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Integral Equations and Operator Theory

The Aronszajn-Donoghue theory for rank one perturbations of the $\mathcal{H}_{-2}$-class

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Abstract. A singular rank one perturbation $A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi$ of a self-adjoint operator A in a Hilbert space $\mathcal{H}$ is considered, where $0 \neq \alpha \in \mathbf{R} \cup \infty$ and $\varphi \in \mathcal{H}_{-2}$ but $\varphi \notin \mathcal{H}_{-1}$, with $\mathcal{H}_s$, $s \in \mathbf{R}$, the usual $A-$scale of Hilbert spaces. A modified version of the Aronszajn-Krein formula is given. It has the form $F_\alpha(z) = \frac{F(z)-\alpha}{1+\alpha F(z)}$ where F_α denotes the regularized Borel transform of the scalar spectral measure of A_α associated with φ. Using this formula we develop a variant of the well known Aronszajn-Donoghue spectral theory for a general rank one perturbation of the $\mathcal{H}_{-2}$ class.

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1. Introduction

Let $A = A^*$ be a self-adjoint unbounded operator in a Hilbert space $\mathcal{H}$ with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Let $\{\mathcal{H}_k(A)\}_{k \in \mathbf{R}}$ denote the associated $A-$scale of Hilbert spaces and $\langle \cdot, \cdot \rangle$ the dual inner product between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$.

The original Donoghue's paper [8] (see also [5]) treats the spectral theory of singular rank one perturbations

$$A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi, \ 0 \neq \alpha \in \mathbf{R} \cup \infty,$$

for the case $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ in terms of von Neumann's theory of self-adjoint extensions of the symmetric operator

$$\dot{A} = A \restriction \{f \in \mathcal{D}(A) : \ \langle f, \varphi \rangle = 0\} \tag{1.1}$$

with deficiency indices (1,1).

If $\varphi \in \mathcal{H}_{-1}$, then the spectral theory has an elegant presentation [12] in terms of the Borel transform

$$\Phi(z) = \left\langle \varphi, (A-z)^{-1}\varphi \right\rangle = \int \frac{d\mu(\lambda)}{\lambda - z}$$

of the spectral measure μ uniquely defined by

$$\langle \varphi, f(A)\varphi \rangle = \int f(\lambda) d\mu(\lambda),$$

where f runs a family of bounded compactly supported measurable functions. The crucial role in the spectral theory of rank one perturbations is played by the classical Aronszajn-Krein formula

$$\Phi_\alpha(z) = \frac{\Phi(z)}{1+\alpha\Phi(z)}, \tag{1.2}$$

where $\Phi_\alpha(z) = \left\langle \varphi, (A_{\,\alpha} - z)^{-1}\varphi \right\rangle$ ($\Phi(z) := \Phi_{\alpha=0}(z)$) is well defined due to $\varphi \in \mathcal{H}_{-1}$.

However in the case where $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ both expressions $\left\langle \varphi, (A-z)^{-1}\varphi \right\rangle$ and $\left\langle \varphi, (A_{\,\alpha} - z)^{-1}\varphi \right\rangle$ fail to exist, since $(A-z)^{-1}\varphi \notin \mathcal{H}_2$. So, in order to extend the formulation of spectral theory to this case, we need at first to make an appropriate change of the Aronszajn-Krein formula.

In this paper for the case $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ we derive a modified version of the Aronszajn-Krein formula

$$F_\alpha\,(z) = \frac{F(z)-\alpha}{1+\alpha F(z)},$$

where $F(z)$ denotes a regularization of the Borel transform of the spectral measure $\mu = \mu_{\,\varphi}$. Then we develop a spectral theory in this case similar to the Aronszajn-Donoghue spectral theory, which was presented in [12] only for $\varphi \in \mathcal{H}_{-1}$.

2. Self-adjoint extensions and Borel transform

Let $A = A^*$ be a self-adjoint operator in a Hilbert space $\mathcal{H}$.

