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Stability and comparison of systems in partially ordered space

Alexey G. Mazko
Institute of mathematics of NAS of Ukraine
3 Tereshchenkivska Str., 01601 Kiev-4, Ukraine

The classes of positive and monotone differential systems with respect to prescribed cone in a phase space are studied. The stability criteria of linear positive systems are formulated in terms of monotonically invertible linear operators. The methods for robust stability analysis and comparison of systems in partially ordered space are developed.

1 Introduction. Various natural systems are positive and monotone. Positivity (monotonicity) of the dynamic system is equivalent to positivity (monotonicity) of some operator describing its motion with respect to the cone of a phase space [1-3]. These properties of systems should be taken into consideration and be used in analysis and synthesis problems especially in stability and spectral characteristics investigation, in numerical procedures of construction of the solutions and appropriate control etc. Stability investigation of linear autonomous positive systems is reduced to solving algebraic equations defined by operator coefficients of the systems [4-6]. The differential Lyapunov and Riccati equations are examples of positive systems concerning the cone of symmetrical positive semi-definite matrices.

In present article, the classes of positive and monotone differential systems in the partially ordered Banach space are studied. The main research results are the criteria of a positivity and asymptotic stability of linear systems formulated in terms of the monotone and monotonically invertible operators. The methods for robust stability study and analogs of comparison systems in partially ordered space are offered.

2. Operators in space with a cone. The convex closed set K of real normalized space Ξ is called a cone if $\mathrm{K} \bigcap -\mathrm{K} = \{0\}$ and $\alpha X + \beta Y \in \mathrm{K}$ for any $X, Y \in \mathrm{K}$ and $\alpha, \beta \geq 0$. A space containing the cone is partially ordered. The inequality $X \leq Y$ $(X < Y)$ means that $X - Y \in \mathrm{K}$ $(X - Y \in \mathrm{K}_0)$, where K_0 is a set of internal points of K. The solid, reproducing and normal cones are defined accordingly by $\mathrm{K}_0 \neq \varnothing$, $\Xi = \mathrm{K} - \mathrm{K}$ and $0 \leq X \leq Y \Rightarrow \|X\| \leq c\|Y\|$, where c is a universal constant.

Let Ξ_1 (Ξ_2) be the Banach space with a cone K_1 (K_2). The linear operator $M : \Xi_1 \to \Xi_2$ is called monotone if from $X \leq Y$ it follows that $MX \leq MY$. The property of monotonicity of a linear operator is equivalent to the positivity: $X \geq 0 \Rightarrow MX \geq 0$. Inequality $M \leq L$ means that the operator $L - M$ is positive. The monotone operator L is called a majorant (minorant) of the monotone operator M if $M \leq L (M \geq L)$. The monotone operator is called extremal if he can't be expressed by a sum of linearly independent minorants. The operator is called strictly monotone (strongly monotone) if $MX > MY$ for $X > Y (X \geq Y, X \neq Y)$. Strongly monotone (extremal) operators are internal (exterior) points of a cone of linear monotone operators. Necessary and sufficient condition for the strictly monotonicity of monotone operator M is the inequality $MX_0 > 0$ for some $X_0 \geq 0$.

The operator M is called monotonically invertible if the equation

$$MX = Y \tag{1}$$

has a solution $X \in \mathrm{K}_1$ for any $Y \in \mathrm{K}_2$. If the cone K_2 is normal and reproducing, then inequalities $M_1 \leq M \leq M_2$ with monotone invertible operators M_1 and M_2 imply that M is also monotone invertible and $M_2^{-1} \le M^{-1} \le M_1^{-1}$[1]. If cone K_2 is normal and solid and $M \le L$, where L is a monotone invertible operator, then monotone invertibility of the operator M is equivalent to a solvability of (1) in the form of $X \ge 0$ for some $Y > 0$.

In the space Ξ with normal reproducing cone K, we eliminate the class of linear operators [6]

$$M = L - P, \quad PK \subset K \subset LK. \tag{2}$$

The monotone invertibility criterion for operator (2) is the inequality

$$\rho(T) < 1, \tag{3}$$

where $\rho(T)$ is a spectral radius of the operator pencil $T(\lambda) = P - \lambda L$. When K is solid, the inequality (3) is equivalent to solvability of (1) in the form of $X \ge 0$, $Y > 0$.

The properties of a monotonicity and positivity are similarly determined for the non-linear operators $F : \Xi_1 \to \Xi_2$. If $F\Xi_1 \subset K_2$, then the operator is called completely positive. If $FX \ge FY$ for $X \ge Y \ge 0$, then F is monotone on the cone K_1.

3. Positive and monotone systems. Let Ξ be the Banach space partially ordered by the cone K. Let $X(t) \in \Xi$ be a state of continuous or discrete time dynamic system. The system is called $(t, t_0)-$positive if $X(t_0) = X_0 \in K$ implies $X(t) \in K$. The system is called positive if it is $(t, t_0)-$positive for $t > t_0 \ge 0$. This property of the system is equivalent to positivity of the operator $V(t, t_0) : \Xi \to \Xi$ determining transition from the state X_0 to the state $X(t) = V(t, t_0) X_0$ for $t > t_0 \ge 0$. The system is called monotone (monotone on the cone K) if its motion operator $V(t, t_0)$ is monotone (monotone on the cone K) for $t > t_0 \ge 0$.

The properties of positivity and monotonicity of control systems are similarly determined with respect to the cones in state and control spaces. If the operator $W(t) : \Xi_1 \to \Xi_2$ determines a system with input $U(t)$ and output $X(t) = W(t)U(t)$, then its positivity (monotonicity) with respect to the cones $K_1 \subset \Xi_1$ and $K_2 \subset \Xi_2$ is equivalent to positivity (monotonicity) of given control system. The operator $W(t)$ can be defined explicit or in form of solutions of differential, difference, integro-differential and other types of systems [1].

