1000–1M19WFM Introduction to Modular Forms
Tutorial 6 – April 12

Written assignment: exercises marked with (H), due on April 26.

Let X be a compact Riemann surface. We denote by $\mathcal{M}(X)$ be the field of meromorphic functions on X. As usually for fields, we denote by $\mathcal{M}(X)^\times = \mathcal{M}(X) \setminus \{0\}$ all non-zero elements. Note that $\mathcal{M}(X)^\times$ is a group under multiplication.

(H)1. The Fundamental Existence Theorem states that there is a non-constant meromorphic function on X. Deduce from this fact that there is a non-zero meromorphic differential form $\omega \neq 0$ on X.

Note that existence of such a form is needed to define the canonical divisor $K = div(\omega)$ in the Riemann–Roch theorem. Do you see why the statement of this theorem is independent of the choice of ω?

(H)2. A divisor $D = \sum_i n_i[P_i] \in Div(X)$ is a finite formal sum of points of X with coefficients in $\mathbb{Z}$. By $D \geq 0$ we mean that all $n_i \geq 0$. In Lecture 6 we consider $\mathbb{C}$-vector spaces

$$L(D) := \{f \in \mathcal{M}(X)^\times \mid div(f) + D \geq 0\} \cup \{0\}$$

for any divisor D. Prove that $\dim L(D) < \infty$.

(H)3. Consider $X = \mathbb{P}^1(\mathbb{C})$, the Riemann sphere. This Riemann surface is covered by two coordinate charts $(\mathbb{C}, z)$ and $(\mathbb{C}, w)$ with the transition map $w = 1/z$. It has genus $g = 0$.

a) Show that $\mathcal{M}(X) = \mathbb{C}(z)$. That is, every meromorphic function on $\mathbb{P}^1(\mathbb{C})$ is rational.

In Lecture 3 we proved that a ratio of two modular forms of the same weight on $\mathrm{SL}_2(\mathbb{Z})$ is a rational function of j-invariant. This exercise gives a conceptual explanation of this fact.

b) Show that there are no non-zero holomorphic differential forms on $\mathbb{P}^1(\mathbb{C})$.
c) Give an example of a canonical divisor on $\mathbb{P}^1(\mathbb{C})$.
d) Compute $\ell(D) = \dim_{\mathbb{C}} L(D)$ for any divisor D. Check that the Riemann – Roch theorem holds for the Riemann sphere.