Simple things we don't know:

reflections on doing research in number theory

Masha Vlasenko

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Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad Re(s) > 1$$

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90}, \ldots \quad \zeta(2k) = (-1)^{k+1} \frac{B_{2k} 2^{2k-1}}{(2k)!} \pi^{2k}$$

$$B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \ldots \quad \text{Bernoulli numbers}$$

... rational numbers that show up *everywhere*, e.g.

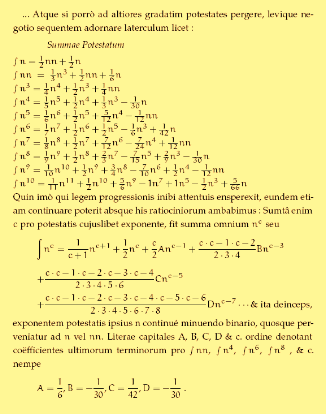
$$1 + 2 + \ldots + n = \frac{1}{2}\left(n^2 + n\right)$$

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{3}\left(n^3 + \frac{3}{2}n^2 + \frac{1}{2}n\right)$$

$$1^m + 2^m + \ldots + n^m$$

$$= \frac{1}{m+1}\left(B_0 \binom{m+1}{0} n^{m+1} + B_1 \binom{m+1}{1} n^m + \ldots + B_m \binom{m+1}{m} n\right)$$

Bernoulli numbers



... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$\int n = \frac{1}{2}nn + \frac{1}{2}n$

$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$

$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$

$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$

$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$

$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$

$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$

$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}nn$

$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

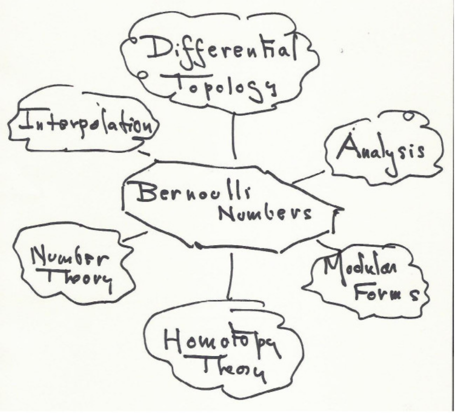
Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c\cdot c-1\cdot c-2}{2\cdot 3\cdot 4}Bn^{c-3}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4\cdot c-5\cdot c-6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7}\cdots \text{\& ita deinceps,}$$

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

Jakob Bernoulli's
"Summae Potestatum", 1713



Barry Mazur's sketch
of the unity of mathematics, 2008

Special values of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad Re(s) > 1$$

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90}, \ldots \quad \zeta(2k) = (-1)^{k+1} \frac{B_{2k} 2^{2k-1}}{(2k)!} \pi^{2k}$$

$$\zeta(3) = 1.2020569031595942853997381615114499908\ldots$$

$$\zeta(5) = 1.0369277551433699263313654864570341681\ldots$$

Theorem (Roger Apéry, 1979) $\zeta(3) \notin \mathbb{Q}$

Theorem (Wadim Zudilin, 2001) Among the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ at least one is irrational.

Conjecture (folklore) The numbers $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$.

Other functions like $\zeta(s)$: Hasse–Weil zeta functions

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 + \tfrac{1}{p^s} + \tfrac{1}{p^{2s}} + \dots\right) = \prod_{p \text{ prime}} \tfrac{1}{1-p^{-s}}$$

X algebraic variety $/\mathbb{Z}$ $\quad\rightsquigarrow\quad$ $\zeta_X(s) := \prod_{p \text{ prime}} \mathcal{Z}_{X/\mathbb{F}_p}(p^{-s})$

$$\mathcal{Z}_{X/\mathbb{F}_p}(T) = \exp\left(\sum_{m=1}^{\infty} \#X(\mathbb{F}_{p^m}) \frac{T^m}{m}\right) = 1 + \#X(\mathbb{F}_p)T + \dots$$

Theorem (Dwork, 1960) The local zeta function $\mathcal{Z}_{X/\mathbb{F}_p}(T)$ is a rational function of T.