Let us consider the differential system

$$\dot{X} + M(t) X = G(t), \tag{4}$$

where $M(t)$ is a linear bounded operator acting in a partially ordered Banach space Ξ with the normal reproducing cone K. Assume that for any initial condition $X(t_0) = X_0$, there is the unique solution

$$X(t) = W(t, t_0) X_0 + \int_{t_0}^{t} W(t, s) G(s) ds, \quad t > t_0 \ge 0, \tag{5}$$

where $W(t, s) = W(t, t_0) W(s, t_0)^{-1}$ is an evolutional operator being the unique solution of the Cauchy problem

$$. \qquad \dot{W} + M(t) W = 0, \quad W(s) = I, \quad t \ge s. \tag{6}$$

The linear operator $W(t, t_0)$ can be represented by uniformly converged on the norm series

$$W(t, t_0) = I - \int_{t_0}^{t} M(t_1) dt_1 + \int_{t_0}^{t} M(t_2) \int_{t_0}^{t_2} M(t_1) dt_1 dt_2 - \cdots, \tag{7}$$

According to (5), $X(t) \in K$ for any initial condition $X_0 \in K$ if

$$W(t,t_0) \geq 0, \quad \int_{t_0}^{t} W(t,s)G(s)ds \geq 0, \quad t \geq t_0 \geq 0. \tag{8}$$

Here first inequality means a monotonicity of the operator with respect to the cone K, and second one is an ownership to the cone. The inverse statement is easily established taking into account a closure of K. Therefore, system (4) is positive if and only if (8) holds.

Using (5) and (8), we establish an equivalence of the following statements: (a) system (4) is (t, 0)-positive for any function $G(t) \geq 0$ and $t > 0$; (b) the operator $W(t,s)$ is monotone for $t \geq s \geq 0$; (c) system (4) is monotone; (d) for any function $Z(t)$ satisfying the inequality $\dot{Z} + M(t)Z \geq 0$, $Z(0) \geq 0$ implies $Z(t) \geq 0$ for any $t > 0$. If $G(t) \geq 0$, then each of the statements (a)-(d) is equivalent to positivity of system (4). If $M(t) = M$ is a constant operator, then $W(t,s) = e^{-M(t-s)}$ and conditions for monotonicity of the operators $W(t,s)$ and $W(t,0)$ for $t \geq s \geq 0$ coincide.

Note the properties of the evolutional operators which follow from (6) and (7). The relations

$$W(t,t) = I, \; W(t,s)^{-1} = W(s,t), \; W(t,\tau) = W(t,s)W(s,\tau)$$

hold for $t \geq s \geq \tau$. If $M(t) = M_1(t) + M_2(t)$, then $W(t,s) = W_{M_1}(t,s)W_{M_3}(t,s)$, where $W_{M_1}(t,s)$ and $W_{M_3}(t,s)$ are evolutional operators corresponding to $M_1(t)$ and $M_3(t) = W_{M_1}(s,t)M_2(t)W_{M_1}(t,s)$. Thus, $W(t,s)$ is monotone if the operators $W_{M_1}(t,s)$ and $W_{M_3}(t,s)$ are monotone.

Lemma 1. *The evolutional operator $W(t,s)$ is monotone for $t \geq s \geq 0$ if and only if the exponential operator $e^{-M(t)h}$ is monotone for $t \geq 0$, $h \geq 0$.*

Proof. We use representation for $W(t,s)$ in the form of multiplicative integral [7]. Breaking the section $[s,t]$ by points $t_{kn} = s + kh_n$, where $h_n = (t-s)/n$, $k = 0,\cdots,n$, for large n, we have

$$W(t,s) = W(t_{nn}, t_{n-1n})\cdots W(t_{1n}, t_{0n}), \; W(t_{kn}, t_{k-1n}) = e^{-M(\theta_{kn})h_n} + o(h_n), \; k = 1,\cdots,n,$$

where $\theta_{kn} \in [t_{k-1n}, t_{kn}]$ are some intermediate points. Therefore,

$$W(t,s) = \lim_{n\to\infty}\left(e^{-M(\theta_{nn})h_n} \cdots e^{-M(\theta_{1n})h_n}\right).$$

If $e^{-M(t)h} \geq 0$ for any $t \geq 0$ and $h \geq 0$, then $W(t,s)$ is a limit of some sequence of monotone operators. It should be monotone due to closure of the cone of linear monotone operators.

The converse is similarly established on basis of the relations

$$W(t, t-h/n) = e^{-M(\theta_n)h/n} + o(1/n), \; e^{-M(t)h} = \lim_{n\to\infty}\left(W(t, t-h/n)\right)^n,$$

where $\theta_n \in [t - h/n, t]$, $n = 1,2,\cdots$. ■

Lemma 2. *If $M(t) = M_1(t) + M_2(t)$ and the operators $W_{M_1}(t,s)$ and $W_{M_2}(t,s)$ are monotone for $t \geq s \geq 0$, then the operator $W_M(t,s)$ is also monotone for $t \geq s \geq 0$.*

Proof. We use Lemma 1 and present the exponential operator by

$$e^{-(M_1+M_2)h} \equiv E(h) - h^2 \sum_{k=3}^{\infty} \frac{(-h)^{k-2}}{k!} S_k,$$

where $E(h) = \frac{1}{2}\left(e^{-M_1 h}e^{-M_2 h} + e^{-M_2 h}e^{-M_1 h}\right)$, $S_k = (M_1 + M_2)^k - \frac{1}{2}\sum_{i=0}^{k} C_k^i \left(M_1^i M_2^{k-i} + M_2^i M_1^{k-i}\right)$.
Here the dependence M_1 and M_2 from t is omitted for a simplicity. If we set $h = \frac{\tau}{k}$, then

$$e^{-(M_1+M_2)\tau} = \lim_{k\to\infty}\left[E\left(\tfrac{\tau}{k}\right)\right]^k .$$

