \eta_1} & \dots & h_{t_\nu t_\nu}^{\xi_\nu \eta_\nu} \end{bmatrix},$$

$$\xi = \{\xi_1, \dots, \xi_\nu\}, \quad i = \{i_1, \dots, i_\nu\}, \quad 1 \le i_1 \le \dots \le i_\nu \le p,$$

$$\eta = \{\eta_1, \dots, \eta_\nu\}, \quad j = \{j_1, \dots, j_\nu\}, \quad 1 \le j_1 \le \dots \le j_\nu \le p,$$

where the summation is made by all sets of indices $\xi(\eta)$, coinciding upon the ordering with $i(j)$. The elements of the vector w_z, as well as the rows (columns) of the matrix Φ_t, corresponding to the sets of indices $i(j)$, are arranged in lexicographical order. From the above relations, the algebraic conditions of the positivity of the operator M follow:

$$\Phi_t \ge 0, \quad t = \{t_1, \dots, t_\nu\}, \quad 1 \le t_1 < \dots < t_\nu \le n, \quad \nu = 1, \dots, n.$$

From Lemma 6.1.3 and the relations (6.1.6) and (6.1.8) the general representation of positive operators follows.

Theorem 6.1.1 *The linear operator M is positive if and only if it is representable in the form*

$$MX = \sum_i A_i X A_i^* + \sum_j B_j X^T B_j^*. \tag{6.1.9}$$

In the expansion (6.1.9) each summand is an extremal operator. Therefore, in accordance with Theorem 6.1.1, positive operators are representable in the form of a sum of their extremal minorants. In the representation (6.1.9) the number of summands of the extremal minorants can be decreased if some of them are linearly expressed through the others, in particular, if the matrices A_i (or B_j) are linearly dependent.

We pass to the description of a class of positively invertible operators $M\colon \mathcal{H}_n \to \mathcal{H}_p$. Since the cone $\mathcal{K}_n$ in the space $\mathcal{H}_n$ is reproducing, then the rank of the positively invertible operator is equal to p^2 and hence $p \leq n$. The operator MM^* is invertible, and $M^+ = M^*(MM^*)^{-1}$ is the right inverse operator of the positively invertible operator M, i.e. $MM^+ = E$.

Let the operator M be represented in the form

$$M \triangleq L - P = L(E - S), \tag{6.1.10}$$

where L and P are given positively invertible and positive operators respectively. Then the operator $S = L^+P$ is positive, and the spectral inequality $\rho(S) < 1$ is the criterion of the positive invertibility of the operator $E - S$. From this inequality the conditions of the positive invertibility of the initial operator (6.1.10) follow, and $M^+ = (E - S)^{-1}L^+$. In the case $p = n$ the inverse proposition is true.

Taking into consideration the structure of the positive operator (6.1.9), take a subclass of positively invertible operators of the form (6.1.10):

$$MX = M_0X - M_1X - \ldots - M_rX, \tag{6.1.11}$$

where

$$M_jX = \begin{cases} A_jXA_j^*, & j \in J_1, \\ A_jX^TA_j^*, & j \in J_2, \end{cases},$$

$A_j \in C^{n\times n}$, J_1, and J_2 are the subsets of indices, for which $J_1 \cap J_2 = \emptyset$, $J_1 \cup J_2 = \{0, \ldots, r\}$. The action of each operator M_j in the space of n^2 vectors is described by the matrix

$$T_j = \begin{cases} A_j \otimes \bar{A}_j, & j \in J_1, \\ (A_j \otimes \bar{A}_j) \sum\limits_{t,\tau=1}^{n} \Delta_{t\tau} \otimes \Delta_{\tau t}, & j \in J_2. \end{cases}$$

Theorem 6.1.2 *The linear operator (6.1.11) is positively invertible if and only if the following relations hold true:*

$$\rho(T) < 1, \quad T(\lambda) = \lambda T_0 - T_1 - \ldots - T_r, \quad \det A_0 \neq 0, \qquad (6.1.12)$$

where $\rho(T)$ is a spectral radius of the matrix pencil $T(\lambda)$.

