1000–1M19WFM Introduction to Modular Forms
Tutorial 1 – March 8

Written assignment: exercises marked with (H), due on March 15.

1. Show that formula
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$
defines a left group action of $\mathrm{SL}_2(\mathbb{R})$ in the upper halfplane $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.
2. Let $k \geq 0$ be an integer and $V = \{f : \mathcal{H} \to \mathbb{C}\}$ be the space of holomorphic functions in the upper halfplane. Show that formula
$$\left(f\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) \;=\; \frac{1}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right)$$
defines a right group action of $\mathrm{SL}_2(\mathbb{R})$ in V.

(H)3. (*Poisson summation on the real line*) Let $g : \mathbb{R} \to \mathbb{C}$ be a differentiable function of rapid decay, that is $|g(x)| = O(|x|^{-c})$ and $|g'(x)| = O(|x|^{-c})$ when $x \to \infty$ for some $c > 1$. Its Fourier transform is defined as
$$\hat{g}(y) = \int_{-\infty}^{+\infty} g(x) e^{-2\pi i x y} dx\,.$$
Prove that
$$\sum_{n\in\mathbb{Z}} \hat{g}(n) \;=\; \sum_{n\in\mathbb{Z}} g(n)\,.$$

(H)4. Use previous exercise to prove the Lipshitz formula
$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} \;=\; \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r z}\,.$$
for integer $k \geq 2$ and $z \in \mathcal{H}$.

(H)5. Use the Lipshitz formula to compute the Fourier coefficients of the Eisenstein series
$$G_k(z) \;=\; \frac{1}{2} \sum_{\substack{m,\, n \,\in\, \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k}\,.$$
($k \geq 4$, even).

6. The Bernoulli numbers $B_0, B_1, \ldots$ are defined by the generating function
$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$
Prove Euler's formula for the values
$$\zeta(k) = -\frac{B_k (2\pi i)^k}{2\,k!}$$
of the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ at even integers $k = 2, 4, 6, \ldots$

Make a conclusion about rationality of Fourier coefficients of the (renormalized) Eisenstein series
$$E_k(z) = G_k(z)/\zeta(k).$$