Taking into account Lemma 1 and our assumptions, it completes the proof of monotonicity for $W_M(t,s)$, $t \ge s \ge 0$. ■

Positivity of system can be utilized for estimation of its solutions. If the functions $X_1(t)$ and $X_2(t)$ satisfy the inequalities $\dot{X}_1 + M(t)X_1 \le G_1(t)$, $\dot{X}_2 + M(t)X_2 \ge G_2(t)$ and $X_1(t_0) \le X_2(t_0)$, then subject to (8) we obtain

$$X_2(t) - X_1(t) \ge W(t,t_0)[X_2(t_0) - X_1(t_0)] + \int_{t_0}^{t} W(t,s)G(s)ds \ge 0$$

where $G(t) = G_2(t) - G_1(t)$. Hence, the following statement holds.

Lemma 3. *Let $X(t)$ be a solution of the positive system (4) and the functions $X_1(t)$ and $X_2(t)$ satisfy the inequalities $\dot{X}_1 + M(t)X_1 \le \alpha_1 G_1(t)$, $\dot{X}_2 + M(t)X_2 \ge \alpha_2 G_2(t)$, where $\alpha_1 \le 1$, $\alpha_2 \ge 1$. Then $X_1(t_0) \le X(t_0) \le X_2(t_0)$ implies that $X_1(t) \le X(t) \le X_2(t)$ for $t \ge t_0$.*

If $\alpha_1 = 0$, then the lower estimation for the solution of (4) in Lemma 3 does not depend on its right part. In the case of $\alpha_1 = \alpha_2 = 1$, Lemma 3 holds provided that the operator $W(t,s)$ is monotone for $t \ge s \ge t_0$.

We now give the examples of linear positive systems with respect to the cones of nonnegative vectors and positive semi-definite matrices.

Example 1. Let us consider the linear differential system

$$\dot{x}(t) + A(t)x(t) = g(t), \quad t \ge 0, \tag{9}$$

where $A(t)$ is a continuous $n \times n$ matrix function. It is known [1], that monotonicity of evolutional operator of system (9) with respect to the cone $\mathrm{K} \subset R^n$ of nonnegative vectors is equivalent to off-diagonal nonpositivity of the matrix $A(t)$. Therefore, system (9) is positive under the restrictions $a_{ij}(t) \le 0$, $i \ne j$, $g(t) \ge 0$, $t \ge 0$.

Example 2. Let us consider the Lyapunov matrix differential equation

$$\dot{X}(t) + A(t)X(t) + X(t)A(t)^T = Y(t), \; t \ge 0, \tag{10}$$

where $A(t)$ and $Y(t) = Y(t)^T \ge 0$ are given matrix functions. In this case $M(t)X = A(t)X + XA(t)^T$ and evolutional operator has the form $W(t,s)X = W_A(t,s)XW_A(t,s)^T$, where $W_A(t,s)$ is an evolutional operator (matriciant) of system (9). Obviously, the operator $W(t,s)$ is monotone with respect to the cone $\mathrm{K} \subset R^{n\times n}$ of symmetric positive semi-definite matrices. Therefore, (10) is a positive system with respect to the cone.

Let's extend the differential system (4) in the form

$$\dot{X} + M(t)X = G(X,t), \; t \ge 0, \tag{11}$$

where $G(X,t)$ is a non-linear operator ensuring an existence and uniqueness of a solution $X(t) \in \Xi$ for $t \ge t_0 \ge 0$, $X(t_0) = X_0 \in \Xi$. Let $V(t,t_0)$ be a shift operator on trajectories of (12) determining a transition from $X(t_0)$ to the state $X(t) = V(t,t_0)X(t_0)$, $t \ge t_0$. Then positivity (monotonicity) of system (11) is equivalent to positivity (monotonicity) of the operator $V(t,t_0)$ for any $t > t_0 \geq 0$.

Solutions of system (11) satisfy the integral equation

$$X(t) = W(t,t_0)X_0 + \int_{t_0}^{t} W(t,s)G(X(s),s)ds\,, \tag{12}$$

where $W(t,s)$ is an evolutional operator of linear system (4). From (12), it follows that system (11) is positive if $G(X,t)$ is completely positive and $W(t,s)$ is monotone for $t \geq s \geq 0$.

Lemma 4. *Let $X(t)$ be a solution of system (11) with monotone operator $W(t,s)$ and the functions $X_1(t)$ and $X_2(t)$ satisfy the relations*

$$\dot{X}_1 + M(t)X_1 = G_1(t),\ \dot{X}_2 + M(t)X_2 = G_2(t),\ G_1(t) \leq G(X(t),t) \leq G_2(t),\ t \geq t_0.$$

Then from $X_1(t_0) \leq X(t_0) \leq X_2(t_0)$ it follows that $X_1(t) \leq X(t) \leq X_2(t),\ t \geq t_0$.

Given statement is established by using (12) and the proof method of a lemma 3.

Example 3. The non-linear differential system

$$\dot{x}.(t) + A(t)x(t) = g(x,t),\ t \geq 0\,, \tag{13}$$

where $A(t)$ is a matrix with nonpositive off-diagonal elements, is positive with respect to the cone $K \subset R^n$ of nonnegative vectors if the vector function $g(x,t)$ satisfies [2]

$$x \geq 0,\ x_i = 0 \ \Rightarrow\ g_i(x,t) \geq 0 \ \ (i = 1,\ldots,n),$$

and the system is monotone with respect to same cone if $g(x,t)$ is semi-monotonic non-decreasing on x (Vazhevsky condition):

$$x \leq y,\ x_i = y_i \ \Rightarrow\ g_i(x,t) \leq g_i(y,t) \ \ (i = 1,\ldots,n).$$

If both above limitations on $g(x,t)$ hold for $0 \leq x \leq y$, then system (13) is monotone on K.