Here we use only a part of the $A-$scale of Hilbert spaces:

$$\mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2, \tag{2.1}$$

where $\mathcal{H}_k \equiv \mathcal{H}_k(A) = \mathcal{D}(|A|^{k/2})$, $k = 1, 2$, in the norm $\|\varphi\|_k := \|(|A|+I)^{k/2}\varphi\|$, where I stands for identity, and $\mathcal{H}_{-k} \equiv \mathcal{H}_{-k}(A)$ is the dual space ($\mathcal{H}_{-k}$ is the completion of $\mathcal{H}$ in the norm $\|f\|_{-k} := \|(|A|+I)^{-k/2}f\|$). Obviously A is bounded as a map from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$ and from $\mathcal{H}$ to $\mathcal{H}_{-2}$, and therefore the expression $\langle f, Ag \rangle$ has sense for any $f, g \in \mathcal{H}_1$.

Let $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, $\|\varphi\|_{-2} = 1$, be fixed.

Define a rank one (singular) perturbation A_α of A, formally written as $A_\alpha = A + \alpha \langle \varphi, \cdot \rangle \varphi$, $0 \neq \alpha \in \mathbf{R} \cup \infty$ ($\infty^{-1} := 0$) by Krein's resolvent formula (see [1, 2, 3, 4, 9, 10, 11]).

$$(A_\alpha - z)^{-1} = (A-z)^{-1} - b_\alpha^{-1}(z)(\eta_{\overline{z}}, \cdot)\eta_z, \ \operatorname{Im} z \neq 0, \tag{2.2}$$

where

$$\eta_z = (A-z)^{-1}\varphi$$

and the scalar function $b_\alpha(z)$ satisfies:

$$b_\alpha(z) - b_\alpha(\zeta) = (\zeta - z)(\eta_z, \eta_{\zeta}), \quad \overline{b}_\alpha(z) = b_\alpha(\overline{z}), \qquad \operatorname{Im} z, \ \operatorname{Im}\zeta \neq 0. \tag{2.3}$$

In particular one can put

$$b_\alpha(z) = \frac{1}{\alpha} + F(z) \tag{2.4}$$

with

$$\begin{aligned} F(z) &= \left\langle \varphi, ((A-z)^{-1} - \tfrac{1}{2}((A-i)^{-1} + (A+i)^{-1}))\varphi \right\rangle \\ &= \left\langle \varphi, \frac{1+zA}{A-z}(A^2+1)^{-1}\varphi \right\rangle. \end{aligned} \tag{2.5}$$

Then (2.3) is obviously fulfilled. So we can write

$$(A_\alpha - z)^{-1} = (A-z)^{-1} - \frac{\alpha}{1+\alpha F(z)}((A-\overline{z})^{-1}\varphi, \cdot)(A-z)^{-1}\varphi, \qquad \operatorname{Im} z \neq 0. \tag{2.6}$$

$$(A_\infty - z)^{-1} = (A-z)^{-1} - \frac{1}{F(z)}((A-\overline{z})^{-1}\varphi, \cdot)(A-z)^{-1}\varphi, \qquad \operatorname{Im} z \neq 0. \tag{2.7}$$

Note that one can consider (2.6) as the generalization of the corresponding formula for the resolvent in the regular case $\varphi \in \mathcal{H}$ (see [8], [12]). In this situation A_α is a bounded rank one perturbation of A and (2.6) is valid if one replaces $F(z)$ by $\Phi(z)$. Moreover, this regular variant of (2.6) remains true for $\varphi \in \mathcal{H}_{-1}$ ([12]), in particular for the case $A_\alpha \geq A \geq 0$, since $\varphi \in \mathcal{H}_{-1}$ with necessity ([10]).

Let $\mathcal{A}(\dot{A})$ denote the family of all self-adjoint extensions of the symmetric operator $\dot{A}$.

Proposition 2.1. *Let $\varphi \in \mathcal{H}_{-2} \backslash \mathcal{H}_{-1}$ and $\dot{A}$ is given by (1.1). Then each $\widetilde{A} \in \mathcal{A}(\dot{A})$, $\widetilde{A} \neq A$, is uniquely defined by Krein's formula (2.2) with $b_\alpha(z)$ given by (2.4) and (2.5), i.e., each $\widetilde{A}$ coincides with some A_α, $0 \neq \alpha \in \mathbf{R} \cup \infty$, where the resolvent of A_α has a form (2.6) or (2.7).*

