Comparison and ordering problems for dynamic systems set

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This paper develops a comparison principle in stability theory of dynamic systems. We consider the comparison and ordering problems for a set of differential systems. The comparison technique suggested is based on mapping of the state spaces of specified set of systems in the partially ordered space with a solid cone. We describe the invariant sets of differential systems in the form of cone inequalities using a differentiation operator along systems trajectories. We also generalize known positivity conditions for linear and nonlinear differential systems with respect to typical classes of cones.

1. Introduction

The well known comparison methods for dynamic systems are essential development and generalization of Lyapunov functions method in stability theory. These methods allow to reduce the stability and state estimation problems for complicated differential and difference systems to studying similar problems for more simple systems in a partially ordered space. Vector, matrix and operator analogs of Lyapunov functions and their derivatives along considered systems solutions are used (see, for example, [1-4]).

In this work, the comparison problem for finite set of differential systems is formulated in the form of a cone inequality for some operator. We construct the conditions ensuring realization of the inequality on the basis of generalized derivation of a comparison operator and using elements of a dual cone. Arrangement problems for a set of systems and, in particular, selection of a dominant system agree with certain structure of a comparison operator. Besides, the technique for construction and research of invariant sets of the differential systems described in the form of cone inequalities is proposed. Known positivity conditions with respect to typical classes of cones are generalized for linear and nonlinear differential systems.

2. Definitions and auxiliary results

A convex closed set K in a real normed spaces Ξ is called *a cone* if $\mathrm{K}+\mathrm{K}\subset\mathrm{K}$, $\alpha\mathrm{K}\subset\mathrm{K}\ \forall\alpha\ge 0$ and $\mathrm{K}\bigcap -\mathrm{K}=\{0\}$. The *dual cone* K^* consists of linear functionals $\varphi\in\Xi^*$ taking nonnegative values on elements of K. A space containing a cone is partially ordered. The inequality $X\overset{\mathrm{K}}{\ge}Y\ \left(X\overset{\mathrm{K}}{>}Y\right)$ means that $X-Y\in\mathrm{K}\ (X-Y\in\operatorname{int}\mathrm{K})$, where $\operatorname{int}\mathrm{K}$ is a set of inner points of K. *Bodily, reproducing* and *normal cone* are defined accordingly by $\operatorname{int}\mathrm{K}\ne\varnothing$, $\Xi=\mathrm{K}-\mathrm{K}$ and $0\le X\le Y\Rightarrow\|X\|\le c\|Y\|$, where c is a universal constant. A cone K is normal $\Leftrightarrow$ K^* is reproducing.

Let $\Xi_1\ (\Xi_2)$ be a Banach space with a cone $\mathrm{K}_1\ (\mathrm{K}_2)$. The linear operator $M:\Xi_1\to\Xi_2$ is called *monotone* if $X\overset{\mathrm{K}_1}{\ge}Y$ implies $MX\overset{\mathrm{K}_2}{\ge}MY$. Monotonic property of a linear operator is equivalent to its *positivity property*: $X\overset{\mathrm{K}_1}{\ge}0\Rightarrow MX\overset{\mathrm{K}_2}{\ge}0$. The operator inequality $M\le L$ means that $L-M$ is *positive*. The operator M is called *positively invertible*, if the inverse operator M^{-1} is positive. If the cone K_2 is normal and reproducing, then the inequalities $M_1\le M\le M_2$ with positively invertible operators M_1 and M_2 imply that M is also positively invertible and $M_2^{-1}\le M^{-1}\le M_1^{-1}$ [5]. If K_2 is normal and solid and $M\le L$, where

L is a positively invertible operator, then positive invertibility of M is equivalent to a solvability of the equation $MX = Y$ in the form of $X \overset{K_1}{\geq} 0$ for any $Y \overset{K_2}{>} 0$.

