Appendix 3 Eisenstein series with characters

$N \geqslant 1$, integer

Def A Dirichlet character modulo N is a homomorphism $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.
A standard convention is that χ is extended to a map $\chi : \mathbb{Z} \to \mathbb{C}$ (traditionally denoted by the same letter) by setting

$$\chi(n) = \begin{cases} \chi(n \bmod N), & (n, N) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Attached to χ, there is a Dirichet series (L-function)

$$L(\chi, s) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad Re(s) > 1.$$

One of the classical results in analytic number theory is that this function has an analytic continuation, that is, it extends to a meromorphic function of s in the whole complex plane $\mathbb{C}$.

In the theory of modular forms, a standard convention is to extend χ to a character of $\Gamma_0(N)$ by

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \chi(a).$$

(Again, this character is denoted by the same letter.)

Exercise: Check that $\chi : \Gamma_0(N) \to \mathbb{C}^\times$ is a character.

Recall (Lecture 7) that the space $M_k(\Gamma, \chi)$ consists of functions $f : \mathcal{H} \to \mathbb{C}$ that are (i) holomorphic, (iii) bounded at cusps and satisfy (ii') $(f|_k g)(z) = \chi(g) f(z) \quad \forall g \in \Gamma$.

Exercise: Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a Dirichlet character and $k \geq 3$ be such that $\chi(-1) = (-1)^k$. Check that

$$\sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{\chi(m)}{(mz+n)^k} \in M_k(\Gamma_0(N), \chi).$$

Using Lipschitz formula (Ex. 4 of Assignment 1) we find that the q-expansion of this function at ∞ is given by

$$2 \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \Big(\sum_{m|n} \chi\left(\tfrac{n}{m}\right) m^{k-1} \Big) q^n.$$

Another example is given by the series

$$G_{k,\chi}(z) = \frac{1}{2} L(\chi, 1-k) + \sum_{n=1}^{\infty} \Big(\sum_{m|n} \chi(m) m^{k-1} \Big) q^n$$

$$\in M_k(\Gamma_0(N), \chi), \qquad k \geq 1, \ \chi(-1) = (-1)^k.$$

The proof of this fact is more elaborate (see Theorem 4.5.1 in Diamond-Shurman)