Appendix 1 Classification of linear fractional transformations

$SL_2(\mathbb{C})$ acts on $\mathbb{C}^2$ in a natural way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + dv \end{pmatrix}$$

Note that each element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by a linear transformation of the vector space $\mathbb{C}^2$, and hence this action descends to the action of $SL_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C}^2 \setminus \{0\} \big/ \mathbb{C}^\times$.

Homogenous coordinates on $\mathbb{P}^1(\mathbb{C})$:

$[u : v]$ denotes the equivalence class of vectors $\begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix} \in \mathbb{C}^2$, $\lambda \in \mathbb{C}^\times$.

$$\mathbb{P}^1(\mathbb{C}) = \mathcal{U} \cup \{\infty\}$$

$$\infty := [1 : 0]$$

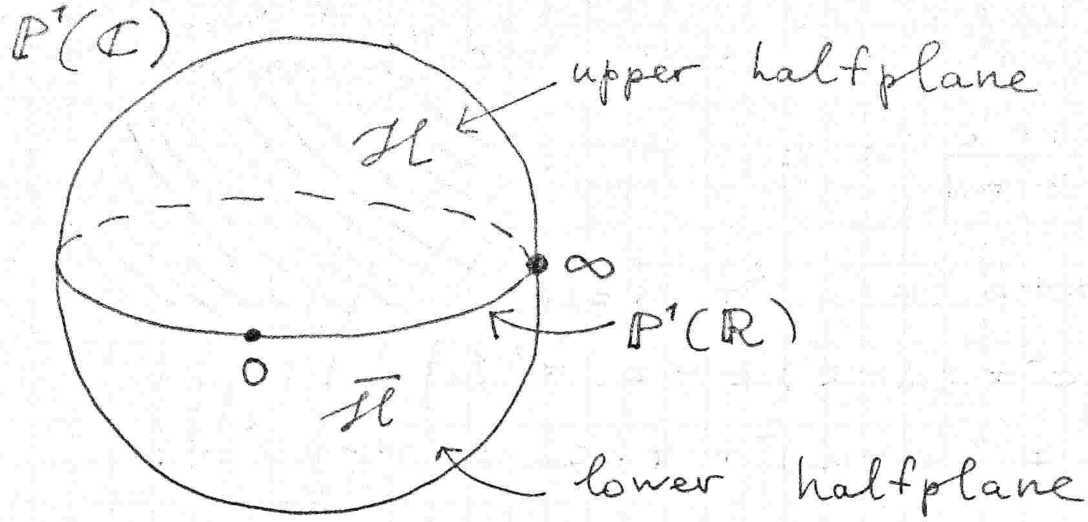
$$\mathcal{U} = \{ [u : v] \mid v \neq 0 \} \xrightarrow{\sim} \mathbb{C}$$

$$[u : v] \longmapsto z = \frac{u}{v}$$

$(\mathcal{U}, z)$ is a coordinate chart on the Riemann surface $\mathbb{P}^1(\mathbb{C})$

Exercise Check that in the coordinate z the above given action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is expressed as

$$z \longmapsto \frac{az + b}{cz + d}.$$



Observe:

$$\begin{array}{ll}
SL_2(\mathbb{C}) & \\
\cup & \\
SL_2(\mathbb{R}) & \text{preserves } \mathbb{P}^1(\mathbb{R}),\ \mathcal{H} \\
 & \quad \text{and the lower halfplane } \overline{\mathcal{H}} \\
\cup & \\
SL_2(\mathbb{Q}) & \text{preserves } \mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1(\mathbb{R}) \\
\cup & \\
SL_2(\mathbb{Z}) & \text{acts on } \mathbb{P}^1(\mathbb{Q}) \\
 & \underline{\text{transitively}}.\ (\text{one orbit})
\end{array}$$

See Exercise 2<u>b</u> in Tutorial 4.

Consider fixed points of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$:

$$\frac{az+b}{cz+d} = z$$

$$az+b = cz^2 + dz$$

$$cz^2 + (d-a)z - b = 0$$

$$D = (d-a)^2 + 4bc = (d+a)^2 - 4$$

Three cases:

- pair of complex roots
  ($\Leftrightarrow$ one fixed point in $\mathcal{H}$, and the other one in the lower halfplane $\bar{\mathcal{H}}$)

  $\Leftrightarrow D < 0 \Leftrightarrow |Tr(g)| < 2$
- unique real root
  $\Leftrightarrow D = 0 \Leftrightarrow Tr(g) = \pm 2$
- pair of real roots
  $\Leftrightarrow D > 0 \Leftrightarrow |Tr(g)| > 2$

In these cases the respective linear fractional transformation $z \mapsto \frac{az+b}{cz+d}$ is called

- elliptic $\quad |Tr(g)| < 2$
- parabolic $\quad Tr(g) = \pm 2$
- hyperbolic $\quad |Tr(g)| > 2$

<u>Exercise</u> Find out the relation between the fixed points of g and eigenvalues of g.

<u>Example</u> Note that any $g \in \Gamma_0(4)$ has $Tr(g) \equiv 2 \bmod 4$, hence there are no ell. points for $\Gamma_0(4)$.