Appendix 2 Elliptic points and ramification

Consider $f : X(\Gamma) \to X(SL_2(\mathbb{Z}))$, the holomorphic map from Lecture 5.

Let $Q = \bar{i}$ or $\bar{\rho}$ be one of the two ell. points on $X(SL_2(\mathbb{Z}))$.

Let $P \in f^{-1}(Q)$. We claim there are exactly two possibilities:

- either P is an elliptic point for $X(\Gamma)$, in which case $e_P = 1$,
- or P is not an ell. point on $X(\Gamma)$, in which case $e_P = 2$ or 3 respectively.

Exercise $X \xrightarrow{f_1} Y \xrightarrow{f_2} Z$ holomorphic maps of Riemann surfaces

Pick $P \in X$; let $Q = f_1(P) \in Y$, $V = f_2(Q) \in Z$.

Then $e_{P/V} = e_{P/Q} \cdot e_{Q/V}$.

ramification degree for $f_1 \circ f_2$ — for f_1 — for f_2

Applying this property to

$$\mathcal{H} \xrightarrow{\pi} Y(\Gamma) \xrightarrow{f} Y(SL_2(\mathbb{Z}))$$

$$z \in \pi^{-1}(P) \longmapsto P \longmapsto Q$$

we obtain:

z is ell. point for Γ $\Rightarrow$ $e_{z/P} = e_{z/Q}$ $(= 2 \text{ or } 3)$ $\Rightarrow e_{P/Q} = 1$

z is not an ell. point for Γ (but ell. point for $SL_2(\mathbb{Z})$) $\Rightarrow$ $e_{z/P} = 1$, $e_{z/Q} = 2$ or 3 $\Rightarrow e_{P/Q} = e_{z/Q} = 2$ or 3 $\boxtimes$

Consequences:

let's compute

$$\sum_{P\in f^{-1}(\bar{i})} (e_P - 1) \quad \text{and} \quad \sum_{P\in f^{-1}(\bar{\rho})} (e_P - 1)$$

(this was done in the proof of the genus formula in Lecture 5, but now we have a more detailed look)

$$\sum_{P\in f^{-1}(\bar{i})} (e_P - 1) = 0\cdot \varepsilon_2 + 1\cdot \#\{P\in f^{-1}(\bar{i}) : P \text{ non-elliptic}\}$$

ell. points of order 2 on $X(\Gamma)$ have $e_P = 1$, so they don't contribute

let's denote this number by X

On the other hand (see Lecture 4) we have

$$d = \sum_{P\in f^{-1}(\bar{i})} e_P = 1\cdot \varepsilon_2 + 2\cdot X$$

$$\Rightarrow X = \frac{d-\varepsilon_2}{2} \Rightarrow \sum_{P\in f^{-1}(\bar{i})} (e_P - 1) = \frac{d-\varepsilon_2}{2}$$

Similarly, $\displaystyle\sum_{P\in f^{-1}(\bar{\rho})} (e_P - 1) = 0\cdot \varepsilon_3 + 2\cdot \#\{P\in f^{-1}(\bar{\rho}) : P \text{ non-ell.}\}$

$$= 2\cdot X$$

$$d = \sum_{P\in f^{-1}(\bar{\rho})} e_P = \varepsilon_3 + 3\cdot X$$

$$\Rightarrow X = \frac{d-\varepsilon_3}{3} \Rightarrow \sum_{P\in f^{-1}(\bar{\rho})} (e_P - 1) = \frac{2}{3}(d-\varepsilon_3)$$