Let us give an example of a positively invertible operator not representable in the form (6.1.11):

$$MX = 6\,A_1XA_1^* + 5\,A_2XA_2^* - 3\,A_3XA_3^* \equiv S \odot X, \quad M^{-1}Y \equiv W \odot Y,$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

$$S = \begin{bmatrix} 6 & 6 & 12 \\ 6 & 3 & 3 \\ 12 & 3 & 2 \end{bmatrix}, \quad i_+(S) = 2,$$

$$W = \begin{bmatrix} 1/6 & 1/6 & 1/12 \\ 1/6 & 1/3 & 1/3 \\ 1/12 & 1/3 & 1/2 \end{bmatrix} > 0.$$

According to Lemma 6.1.2, the operator M is positively invertible. However, it is not representable in the form (6.1.11) due to the linear independence of the matrix coefficients A_1, A_2, and A_3.

It can be proved that operators representable in the form (6.1.11) with linearly independent matrix coefficients $A_0, \ldots, A_r$ are not positive. If an operator M is at the same time positive and positively invertible, then it is an extremal operator of the type AXA^* or AX^TA^*, where A is some matrix of full rank by column. The relations (6.1.11), (6.1.12) determine some class of positively invertible operators. The conditions (6.1.12) can be applied to the case $p \leq n$ under the limitation $\operatorname{rank} A_0 = p$. The most general representation of linear positively invertible operators has not been found yet.

6.2 Linear Equations in Partially Ordered Space

Let the Banach space $\mathcal{E}$ be partially ordered by a normal reproducing cone $\mathcal{K}$. Consider the class of equations

$$MX \triangleq LX - PX = Y, \tag{6.2.1}$$

where $X, Y \in \mathcal{E}$, and L and P are linear operators satisfying the condition $P\mathcal{K} \subseteq L\mathcal{K}$. In particular, suppose that the operator P is positive and the operator L is positively invertible with respect to $\mathcal{K}$:

$$P\mathcal{K} \subseteq \mathcal{K} \subseteq L\mathcal{K}. \tag{6.2.2}$$

In wide assumptions, equations with linear operators, occurring in applications, are described in the form (6.2.1), (6.2.2). Thus, the matrix equations underlying the Lyapunov second method in stability theory of linear differential, differential-difference, and some stochastic systems are representable in the form (6.2.1). In this case $\mathcal{K}$ is a cone of Hermitian nonnegative definite $n \times n$ matrices.

Define analogues of the concepts of rank, signature, and inertia of Hermitian matrices for the elements of the space $\mathcal{E}$ with a reproducing cone $\mathcal{K}$.

Let $Z \geq 0$ be an arbitrary element of the cone $\mathcal{K}$. Let $\mathcal{Z}^-$ ($\mathcal{Z}^+$) denote a set of elements $X \in \mathcal{K}$, for which there exists a number $\alpha > 0$ such that $\alpha X \leq Z$ ($\alpha X \geq Z$). The set $\mathcal{Z}^0 = \mathcal{Z}^+ \cap \mathcal{Z}^-$ generates an equivalence relation: $X \sim Y \Longleftrightarrow X, Y \in \mathcal{Z}^0$. If $Z > 0$ is an internal element of the cone $\mathcal{K}$, then $\mathcal{Z}^- = \mathcal{K}$ and $\mathcal{Z}^+ = \mathcal{K}_0$. The cone $\mathcal{K}$ splits into disjoint classes of equivalent elements of the type $\mathcal{Z}^0$. The class $\mathcal{Z}^0$ is called extreme if $\mathcal{Z}^- = \mathcal{Z}^0 \bigcup \{0\}$. If the class $\mathcal{Z}^0$ is not extreme, then the set $\mathcal{Z}^-$ contains nontrivial classes different from $\mathcal{Z}^0$. For a given element Z select an arbitrary sequence of classes according to the following rule:

$$\mathcal{Z}^0_0 = \mathcal{Z}^0, \quad \mathcal{Z}^0_t \subset \mathcal{Z}^-_{t-1} \backslash \mathcal{Z}^0_{t-1}, \quad t = 1, 2, \ldots . \tag{6.2.3}$$

If for some step t an extreme class $\mathcal{Z}^0_t = \{0\}$ is selected, then the sequence (6.2.3) has a finite length t. The element Z has a finite *rank* $r = r(Z)$ if all sequences of the classes selected in accordance with

(6.2.3) are finite, and their maximum allowable length equals r. The cone $\mathcal{K}$ has order *order* n, if

$$\max_{Z \in \mathcal{K}} r(Z) = n < \infty.$$

Let $Z \in \mathcal{E}$. If the cone $\mathcal{K}$ is reproducing, then there exists the decomposition

$$Z = Z_+ - Z_-, \quad Z_+ \in \mathcal{K}, \quad Z_- \in \mathcal{K}. \tag{6.2.4}$$

