1000–1M19WFM Introduction to Modular Forms
Tutorial 4 – March 29

Written assignment: exercises marked with (H), due on April 5.

1. Let $f(z) \in \mathbb{C}(z)$ be a non-constant rational function. Regard it as a holomorphic map of the Riemann sphere to itself $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. What is the degree of this map? What does the Riemann–Hurwitz formula say?

(H)2. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index such that $-1 \in \Gamma$. We denote $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$, $\widetilde{\Gamma} = \Gamma/\{\pm 1\}$. Consider the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$ by linear fractional transformations.

For a group G acting on a set X, for any $x \in X$ we denote by $I_G(x) = \{g \in G : gx = x\}$, the stabilizer *of x in G. It is clear that $I_G(x) \subset G$ is a subgroup.*

a) Show that $I_{\mathrm{PSL}_2(\mathbb{Z})}(\infty)$ is $\langle T \rangle$, the subgroup generated by $T = \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. (This subgroup is then isomorphic to the additive group $\mathbb{Z}$ via the map $T^m \mapsto m$.)

b) For any $\alpha \in \mathbb{P}^1(\mathbb{Q})$, check that there exists $g \in \mathrm{PSL}_2(\mathbb{Z})$ such that $g(\infty) = \alpha$. Check that $g^{-1} I_{\widetilde{\Gamma}}(\alpha) g \subseteq I_{\mathrm{PSL}_2(\mathbb{Z})}(\infty)$ and show that the index

$$h = \left[I_{\mathrm{PSL}_2(\mathbb{Z})}(\infty) : g^{-1} I_{\widetilde{\Gamma}}(\alpha) g \right]$$

depends only on the orbit $\Gamma\alpha$.

The orbits ${}_{\Gamma}\backslash\mathbb{P}^1(\mathbb{Q})$ are called *cusps* of Γ, and the above number h is called the *width* of the respective cusp $\Gamma\alpha$.

(H)3. Consider $\Gamma = \Gamma_0(4)$. By Ex. 5 of Assignment 3 we know that $[\mathrm{PSL}_2(\mathbb{Z}) : \widetilde{\Gamma}] = [\mathrm{SL}_2(\mathbb{Z}), \Gamma] = 6$.

a) Describe a connected fundamental domain for the action of Γ in the upper halfplane $\mathcal{H}$. (It should consist of 6 images of the fundamental domain $\mathcal{D}$ for $\mathrm{SL}_2(\mathbb{Z})$.)

b) Show that Γ has 3 cusps and find their widths.
Can you see the width of a cusp on a sketch of a fundamental domain?

c) Note that the quotient $X = {}_{\Gamma}\backslash\mathcal{H}$ can be turned into a compact surface $\overline{X}$ by adding 3 points corresponding to the cusps. Compute the genus of $\overline{X}$.
You could use the triangulation from part a) and the formula $2 - 2g = V - E + F$, see Lecture 4.