1000–1M19WFM Introduction to Modular Forms
Tutorial 12 – June 7

Written assignment: exercises marked with (H), due on June 14.

(H)1. Let $k \geq 4$ be even and let $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ be the Eisenstein series of weight k. By $\langle \cdot, \cdot \rangle$ we denote the Petersson inner product on $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

a) Prove that $\langle E_k, g \rangle = 0$ for every cusp form $g \in S_k$.

b) Show that E_k is a Hecke eigenform and find the respective engenvalues $\{\lambda_n = \lambda_n(E_k); n \geq 1\}$:

$$\mathbb{T}_n \, E_k \; = \; \lambda_n \, E_k \, .$$

c) Let $G_k = const \cdot E_k$ be the respective normalized Hecke eigenform. Show that

$$L(G_k, s) \; = \; \zeta(s)\zeta(s-k+1)$$

where $\zeta(s)$ is the Riemann zeta function given for $\mathrm{Re}(s) > 1$ by the Dirichlet series

$$\zeta(s) \; = \; \sum_{n=1}^{\infty} \frac{1}{n^s} \, . \qquad (*)$$

d) Write down the Euler product for $L(G_k, s)$.

(H)2. a) Prove that the Dirichlet series (*) converges when $\mathrm{Re}(s) > 1$.

b) Let $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$ be the Jacobi theta function. Compute the Mellin transform of the restriction of θ to the imaginary axis

$$\phi(s) \; = \; \int_0^{\infty} t^{s-1} \big(\theta(it) - 1\big) dt \, . \qquad (**)$$

c) A while ago we proved in class that

$$\theta(it) \; = \; \frac{1}{\sqrt{2t}} \, \theta\big(\frac{i}{4t}\big) \, . \qquad (***)$$

Use this modular property of θ to construct the analytic continuation of $\zeta(s)$ to the entire complex plane, show that the only pole of $\zeta(s)$ is at $s = 1$ and prove the functional equation

$$\widehat{\zeta}(s) \; = \; \widehat{\zeta}(1-s),$$

where $\widehat{\zeta}(s) \; = \; \dfrac{\Gamma(s/2)}{\pi^{s/2}} \zeta(s)$.

d) Use the functional equation given in part c) to evaluate $\zeta(k)$ for $k \in \mathbb{Z}_{\leq 0}$.

Hint for part c): The problematic point in the integral (**) is $t = 0$. The same integral taken from some $c \in \mathbb{R}_{>0}$ to ∞ would converge for any $s \in \mathbb{C}$, thus defining an entire function. Break (**) at the symmetry point c for $\theta(it)$ and use (***) to transform integration along $\int_0^c$ into integration along $\int_c^\infty$.

If you do part d) correctly, you should see that the following folklore 'identities' make some sense:

$$\begin{aligned} 1+1+1+1+1+1+\ldots &= -\frac{1}{2} \\ 1+2+3+4+5+6+\ldots &= -\frac{1}{12} \end{aligned}$$