Zeta functions of algebraic varieties

Example 1: $X =$ one point, $\#X(\mathbb{F}_{p^m}) = 1$

$$\mathcal{Z}_{X/\mathbb{F}_p}(T) = \exp\Big(\sum_{m\geq 1} \tfrac{T^m}{m}\Big) = \tfrac{1}{1-T}, \quad \zeta_X(s) = \prod_{p \text{ prime}} \tfrac{1}{1-p^{-s}}$$

Example 2: $X =$ elliptic curve

$$\mathcal{Z}_{X/\mathbb{F}_p}(T) = \frac{1-\alpha_p T + pT^2}{(1-T)(1-pT)}, \quad \alpha_p = p - \#X(\mathbb{F}_p)$$

Conjecture. $\zeta_X(s)$ can be analytically continued to a meromorphic function of s in the whole $\mathbb{C}$.

This is known only for very special classes of varieties. Orders of poles and zeroes, and special values of $\zeta_X(s)$ should "know" a lot about geometry and arithmetic of X: Birch and Swinnerton–Dyer conjecture, Beilinson–Deligne conjectures, Langlands program...

What we know about local zeta functions

$X =$ elliptic curve $\{y^2 = x^3 + ax + b\}$, $\quad \Delta := 4a^3 + 27b^2 \neq 0$

$$\mathcal{Z}_{X/\mathbb{F}_p}(T) = \frac{1 - \alpha_p T + pT^2}{(1-T)(1-pT)} \quad \text{for all} \quad p \nmid \Delta$$

$$\alpha_p = p - \#\{(x,y) \in \mathbb{F}_p^2 : y^2 = x^3 + ax + b\}$$

Theorem (Helmut Hasse, 1933) $\frac{\alpha_p}{2\sqrt{p}} \in [-1,1]$

Theorem (former Sato–Tate conjecture: Clozel, Harris, Shepherd-Barron, Taylor 2008, Barnet-Lamb, Geraghty, Harris, Taylor 2011)
Numbers $\frac{\alpha_p}{2\sqrt{p}} =: \cos(\theta_p)$ are distributed in $[-1,1]$ according to the law

$$\lim_{N\to\infty} \frac{\#\{p < N : t_1 < \theta_p < t_2\}}{\#\{p < N\}} = \frac{2}{\pi}\int_{t_1}^{t_2} \sin^2(\theta)d\theta.$$

*More precisely, this statement concerns elliptic curves without *complex multiplication*.

Punchline: could there be a formula for α_p?

$X_t \; : \; y^2 = x(x-1)(x-t), \qquad t \neq 0,1$ parameter
$\alpha_p(t) = p - \#\{(x,y) \in \mathbb{F}_p^2 : y^2 = x(x-1)(x-t)\}$

- The elliptic integral

$$F(t) = \frac{1}{\pi} \int_1^{\infty} \frac{dx}{\sqrt{x(x-1)(x-t)}} = \sum_{n=0}^{\infty} \frac{1}{16^n} \binom{2n}{n}^2 t^n$$

is annihilated by $L = t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}$.

- For any $p \neq 2$ the truncation $F_p(t) = \sum_{n=0}^{p-1} \frac{1}{16^n} \binom{2n}{n}^2 t^n$ is a solution to L modulo p, and

$$\alpha_p(t) \equiv F_p(t) \mod p.$$

- The function $\lambda_p(t) := \frac{F(t)}{F(t^p)}$ admits a *p-adic analytic continuation* to the set $\{t : F_p(t) \not\equiv 0 \mod p\}$ and for such t one has

$$\alpha_p(t) = \lambda_p(t) + \frac{p}{\lambda_p(t)} \qquad \text{(Dwork, 1969).}$$

Deformation theory of local zeta functions, after Dwork

$$\#\{(x,y) \in \mathbb{F}_p^2 : y^2 = x(x-1)(x-t)\} = p - \lambda_p(t) - \frac{p}{\lambda_p(t)}$$

$$\lambda_p(t) = \frac{F(t)}{F(t^p)}, \qquad F(t) = \int_1^\infty \frac{dx}{\sqrt{x(x-1)(x-t)}}$$

What you see here is a glimpse of an explicit deformation theory for zeta functions, which was anticipated by Bernard Dwork. It relies on fine arithmetic properties of solutions of differential equations arising from geometry, like $F(t)$. Together with Frits Beukers we started to explore and generalize these properties in a recent series of papers, which we call "Dwork crystals I, II, III"...