Example 4. Let us consider a matrix system extending the Lyapunov and Riccati differential equations in the form

$$\dot{X}. + A(t)X + XA(t)^T - \sum_k B_k(t)XB_k(t)^T = XC(t)X + D(t),\ t \geq 0\,, \tag{14}$$

where $A(t), B_k(t), C(t)$ and $D(t)$ are given matrix functions. The operator $M(t)$ has the form

$$M(t) = L(t) - P(t),\ \ L(t)X = A(t)X + XA(t)^T,\ \ P(t) = \sum_k B_k(t)XB_k(t)^T\,,$$

and $W(t,s)$ is monotone with respect to a cone of symmetric positive semi-definite matrices (see Example 2 and Lemma 2). If $C(t) = C(t)^T \geq 0$ and $D(t) = D(t)^T \geq 0$, then the operator in right side of (14) is completely positive and the system is positive. In the case of $C(t) \equiv 0$, the system is also monotone.

Note, that matrix equation (14) with zero right side is known as equation of moments for the stochastic system Ito

$$dx(t) + A(t)x(t)dt = \sum_k B_k(t)x(t)dw_k(t),$$

where w_k are components of standard Wiener process. This equation is positive, and it is used in studying the mean quadratic stability of mentioned stochastic system.

4. Stability of linear positive systems. Stability problem for class of the non-stationary systems (4) without additional constraints is enough complicated. There is not to date constructive methods for its solving. Stability of the system has been studied by using the Lyapunov functions method, theory of characteristic indexes, methods of compare systems, theory of reducible and periodic systems etc. (see, for example, [6-9]). We show that an asymptotic stability analysis of system (4) under some additional conditions can be reduced to solution of simple equations such as (1) with monotonically invertible operator. One of such conditions is a positivity of original or auxiliary system with respect to normal reproducing cone of a phase space.

Stability study of system (4) for any bounded function $G(t)$ is reduced to learning stability conditions of a trivial solution of the homogeneous system

$$\dot{X} + M(t)X = 0,\ \ t \ge 0 . \qquad (15)$$

Stability and asymptotic stability of system (15) are equivalent to the appropriate conditions $\sup\limits_{t \ge t_0} \|W(t,t_0)\| < \infty$, $\|W(t,t_0)Y\| \to 0$, $Y \in \Xi$, $t \to \infty$. System (15) is exponential stable if its arbitrary solution satisfies the estimation

$$\|X(t)\| \le a e^{-b(t-t_0)} \|X_0\|,\ t \ge t_0 , \qquad (16)$$

where $a > 0$ and $b > 0$ are some constants. Asymptotic stability of system (15) follows from its exponential stability. For classes of autonomous and periodic systems, the properties of exponential and asymptotic stability are equivalent.

Solutions of asymptotically stable system (15) satisfy the equality

$$\int\limits_{t_0}^{t} M(t)X(t)dt = Y , \qquad (17)$$

where $X(t_0) = Y$. Thus, $X(t) \ge 0$ if $Y \ge 0$ and system (15) is positive. We consider a class of the linear autonomous systems

$$\dot{X} + MX = 0,\ \ t \ge 0 , \qquad (18)$$

where M is a bounded operator, $\sigma(M) \ne \varnothing$. The equality (17) is reduced to form (1), where

$$X = \int\limits_{t_0}^{t} W(t,t_0)Y dt ,\ W(t,t_0) = e^{-M(t-t_0)} .$$

Positivity of system (18) is equivalent to monotonicity of the exponential operator e^{-Mt} for $t \ge 0$. For operator exponent, we define the growth border by $\gamma_M = \lim\limits_{t\to\infty} \frac{1}{t} \ln\|e^{-Mt}\| < \infty$ and the spectral edge $\alpha_M = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(M)\}$. From theorem on transformation of spectrum for bounded operators, it follows that $\gamma_M = -\alpha_M$ [3]. Spectral radius of monotone operator is a point of its spectrum (Kreyn-Bonsall-Carlin theorems [1]). Therefore, for positive system (18), $\alpha_M \in \sigma(M)$.

Lemma 5. *For positive system (18), the operator $M + \gamma I$ is monotonically invertible ft and only if $\gamma > \gamma_M$. If the operator $M + \gamma I$ is monotonically invertible for any $\gamma \ge \gamma_0$, then system (18) is positive and $\gamma_0 > \gamma_M$.*

Proof. If system (18) is positive, then for any $\gamma > \gamma_M$, we have

$$(M + \gamma I)^{-1} = \int\limits_0^\infty e^{-\gamma t} e^{-Mt} dt \ge 0 .$$

Back, if the operator $M + \gamma I$ is monotonically invertible for any value $\gamma \ge \gamma_0$, then

$$e^{-Mt} = \lim_{k\to\infty} \left[t_k (M + t_k I)^{-1} \right]^k \ge 0 ,\ t_k = k/t ,\ t \ge 0 .$$

For positive system (18), we show that the operator $M+\gamma I$ is not monotonically invertible for $\gamma \le \gamma_M$. Assume that for some γ_1 and γ_2, the operators $M_1 = M+\gamma_1 I$ and $M_2 = M+\gamma_2 I$ are monotonically invertible and $\gamma_1 < \gamma_M < \gamma_2$. Since $M_1 \le M+\gamma_M I \le M_2$, then according to the theorem on two-sided estimation of monotonically invertible operators, the operator $M+\gamma_M I$ should be also monotonically invertible [1]. However, it contradicts to the condition $\alpha_M = -\gamma_M \in \sigma(M)$. Therefore, for positive system (18), the operator $M+\gamma I$ is monotonically invertible only for $\gamma > \gamma_M$.■