Let $i_+(Z)$ ($i_-(Z)$) denote the least value of the rank $r(Z_+)$ ($r(Z_-)$) which the components Z_+ (Z_-) can have in the decompositions of the element Z of the type (6.2.4). The decomposition (6.2.4) is called *inertial*, if $r(Z_+) = i_+(Z)$ and $r(Z_-) = i_-(Z)$. The numbers $r(Z) = i_+(Z) + i_-(Z)$ and $s(Z) = i_+(Z) - i_-(Z)$ determine respectively the rank and the *signature* of the element Z. If n is the order of the cone $\mathcal{K}$, then the triple of numbers $i_+(Z)$, $i_-(Z)$, and $i_0(Z) = n - r(Z)$ is the *inertia* $i(Z)$ of the element Z.

The equivalence relation determined in the cone $\mathcal{K}$ is true for the whole space $\mathcal{E}$. The elements X and Y are equivalent if $X_+ \sim Y_+$ and $X_- \sim Y_-$, where $X_\pm$ and $Y_\pm$ are components of the inertial decompositions X and Y. Obviously, inertias of all elements equivalent to each other coincide.

Note that a minimal decomposition of the form (6.2.4) for elements of space with a *minihedral cone* is inertial, and its components are uniquely determined by supremum and infimum operations: $Z_+ = \sup(Z, 0)$, $Z_- = -\inf(Z, 0)$. If $\mathcal{K}$ is a cone of nonnegative definite matrices, then the inertial decomposition of a Hermitian matrix describes its inertia and is determined by congruent transformation to diagonal form.

The introduced inertial characteristics and invariants of Hermitian matrices have similar properties. In particular, if $X \le Y$, then $i_+(X) \le i_+(Y)$ and $i_-(X) \ge i_-(Y)$. The latter inequalities follow from the definition of the numbers $i_\pm(\cdot)$ and the relation $X_+ - X_- = Y_+ - Y_- - Z$, where $Z \in \mathcal{K}$, $X_\pm$ ($Y_\pm$) are components of the inertial decomposition $X(Y)$. If $X \ge Y \ge 0$, then the relations $r(X) = r(Y)$ and $X \sim Y$ are equivalent. For any $X, Y \in \mathcal{E}$ the inequality $r(X + Y) \le r(X) + r(Y)$ holds true.

We will set out some properties of inertial characteristics determined for the solutions of the equation (6.2.1) and the elements of the iterative process of the convergence method

$$X_0 = G, \quad LX_{t+1} = PX_t + Y, \quad t = 0, 1, \dots , \tag{6.2.5}$$

where L and P are linear operators satisfying the conditions (6.2.2). For any initial approximation $G \in \mathcal{E}$ the inequality

$$\rho(T) < 1, \tag{6.2.6}$$

where $\rho(T)$ is the spectral radius of the operator pencil $T(\lambda) = P - \lambda L$, ensures the convergence of the sequence (6.2.5) to the unique solution X of the equation (6.2.1). If $MG \leq Y$, then this series monotone tends to X "from the bottom":

$$X_0 \leq X_1 \leq \dots \leq X.$$

Similarly, for $MG \geq Y$ we have the estimates "from the top":

$$X_0 \geq X_1 \geq \dots \geq X.$$

These statements follow from the assumption (6.2.2) and the known results in the case of the identity operator $L = E$. We will formulate more general statements.

Theorem 6.2.1 *Let the initial parameters of the process (6.2.5) satisfy the conditions*

$$Y + T(\alpha)G \geq 0, \quad Y - \alpha Y \in L\mathcal{K}, \tag{6.2.7}$$

where $\alpha > 0$ is some real number. Then there exists a sequence of positive numbers α_t such that

$$\alpha_0 X_0 \leq \alpha_1 X_1 \leq \dots \leq \alpha_t X_t \leq \dots . \tag{6.2.8}$$

If

$$Y + T(\beta)G \leq 0, \quad \beta Y - Y \in L\mathcal{K}, \tag{6.2.9}$$

where $\beta > 0$, then for some $\beta_t > 0$ the following inequalities

$$\beta_0 X_0 \geq \beta_1 X_1 \geq \dots \geq \beta_t X_t \geq \dots \tag{6.2.10}$$

hold true.

