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Algebraic hypergeometric functions I

We will consider functions of the form

$$u_n = \prod_{m>0} (nm)!^{\gamma_m} \qquad \gamma = (\gamma_1, \gamma_2, \dots) \in \mathbb{Z}$$

finitely many non-zero terms

$$X(t) = \sum_{n\geq 0} u_n t^n$$

we assume regularity condition: $\sum_{m>0} \gamma_m \cdot m = 0$

$\Leftrightarrow$ $X(t)$ has a finite non-zero radius of convergence

When is $X(t)$ algebraic?

We will prove that if it is algebraic then

(1) $d := -\sum_{m>0} \gamma_m = 1$

(2) all $u_n \in \mathbb{Z}$

E.g. $X(t) = \sum_{n=0}^{\infty} \underbrace{\binom{2n}{n}}_{\in \mathbb{Z}} t^n \qquad \gamma = (-2, 1, 0, \dots)$

$d = 1$

$$= \frac{1}{\sqrt{1-4t}}$$

(I) Eisenstein's theorem

- $X(t) \in \mathbb{C}[[t]]$
- $\exists F \in \overline{\mathbb{Q}}[x,y] : F(X(t), t) = 0$

Then $\exists c, N \in \mathbb{Z} : \quad c \cdot X(Nt) \in \mathbb{Z}[[t]]$

(II) We will prove that the monodromy group of the respective hypergeometric diff. equation is finite...

and $d = \#\{$ local exponents at 0 which are $= 1\}$

$\quad - \#\{$ local exps at ∞ which are $= 1\}$

(I) $x(t) = \underbrace{a_0 + a_1 t + a_2 t^2 + \dots + a_k t^k}_{=:\, x_k(t)} + \underbrace{a_{k+1} t^{k+1} + \dots}_{=\, y_k(t)}$

$F(x(t), t) \equiv 0$

(1) $a_0 \in \overline{\mathbb{Q}}$

(2) can assume that $a_0 = 0$ by considering $\tilde{F}(x, y) = F(x + a_0, y)$

(3) can assume that $F \in \overline{\mathbb{Z}}[x, y]$

Now we will prove the so-called non-degenerate case: assume that $\delta := F_x(0,0) \neq 0$.

Claim: $a_{k+1} = - \left. \dfrac{F(x_k(t), t)}{t^{k+1}\, \delta} \right|_{t=0}$

Proof: $0 = F(x_k(t) + y_k(t), t) =$

$F(x_k(t), t) + \underbrace{y_k(t)\, F_x(x_k(t), t)}_{=\, \delta\, a_{k+1} t^{k+1} + O(t^{k+2})} + O(t^{2k+2})$

$\boxtimes$

So, $a_1 = - \left. \frac{F(0,t)}{t\,\delta} \right|_{t=0} \in \overline{\mathbb{Q}}$

and by induction on $\mathbb{N}$ we get $a_n \in \overline{\mathbb{Q}}$ $\forall n$.

Note that if $\delta = 1$ then we obtain $a_n \in \overline{\mathbb{Z}}$ $\forall n$.

To prove the case of general δ, we consider the polynomial

$$G(x,y) := \delta^{-2} \cdot F(\delta x, \delta^2 y).$$

Note that $G_x(0,0) = 1$.

and $z(t) := \delta^{-1} x(\delta^2 t)$

satisfies $G(z(t), t) = 0$.

Therefore $z(t) \in \overline{\mathbb{Z}}[\![t]\!]$ by what was proved above.

$$z(t) = \sum_{k=1}^{\infty} \delta^{2k-1} a_k t^k, \quad \text{so} \quad \delta^{2k-1} a_k \in \overline{\mathbb{Z}} \ \forall k$$

Take $N := N_\delta \in \mathbb{Z}$ s.t. $\frac{N}{\delta^2} \in \overline{\mathbb{Z}}$.

Then

$$\underset{\overline{\mathbb{Z}}}{\underset{\cap}{\delta}} \cdot \underset{\overline{\mathbb{Z}}}{\underset{\cap}{\left(\frac{N^k}{\delta^{2k}}\right)}} \cdot \underset{\overline{\mathbb{Z}}}{\underset{\cap}{\left(\delta^{2k-1} a_k\right)}} \in \overline{\mathbb{Z}}$$

$$\| \\ N^k a_k \in \overline{\mathbb{Z}}.$$

Non-degenerate case ($\delta \neq 0$) is now proved.

Let us prove that any case can be reduced to the non-degenerate case.

We will prove that

$\exists M: \forall m > M \quad y_m$ satisfies condition $\delta \neq 0$.

$F(x(t), t) = 0$

field of Laurent series

The algebraic closure of $\overline{\mathbb{Q}}((t))$ is the field of Puiseux series over $\overline{\mathbb{Q}}$: these are Laurent series allowing rational exponents with bounded denominators.

This field is equipped with the t-adic valuation:

$$\nu\left(a t^{\frac{m}{n}} + \dots\right) = \frac{m}{n}$$

smallest exponent

The respective norm is $|\cdot| = e^{-\nu(\cdot)}$.