Note. From the theorems on spectrum mapping and spectral radius of a monotone operator, it follows that under the conditions $e^{-Mt} \ge 0$ and $(M-\alpha I)^{-1} \ge 0$, there are such points of spectrum $\alpha_*, \beta_* \in \sigma(M)$, that for any $\lambda \in \sigma(M)$, the inequalities

$$e^{-\mathrm{Re}\lambda t} \le e^{-\alpha_* t}, \quad e^{-\mathrm{Re}\lambda t} \big/ |\lambda-\alpha| \le e^{-\beta_* t} \big/ |\beta_* - \alpha| .$$

are hold. The right sides of these inequalities are real positive values for $t \ge 0$. The first inequality (for small $t < 2\pi\rho(M)$) implies that α_* is a real point of spectrum and $\mathrm{Re}\lambda \ge \alpha_*$, $\forall\lambda \in \sigma(M)$. According to the second inequality for large value t and $\forall\lambda \in \sigma(M)$, we have to put $\beta_* = \alpha_* = \alpha_M > \alpha$.

Lemma 6. *If the operator $M-\alpha I$ is monotonically invertible for any $\alpha \le \alpha_0$, then the spectrum $\sigma(M)$ lies in the half-plane $\mathrm{Re}\lambda > \alpha_0$.*

Proof. Since $M-\alpha I$ is invertible operator for $\alpha \le \alpha_0$, the operator M has no real points of a spectrum in the interval $(-\infty, \alpha_0]$. Spectral radius of the monotone operator $(M-\alpha I)^{-1}$ is equal to $1/(\alpha_* - \alpha)$, where α_* is a real point of the spectrum $\sigma(M)$ such that $|\lambda-\alpha| \ge \alpha_* - \alpha > 0$. Thus, $\alpha_* > \alpha_0 \ge \alpha$ and α_* does not depend of α. If $\mathrm{Re}\lambda \le \alpha_0$ then for some value α, the inverse inequality $|\lambda-\alpha| < \alpha_* - \alpha$ holds. Therefore, $\mathrm{Re}\lambda > \alpha_0$, $\forall\lambda \in \sigma(M)$ and α_* coincides with α_M.■

If $\alpha_M > 0$, then for any solution of (18), the estimation (16) holds with $0 < \gamma < \alpha_M$. Conversely, if system (18) is exponential stable and positive, then inequality (16) for the partial solution $X(t) = e^{-\alpha_M (t-t_0)} V$ ($V \ne 0$) implies that $\alpha_M > 0$. Using lemmas 8 and 9, we obtain the following result.

Theorem 1. *Positive system (18) is exponential stable if and only if the operator M is monotonically invertible. If the operator $M+\gamma I$ is monotonically invertible for any $\gamma \ge 0$, then system (18) is positive and exponential stable.*

Note, if the operators M and $M+\gamma I$ with enough large γ are monotonically invertible, then system (18) is exponential stable (see proofs of Lemma 5 and Lemma 6). Well-known criteria for a mean quadratic asymptotic stability of the Ito stochastic systems are corollaries of Theorem 1.

Now, we consider the classes of non-stationary systems (15). System (15) is called positively reducible if there is the Lyapunov transformation $X = Q(t)H$ reducing to the positive stationary system

$$\dot{H} + M_0 H = 0, \tag{19}$$

where M_0 is a constant operator. In this definition, $Q(t)$ is uniformly bounded differentiable operator having uniformly bounded inverse $Q^{-1}(t)$ and satisfying the operator differential equation

$$\dot{Q} + M(t)Q - QM_0 = 0 .$$

Theorem 2. *Positively reducible system (15) is exponential stable if and only if the operator M_0 is monotonically invertible.*

The $\varpi-$periodic systems of type (15) is reducible and

$$M(t+\varpi) = M(t),\ W(t+\varpi) = W(t)W(\varpi),\ t \ge 0 ,$$

where $W(t) = W(t,0)$. In addition, the spectrum of the monodromy operator $W(\omega)$ does not enclose zero. Operator of the Lyapunov transformation has the form $Q(t) = W(t)e^{M_0 t}$, where $M_0 = -\varpi^{-1}\ln W(\varpi)$. Therefore, positively reducible $\omega-$periodic system (15) is exponential stable if and only if the operator M_0 is monotonically invertible.

Consider the systems (15) with functionally commutative operator $M(t)$:

$$M(t)M(s) = M(s)M(t),\ \forall t,s \ge 0 . \tag{20}$$

In this case, the evolutional operator is determined by

$$W(t,s) = e^{-N(t,s)},\ N(t,s) = \int_s^t M(\tau)d\tau ,\ t \ge s . \tag{21}$$

Assume that there is the bounded limit operator

$$M_0 = \lim_{t\to\infty} \frac{1}{\varphi(t)} \int_{t_0}^t M(\tau)d\tau , \tag{22}$$

where $\varphi(t) > 0$ is some function such that $\varphi(t) \to \infty$ for $t \to \infty$.