Proof. Let $\widetilde{A} \in \mathcal{A}(\dot{A})$. Then its resolvent has the form

$$(\widetilde{A} - z)^{-1} = (A-z)^{-1} - \widetilde{b}^{-1}(z)(\eta_{\overline{z}}, \cdot)\eta_z, \ \operatorname{Im} z \neq 0,$$

with a scalar function $\widetilde{b}(z)$ which satisfies (2.3). Therefore

$$\operatorname{Im}\widetilde{b}(z) = - \operatorname{Im} z \|\eta_z\|^2$$

and we have

$$\widetilde{b}(z) = c - \operatorname{Im} z \|\eta_z\|^2 ,$$

with some $c = c(z) \in \mathbf{R}$. We observe now that

$$\widetilde{b}(z) = b_\alpha(z), \qquad \text{if} \qquad c = \frac{1}{\alpha} + \mathrm{Re}F(z).$$

□

Let $\mathbf{E}(\cdot)$ be the operator spectral measure (the resolution of the identity) of A, and $\mu(\Delta) \equiv \mu_\varphi(\Delta) = (\varphi, \mathbf{E}(\Delta)\varphi)$ denote the scalar spectral measure of A associated with φ. This measure is not finite as $\varphi \notin \mathcal{H}$. One can introduce a regularization of this measure by

$$d\mu^{\mathrm{reg}}(x) := \frac{d\mu(x)}{1+x^2},$$

so that $\mu^{\mathrm{reg}}(\mathbf{R}) = \int d\mu^{\mathrm{reg}}(x) = 1$. Clearly the measures μ and μ^{reg} are equivalent. It is convenient to introduce the regularized version of the Borel transform of μ as follows (cf. (2.5))

$$F(z) = \int \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu(x) = \int \frac{1+zx}{x-z} d\mu^{\mathrm{reg}}(x). \tag{2.8}$$

Consider the operator spectral measure $\mathbf{E}_\alpha(\cdot)$ for A_α. Similarly to above constructions one can introduce

$$\mu_\alpha^{\mathrm{reg}}(\Delta) := ((A+i)^{-1}\varphi, \mathbf{E}_\alpha(\Delta)(A+i)^{-1}\varphi).$$

Define

$$d\mu_\alpha(x) := (1+x^2) d\mu_\alpha^{\mathrm{reg}}(x). \tag{2.9}$$

Henceforth, we assume that φ is a cyclic vector for A, i.e. $\{(A-z)^{-1}\varphi : \mathrm{Im}z \neq 0\}$ is a total set of $\mathcal{H}$. In general, if $\mathcal{H}_\varphi$ denotes the closed subspace in $\mathcal{H}$ generated by vectors from this set, then $\mathcal{H}_\varphi$ is an invariant subspace for each A_α and $A_\alpha = A$ on the orthogonal complement to $\mathcal{H}_\varphi$. Thus the extension from the cyclic to general case is trivial. It is easy to see that $(A+i)^{-1}\varphi$ is a cyclic vector for A_α (cf. [8]) and μ_α is equivalent to the spectral measure $\mathbf{E}_\alpha(\cdot)$. In the following we shall say that μ_α is a scalar spectral measure of A_α associated with φ.

Let F_α be the regularized Borel transform of μ_α (cf. (2.8))

$$F_\alpha(z) := \int \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\mu_\alpha(x) = \int \frac{1+zx}{x-z} d\mu_\alpha^{\mathrm{reg}}(x). \tag{2.10}$$

Clearly $F_0(z) = F(z)$. Note that one can rewrite (2.10) as

$$\begin{aligned} F_\alpha(z) &= ((A+i)^{-1}\varphi, (A_\alpha - z)^{-1}(1+zA_\alpha)(A+i)^{-1}\varphi) \\ &= (1+z^2)((A+i)^{-1}\varphi, (A_\alpha - z)^{-1}(A+i)^{-1}\varphi) + z. \end{aligned} \tag{2.11}$$

We recall that the classical Aronszajn-Krein formula has the form (1.2) where Φ_α is the Borel transform of the measure μ_α. In the considered situation where $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, the Borel transform of μ_α is not well defined and we have the following modification of (1.2).