Monotonic and positivity properties are similarly defined for nonlinear operators $F : \Xi_1 \to \Xi_2$. If $F\Xi_1 \subset K_2$, then an operator F is called *everywhere positive*. If $FX \overset{K_2}{>} FY$ for $X \overset{K_1}{\geq} Y \overset{K_1}{\geq} 0$, then F is monotone on the cone K_1.

A dynamic system, whose state $X(t) = \Omega(t,t_0)X_0$ for every $t > t_0 \geq 0$ is defined by a positive (monotone) operator $\Omega(t,t_0) : X \to X$, is positive (monotone) with respect to a cone. The system has a time-varying invariant set $I_t \subset X$ if for any $t_0 \geq 0$ from $X_0 \in I_{t_0}$ follows $X(t) \in I_t$ at $t > t_0$. The system with invariant cone K_t is positive with respect to K_t. Positivity and monotonic properties of the differential system

$$\dot{X} = F(X,t), \quad X \in X, \quad t \geq 0 \tag{1}$$

with respect to a solid cone K_t under the restrictions

$$\operatorname{int} K_0 \neq \varnothing, \quad K_t \subseteq K_\tau, \quad t < \tau \tag{2}$$

follow from corresponding conditions

$$X \overset{K_t}{\geq} 0, \quad \varphi \in K_t^*, \quad \varphi(X) = 0 \quad \Rightarrow \quad \varphi(F(X,t)) \geq 0, \tag{3}$$

$$X \overset{K_t}{\geq} Y, \quad \varphi \in K_t^*, \quad \varphi(X) = \varphi(Y) \quad \Rightarrow \quad \varphi(F(X,t)) \geq \varphi(F(Y,t)) \tag{4}$$

where K_t^* is a dual cone and $t \geq 0$ [4]. In the case of a constant cone $K_t \equiv K$ the condition (3) ((4)) is equivalent to positivity (monotonic) of the system (1).

Through $\overline{M}$ we denote a class of systems (1) between whose solutions and solutions of the differential inequalities $\dot{Z} \overset{K_t}{\leq} F(Z,t)$ it is possible to establish such correspondence, that for any $t_0 \geq 0$ from $Z(t_0) \overset{K_{t_0}}{\leq} X(t_0)$ follows $Z(t) \overset{K_t}{\leq} X(t)$ for $t > t_0$. The extensions $\overline{M}_1$ and $\overline{M}_2$ of $\overline{M}$ are defined under additional requirements $X(t_0) \in K_{t_0}$ and $Z(t_0) \in K_{t_0}$ accordingly. The classes of systems $\underline{M}$, $\underline{M}_1$ and $\underline{M}_2$ are similarly defined using instead of K_t a negative cone $-K_t$.

Obviously, $\overline{M} \subseteq \overline{M}_1 \subseteq \overline{M}_2$ and $\underline{M} \subseteq \underline{M}_1 \subseteq \underline{M}_2$. A system from $\overline{M} \cup \underline{M}$ is monotone with respect to K_t. If $F(0,t) \in K_t$ ($F(0,t) \in -K_t$), then a system of the class $\overline{M}_2$ ($\underline{M}_2$) is positive with respect to K_t ($-K_t$) and it is monotone in K_t ($-K_t$). Under conditions (2) and (4) with a solid cone K_t, the system (1) belongs to both classes $\overline{M}$ and $\underline{M}$ [4].

The isolated equilibrium state $X \equiv 0$ of a dynamic system is called stable in I_t if for any $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta > 0$ such that from $X_0 \in S_\delta(t_0)$ follows $X(t) \in S_\varepsilon(t)$ at $t > t_0$, where $S_\varepsilon(t) = \{X \in I_t : \|X\| < \varepsilon\}$. If, in addition, for some $\delta > 0$ from $X_0 \in S_\delta(t_0)$ follows $\|X(t)\| \to 0$ as $t \to \infty$, the state $X \equiv 0$ is asymptotically stable in I_t. If the state $X \equiv 0$ of the system with an invariant set I_t is stable (asymptotically stable) by Lyapunov, then it is stable (asymptotically stable) in I_t.

