1000–1M19WFM Introduction to Modular Forms
Projects

1. For $n \geq 1$ we denote by $p(n)$ the number of partitions of n into a sum of positive integers. For example, the list of partitions of $n = 5$ is given by:

$$\begin{aligned} 5 &= 5 \\ 5 &= 4+1 \\ 5 &= 3+2 \\ 5 &= 3+1+1 \\ 5 &= 2+2+1 \\ 5 &= 2+1+1+1 \\ 5 &= 1+1+1+1+1 \end{aligned}$$

and therefore $p(5) = 7$. Start with proving the following identity for the generating function of these numbers:

$$\sum_{n=0} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \ldots = \prod_{m=1}^{\infty} \frac{1}{1-q^m}.$$

Prove Ramanujan's congruences:

$$\begin{aligned} p(5k+4) &\equiv 0 \mod 5 \\ p(7k+5) &\equiv 0 \mod 7 \\ p(11k+6) &\equiv 0 \mod 11 \end{aligned}$$

The best moment to start working on this project is after you are done with the third assignment.

2. Consider the elliptic curve given by the equation

$$E \ : \ y^2 = x^3 + x$$

and the q-series given by

$$f(q) = \sum_{\substack{m,\, n \,\in\, \mathbb{Z} \\ m \,\equiv\, 1 \mod 4}} m\, q^{m^2+4n^2} =: \sum_{n=1}^{\infty} a_n q^n.$$

(i) Prove that for every prime $p \neq 2$ we have $a_p = p - \#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p) = \{(x,y) \in \mathbb{F}_p^2 : y^2 = x^3 + x\}$ is the number of points on E over the finite field $\mathbb{F}_p$.

(ii) Prove that $f(q)$ is a modular form of weight 2 on a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. (This task of course involves identifying Γ.)

The formula in part (i) goes back to Gauss; you could work on it at any time. Attempt part (ii) after we do theta functions in class.

3. Consider the elliptic curve given by the equation

$$E \ : \ y^2 + y = x^3 - x^2.$$

(i) We will see in Part III of this course that the set of complex points of E can be turned into a compact Riemann surface by adding one point 'at infinity'. The compactified curve can be given by the homogeneous equation

$$\{[X : Y : Z] \in \mathbb{P}^2(\mathbb{C}) \mid Y^2Z + YZ^2 = X^3 - X^2Z\}.$$

Prove that this compact Riemann surface is isomorphic to $X_0(11) = X(\Gamma_0(11))$.

(ii) Show that

$$f(q) = q \prod_{m=1}^{\infty} (1 - q^m)^2 (1 - q^{11m})^2 =: \sum_{n=1}^{\infty} a_n q^n$$

is a modular form of weight 2 on $\Gamma_0(11)$.

(iii*) Prove that for every prime $p \neq 11$ we have $a_p = p - \#E(\mathbb{F}_p)$, where $\#E(\mathbb{F}_p) = \{(x, y) \in \mathbb{F}_p^2 : y^2 + y = x^3 - x^2\}$ is the number of points on E over the finite field $\mathbb{F}_p$.

The essential difference between the elliptic curve considered here and the one in Project 2 is that the curve $y^2 = x^3 + x$ has more automorphisms than a generic elliptic curve. Such curves are said to have complex multiplication. *That particular curve has an automorphism $(x, y) \mapsto (-x, iy)$, which makes it possible to compute its points over finite fields explicitly. In contrast to part (i) of Project 2, part (iii) of Project 3 might be difficult.*