Lemma 2.2. *For $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ the function $F_\alpha(z)$ admits the representation*

$$F_\alpha(z) = \frac{F(z) - \alpha}{1 + \alpha F(z)}, \text{ Im } z \neq 0, \ 0 \neq \alpha \in \mathbf{R} \cup \infty. \tag{2.12}$$

Remark 2.3. For $\alpha = \infty$ (2.12) means that

$$F_\infty(z) = -\frac{1}{F(z)}$$

Proof. By (2.11), (2.6)

$$\begin{aligned} F_\alpha(z) &= F(z) - \frac{\alpha(1+z^2)}{1+\alpha F(z)}((A-\overline{z})^{-1}\varphi, (A+i)^{-1}\varphi)((A+i)^{-1}\varphi, (A-z)^{-1}\varphi) \\ &= F(z) - \frac{\alpha(F(z)-F(-i))(F(z)-F(i))}{1+\alpha F(z)} = \frac{F(z)-\alpha}{1+\alpha F(z)}. \end{aligned}$$

Here we have used the following simple identies

$$((A-\overline{z})^{-1}\varphi, (A+i)^{-1}\varphi) = \frac{F(z)-F(-i)}{z+i},$$

$$((A+i)^{-1}\varphi, (A-z)^{-1}\varphi) = \frac{F(z)-F(i)}{z-i},$$

$$F(i) = i(\varphi, (A^2+1)^{-1}\varphi) = \|\varphi\|_{-2} = i, \quad F(-i) = -i.$$

□

3. Spectral theory

Although the classical Aronszajn-Krein formula (1.2) in the considered case with $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ is changed into its modified version (2.12), the main features of Donoghue's spectral theory are preserved in the same form as for the case $\varphi \in \mathcal{H}_{-1}$.

Recall that a holomorphic function $G : \mathbf{C}^+ \to \mathbf{C}^+$ ($\mathbf{C}^+$ denotes the open upper halfplane) is said to be an R-function (or Herglotz, or Nevanlinna function). Each R-function admits the following representation (see, e.g., [5, 6]):

$$G(z) = a + bz + \int \left(\frac{1}{x-z} - \frac{x}{1+x^2} \right) d\sigma(x), \ z \in \mathbf{C}^+.$$

Here $a \in \mathbf{R}$, $b \geq 0$, and σ is a Borel measure on $\mathbf{R}$ such that

$$\int \frac{d\sigma(x)}{1+x^2} < \infty.$$

First of all recall that $\lim_{\varepsilon \downarrow 0} G(x + i\varepsilon)$ exists and is finite for (Lebesgue) a.e. x. Moreover one can derive the properties of the measure σ from the boundary behavior of the corresponding Herglotz function on the real axis. According to the Lebesgue-Jordan decomposition $\sigma = \sigma_{\rm ac} + \sigma_{\rm sing}$, $\sigma_{\rm sing} = \sigma_{\rm sc} + \sigma_{\rm p}$, where $\sigma_{\rm ac}, \sigma_{\rm sing}, \sigma_{\rm sc}, \sigma_{\rm p}$ are the absolutely continuous, singular, singular continuous, and pure point parts of σ, respectively. We need the following well known result (see, e.g., [5, 6, 8, 7, 12]).

Lemma 3.1. *Introduce the sets*

$$\begin{aligned} S(\sigma) &= \left\{x \in \mathbf{R} : \mathrm{Im}G(x+i0) = \infty, \lim_{\varepsilon \downarrow 0} \varepsilon \mathrm{Im}G(x+i\varepsilon) = 0\right\}, \\ P(\sigma) &= \left\{x \in \mathbf{R} : \mathrm{Im}G(x+i0) = \infty, \lim_{\varepsilon \downarrow 0} \varepsilon \mathrm{Im}G(x+i\varepsilon) > 0\right\}, \\ L(\sigma) &= \{x \in \mathbf{R} : 0 < \mathrm{Im}G(x+i0) < \infty\}. \end{aligned}$$

Then

(i) σ_{ac} is supported on $L(\sigma)$,

(ii) σ_{sc} is supported on $S(\sigma)$,

(iii) σ_p is supported on $P(\sigma)$ and for each $x \in \mathbf{R}$ one has

$$\sigma(\{x\}) = \lim_{\varepsilon \downarrow 0} \varepsilon \mathrm{Im}G(x+i\varepsilon).$$