Proof. Form a sequence of numbers α_t satisfying the conditions

$$0 < \alpha_0 \le \alpha\alpha_1, \quad \alpha_t^2 \le \alpha_{t-1}\alpha_{t+1}, \quad t = 1, 2, \dots . \tag{6.2.11}$$

In the case $G \notin \mathcal{K}$ require that the equality $\alpha_0 = \alpha\alpha_1$ hold true. If $X_{t+1} \notin \mathcal{K}$ for some t, then assume $\alpha_t^2 = \alpha_{t-1}\alpha_{t+1}$. (6.2.7) and (6.2.11) imply $\alpha_0 X_0 \le \alpha_1 X_1$. The last inequality is equivalent to (6.2.7) if, e.g., the operator L is positively invertible and positive at the same time.

Show that $\alpha_{t-1}X_{t-1} \le \alpha_t X_t$ implies $\alpha_t X_t \le \alpha_{t+1}X_{t+1}$. According to (6.2.5), we have

$$L(\alpha_t X_{t+1} - \alpha_{t-1}X_t) = P(\alpha_t X_t - \alpha_{t-1}X_{t-1}) + (\alpha_t - \alpha_{t-1})Y.$$

If $\alpha < 1$ ($\alpha > 1$), then assume $\alpha_t > \alpha_{t-1}$, $Y \in L\mathcal{K}$ ($\alpha_t \le \alpha_{t-1}$, $-Y \in L\mathcal{K}$). If $\alpha = 1$, then the second condition (6.2.7) is automatically true irrespective of Y.

Taking into consideration the properties (6.2.2) of the operators L, P and of the sequence (6.2.11), obtain

$$\alpha_{t+1}X_{t+1} - \alpha_t X_t \ge \frac{\alpha_t}{\alpha_{t-1}}(\alpha_t X_{t+1} - \alpha_{t-1}X_t) \ge 0.$$

Hence, $\alpha_t X_t \le \alpha_{t+1}X_{t+1}$, $t = 0, 1, \dots$.

Similarly, proceeding from (6.2.5), (6.2.9) and the sequence β_t of the form

$$\beta_0 \ge \beta\beta_1 > 0, \quad \beta_t^2 \ge \beta_{t-1}\beta_{t+1}, \quad t = 1, 2, \dots , \tag{6.2.12}$$

one can find the sequence of inequalities (6.2.10). While finding β_{t+1} in (6.2.12) a strict inequality is possible if $X_{t+1} \in \mathcal{K}$.

The theorem is proved.

Remark 6.2.1 All second-order minors of the infinite *Hankel matrices*

$$\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \dots \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}, \quad \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \dots \\ \beta_1 & \beta_2 & \beta_3 & \dots \\ \beta_2 & \beta_3 & \beta_4 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix},$$

constructed according to (6.2.11) and (6.2.12) are respectively nonnegative and nonpositive. If $\alpha = 1$ ($\beta = 1$), then in (6.2.8) ((6.2.10)) one can assume $\alpha_t = 1$ ($\beta_t = 1$) for each t. Here there are no limitations on Y. In the other cases $Y \in L\mathcal{K}$ or $-Y \in L\mathcal{K}$.

Remark 6.2.2 If $Y \in \mathcal{K}$, then as G in the conditions $T(\lambda)$ an arbitrary eigenvector of the operator pencil $T(\lambda)$ can be chosen, which corresponds to the eigenvalue α. Taking into consideration the Krein–Rutman theorem on the spectral radius of a positive operator, one can assume $\alpha = \rho(T)$, $G \in \mathcal{K}$. If in (6.2.8) and (6.2.11) $\alpha_t = 1/\alpha^t$, then $X_{t+1} \geq \alpha X_t$, $X_t \geq \alpha^t G$, $t = 0, 1, \ldots$. In the case of convergence of (6.2.6) while $t \to \infty$ we have $X \geq 0$. It means that the operator $M = L - P$ is positively invertible. Note that for $\alpha = 1$ and $Y = LG$ from (6.2.7) and (6.2.8) the estimate $X \geq G$, $PX \geq 0$ follows.

Corollary 6.2.1 *Under the conditions (6.2.7) the signature of elements of the process (6.2.5) does not decrease:*

$$s(X_0) \leq s(X_1) \leq \ldots \leq s(X_t).$$

If its maximum value is achieved for the k-th iteration, then all the elements X_t for $t \geq k$ have the same inertia. Similarly, the conditions (6.2.9) ensure the sequence of inequalities

$$s(X_0) \geq s(X_1) \geq \ldots \geq s(X_t).$$

Upon attainment of the minimum value of the signature, the inertia of the elements of the process (6.2.5) does not change.