Invariant sets, positivity, monotonic and stability in I_t are similarly defined for the discrete time dynamic systems.

3. Constructing of invariant sets of differential systems

We define invariant sets of the differential system of order $s \geq 2$

$$G(X, \dot{X}, \ldots, X^{(s)}, t) = 0, \quad X \in \mathrm{X}, \quad t \geq 0,$$

in a phase space $\hat{\mathrm{X}} = \mathrm{X} \times \cdots \times \mathrm{X}$ of the corresponding differential first order system

$$\dot{X}_1 = X_2, \ \ldots \ , \dot{X}_{s-1} = X_s, \ G(X_1, \ldots, X_s, \dot{X}_s, t) = 0,$$

where $X_1 = X$. If operator G admits to eliminate the higher derivative $X^{(s)}$, then given system can be represented in the form of (1).

We consider in a Banach space the system (1) with an operator F ensuring existence of a unique solution $X(t)$ in some domain $\Theta \subseteq \mathrm{X}$ with $X(t_0) = X_0 \in \Theta$. We describe invariant sets of the system in the form

$$\mathrm{I}_t = \left\{ X \in \Theta : V(X,t) \overset{\mathrm{K}_t}{\geq} 0 \right\} \tag{5}$$

where $V : \mathrm{X} \times [0,\infty) \to \mathrm{E}$ is an operator-function continuously differentiated in $\Theta \times [0,\infty)$, and K_t is some cone in space E.

Let's define an operator of derivation along the system trajectories as a strong derivative of a composite function

$$D_t V(X,t) = \frac{d}{dt} V(\Psi(\tau,t,X),\tau)\big|_{\tau=t} \tag{6}$$

where $X(\tau) = \Psi(\tau,t,X)$ is a solution of system (1) with $X(t) = X$. If $\mathrm{X} = C^n$ and $\mathrm{E} = C^m$, then

$$D_t V(X,t) = V'_X(X,t) F(X,t) + V'_t(X,t),$$

where $V'_X(X,t)$ is a Jacobi $m \times n$ matrix composed of partial derivatives V with respect to X. We shall consider generalization of this relation using Gateaux and Freshet types derivatives of a nonlinear operator [6]. For example, it is possible to consider that $V'_t(X,t)$ is a strong time derivative, and $V'_X(X,t)$ is the Gateaux derivative with respect to X, i.e. a linear bounded operator of the type $V'_X(X,t)H = \frac{d}{d\tau} V(X + \tau H, t)\big|_{\tau=0}$.

Lemma 1. *Let the cone* K_t *satisfies (2). Then (5) is an invariant set of system* (1) *if and only if for any* $t \geq 0$

$$X \in \mathrm{I}_t, \ \varphi \in \mathrm{K}_t^*, \ \varphi(V(X,t)) = 0 \ \Rightarrow \ \varphi(D_t V(X,t)) \geq 0. \tag{7}$$

Proof. Let $X(t)$ be a solution of (1) with the initial conditions $X(t_9) = X_0 \in \mathrm{I}_{t_0}$. Then by definition of D_t we have the relation

$$\int_{t_0}^{t} D_s V(X(s),s) ds = V(X(t),t) - V(X_0,t_0).$$

Let's assume that for some $\tau \geq t_0$ the value of $V(X_\tau,\tau)$, where $X_\tau = X(\tau)$, reaches the boundary of K_τ. Then $\varphi(V(X_\tau,\tau)) = 0$ for some $\varphi \neq 0 \in \mathrm{K}_\tau^*$.

We define a neighborhood of the set (5) in the form

$$\mathrm{I}_t^\varepsilon = \left\{ X \in \Theta : V_\varepsilon(X,t) \overset{\mathrm{K}_t}{\geq} 0 \right\}, \quad V_\varepsilon(X,t) = V(X,t) + \varepsilon \varpi(t) Y,$$

where $\varepsilon > 0$, $Y \overset{K_\tau}{>} 0$, and $\varpi(t) \geq 0$ is a scalar continuously differentiable function such that $\varpi(\tau) = 0$ and $\dot{\varpi}(\tau) > 0$. For example, we can suppose that $\varpi(t) = \arctan(t - \tau)$. Then $I_t \subset I_t^\varepsilon$ and $I_t^\varepsilon \to I_t$ as $\varepsilon \to 0$, $t \geq \tau$.

