1000–1M19WFM Introduction to Modular Forms
Tutorial 11 – May 31

Written assignment: exercises marked with (H), due on June 7.

(H)1. In this exercise we construct the so called *Petersson inner product*, which is important for the study of arithmetic properties of modular forms. We denote the real and imaginary parts of $z \in \mathcal{H}$ by x and y respectively: $z = x+iy$. Complex conjugation is denoted by $\overline{z} = x - iy$.

a) Show that the differential 2-form $y^{-2}dxdy$ is $\mathrm{SL}_2(\mathbb{R})$-invariant.

b) Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index. For $f, g \in M_k(\Gamma)$, show that the real analytic function $y^k f(z)\overline{g(z)}$ is Γ-invariant.

c) It follows from a) and b) that the following double integral is independent of the choice of a fundamental domain $\mathcal{D}_\Gamma \subset \mathcal{H}$ for the action of Γ in $\mathcal{H}$:

$$\langle f, g\rangle := \iint\limits_{\mathcal{D}_\Gamma} f(z)\overline{g(z)}y^{k-2}dxdy.$$

Show that this integral is convergent when either f or g is a cusp form.

Simplification: if you wish you can do a simpler version of c): assume that $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

(H)2. For a modular form $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, compute $\lim_{t\to 0_+} f(it)$.

(H)3. For a $\Lambda \subset \mathbb{C}$ let us denote the respective Weierstrass p-function by

$$\wp(\Lambda, u) = \frac{1}{u^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(u-\lambda)^2} - \frac{1}{\lambda^2}\right), \qquad u \in \mathbb{C}/\Lambda.$$

Consider the following functions of lattices Λ, that are given by evaluation of their Weierstrass p-function at the points of finite order:

$$f_{\binom{\alpha}{\beta}}(z) := \wp(\mathbb{Z}z + \mathbb{Z}, \alpha z + \beta), \qquad \begin{pmatrix}0\\0\end{pmatrix} \neq \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in \mathbb{Q}^2/\mathbb{Z}^2.$$

Remark: in Lecture 10 we proved that isomorphism classes of complex tori correspond to $\mathrm{SL}_2(\mathbb{Z})$-orbits in the upper halfplane $\mathcal{H}$. What do points $z \in \mathcal{H}$ correspond to? Here is the answer: a point z corresponds to an isomorphism class of tori $\mathbb{C}/\Lambda$ with a fixed basis (ω_1, ω_2) in the lattice $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ (via $z = \omega_1/\omega_2$).

a) For $g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, show that

$$f_{\binom{\alpha}{\beta}}\Big|_2 g(z) \;=\; f_{\binom{\alpha'}{\beta'}}(z)$$

for some $\binom{\alpha'}{\beta'} \in \mathbb{Q}^2/\mathbb{Z}^2$. Give a formula for $\binom{\alpha'}{\beta'}$.

b) Prove that functions $f_{\binom{\alpha}{\beta}}$ with $\binom{\alpha}{\beta} \in \frac{1}{N}\mathbb{Z}^2/\mathbb{Z}^2$ are modular forms of weight 2 for $\Gamma(N)$.

c) Take $N = 2$ and consider $f_1(z) := f_{\binom{1/2}{1/2}}(z)$, $f_2(z) := f_{\binom{1/2}{0}}(z)$ and $f_3(z) := f_{\binom{0}{1/2}}(z)$. Since $\dim M_2(\Gamma(2)) = 2$, there must be a linear relation between these three functions. Show that

$$f_1(z) + f_2(z) + f_3(z) = 0.$$

Hint: Look at Lemma 3 in Lecture 11.

d) In Lecture 8 we constructed Jacobi's theta series and showed that $\theta_1^4(z)$, $\theta_2^4(z)$ and $\theta_3^4(z)$ belong to $M_2(\Gamma(2))$. Express f_1, f_2, f_3 as linear combinations of $\theta_1^4(z)$ and $\theta_2^4(z)$ (this is a basis). Check that your result agrees with the identity from part b).

Hint: Write down 3×3 matrices that correspond to the action of the generators $S, T \in \mathrm{SL}_2(\mathbb{Z})$ on the vector $\overrightarrow{f}(z) = (f_1(z), f_2(z), f_3(z))^T$:

$$\overrightarrow{f}|_2 S = (?)\overrightarrow{f}, \quad \overrightarrow{f}|_2 T = (?)\overrightarrow{f}.$$

In Lecture 8 we've done the same matrices for $\overrightarrow{\theta} = (\theta_1^4(z), \theta_2^4(z), \theta_3^4(z))$. Now the question is reduced to linear algebra. You could identify linear combinations of $\theta_i^4(z)$ that are stable under S and T respectively, which would give you two of the three functions f_i (up to a constant).