The proof of the statements of Corollary 6.2.1 follows from the relations (6.2.8) and (6.2.10), the obvious equality $s(\alpha X) = \operatorname{sign} \alpha \; s(X)$, and the monotonicity of the signature: $X \leq Y \Longrightarrow s(X) \leq s(Y)$.

Corollary 6.2.2 *Let $G \in \mathcal{K}$ and the conditions (6.2.7) hold true. Then*

$$r(X_0) < r(X_1) < \ldots < r(X_k) = r(X_{k+1}) = \ldots = m. \qquad (6.2.13)$$

Here from the inequality

$$c_0 X_0 + \ldots + c_t X_t + X_{t+1} \leq 0, \qquad (6.2.14)$$

where $c_0, \ldots, c_t$ are real numbers, the estimate follows:

$$k \leq min\{t, m - r(G)\}. \tag{6.2.15}$$

If $G \in \mathcal{K}$, then $X_t \in \mathcal{K}$ and in Corollary 6.2.1 $s(X_t) = r(X_t)$, $t = 0, 1, \ldots$. The maximum value of the rank in (6.2.13) is achieved for the k-th iteration, when $X_k \sim X_{k+1}$. Here $X_t \sim X_k$ for all $t \geq k$. This fact is observed at successive comparison of two neighbouring iterations (6.2.5), taking into account the following properties of the equivalence relation. If P is a positive operator, then $U \sim V$ implies $PU \sim PV$; if the operator L is positively invertible, then $LU \sim LV$ implies $U \sim V$ $(U, V \in \mathcal{K})$. The estimate (6.2.15) follows from the relations (6.2.8) and (6.2.14). Indeed, if the inequality (6.2.14) is solvable with respect to $c_0, \ldots, c_t$, then $X_{t+1} \leq cX_t$, where

$$c = |c_0| \frac{\alpha_t}{\alpha_0} + \ldots + |c_{t-1}| \frac{\alpha_t}{\alpha_{t-1}} + |c_t| > 0.$$

On the other hand, $X_{t+1} \geq (\alpha_t/\alpha_{t+1})X_t$, hence $X_{t+1} \sim X_t$.

Corollary 6.2.3 *Let the conditions (6.2.6), (6.2.7) be true, as well as the inequality*

$$h(S)\, Z = (h_0 E + h_1 S + \ldots + h_t S^t)\, Z \geq 0, \tag{6.2.16}$$

where $S = PL^{-1}$, $Z = MG - Y$, $G \geq 0$, $h_0, \ldots, h_t$ are real coefficients, $h_t > 0$. Then the solution $X \in \mathcal{K}$ of the equation (6.2.1) is equivalent to X_t, and the relations (6.2.13) – (6.2.15) hold true.

The proof of this proposition is made based on Corollary 6.6.2 and the relation

$$L^{-1}h(S)Z = h_0X_0 + (h_1 - h_0)X_1 + \ldots + (h_t - h_{t-1})X_t - h_tX_{t+1} \geq 0.$$

Note that the condition (6.2.16) holds true if h is an annihilating polynomial of the operator S. In the case $Z \in \mathcal{K}$ for the condition (6.2.16) to hold true it is sufficient that the operator $h(S)$ be positive. If the operators L and P commute, then instead of (6.2.16) in Corollary 6.2.3 one can use the inequality

$$(h_0 L^t + h_1 PL^{t-1} + \ldots + h_t P^t)\, Z \geq 0.$$

The check of the last inequality is not connected with the inversion of the operator L.

In conclusion we will describe some general properties of the solutions of the equation (6.2.1) and a special technique connecting the analysis of the class of equations (6.2.1) with the theory of linear matrix equations.

Lemma 6.2.1 *Let L be an invertible operator and the inclusion $P\mathcal{K} \subseteq L\mathcal{K}$ be true. Then for any $Y \in L\mathcal{K}$ the equation (6.2.1) has a solution $X \in \mathcal{K}$ if and only if the inequality (6.2.6) holds true.*

Proof. Under the condition $P\mathcal{K} \subseteq L\mathcal{K}$ the operator $S = L^{-1}P$ is positive, and its spectral radius $\rho(S)$ coincides with $\rho(T)$. Therefore the inequality (6.2.6) ensures the existence of an inverse operator

$$(E - S)^{-1} = E + S + S^2 + \ldots = M^{-1}L,$$

where $M = L - P$. This operator is positive, which is equivalent to the inclusion $L\mathcal{K} \subseteq M\mathcal{K}$. Here $\mathcal{K}$ can be a wedge, an arbitrary cone in particular.