Let $\mu(\cdot) = (\varphi, \mathbf{E}(\cdot)\varphi)$ be the scalar spectral measure of A associated with φ, $F(\cdot)$ be a regularized transform of μ (see (2.8)). Introduce the function

$$H(x) = \int \frac{d\mu(y)}{(x-y)^2}, \qquad x \in \mathbf{R}.$$

We remark that

$$\mathrm{Im}F(x+i\varepsilon) = \int \frac{\varepsilon}{(x-y)^2+\varepsilon^2} d\mu(y).$$

By the monotone convergence theorem we have

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \mathrm{Im}F(x+i\varepsilon) = H(x). \tag{3.1}$$

It is easy to see that if $H(x) < \infty$ then (cf. [12]) $\lim_{\varepsilon \downarrow 0} F(x+i\varepsilon)$ exists and is real. Moreover,

$$\lim_{\varepsilon \downarrow 0} (i\varepsilon)^{-1}[F(x+i\varepsilon) - F(x+i0)] = H(x). \tag{3.2}$$

Our main result is as follows:

Theorem 3.2. *Suppose that $\varphi \in \mathcal{H}_{-2} \backslash \mathcal{H}_{-1}$ and μ_α (see (2.9)) be the scalar spectral measure of A_α associated with φ . For $\alpha \neq 0$, define the sets*

$$\begin{aligned} S_\alpha &= \left\{x \in \mathbf{R} : F(x+i0) = -\alpha^{-1}, H(x) = \infty\right\}, \\ P_\alpha &= \left\{x \in \mathbf{R} : F(x+i0) = -\alpha^{-1}, H(x) < \infty\right\}, \\ L &= \{x \in \mathbf{R} : 0 < \mathrm{Im}F(x+i0) < \infty\}. \end{aligned}$$

Then

(i) P_α is the set of eigenvalues of A_α , $(\mu_\alpha)_{ac}$ is supported on L , $(\mu_\alpha)_{sc}$ is supported on S_α .

(ii) $\{S_\alpha\}_{\alpha \neq 0}$, $\{P_\alpha\}_{\alpha \neq 0}$ and L are mutually disjoint.

(iii) For $\alpha \neq \beta$, $(\mu_\alpha)_{sing}$ and $(\mu_\beta)_{sing}$ are mutually singular.

Proof. (ii) is obvious and (iii) follows from (i) and (ii). By the modified Aronszain-Krein formula (2.12) (cf. [12])

$$\mathrm{Im}F_\alpha\,(z) = (1+\alpha^2)\frac{\mathrm{Im}F(z)}{|1+\alpha F(z)|^2}, \qquad \mathrm{Im}\, z \neq 0, \quad \alpha \neq \infty, \tag{3.3}$$

$$\mathrm{Im}F_\infty\,(z) = \frac{\mathrm{Im}F(z)}{|F(z)|^2}, \qquad \mathrm{Im}\, z \neq 0. \tag{3.4}$$

Then,

$$L = \{x \in \mathbf{R} : \mathrm{Im}F(x+i0) \neq 0\} = \{x \in \mathbf{R} : \mathrm{Im}F_\alpha\,(x+i0) \neq 0\} = L(\mu_\alpha\,).$$

This proves that $(\mu_\alpha)_{\mathrm{ac}}$ is supported on L. By Lemma 3.1 $(\mu_\alpha)_{\mathrm{sing}}$ is supported by

$$\{x \in \mathbf{R} : \mathrm{Im}F_\alpha(x+i0) = \infty\}\,.$$

If we suppose that $F(x+i0) = -\alpha^{-1}$ $(0 \neq \alpha \in \mathbf{R} \cup \infty)$, then by (3.1) − (3.4)

$$\mu_\alpha(\{x\}) = \lim_{\varepsilon \downarrow 0} \varepsilon \mathrm{Im}F_\alpha\,(x+i\varepsilon) = \frac{1+\alpha^2}{\alpha^2 H(x)} \quad (0 < |\alpha| < \infty),$$

and for $\alpha = \infty$,

$$\mu_\infty(\{x\}) = \lim_{\varepsilon \downarrow 0} \varepsilon \mathrm{Im}F_\infty\,(x+i\varepsilon) = \frac{1}{H(x)}.$$

Now the proof follows from the Lemma 3.1. □

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