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THEOREM ON CONFLICT FOR A PAIR OF STOCHASTIC VECTORS

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We investigate a mathematical model of conflict with a discrete collection of positions.

1. Introduction

In the present paper, we propose a mathematical model of conflict with a finite collection of positions for two opponents. The combined distribution of the probability of capturing an arbitrary position is equal to one for each opponent, i.e., *a priori,* the opponents are assumed to be indestructible. The opponents only struggle for a "just" redistribution of conflict positions.

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_d\}$, $d \geq 2$, be a finite set of positions that can be occupied by each of two opposing sides (opponents) denoted by A and B. We assume that, for both sides, the total sojourn probability of presence on the set Ω is constant and normalized to unity, i.e., $\mathrm{P}_A(\Omega) = \mathrm{P}_B(\Omega) = 1$. The initial probability distribution (independent of the presence of an opponent) of the presence of each of the sides A and B in the positions ω_i, $i = 1, 2, \ldots, d$, is arbitrary:

$$1 = \sum_{i=1}^{d} \mathrm{P}_A(\omega_i) = \sum_{i=1}^{d} p_i, p_i \geq 0,$$

$$1 = \sum_{i=1}^{d} \mathrm{P}_B(\omega_i) = \sum_{i=1}^{d} q_i, q_i \geq 0.$$

The essence of the conflict is that the opponents A and B cannot simultaneously occupy a questionable position ω_i. The problem is to construct a mathematical composition of the conflict between the vectors $\mathbf{p} = (p_1, p_2, \ldots, p_d)$ and $\mathbf{q} = (q_1, q_2, \ldots, q_d)$, to investigate the evolution of the redistribution of the probabilities $\mathrm{P}_A(\omega_i) = p_i$ and $\mathrm{P}_B(\omega_i) = q_i$, $i = 1, 2, \ldots, d$, with respect to the conflict composition, and to determine invariant states.

2. Composition of a Conflict for Stochastic Vectors

The vector $\mathbf{p} = (p_1, p_2, \ldots, p_d) \in R^d$, $d \geq 2$, is called stochastic if its coordinates are nonnegative, $p_i \geq 0$, $i = 1, 2, \ldots, d$, and its l^1-norm is unit:

$$|\mathbf{p}| = \sum_{i=1}^{d} p_i = 1.$$

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Each pair of stochastic vectors $\mathbf{p}, \mathbf{q} \in R^d$ forms a conflict system, which is nontrivial if the scalar product $(\mathbf{p}, \mathbf{q}) \neq 0$ and trivial if $\mathbf{p} \perp \mathbf{q}$. The case $\mathbf{p} = \mathbf{q} = \mathbf{1}_i$, where

$$\mathbf{1}_i = \Big(\underbrace{0, \ldots, 0}_{i-1}, 1, 0, \ldots, 0\Big),$$

corresponds to the state of collapse and is not considered in the present paper.

A noncommutative conflict composition (denoted by $\ast$) between the stochastic vectors $\mathbf{p}^0$ and $\mathbf{q}^0$ with coordinates $p_i^{(0)}$ and $q_i^{(0)}$, $i = 1, 2, \ldots, d$, is defined as follows: The pair $\mathbf{p}^0$, $\mathbf{q}^0$ is associated with a new pair of stochastic vectors

$$\mathbf{p}^1 := \mathbf{p}^0 \ast \mathbf{q}^0 \equiv \mathbf{p}^0 \overset{1}{\ast} \mathbf{q}^0, \qquad \mathbf{q}^1 := \mathbf{q}^0 \ast \mathbf{p}^0 \equiv \mathbf{q}^0 \overset{1}{\ast} \mathbf{p}^0$$

with the coordinates

$$p_i^{(1)} := \frac{1}{z_1} p_i^{(0)} (1 - q_i^{(0)}) \equiv \frac{1}{z_1} p_i^{(0)} q_i^{(0),c}, \qquad p_i^{(0)} \equiv p_i, \qquad q_i^{(0)} \equiv q_i,$$
$$q_i^{(1)} := \frac{1}{z_1} q_i^{(0)} (1 - p_i^{(0)}) \equiv \frac{1}{z_1} q_i^{(0)} p_i^{(0),c}, \tag{1}$$