Let the inclusions $P\mathcal{K} \subseteq L\mathcal{K} \subseteq M\mathcal{K}$ hold true. Since L is an invertible operator and the cone $\mathcal{K}$ is reproducing, then the operator M is also invertible. Here S is a positive operator, and $E - S$ is a positively invertible operator. Hence, the estimate $\rho(S) < 1$ holds true.

The lemma is proved.

If L is a positively invertible operator, then under the condition $L\mathcal{K} \subseteq M\mathcal{K}$ the operator M is also positively invertible. Therefore, from Lemma 6.2.1 and the known properties of positively invertible operators (see Section 6.3) the next propositions follow.

Theorem 6.2.2 *Let the operators L and P satisfy the conditions (6.2.2) with a normal reproducing cone $\mathcal{K}$. Then the following statements are equivalent:*

1) the operator $M = L - P$ is positively invertible ($\mathcal{K} \subseteq M\mathcal{K}$);

2) the spectral inequality (6.2.6) is true.

If $\mathcal{K}_0 \neq \emptyset$ is the set of inner points of the cone $\mathcal{K}$, then the statements 1) and 2) are equivalent to each of the following statements:

3) For any $Y \in \mathcal{K}_0$ the equation (6.2.1) has the solution $X \in \mathcal{K}_0$;

4) there exist $X \in \mathcal{K}_0$ and $Y \in \mathcal{K}_0$ satisfying the equation (6.2.1).

Formulate an analogue of Theorem 4.6.2 for the equation (6.2.1).

Theorem 6.2.3 *Let the exponential operator e^{-Mt} be positive with respect to a normal reproducing cone $\mathcal{K}$ for all $t \geq 0$. Then the operator M is positively invertible if and only if its spectrum is located on the open right half-plane* $\operatorname{Re} \lambda > 0$.

The proof of this proposition is made in the same way as in the case of matrix equations with the use of the theorem on the spectral radius of a positive operator (see the proof of Theorem 4.6.2 and Section 5.3).

If in Theorem 6.2.3 $M = L - P$, and the operators e^{-Lt} and e^{Pt} are positive, then the operator e^{-Mt} is also positive (see Lemma 5.2.2). In particular, one can assume $M = aE - P$, $a > 0$. Note that the positivity of the operator e^{-Lt} is equivalent to the positive invertibility of the operator e^{Lt}, and the operator e^{Pt} for $t \geq 0$ is positive if such is the operator P. This follows from the corresponding relations

$$e^{-Lt} e^{Lt} \equiv E, \quad e^{Pt} = \sum_{k=0}^{\infty} \frac{t^k}{k!} P^k.$$

Suppose that the positive operator P in (6.2.1) has the following structure:

$$PX \equiv QRX = \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij}(X)\, Q_{ij}, \qquad (6.2.17)$$

where

$$QZ = \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij}\, Q_{ij}, \quad Q\hat{\mathcal{K}} \subseteq \mathcal{K},$$

$$RX = \begin{bmatrix} r_{11}(X) & \dots & r_{1m}(X) \\ \cdots & \cdots & \cdots \\ r_{n1}(X) & \dots & r_{nm}(X) \end{bmatrix}, \quad R\mathcal{K} \subseteq \hat{\mathcal{K}},$$

$r_{ij} \in \mathcal{E}^*$ are linear functionals, $Q_{ij} \in \mathcal{E}$, $\mathcal{K} \subset \mathcal{E}$ and $\hat{\mathcal{K}} \subset C^{n \times m}$ are given normal reproducing cones. As $\hat{\mathcal{K}}$ can be, e.g., cones of nonnegative and nonnegative definite matrices.

Construct a matrix equation

$$Z - WZ = G, \tag{6.2.18}$$

where W is a linear operator acting in the space of matrices $C^{n\times m}$ and determined by

$$WZ = \sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}W_{ij}, \quad W_{ij} = RH_{ij}, \quad LH_{ij} = Q_{ij}.$$

This operator is representable in the form $W = RL^{-1}Q$, and under the conditions (6.2.2) and (6.2.17) it is positive with respect to the cone $\hat{\mathcal{K}}$.