Since $\varphi(Y) > 0$ and $V_\varepsilon(X_\tau, \tau) = V(X_\tau, \tau)$ according to (7) for the some $\delta > 0$ we have

$$\varphi(D_t V_\varepsilon(X,t)) = \varphi(D_t V(X,t)) + \frac{\varepsilon}{1+(t-\tau)^2}\varphi(Y) \geq 0, \quad \tau \leq t \leq \tau + \delta,$$

$$\int_\tau^{\tau+\delta} \varphi(D_t V_\varepsilon(X(t),t))dt = \varphi(V_\varepsilon(X(\tau+\delta), \tau+\delta)) \geq 0.$$

It means that the trajectory $X(t)$ at the instant τ cannot leave the limits of I_τ^ε, i.e. $V_\varepsilon(X(t),t) \overset{K_\tau}{\geq} 0$ for $\tau \leq t \leq \tau + \delta$. Otherwise for some $\varphi \in K_\tau^*$ and arbitrarily small $\delta > 0$ it would be hold the relations $\varphi(V(X_\tau, \tau)) = 0$ and $\varphi(V_\varepsilon(X(\tau+\delta), \tau+\delta)) < 0$. Taking into account (2), we have $X(t) \in I_t^\varepsilon$ for $\tau \leq t \leq \tau + \delta$. Since K_t is a closed set, for $\varepsilon \to 0$ we have $V_\varepsilon(X(t),t) \to V(X(t),t) \overset{K_t}{\geq} 0$, $\tau \leq t \leq \tau + \delta$.

Thus, (5) is an invariant set of the system (1).

The converse follows from the Lagrange's theorem:

$$\varphi(V(X(\tau+\delta), \tau+\delta)) - \varphi(V(X(\tau),\tau)) = \delta\varphi(D_\xi V(X(\xi),\xi)),$$

where $\xi \in (\tau, \tau+\delta)$. If $\varphi(V(X(\tau),\tau)) = 0$ and $X(\tau+\delta) \in I_{\tau+\delta}$ for small enough $\delta > 0$, then it is necessary that $\varphi(D_\tau V(X(\tau),\tau)) \geq 0$.

Note 1. The condition (7) holds if for some continuous scalar function $\alpha(X,t)$

$$D_t V(X,t) + \alpha(X,t)V(X,t) \overset{K_t}{\geq} 0, \quad X \in \partial I_t, \quad t \geq 0 \tag{8}$$

Example 1. For the nonlinear system

$$\dot{x} = f(x,t), \quad x \in C^n, \quad t \geq 0 \tag{9}$$

we construct invariance conditions of the time-varying ellipsoidal cone I_t described by (5) with the operator $V(x,t) = \left[x^* Q(t)x, x^* h(t)\right]^T$, where $Q(t)$ is a nonsingular Hermit matrix having only one positive eigenvalue $q(t)$, and $h(t)$ is an eigenvector of $Q(t)$ corresponding to $q(t)$. Here as K_t we use the constant cone of nonnegative vectors R_+^2. Therefore in (7), where

$$D_t V(x,t) = \left[x^* \dot{Q} x + f^*(x,t)Q(t)x + x^* Q(t) f(x,t), x^* \dot{h}(t) + f^*(x,t)h(t)\right]^T,$$

we can use only two functionals from K^*. If $\varphi(y) = y_1$ with $y = [y_1, y_2]^T$, then according to (7) we have

$$x^* \dot{Q} x + f^*(x,t)Q(t)x + x^* Q(t) f(x,t) \geq 0, \quad x \in \partial I_t, \quad t \geq 0 \tag{10}$$

where $\partial I_t = \{x \in I_t : x^* Q(t)x = 0\}$. If $\varphi(y) = y_2$, then

$$f^*(0,t)h(t) \geq 0, \quad t \geq 0 \tag{11}$$

Here we use the fact that under specified spectral restrictions on $Q(t)$ from $x^* Q(t)x \geq 0$ and $x^* h(t) = 0$ follows that $x = 0$.