where the normalizing coefficient z_1 is fixed by the stochasticity condition, $|\mathbf{p}^1| = |\mathbf{q}^1| = 1$:

$$z_1 = 1 - (\mathbf{p}^0, \mathbf{q}^0). \tag{2}$$

The coordinates $p_i^{(1)}, q_i^{(1)}$ in (1) are well defined except for the case where $\mathbf{p}^0 = \mathbf{1}_i = \mathbf{q}^0$. The degree of a conflict composition (denoted by $\overset{n}{\ast}$, $n = 1, 2, \ldots$) between the vectors $\mathbf{p}^0$ and $\mathbf{q}^0$ is defined by induction:

$$\mathbf{p}^n = \mathbf{p}^0 \overset{n}{\ast} \mathbf{q}^0 := \mathbf{p}^{n-1} \ast \mathbf{q}^{n-1},$$

$$\mathbf{q}^n = \mathbf{q}^0 \overset{n}{\ast} \mathbf{p}^0 := \mathbf{q}^{n-1} \ast \mathbf{p}^{n-1},$$

where the coordinates of the vectors $\mathbf{p}^n$ and $\mathbf{q}^n$ are defined by the relations

$$p_i^{(n)} := \frac{1}{z_n} p_i^{(n-1)} (1 - q_i^{(n-1)}) \equiv \frac{1}{z_n} p_i^{(n-1)} q_i^{(n-1),c},$$
$$q_i^{(n)} := \frac{1}{z_n} q_i^{(n-1)} (1 - p_i^{(n-1)}) \equiv \frac{1}{z_n} q_i^{(n-1)} p_i^{(n-1),c}, \tag{3}$$

where

$$z_n = 1 - (\mathbf{p}^{n-1}, \mathbf{q}^{n-1}).$$

For a fixed pair of the stochastic vectors $\mathbf{p}^0$ and $\mathbf{q}^0$, the matrix

$$M^n = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} & \dots & p_d^{(n)} \\ q_1^{(n)} & q_2^{(n)} & \dots & q_d^{(n)} \end{pmatrix} \tag{4}$$

is called the state of the conflict system at the nth step of the conflict. The evolution of the initial state M^0 associated with the pair of vectors $\mathbf{p}^0$ and $\mathbf{q}^0$ is given by the transformation

$$U\left(\overset{n}{\ast}\right)M^0 := M^n, \qquad U\left(\overset{1}{\ast}\right) \equiv U(\ast).$$

In the present paper, for a conflict system with arbitrary initial vectors $\mathbf{p}^0, \mathbf{q}^0 \in R^d$, we prove the existence of limit invariant states $M^\infty = \lim_{n\to\infty} M^n$, $U(\ast)M^\infty = M^\infty$, and partially describe their structure.

3. Example

In R^2, we consider a pair of stochastic vectors $\mathbf{p}^0 = (p_1^{(0)}, p_2^{(0)})$ and $\mathbf{q}^0 = (q_1^{(0)}, q_2^{(0)})$, $p_i^{(0)} \neq 0 \neq q_i^{(0)}$. By virtue of (1), even the first step of the conflict composition leads to the symmetric state

$$U\left(\overset{1}{\ast}\right)\begin{pmatrix} p_1^{(0)} & p_2^{(0)} \\ q_1^{(0)} & q_2^{(0)} \end{pmatrix} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \qquad 0 \le a, \quad b \le 1.$$

Proposition 1. *For an arbitrary pair of stochastic vectors* $\mathbf{p}^0 \neq \mathbf{q}^0$ *from* R^2, *the sequence of states*

$$M^n = U\left(\overset{n}{\ast}\right)M^0 = \begin{pmatrix} p_1^{(n)} & p_2^{(n)} \\ q_1^{(n)} & q_2^{(n)} \end{pmatrix}$$

converges to one of the invariant states $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *If* $\mathbf{p}^0 = \mathbf{q}^0 \neq \mathbf{1}_i$, $i = 1, 2$, *then the limit state is an equilibrium state:*

$$M^\infty = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.$$

Proof. Without loss of generality, we can assume that

$$M^0 = \begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where $0 < a < 1/2$ and $b = 1 - a$. Then