Theorem 3. *Let conditions (20) hold and system (19) with operator (22) be positive. Then asymptotic stability of system (15) follows from monotone invertibility of operator (22).*

Proof. Following [8], from (20)-(22) we obtain the relations

$$M(t)N(t,\tau) = N(t,\tau)M(t),\ M_0 N(t,t_0) = N(t,t_0)M_0 ,$$

$$M_0\Delta(t,t_0) = \Delta(t,t_0)M_0 ,\ \Delta(t,t_0) = \psi(t)^{-1}N(t,t_0) - M_0 ,$$

where $\Delta(t,t_0) \to 0$, $t \to \infty$. Therefore, arbitrary solution of (15) subject to (21) can be presented by

$$X(t) = e^{-\psi(t)[M_0+\Delta(t,t_0)]}X_0 = e^{-\psi(t)M_0}e^{-\psi(t)\Delta(t,t_0)}X_0 .$$

Let $\alpha_0 = \inf\{\mathrm{Re}\,\lambda : \lambda \in \sigma(M_0)\}$. Then according to Lemma 6, $\alpha_0 > 0$ and for any positive number $\varepsilon < \alpha_0/2$ there is an instant t_1 such that $\|\Delta(t,t_0)\| < \varepsilon$ for $t > t_1$ and

$$\|X(t)\| \le \beta e^{-\psi(t)(\alpha_0 - t)}e^{\psi(t)\varepsilon}\|X_0\| = \beta e^{-\psi(t)(\alpha_0-2\varepsilon)}\|X_0\| ,$$

where $\beta > 0$ is some constant. Since $\varphi(t) \to \infty$ and $\alpha_0 > 2\varepsilon$, we have $\|X(t)\| \to 0$ for $t \to \infty$. Therefore, system (15) is asymptotically stable. ■

Example. We consider the matrix system (15) and assume that

$$M(t) = \begin{bmatrix} a(t) & -b(t) \\ -b(t) & a(t) \end{bmatrix},\ \varphi(t) = \int_{t_0}^t b(s)ds \to \infty ,\ \frac{1}{\varphi(t)}\int_{t_0}^t a(s)ds \to c ,\ t \to \infty,$$

where $a(t)$ and $b(t)$ are given functions. Obviously, the matrix $M(t)$ satisfies the functional commutability condition (20) and system (19) with the extreme matrix

$$M_0 = \begin{bmatrix} c & -1 \\ -1 & c \end{bmatrix}$$

is positive with respect to a cone of nonnegative vectors (see item 2). Monotone invertibility of the matrix M_0 is reduced to the inequality $c > 1$. Thus, according to Theorem 3, an original system (15) is asymptotically stable.

5. Robust stability of positive systems. Some applied investigations give rise a stability problem for the family of dynamic systems with uncertain parameters (robust stability problem).

We consider the family of dynamic systems defined by

$$\dot{X} + M(t)X = G(X,t), \ \ t \geq 0, \tag{23}$$

$$\underline{M}(t) \leq M(t) \leq \overline{M}(t), \tag{24}$$

$$G_1(t) - M_1(t)X \leq G(X,t) \leq G_2(t) - M_2(t)X, \tag{25}$$

where all operators are bounded on a set of variables. Inequalities are defined with respect to the normal reproducing cone $\mathrm{K} \subset \Xi$. Generally, the double-sided estimation (25) should hold for any point $X \in \Xi$ in a phase space where solution of (23) is determined. If we consider the solutions $X(t) \geq 0$, then in (25), $X \in \mathrm{K}$.

In (23) - (25), we eliminate two linear systems

$$\dot{X}_1 + \left(\overline{M}(t) + M_1(t)\right)X_1 = G_1(t), \tag{26}$$

$$\dot{X}_2 + \left(\underline{M}(t) + M_2(t)\right)X_2 = G_2(t). \tag{27}$$

Lemma 7. *Let evolutional operator of the system (26) be monotone and the inequalities (25) hold for* $X \in \mathrm{K}$. *Then the solution* $X(t) \geq 0$ *of each system (23)-(25) are bounded by suitable solutions of (26) and (27):*

$$X_1(t_0) \leq X(t_0) \leq X_2(t_0) \Rightarrow X_1(t) \leq X(t) \leq X_2(t), \ t > t_0 \geq 0.$$

If inequalities (25) hold for $X \in \Xi$, *then positivity of system (26) leads to positivity of each system (23)-(25) and*

$$0 \leq X_1(t_0) \leq X(t_0) \leq X_2(t_0) \Rightarrow 0 \leq X_1(t) \leq X(t) \leq X_2(t), \ t > t_0 \geq 0.$$

Proof. Subtracting (26) from (23) and (23) from (27) subject to (24) and (25), we obtain the differential inequalities

$$\dot{H}_1 + \left[\overline{M}(t) + M_1(t)\right]H_1 \geq \left[\overline{M}(t) - M(t)\right]X(t), \ \ \dot{H}_1 + \left[M(t) + M_1(t)\right]H_1 \geq \left[\overline{M}(t) - M(t)\right]X_1(t),$$

$$\dot{H}_2 + \left[M(t) + M_2(t)\right]H_2 \geq \left[M(t) - \underline{M}(t)\right]X_1(t), \ \ \dot{H}_2 + \left[\underline{M}(t) + M_2(t)\right]H_2 \geq \left[M(t) - \underline{M}(t)\right]X(t),$$

where $H_1(t) = X(t) - X_1(t)$, $H_2(t) = X_2(t) - X(t)$ and

$$\overline{M}(t) + M_1(t) \leq M(t) + M_1(t) \leq M(t) + M_2(t) \leq \underline{M}(t) + M_2(t).$$

If system (26) is positive, then its evolutional operator should be monotone. Monotonicity of the operators $W_{M+M_1}(t,s)$ follows from monotonicity of the operators $W_{\overline{M}+M_1}(t,s)$, $W_{M+M_2}(t,s)$ and $W_{\underline{M}+M_2}(t,s)$ (see item 2).