Conditions (10) and (11) are equivalent to an invariance of the cone I_t for system (9). The inequality (11) holds always for systems with zero equilibrium state, i.e. $f(0,t)\equiv 0$. Such is, for example, the differential system

$$\dot{x}=A(x,t)x, \tag{12}$$

where $A(x,t)$ is a continuous matrix function. In this case, according to (8) we have the matrix inequality

$$\dot{Q}(t)+\alpha(x,t)Q(t)+A^{*}(x,t)Q(t)+Q(t)A(x,t)\geq 0,\quad x\in\partial I_t,\quad t\geq 0 \tag{13}$$

If there exists a continuous function $\alpha(x,t)$ which satisfies (13), then (12) is a positive system with respect to I_t.

Note, that the matrix inequality (13) is a generalization of known invariance conditions of ellipsoidal cone for linear systems [7, 8].

Example 2. For the linear system

$$\dot{x}=A(t)x+B(t)u,\quad \dot{u}=C(t)x+D(t)u,\quad x\in R^{n},\quad u\in R^{m},\quad t\geq 0 \tag{14}$$

we define the set $I_t:\max_k|x_k|\leq\alpha(t)\min_s u_s$, where $\alpha(t)>0$ is a differentiable function. I_t is a normal solid cone described in the form of (5) with the operator

$$V(X,t)=\begin{bmatrix}\alpha^2u^2\otimes e_n-e_m\otimes x^2\\ u\end{bmatrix},\quad X=\begin{bmatrix}x\\ u\end{bmatrix},\quad e_n=[1,\dots,1]^T\in R^n,$$

where $\otimes$ is Kronecker product and x^2 is an element wise vector operation. Here as K_t we use a cone of nonnegative vectors $R_+^{m(n+1)}$. The criterion (7) for positivity property of the systems (14) with respect to I_t is reduced to the system of inequalities

$$\alpha d_{sj}\geq|b_{kj}|,\quad j\neq s,\qquad \dot{\alpha}\pm\alpha(\alpha c_{sk}\mp a_{kk})+\sum_j(\alpha d_{sj}\mp b_{kj})\geq\alpha\left(\sum_{i\neq k}\left|\alpha c_{si}\mp a_{ki}\right|\right),\quad t\geq 0,$$

where $k,i=\overline{1,n}$, $s,j=\overline{1,m}$. Here for simplicity a time dependence is omitted for α and elements of the matrices A,B,C and D.

Example 3. The autonomous second order differential system

$$\ddot{x}+B\dot{x}+Ax=0,\quad x\in R^n,\quad t\geq 0 \tag{15}$$

has the invariant set $I:x^T(S+B^TRB)x+2\dot{x}^TRBx+\dot{x}^TR\dot{x}\geq 0$ if for some $\alpha<0$ the system of matrix correlations

$$\alpha^2S-\alpha(B^TS+SB)-(S-A^TR)R^{-1}(S-RA)\leq 0,\quad S=S^T,\quad R=R^T<0,$$

holds. If in addition $A^TR+RA=S>0$, then by the Lyapunov theorem the matrices A and B should be asymptotically stable.

The system (15) has the invariant cone $I_t:\max_k|x_k|\leq\alpha(t)\min_s\dot{x}_s$, where $\alpha(t)>0$ if

$$\alpha b_{sj}+1\leq 0,\quad j\neq s,\quad \dot{\alpha}-\alpha\sum_j b_{sj}\geq\left|1+\alpha^2a_{sk}\right|+\alpha^2\sum_{i\neq k}|a_{si}|,\quad t\geq 0,$$

where $i,j,k,s=\overline{1,n}$. Last statement is true also for a non-autonomous system of the type (15).