$$M^1 = \begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix},$$

where $a_1 = \dfrac{a^2}{a^2+b^2} = ak_1$ and $k_1 = \dfrac{a}{2a^2-2a+1} < 1$ because $a < a^2 + b^2$ for $a < 1/2$. Thus, we get $a_1 < a$. By induction, for arbitrary $n \geq 1$ we obtain

$$a_n = a_{n-1}k_n \equiv a\prod_{i=1}^{n} k_i,$$

where all $k_i < 1$ and $k_i \nrightarrow 1$. This means that $a_n \to 0$ (respectively, $b_n \to 1$) as $n \to \infty$. For $\mathbf{p}^0 = \mathbf{q}^0 \neq \mathbf{1}$, we directly verify that, not later than at the first step, the conflict composition leads to the equilibrium state $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$.

4. Theorem on Conflict

Theorem 1. *For an arbitrary pair of stochastic vectors* $\mathbf{p}^0, \mathbf{q}^0 \in R^d$, $d \geq 2$, $\mathbf{p}^0 \neq \mathbf{q}^0$, *that form a nontrivial conflict system,* $(\mathbf{p}^0, \mathbf{q}^0) \neq 0$, *there exist the limits*

$$p_i^{(\infty)} = \lim_{n\to\infty} p_i^{(n)}, \qquad q_i^{(\infty)} = \lim_{n\to\infty} q_i^{(n)}, \quad i = 1, \dots, d$$

and the sequence of states M^n *defined by (1)–(4) converges to the invariant state*

$$M^\infty = \lim_{n\to\infty} M^n = \begin{pmatrix} p_1^{(\infty)} & p_2^{(\infty)} & \dots & p_d^{(\infty)} \\ q_1^{(\infty)} & q_2^{(\infty)} & \dots & q_d^{(\infty)} \end{pmatrix}, \qquad U(\circledast)M^\infty = M^\infty. \tag{5}$$

In this case, the limit vectors $\mathbf{p}^\infty = (p_1^{(\infty)}, p_2^{(\infty)}, \dots, p_d^{(\infty)})$ *and* $\mathbf{q}^\infty = (q_1^{(\infty)}, q_2^{(\infty)}, \dots, q_d^{(\infty)})$ *associated with the state* M^∞ *are orthogonal:*

$$\mathbf{p}^\infty \perp \mathbf{q}^\infty. \tag{6}$$

If the initial vectors are equal, $\mathbf{p}^0 = \mathbf{q}^0$, *and, for all* $i = 1, 2, \dots, d$, *the coordinates* $p_i^{(0)} = q_i^{(0)} \neq 0$, *then the limit vectors also exist, are equal,* $\mathbf{p}^\infty = \mathbf{q}^\infty$, *and have the uniform distribution*

$$p_i^{(\infty)} = q_i^{(\infty)} = 1/d. \tag{7}$$

Proof. For $d = 2$, the validity of the theorem follows from the proposition. Let $d \geq 3$. Assume that $\mathbf{p}^0 \neq \mathbf{q}^0$. If, for certain i, one has $p_i^{(0)} > q_i^{(0)}$, then, by virtue of (1), $p_i^{(1)} > q_i^{(1)}$ and, hence, $p_i^{(n)} > q_i^{(n)}$ for all n. Moreover, if, for fixed i, one has $p_i^{(0)} > q_i^{(0)}$, then, necessarily,

$$q_i^{(n)} \to 0, \qquad n \to \infty. \tag{8}$$

To prove (8), it suffices to show that

$$c_i^{(n)} := \frac{p_i^{(n)}}{q_i^{(n)}} \to \infty, \qquad n \to \infty.$$

It is obvious that $c_i^{(0)} > 1$. Since $p_i^{(0)} > q_i^{(0)}$, we conclude that $k_i^{(0)} := \frac{1 - q_i^{(0)}}{1 - p_i^{(n)}} > 1$. Therefore, $c_i^{(0)} < c_i^{(1)} = c_i^{(0)} k_i^{(0)}$. By induction, we get

$$1 < c_i^{(0)} < c_i^{(1)} < \ldots < c_i^{(n)} < c_i^{(n+1)} = c_i^{(n)} k_i^{(n)} = c_i^{(0)} k_i^{(0)} \ldots k_i^{(n)}.$$