<u>Lemma</u> If ν is non-archimedean valuation on an algebraically closed field K, and we have a polynomial

$$x^n + b_{n-1}x^{n-1} + \dots + b_0 = \prod_{k=1}^{n} (x - \beta_k) \in K[x].$$

If $|\beta_1| < 1 < |\beta_2| \leq \dots \leq |\beta_n|$ then $|b_1| \geq |b_0|$ and $|b_1| > \max\{|b_2|, \dots, |b_{n-1}|\}$

Remark: $F(x(t), t) \equiv 0$

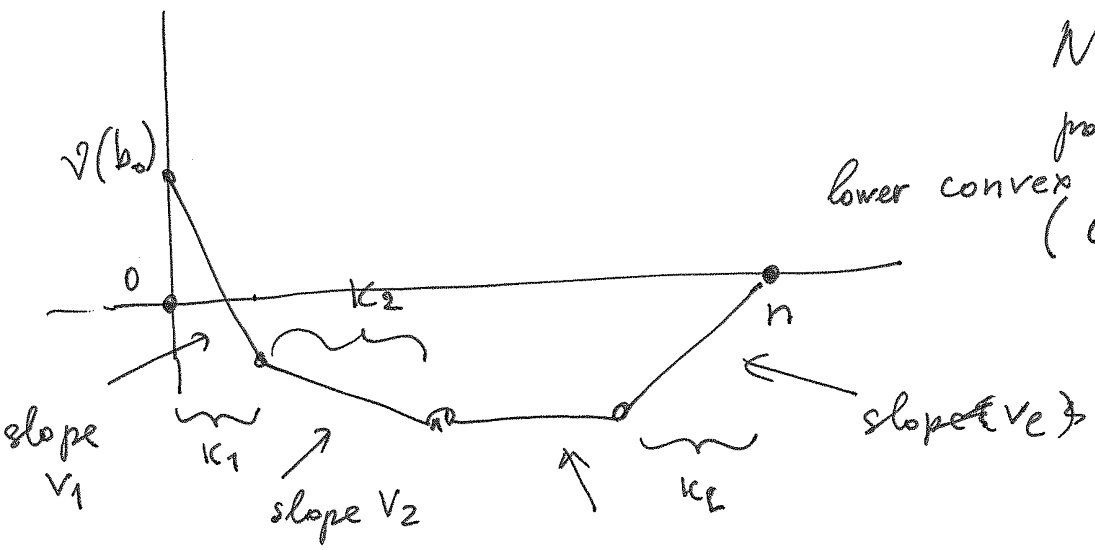
$$F(x, t) = \prod_{i=1}^{n} (x - \alpha_i(t))$$

$\alpha_i(t)$: Puiseux series

Back to the lemma:

$$x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0 = \prod_{i=0}^{n} (x - \beta_i)$$

Newton polygon = lower convex hull of $(i, \nu(b_i))$



Theorem about the Newton polygon: the roots can be numbered so that

$$\underbrace{\nu(\beta_1) = \ldots = \nu(\beta_{\kappa_1})}_{\kappa_1} = -v_1$$

$$\ldots$$

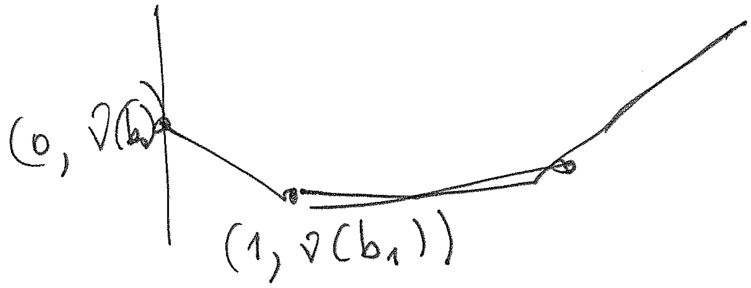
$$\underbrace{\nu(\beta) = \quad = \nu(\beta_n)}_{\kappa_\ell} = -v_\ell .$$

Our lemma immediately follows from this:

In our case:

$$\nu(\beta_1) > 0 > \nu(\beta_2) \geq \ldots \geq \nu(\beta_n)$$

so the Newton poly looks like



So $\nu(b_1)$ is the smallest among all coefficients

~~Take M such that $\nu(\alpha_1 - \alpha_i) < M \quad \forall i > 1$.~~

$$F(x(t), t) = 0 \qquad x_M = a_1 t + \ldots + a_M t^M$$

$$\alpha_1(t) = x(t)$$

$$H(x, t) := \prod_{i=1}^{n} \left(x - \frac{\alpha_i - x_M}{t^M} \right)$$

$$1 > \left| \frac{\alpha_1 - x_M}{t^M} \right|$$

$$\parallel \frac{y_M}{t^M}$$

$$\left| \frac{\alpha_i - x_M}{t^M} \right| > 1 \qquad \forall i > 1$$

So, $H(x,t) = b_0 + b_1 x + \ldots + x^n$
$b_1 = H_x(0,t)$

has $H_x(0,0) \neq 0$