4. Comparison methods for a set of systems

In stability theory the comparison methods based on a map of the state space of a complicated system in the state space of auxiliary system are applied [1-3]. The comparison systems are

constructed in the classes of positive and monotone systems with respect to some cones. The time-varying cones in comparison problems are offered in [4].
We state the generalized comparison technique based on construction of invariant sets (see section 3). This technique allows to compare the dynamics of a finite set of dynamic systems functioned in different spaces.
We consider a set of the independent systems

$$(\Sigma_i): \quad \dot{X}_i = F_i(X_i, t), \quad X_i \in \mathrm{X}_i, \quad t \geq 0, \quad i = \overline{1, s} \tag{16}$$

For simplicity of presentations, we introduce the following notations

$$X = (X_1, \ldots, X_s), \quad F(X,t) = (F_1(X_1,t), \ldots, F_s(X_s,t)), \quad \mathrm{X} = \mathrm{X}_1 \times \cdots \times \mathrm{X}_s.$$

We assume that for each initial condition $X(t_0) = X_0 \in \Theta$ there corresponds a unique solution $X(t)$ of (16) in some domain $\Theta \subseteq \mathrm{X}$, $t > t_0 \geq 0$.
Let E be a space with the wedge K_t. We specify the map $W: \mathrm{X} \times [0, \infty) \to \mathrm{E}$ which is continuously differentiated in $\hat{\Theta} = \Theta \times [0, \infty)$. We assume also that $W(X,t)$ is not everywhere positive with respect to K_t operator-function.

Definition. The set of systems (16) is *comparable* if for any $t_0 \geq 0$

$$W(X_0, t_0) \overset{\mathrm{K}_{t_0}}{\geq} 0 \;\Rightarrow\; W(X(t), t) \overset{\mathrm{K}_t}{\geq} 0, \quad t > t_0 \tag{17}$$

At the same time W is a *comparison operator* of the set (16).
Now, we define the operator $D_t W(X,t)$ as generalized derivation along the trajectories of (16) and formulate a corollary of Lemma 1.

Theorem 1. Let K_t be a solid cone satisfying (2). Then the set of systems (16) is comparable if and only if for any $t \geq 0$

$$W(X,t) \overset{\mathrm{K}_t}{\geq} 0, \quad \varphi \in \mathrm{K}_t^*, \quad \varphi(W(X,t)) = 0 \;\Rightarrow\; \varphi(D_t W(X,t)) \geq 0 \tag{18}$$

We formulate the basic statements of well known comparison principle for two and three systems with zero states equilibrium. In a phase spaces of the comparison systems we shall use only normal reproducing cones with bounded normality constants.
Let $s = 2$ and $W(X,t) = X_2 - V(X_1, t)$, where $V: \mathrm{X}_1 \times [0, \infty) \to \mathrm{X}_2$ is an everywhere positive operator with respect to the cone $\mathrm{K}_t \subset \mathrm{X}_2$. If (Σ_2) is a system of the class $\overline{\mathrm{M}}_2$, then from the inequality

$$D_t V(X_1, t) \overset{\mathrm{K}_t}{\leq} F_2(V(X_1, t), t) \tag{19}$$

follows that

$$0 \overset{\mathrm{K}_{t_0}}{\leq} V(X_1(t_0), t_0) \overset{\mathrm{K}_{t_0}}{\leq} X_2(t_0) \;\Rightarrow\; 0 \overset{\mathrm{K}_t}{\leq} V(X_1(t), t) \overset{\mathrm{K}_t}{\leq} X_2(t), \quad t > t_0 \geq 0.$$

It means that (17) holds, i.e. two systems (16) are comparable.
We assume that the operator V has the additional properties

$$V(0,t) \equiv 0, \quad \|V(X,t)\| \geq v(X) > 0, \quad X \neq 0, \quad t \geq 0 \tag{20}$$

where $v(X)$ is a continuous nonnegative function such that $v(0) = 0$ $v(X) \leq v(Y)$ $\Rightarrow \|X\| \leq \|Y\|$. Then the following statement is true.

