1000–1M19WFM Introduction to Modular Forms
Tutorial 2 – March 15

Written assignment: exercises marked with (H), due on March 22.

In Exercises 1-3 we follow some ideas due to Erich Hecke to construct the Eisenstein series of weight 2 and prove its transformation properties under $\mathrm{SL}_2(\mathbb{Z})$, which are close to those of a modular form of weight 2 but with a correction term.

1. Show that the series

$$G_2(z) \;=\; \zeta(2) \;-\; 4\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

is convergent for $|q| < 1$.

Remark: since the Lipschitz formula (Ex. 4, assignment 1) holds for $k = 2$, we still have

$$G_2(z) \;=\; \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} \;+\; \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2}\,,$$

if we agree to carry the summation over n first and then over m. However, because of the non-absolute convergence of the double series, we can no longer interchange the order of summation to get the transformation formula $G_2(-\frac{1}{z}) = z^2 G_2(z)$. (The equation $G_2(z+1) = G_2(z)$, of course, still holds.)

2. Define

$$G_{2,\epsilon}(z) \;=\; \frac{1}{2} \sum_{\substack{m,\,n\,\in\,\mathbb{Z} \\ (m,n)\,\neq\,(0,0)}} \frac{1}{(mz+n)^2|mz+n|^{2\epsilon}}$$

for $\epsilon > 0$ and $z \in \mathcal{H}$. This series converges absolutely due to the argument which was given in class for the convergence of Eisenstein series. Show that

$$G_{2,\epsilon}\Big(\frac{az+b}{cz+d}\Big) \;=\; (cz+d)^2|cz+d|^{2\epsilon} G_{2,\epsilon}(z).$$

(Here and below $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.)

(H)3. It is proved in [1-2-3] in Section 2.3 (p.20) that

$$\lim_{\epsilon\to 0} G_{2,\epsilon}(z) \;=\; G_2(z) - \frac{\pi}{2\,\mathrm{Im}(z)}\,.$$

a) Use this formula and the previous exercise to show that the non-holomorphic function in the right-hand side transforms like a modular form of weight 2.

b) Next, deduce the transformation formula

$$G_2\Big(\frac{az+b}{cz+d}\Big) \;=\; (cz+d)^2 G_2(z) \;-\; \pi\, i\, c\,(cz+d).$$

(H)4. Let $f(z)$ be a modular form of weight k, let $f'(z) = \frac{d}{dz}f(z)$. Find out the missing term in the formula

$$f'\Big(\frac{az+b}{cz+d}\Big) \;=\; (cz+d)^{k+2} f'(z) + ??\,.$$

Write a transformation formula for the logarithmic derivative $g(z) = \frac{f'(z)}{f(z)}$:

$$g\Big(\frac{az+b}{cz+d}\Big) \;=\; (cz+d)^2\, g(z) + ??\,.$$

In Exercises 5-6 we prove that $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is a modular form. We define $E_2(z) = G_2(z)/\zeta(2) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$.

(H)5. Prove that $\dfrac{1}{2\pi i}\dfrac{\Delta'(z)}{\Delta(z)} = E_2(z)$.

Hint: Observe that $\dfrac{\Delta'(z)}{\Delta(z)} = \frac{d}{dz}\log\Delta(z)$. Use multiplicativity of the logarithm function and expansion $\log(1-x) = -\sum_{m=1}^{\infty}\frac{x^m}{m}$.

6. Use the result of the previous exercise to show that

$$\frac{d}{dz}\log\Big(\frac{\Delta\big(\frac{az+b}{cz+d}\big)}{(cz+d)^{12}\Delta(z)}\Big) \;=\; 0\,.$$

Prove that $\Delta(z)$ is a modular form of weight 12.

7. Using the fact that $\dim M_{12} = 2$, prove the identity

$$1728\,\Delta \;=\; E_4^3 \;-\; E_6^2\,.$$

References

[1-2-3] Don Zagier, *Elliptic modular forms and their applications*, in *The 1-2-3 of Modular Forms*