Let $\rho(W)$ be a spectral radius of the operator W and its eigenvalue. If $\lambda \in \sigma(T)$ is an eigenvalue of the pencil of operators $T(\lambda) = P - \lambda L$, then $\lambda \in \sigma(W)$ or $\lambda = 0$. Indeed, the equality $PV = \lambda LV$ entails $WU = \lambda U$, where $U = RV$. If $U \neq 0$, then $\lambda \in \sigma(W)$. If $U = 0$, then $\lambda = 0$, since $V \neq 0$. Similarly, if $\lambda \in \sigma(W)$, then either $\lambda \in \sigma(T)$, or $\lambda = 0$. Consequently, $\rho(W) = \rho(T)$, and, taking into consideration Theorem 6.2.2, obtain the following proposition.

Theorem 6.2.4 *Let the relations (6.2.2) and (6.2.17) hold true. Then the following statements are equivalent:*

1) for any $Y \in \mathcal{K}$ the equation (6.2.1) has the solution $X \in \mathcal{K}$;

2) for any matrix $G \in \hat{\mathcal{K}}$ the equation (6.2.18) has the solution $Z \in \hat{\mathcal{K}}$;

3) $\rho(W) < 1$.

Construct a matrix of the operator W with respect to a unit basis:

$$\Sigma = \begin{bmatrix} r_{11}(H_{11})\dots r_{11}(H_{1m}) & \dots & r_{11}(H_{n1})\dots r_{11}(H_{nm}) \\ \dots & \dots & \dots \\ r_{1m}(H_{11})\dots r_{1m}(H_{1m}) & \dots & r_{1m}(H_{n1})\dots r_{1m}(H_{nm}) \\ \dots & \dots & \dots \\ r_{n1}(H_{11})\dots r_{n1}(H_{1m}) & \dots & r_{n1}(H_{n1})\dots r_{n1}(H_{nm}) \\ \dots & \dots & \dots \\ r_{nm}(H_{11})\dots r_{nm}(H_{1m}) & \dots & r_{nm}(H_{n1})\dots r_{nm}(H_{nm}) \end{bmatrix}.$$

The eigenvalues of this matrix form the spectrum of the operator W. If in the decomposition (6.2.17) $Q_{ij} \in \mathcal{K}$, and $\hat{\mathcal{K}}$ is a cone of nonnegative $n \times m$ matrices, then all elements of the matrix Σ are nonnegative. In this case one can use the known methods of estimate of the spectral radius of nonnegative matrices. Thus, the inequality $\rho(\Sigma) < r$ holds true if and only if all the successive principal minors of the matrix $rI - \Sigma$ are positive.

Corollary 6.2.4 *Let the conditions (6.2.2), (6.2.17) hold true and all the elements of the matrix Σ be nonnegative. Then the operator $M = L - P$ is positively invertible if and only if all successive principal minors of the matrix $I - \Sigma$ are positive.*

Corollary 6.2.5 *Let the conditions of Corollary 6.2.4 hold true, as well as*

$$\max_{i,j} r_{ij}(H) < 1, \quad LH = \sum_{i=1}^{n} \sum_{j=1}^{m} Q_{ij}.$$

Then for any $Y \in \mathcal{K}$ the equation (6.2.1) has the solution $X \in \mathcal{K}$.

In the case of solid cones $\mathcal{K}$ and $\hat{\mathcal{K}}$, Theorem 6.2.4 can be strengthened and supplemented by statements related to the usage of the sets of inner points of $\mathcal{K}_0$ and $\hat{\mathcal{K}}_0$ (see Theorem 6.2.2). Determine the connection between the conditions of solvability of the respective equations (6.2.1) and (6.2.18) over $\mathcal{K}_0$ and $\hat{\mathcal{K}}_0$, supposing that $Y \in \mathcal{K}$, $G \in \hat{\mathcal{K}}$ and the relations (6.2.2) and (6.2.17) hold true.