Theorem 2. Let the everywhere positive operator V satisfies (19), (20), and (Σ_2) is a system of the class $\overline{\mathrm{M}}_2$. Then the solution $X_1\equiv 0$ of (Σ_1) is stable (asymptotically stable) by Lyapunov, if the solution $X_2\equiv 0$ of (Σ_2) is stable (asymptotically stable) in K_t.
Let's consider the case $s=3$ assuming that the spaces X_1 and X_3 coincide and contain the cone K_t^1. We define a comparison operator in the block form as follow

$$W(X,t)=\left[V(X_2,t)-X_1, X_3-V(X_2,t)\right], \quad V:\mathrm{X}_2\times[0,\infty)\to \mathrm{X}_1.$$

If the cone inequalities

$$F_1(V(X_2,t),t)\overset{\mathrm{K}_t^1}{\leq} D_tV(X_2,t)\overset{\mathrm{K}_t^1}{\leq} F_3(V(X_2,t),t) \tag{21}$$

hold and the systems (Σ_1) and (Σ_3) belong to corresponding classes $\underline{\mathrm{M}}_1$ and $\overline{\mathrm{M}}_1$, then the solutions of (Σ_2) can be compared with the solutions of (Σ_1) and (Σ_3) in the form of

$$X_1(t_0)\overset{\mathrm{K}_{t_0}^1}{\leq} V(X_2(t_0),t_0)\overset{\mathrm{K}_{t_0}^1}{\leq} X_3(t_0) \;\Rightarrow\; X_1(t)\overset{\mathrm{K}_t^1}{\leq} V(X_2(t),t)\overset{\mathrm{K}_t^1}{\leq} X_3(t), \quad t>t_0\geq 0.$$

It means that (17) with the cone $\mathrm{K}_t=\mathrm{K}_t^1\times\mathrm{K}_t^1$ holds, i.e. three systems (16) are comparable in specified sense. In this case the condition (18) of Theorems 1 follows from (21) and definitions of the classes $\underline{\mathrm{M}}_1$ and $\overline{\mathrm{M}}_1$. In view of (17) (Σ_1) is a lower comparison system and (Σ_3) is an upper comparison system for (Σ_2).
Theorem 3. Let the operator V satisfies (20), (21), and the systems (Σ_1) and (Σ_3) belong to corresponding classes $\underline{\mathrm{M}}_1$ and $\overline{\mathrm{M}}_1$. Then the solution $X_2\equiv 0$ of (Σ_2) is stable (asymptotically stable) by Lyapunov, if the solution $X_1\equiv 0$ of (Σ_1) is stable (asymptotically stable) in $-\mathrm{K}_t^1$ and the solution $X_3\equiv 0$ of (Σ_3) is stable (asymptotically stable) in K_t^1.
The problems of *arrangement* and determination of a *dominating* system in (16) for $s\geq 2$ can be formulated in the form of a general comparison problem using the block comparison operator

$$W(X,t)=\left[V_2(X_2,t)-V_1(X_1,t),\dots,V_s(X_s,t)-V_{s-1}(X_{s-1},t)\right], \quad V_i:\mathrm{X}_i\times[0,\infty)\to \mathrm{E}_1 \tag{22}$$

Assume that (17) hold with $\mathrm{K}_t=\mathrm{K}_t^1\times\cdots\times\mathrm{K}_t^1$, where K_t^1 is a wedge in the space E_1. Then the solutions of (16) are ordered in the form of

$$V_1(X_1(t_0),t_0)\overset{\mathrm{K}_{t_0}^1}{\leq}\cdots\overset{\mathrm{K}_{t_0}^1}{\leq} V_s(X_s(t_0),t_0) \;\Rightarrow\; V_1(X_1(t),t)\overset{\mathrm{K}_t^1}{\leq}\cdots\overset{\mathrm{K}_t^1}{\leq} V_s(X_s(t),t), \quad t>t_0\geq 0 \tag{23}$$