If $X(t) \geq 0$ or $X_1(t) \geq 0$, then $H_1(t_0) \geq 0$ implies that $H_1(t) \geq 0$ for $t \geq t_0$. Similarly, if $X(t) \geq 0$ or $X_2(t) \geq 0$, then $H_2(t_0) \geq 0$ implies that $H_2(t) \geq 0$ for $t \geq t_0$. Therefore, positivity of system (26) implies positivity of each system (23)-(25). In the case of $X(t) \geq 0$, the inequalities (25) are used above only for $X \in \mathrm{K}$.

Lemma 7 can be used for studying a robust stability of the family of differential systems (23). For example, we consider the family of linear systems

$$\dot{X} + M(t)X = 0,\ \underline{M}(t) \le M(t) \le \overline{M}(t),\ t \ge 0. \tag{28}$$

In this case, systems (26) and (27) take the form

$$\dot{X}_1 + \overline{M}(t)X_1 = 0,\ t \ge 0, \tag{29}$$

$$\dot{X}_2 + \underline{M}(t)X_2 = 0,\ t \ge 0. \tag{30}$$

If initial conditions in (28) and (30) satisfy the inequalities $0 \le X(t_0) \le X_2(t_0)$, then monotonicity of an evolutional operator of system (29) implies that $0 \le X(t) \le X_2(t)$, $t \ge t_0$. Furthermore, if system (30) is asymptotically stable, then from normality of the cone K, we obtain $\|X(t)\| \to 0$, $t \to \infty$. If K is a reproducing cone, then the solutions $X(t)$ possess this property for any initial conditions $X_0 \in \Xi$ and $X(t) = X_+(t) - X_-(t)$, where functions $X_\pm(t) \in \mathrm{K}$ satisfy (28) and for which the above stated reasoning hold. Therefore, an asymptotic stability of each positive system of the family (28) follows from an asymptotic stability of a trivial solution of system (30).

For linear systems from family (23)-(25), in particular, (28), we formulate the following result.

Theorem 4. *If system (27) is asymptotically stable and system (26) is positive, then each linear system from the family (23) - (25) is asymptotically stable and positive.*

Note, that for the family of stationary systems (28), monotone invertibility of the operators $\underline{M}$ and $\overline{M}$ implies monotone invertibility of the operator segment $\underline{M} \le M \le \overline{M}$ [1]. From Theorems 1 and 4, in particular, follows that the operator M is monotone invertible if the operator $e^{-\overline{M}t}$ is monotone for $t \ge 0$ and spectrum of $\underline{M}$ lies in the half-plane $\operatorname{Re}\lambda > 0$.

6. Differential comparison systems. The methods for comparison based on mapping state space of an original (complicated) system in state space of an auxiliary (investigated) system are used in various applied and theoretical problems (see, for example, [9-11]). In stability study, it is expedient to use as comparison systems the classes of positive and monotone systems, and also non-linear systems satisfying conditions of the Chaplygin and Wazhevsky type theorems. Thus, there can be useful statements of Theorems 1-4 and Lemmas 3, 4 and 7.

In a Banach space X, we consider the differential system

$$\dot{x} = f(x,t),\ x \in \mathrm{X},\ t \ge 0, \tag{31}$$

where f is an operator ensuring an existence of unique solution $x(t) \in \mathrm{X}$. Let Ξ be a Banach space partially ordered by the normal reproducing cone K. In Ξ, we construct the class of differential systems

$$\dot{X} = F(X,t),\ X \in \Xi,\ t \ge 0, \tag{32}$$

as comparison systems for an original system (31). By Σ_+ we denote such class of systems (32) that between their solutions and solutions of the differential inequalities

$$\dot{Z} \le F(Z,t),\ Z \in \Xi,\ t \ge 0, \tag{33}$$

there is a conformity for which $X(t_0) \ge Z(t_0)$ implies $X(t) \ge Z(t)$, $t > t_0$. Apparently, each system of Σ_+ is monotone. If $F(0,t) \ge 0$ for $t \ge 0$, then system (32) from Σ_+ is positive.

Let $E(x,t)$ be an operator continuously mapping some neighbourhood of the point $x = 0 \in \mathrm{X}$ for $t \ge 0$ in Ξ. If $E(x,t)$ and its generalized derivative by virtue of (31) satisfy

$$D_t E(x,t)\big|_{(31)} \le F\big(E(x,t),t\big), \tag{34}$$

then system (32) from Σ_+ is an upper comparison system , i.e.

$$E\big(x(t_0),t_0\big)\le X(t_0) \;\Rightarrow\; E\big(x(t),t\big)\le X(t)\,,\; t>t_0\,. \tag{35}$$

In (34), the derivative by virtue of (31) can be defined by

$$D_t E(x,t)\Big|_{(31)} = \overline{\lim_{h\to 0+}} \tfrac{1}{h}\big[E\big(x+hf(x,t),t+h\big)-E(x,t)\big].$$

Similarly, we determine the class of systems Σ_- and lower comparison systems (32) for (31) by changing all used inequalities in Ξ by inverse. If we require instead of (34) the equality

$$D_t E(x,t)\Big|_{(31)} = F\big(E(x,t),t\big), \tag{36}$$

then due to a monotonicity of system (32), we have

$$X_1(t_0)\le E\big(x(t_0),t_0\big)\le X_2(t_0) \;\Rightarrow\; X_1(t)\le E\big(x(t),t\big)\le X_2(t)\,,\; t>t_0\,, \tag{37}$$

where $X_1(t)$ and $X_2(t)$ are some solutions of (32). It means, that (36) determines a class of monotone systems (32) used as lower and upper comparison systems simultaneously for (31).

The estimations (35) and (37) can be used for comparison of dynamic properties of systems (31) and (32), and also for construction of attraction region in a phase space of (31). For example, if given operator E, the inequality $E(x,t)\le 0$ is possible only for $x=0$, then under conditions (35) and $X(t)\to 0$, we have $x(t)\to 0$, $t\to\infty$. Constructing the positive or monotone on a cone upper comparison systems, we can choose E from a class of everywhere positive operators.