If $\rho(W) < 1$, then the equation (6.2.18) has a solution $Z \in \hat{\mathcal{K}}_0$ if and only if for some k the condition

$$G + WG + \cdots + W^k G \in \hat{\mathcal{K}}_0$$

holds true. For the estimate of the number k it is possible to use Corollary 6.2.2. If $Z \in \hat{\mathcal{K}}_0$ is a solution of the equation (6.2.18), then under the condition $Q\hat{\mathcal{K}}_0 \subseteq L\mathcal{K}_0$ the equation (6.2.1) is solvable in the form

$$X = L^{-1}QZ \in \mathcal{K}_0, \quad Y = QG \in \mathcal{K}.$$

Let the right-hand sides of the equations (6.2.1) and (6.2.18) be connected by the relations

$$Y = LH \in \mathcal{K}, \quad G = RH \in \hat{\mathcal{K}},$$

where $H \in \mathcal{K}$. Then, if $X \in \mathcal{K}_0$ is a solution of the equation (6.2.1), and the condition $R\mathcal{K}_0 \subseteq \hat{\mathcal{K}}_0$ holds true, then the matrix

$Z = RX \in \hat{\mathcal{K}}_0$ satisfies the equation (6.2.18). Conversely, if the matrix $Z \in \hat{\mathcal{K}}_0$ satisfies the equation (6.2.18) under the condition

$$Y + Q\hat{\mathcal{K}}_0 \subseteq L\mathcal{K}_0, \tag{6.2.19}$$

then the equation (6.2.1) has a solution $X \in \mathcal{K}_0$ for which $Z = RX$. The inclusion (6.2.19) means that the equation $L\tilde{X} = \tilde{Y}$ has a solution $\tilde{X} \in \mathcal{K}_0$, as soon as $\tilde{Y} = Y + QZ$ and $Z \in \hat{\mathcal{K}}_0$. For it to hold true, one of the following conditions is sufficient: $Y \in \mathcal{K}_0$, $H \in \mathcal{K}_0$, $Q\hat{\mathcal{K}}_0 \subseteq \mathcal{K}_0$ or $Q\hat{\mathcal{K}}_0 \subseteq L\mathcal{K}_0$. In the case $\ker R = 0$ the equation (6.2.1) has a solution X if and only if the expression $Z = RX$ is a solution of the matrix equation (6.2.18).

6.3 Notes and References

6.1 The statements of Lemmas 6.1.1–6.1.3 and Theorems 6.1.1 and 6.1.2 were obtained by Mazko [19, 23, 29, 30] on the basis of a number of facts of the matrix theory, taken from Gantmacher [1], Voevodin, Kuznetsov [1], Lancaster [1], and Horn, Johnson [1]. Representation of linear operators in a matrix space see also in Vetter [1], Schneider [1], Hill [1], and others.

6.2 The class of equations (6.2.1) – (6.2.2) and the respective iterative process (6.2.5) in a partially ordered space were studied in Mazko [18] and Korenevskii, Mazko [1]. The known results, in the case of an identity operator $L = E$, are given in Krasnoselskii, Lifschits, Sobolev [1], Ran, Reurings [1], and others. Theorems 6.2.2 and 6.2.3 are formulated on the basis of the spectral theory of monotone and monotone invertible operators (see Krasnoselskii, Lifschits, Sobolev [1]). Theorem 6.2.4 and its corollaries are formulated in consideration of the special structure of the monotone operator (6.2.17). In this case one can use the spectral properties of nonnegative matrices (see, e.g., Gantmacher [1], Horn, Johnson [1], and others).

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Volume 2

MATRIX EQUATIONS, SPECTRAL PROBLEMS AND STABILITY OF DYNAMIC SYSTEMS

A. G. Mazko

This book contains the methods for localization of eigenvalues of matrices and matrix functions, based on the construction and study of the generalized Lyapunov equation. The theory of linear equations and operators in a matrix space is developed and the known theorems on the inertia of Hermitian solutions of matrix equations are generalized. New algebraic methods for stability analysis, an evaluation of spectrum and representation of solutions of linear arbitrary order differential and difference systems are worked out. The methods for research and comparison of dynamic systems in partially ordered Banach space are developed. The book is intended for researchers, engineers, and postgraduates interested in the theory of stability and stabilization of dynamic systems, matrix analysis and applications.

Enriched with over 1200 mathematical expressions, **Matrix equations, Spectral problems and Stability of Dynamic Systems** evolves the **inertia theory** for matrix equations ... presents new **Lyapunov type equations** for various classes of dynamic systems ... advances the stability theory of **positive and monotone dynamic systems** with respect to some classes of cones ... demonstrates the effective application of the matrix equations approaches in stability analysis and control theory ... and more.

About the Author

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