In particular, the solutions are ordered by norms if $V_i(X_i,t)=\|X_i\|_{\mathrm{X}_i}$ is a norm in X_i. If all $V_i=E$ are identical operators, then (23) estimates the solutions ordering by K_t^1. In the case of a solid cone K_t^1 satisfying (2), Theorem 1 gives an arrangement criterion for the systems (16) in the form of (23).
Example 4. Consider a set of the differential systems

$$\dot{X}_i=A_i(X_i,t)X_i, \quad X_i\in C^{n_i}, \quad t\geq 0, \quad i=\overline{1,s} \tag{24}$$

where A_i are $n_i\times n_i$ matrices continuously depending on X_i and t. We define the operator (22) with $V_i(X_i,t)=X_i^*Q_i(t)X_i$, where $Q_i(t)\equiv Q_i^*(t)$ are Hermit matrices. Then

$$\lambda_{\min}(H_i)X_i^*X_i\leq D_tV_i(X_i,t)=X_i^*H_iX_i\leq\lambda_{\max}(H_i)X_i^*X_i, \quad i=\overline{1,s},$$

where $H_i=A_i^*Q_i+Q_iA_i+\dot{Q}_i$. Using Theorem 1 it is possible to establish that the systems (24) are ordered in the form of (23) if in $\hat{\Theta}$

$$H_j \le \beta_j Q_j, \quad \alpha_{j+1} Q_{j+1} \le H_{j+1}, \quad \beta_j \le \alpha_{j+1}, \quad j=1, \ ,s-1 \tag{25}$$

where $\beta_j\ (X_j,\ t)$ and $\alpha_{j+1}(X_{j+1},\ t)$ are some continuous scalar functions. If all $Q_i>0$ are positively defined, then in (25)

$$\beta_j \ge \lambda_{\max}(H_j-\lambda Q_j), \quad \alpha_{j+1} \le \lambda_{\min}(H_{j+1}-\lambda Q_{j+1})$$

where $\lambda_{\max}(\cdot)$ ($\lambda_{\min}(\cdot)$)is a maximal (minimal) eigenvalue of a matrix pencil. In this case we have sufficient arrangement conditions for the systems (24) as follow

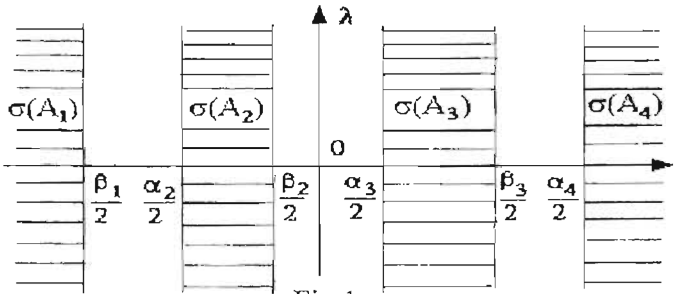

Fig.1.

$$\lambda_{\max}(H_j - \lambda Q_j) \le \lambda_{\min}(H_{j+1} - \lambda Q_{j+1}), \quad j = \overline{1, s-1}.$$

Let the matrices Q_i be constant and positively defined. Then under conditions (25) the spectrum $\sigma(A_i)$ of A_i should belong to corresponding zone (see, for example, [9] and fig.1). If all matrices $Q_i \equiv I$ are identity and the inequalities $\lambda_{\max}(A_j^* + A_j) \le \lambda_{\min}(A_{j+1}^* + A_{j+1})$, $j = \overline{1, s-1}$, hold in $\hat{\Theta}$, then the solutions of (24) are ordered by norm, i.e.

$$\|X_1(t_0)\| \le \cdots \le \|X_s(t_0)\| \ \Rightarrow\ \|X_1(t)\| \le \cdots \le \|X_s(t)\|, \quad t > t_0 \ge 0.$$

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