1000–1M19WFM Introduction to Modular Forms
Tutorial 5 – April 5

Written assignment: exercises marked with (H), due on April 12.

(H)1. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index such that $-1 \in \Gamma$. *Cusps* of Γ are defined as orbits of the Γ-action on $\mathbb{P}^1(\mathbb{Q})$. For any $\alpha \in \mathbb{P}^1(\mathbb{Q})$ the index

$$h_\alpha = [I_{\mathrm{SL}_2(\mathbb{Z})}(\alpha) : I_\Gamma(\alpha)]$$

depends only on the Γ-orbit of α (see Ex. 2 of Assignment 4.) This number is called the *width* of the respective cusp $[\alpha] := \Gamma\alpha$.

a) Show that the number of cusps of Γ is finite.
b) Show that the width of a cusp is finite.
c) Denote $d = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$ and choose any set $\{\gamma_i : 1 \le i \le d\}$ of representatives of right cosets, so that $\mathrm{SL}_2(\mathbb{Z}) = \cup_{i=1}^d \Gamma\gamma_i$. Prove that

$$h_\alpha = \#\{1 \le i \le d : \gamma_i(\infty) \in [\alpha]\}.$$

d) Conclude that the sum of widths of all cusps of Γ equals d:

$$\sum_{[\alpha] \in \Gamma\backslash\mathbb{P}^1(\mathbb{Q})} h_\alpha = d.$$

(See also the question after Ex.3 b) in Assignment 4.)

(H)2. Show that if $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a normal subgroup then all cusps have equal width.

3. Let p be a prime, let $X_0(p) := X(\Gamma_0(p))$.

a) Show that

$$\gamma_j = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix} \text{ for } j = 0, \ldots, p-1, \quad \gamma_\infty = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is a set of right coset representatives for $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{Z})$.

b) Show that $X_0(p)$ has exactly two cusps.
c) Show that $g\gamma_j(i) = \gamma_j(i)$ for some $g \in \Gamma_0(p)$ of order 4 if and only if $j^2 + 1 \equiv 0 \mod p$. Thus the number of elliptic points of order 2 in $X_0(p)$ equals

$$\varepsilon_2 = \begin{cases} 2, & p \equiv 1 \mod 4, \\ 0, & p \equiv 3 \mod 4, \\ 1, & p = 2. \end{cases}$$

d) Show that $g\gamma_j(\rho) = \gamma_j(\rho)$ for some $g \in \Gamma_0(p)$ of order 6 if and only if $j^2 - j + 1 \equiv 0 \mod p$. Thus the number of elliptic points of order 3 in $X_0(p)$ equals

$$\varepsilon_3 = \begin{cases} 2, & p \equiv 1 \mod 3, \\ 0, & p \equiv 2 \mod 3, \\ 1, & p = 3. \end{cases}$$

Along with part c), this shows that the number of elliptic points is determined by $p \mod 12$*. In particular,* $p = 13$ *is the smallest prime such that all four possible elliptic points exist.*

e) Conclude that the genus of $X_0(p)$ is equal to

$$g = \begin{cases} \lfloor \frac{p+1}{12} \rfloor - 1, & p \equiv 1 \mod 12, \\ \lfloor \frac{p+1}{12} \rfloor, & \text{otherwise.} \end{cases}$$

MAGMA is a computer algebra system which is convenient for computations with modular forms. Its online calculator can be found at *http : //magma.maths.usyd.edu.au/calc/* Here is a sample code to enter:

```
G:=Gamma0(11);
Index(G);
Cusps(G);
EllipticPoints(G);
Genus(G);
```