As an example, for the linear system

$$\dot{x} = A(t)x\,,\; x\in R^n\,, \tag{38}$$

we give the upper matrix comparison system

$$\dot{X} - A(t)X - XA(t)^T = G(X,t)\ge 0\,,\; X\in R^{n\times n}\,, \tag{39}$$

constructed by using (34) with $E(x,t)=xx^T\ge 0$ [10]. Here K is a cone of symmetric positive semi-definite matrices and system (39) is positive. If $G(X,t)=XP(t)X$, where $P(t)=P(t)^T\ge 0$, then (39) is the Riccati differential equation. Asymptotic stability of system (38) follows from asymptotic stability of the matrix differential equation (39).

Studying a system (32), we can use comparison systems in a phase space Ξ. Assume, that right side of (32) satisfies the estimation

$$G_1(t)-M(t)X\le F(X,t)\le G_2(t)-M(t)X\,,\; X\in\Xi\,,\; t\ge 0, \tag{40}$$

where $M(t)$ is a linear operator describing the differential systems

$$\dot{X}_1 + M(t)X_1 = G_1(t),\; X_1\in\Xi\,,\; t\ge 0, \tag{41}$$

$$\dot{X}_2 + M(t)X_2 = G_2(t),\; X_2\in\Xi\,,\; t\ge 0, \tag{42}$$

with monotone evolutional operator $W(t,s)$. If in Lemma 4, $G(X,t)=F(X,t)+M(t)X$, then

$$X_1(t_0)\le X(t_0)\le X_2(t_0)\Rightarrow X_1(t)\le X(t)\le X_2(t)\,,\; t>t_0\,.$$

It means, that double-sided estimation (40) determines for (32) accordingly the lower and upper comparison systems (41) and (42). Here $E=I$, i.e. a phase space transformation is not used.

Note that we can construct the lower and upper systems of comparison in different partially ordered spaces Ξ_1 and Ξ_2 as follows

$$\dot{X}_1 = F_1(X_1,t),\;\; X_1\in\Xi_1\,,\; t\ge 0, \tag{43}$$

$$\dot{X}_2 = F_2(X_2,t),\;\; X_2\in\Xi_2\,,\; t\ge 0, \tag{44}$$

In addition, properties of suitable operators $E_1(x,t)$ and $E_2(x,t)$, and also the order relations defined by cones $K_1 \subset \Xi_1$ and $K_2 \subset \Xi_2$ in

$$E_1(x(t),t) \geq X_1(t),\ E_2(x(t),t) \leq X_2(t), \tag{45}$$

should be defined by the studying characteristics of original system (31). For example, we can require that the system of inequalities $E_1(x,t) \geq 0$ and $E_2(x,t) \leq 0$ holds only for $x = 0$. In this case, we have to expect that in (45), $x(t) \to 0$ for $t \to \infty$ if $X_1(t) \to 0$ and $X_2(t) \to 0$, where $X_1(t)(X_2(t))$ is a solution of lower (upper) comparison system (43) ((44)). If $E_1(x,t) \equiv E_2(x,t)$, then this follows from a lemma on two militiamen in a partially ordered space [1].

References

1. M.A.Krasnoselsky, E.A.Lifshits, A.V.Sobolev. Positive linear systems.- Moskow: Nauka, 1985.- 256 p. (Russ.).
2. M.A.Krasnoselsky. Shift operator on trajectories of differential equations.- Moskow: Nauka, 1966.- 332 p. (Russ.).
3. Ph.Clement, H.Heijmans, S.Angenent, C. van Duijn, B. De Pagter. One-parameter semigroups .- Moskov: Mir, 1992.- 352 p. (Russ.).
4. Milshtein G.N. Exponential stability of positive semigroups in linear topological space // Izv. vuzov. Matematika.- 1975.- № 9.- P.35-42 (Russ.).
5. A.G.Mazko. Stability of linear positive systems // Ukrain. Math. Zhurn..- 2001.- vol. 53, № 3.- P. 323-330 (Russ.).
6. A.G.Mazko. Localization of a spectrum and stability of dynamic systems.- Kiev: Institute of Mathematics of NAS of Ukraine, 1999.- 216 p. (Russ.).
7. Ju.L.Daletsky, M.G.Kreijn. Stability of the solutions of differential equations in a Banach space.- Moskow: Nauka, 1970.- 535 p. (Russ.).
8. B.P.Demidovich. The lectures on a mathematical stability theory.- Moskow: Nauka, 1967.- 472 p. (Russ.).
9. V.M.Matrosov, L.Ju.Anapolsky, S.N.Vasiljev. Method of comparison in mathematical systems theory. - Novosibirsk: Nauka, 1980.- 480 p. (Russ.).
10. V.Lakshmikantham, S.Leela, A.A.Martynyuk. Motion stability: method of comparison.- Kiev: Nauk. dumka, 1991.- 248 c. (Russ.).
11. N.S.Postnikov, E.F.Sabaev. Matrix comparison systems and their applications in automatic control problems // Avtomatika & telemechanika.- 1980.- № 4.- P. 24-34 (Russ.).

Author information: Mazko Alexey Grigorjevich, doctor of physical and mathematical sciences, head scientist of Institute of mathematics of Ukraine National Academy of Sciences; area of scientific interests: stability and control theory; tel.: (044) 224-02-95; e-mail: mazko@imath.kiev.ua.

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Alexey G. Mazko

Stability and comparison of systems in partially ordered space

Abstract

The classes of positive and monotone differential systems with respect to prescribed cone in a phase space are studied. The stability criteria of linear positive systems are formulated in terms of monotonically invertible linear operators. The methods for robust stability analysis and comparison of systems in partially ordered space are developed.