1000–1M19WFM Introduction to Modular Forms
Tutorial 9 - May 17

Written assignment: exercises marked with (H), due on May 24.

1. In class we proved (Lemma 2 and Corollary 3) that every holomorphic homomorphism of complex tori

$$\phi : \mathbb{C}/\Lambda_1 \to \mathbb{C}/\Lambda_2$$

is given by

$$\phi(z + \Lambda_1) = \alpha\, z + \Lambda_2$$

for a complex number $\alpha \in \mathbb{C}$ such that $\alpha\Lambda_1 \subseteq \Lambda_2$. Let us denote the set of homomorphisms by

$$Hom(\Lambda_1, \Lambda_2) = \{\alpha \in \mathbb{C} : \alpha\Lambda_1 \subseteq \Lambda_2\}\,.$$

The set of endomorphisms (homomorphisms from a torus to itself) is denoted by $End(\Lambda) = Hom(\Lambda, \Lambda)$.

a) Show that $Hom(\Lambda_1, \Lambda_2)$ is an additive subgroup of $\mathbb{C}$.

b) Show that $End(\Lambda)$ is a subring of $\mathbb{C}$. In particular, $\mathbb{Z} \subseteq End(\Lambda)$.

c) For a nonzero $n \in \mathbb{Z}$, consider the multiplication-by-n endomorphism

$$[n] : \mathbb{C}/\Lambda \to \mathbb{C}/\Lambda$$
$$z + \Lambda \mapsto nz + \Lambda$$

Show that $Ker([n]) \cong \frac{1}{n}\Lambda/\Lambda$. Show that every point has n^2 preimages.

(H)2. A number $z \in \mathcal{H}$ is called a quadratic irrationality if it is a root of a definite quadratic form with integral coefficients:

$$Az^2 + Bz + C = 0, \qquad A, B, C \in \mathbb{Z}\,, \quad B^2 - 4AC < 0\,.$$

When A, B, C are coprime, the number $D = B^2 - 4AC$ is called the *discriminant* of z (denoted $D(z)$).

a) Show that if z is a quadratic irrationality, then gz is also a quadratic irrationality for any $g \in \mathrm{SL}_2(\mathbb{Z})$. Show that discriminants of z and gz are equal.

Observe that a) allows us to define a class of lattices $\Lambda = \mathbb{Z}w_1 + \mathbb{Z}w_2$ for which $\frac{w_1}{w_2}$ is a quadratic irrationality. They are called *CM lattices*, and respective tori are called *CM tori.* "CM" is an abbreviation for "complex multiplication", and the following exercise explains this term. For a CM lattice the discriminant $D(\frac{w_1}{w_2})$ is independent of the choice of basis and is denoted by $D(\Lambda)$.

b) Describe $End(\Lambda)$ for $\Lambda = \mathbb{Z}i + \mathbb{Z}$.
c) Show that $End(\Lambda) \neq \mathbb{Z}$ if and only if Λ is a CM lattice.
d) Show that for a CM lattice we have $End(\Lambda) \subset \mathbb{Q}(\sqrt{D})$ where $D = D(\Lambda)$.

(H)3. In this exercise we introduce an important elliptic function, which is called Weierstrass' p-function. Let Λ be a lattice in $\mathbb{C}$.

a) Prove that the series

$$\wp(z) \; = \; \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

converges absolutely and uniformly on compact sets not containing points in the lattice Λ. Conclude that $\wp(z)$ is a meromorphic doubly periodic function for Λ. Note that the poles of $\wp(z)$ are located at $z \in \Lambda$.

Hint: In Lecture 1 we proved that the sum $\sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^s}$ *is convergent for any* $s > 2$.

b) Show that the Laurent expansion of $\wp(z)$ at $z = 0$ is given by

$$\wp(z) \; = \; \frac{1}{z^2} + \sum_{m=2}^{\infty} (m+1) G_{m+2}(\Lambda) z^m,$$

where $G_k(\Lambda) = \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-k}$ is the Eisenstein series of weight k (Lecture 10).

c) Show that $\wp(z)$ and its derivative $\wp'(z)$ satisfy the algebraic relation

$$(\wp'(z))^2 \; = \; 4\,\wp(z)^3 \, + \, a\,\wp(z) \, + \, b,$$

with certain coefficients $a = a(\Lambda)$, $b = b(\Lambda)$.