Moreover, the numbers $k_i^{(n)}$ are strictly increasing as $n \to \infty$:

$$1 < k_i^{(0)} < k_i^{(1)} < \ldots < k_i^{(n)}. \tag{9}$$

Indeed, for $k_i^{(1)}$, we have

$$k_i^{(1)} = \frac{1 - q_i^{(1)}}{1 - p_i^{(1)}} = \frac{1 - \frac{1}{z_1} q_i^{(0)}(1 - p_i^{(0)})}{1 - \frac{1}{z_1} p_i^{(0)}(1 - q_i^{(0)})} = \frac{z_1 - q_i^{(0)}(1 - p_i^{(0)})}{z_1 - p_i^{(0)}(1 - q_i^{(0)})}$$

$$= \frac{1 - q_i^{(0)} - (\mathbf{p}^0, \mathbf{q}^0) + q_i^{(0)} p_i^{(0)}}{1 - p_i^{(0)} - (\mathbf{p}^0, \mathbf{q}^0) + q_i^{(0)} p_i^{(0)}} = \frac{1 - q_i^{(0)} - I_i^0}{1 - p_i^{(0)} - I_i^0},$$

where

$$0 < I_i^0 := (\mathbf{p}^0, \mathbf{q}^0) - p_i^{(0)} q_i^{(0)} = \sum_{k \neq i} p_k^{(0)} q_k^{(0)} < \sum_{k \neq i} p_k^{(0)} = 1 - p_i^{(0)}.$$

It is now obvious that

$$k_i^{(0)} = \frac{1 - q_i^{(0)}}{1 - p_i^{(0)}} < k_i^{(1)} = \frac{1 - q_i^{(0)} - I_i^0}{1 - p_i^{(0)} - I_i^0}.$$

By analogy, we prove that $k_i^{(1)} < k_i^{(2)}$ and, by induction, we get (9). Thus, it follows from the relation $p_i^{(0)} > q_i^{(0)}$ that $c_i^{(n)} \to \infty$ and, hence, $q_i^{(n)} \to 0$ because all $p_i^{(n)}$ are bounded. By analogy, if $p_k^{(0)} < q_k^{(0)}$ for certain fixed k, then $p_k^{(n)} \to 0$. By virtue of the relation $\mathbf{p}^0 \neq \mathbf{q}^0$, this means that $(\mathbf{p}^n, \mathbf{q}^n) \to 0$ and $z_n \to 1$, provided that all corresponding coordinates of the vectors $\mathbf{p}^n$ and $\mathbf{q}^n$ are different, i.e., there exists an empty set of indices Ω:

$$\Omega_0 := \{ i : p_i^{(0)} = q_i^{(0)} \neq 0 \} = \varnothing.$$

In turn, this means that the sequences $p_i^{(n)}$ and $q_k^{(n)}$ for $p_i^{(0)} > q_i^{(0)}$ and $q_k^{(0)} > p_k^{(0)}$ also converge on the segment $[0, 1]$:

$$p_i^{(n)} \to p_i^{(\infty)}, \qquad q_k^{(n)} \to q_k^{(\infty)}, \qquad n \to \infty.$$

Taking into account that $q_i^{(n)} \to 0$ and $p_k^{(n)} \to 0$, we establish that the limit state of the system is invariant under the conflict composition, which proves relation (5). Since $(\mathbf{p}^\infty, \mathbf{q}^\infty) = 0$, relation (6) is true.

For $\Omega_0 \neq \varnothing$, the proof of the theorem is more complicated. Of course, the limit values $p_i^{(\infty)}$ and $q_k^{(\infty)}$ for $i, k \notin \Omega_0$ exist as in the previous case. This follows from relations (3) because, as before, $q_i^{(n)} \to 0$ for $p_i^{(0)} > q_i^{(0)}$ and $p_k^{(n)} \to 0$ for $q_k^{(0)} > p_k^{(0)}$. Therefore, in this case, the normalizing coefficient can be written in the form

$$z_n = 1 - \sum_{j \in \Omega_0} (p_j^{(n)})^2 - \varepsilon_n,$$

where $\varepsilon_n \to 0$. It follows from relations (3) that, at least for a certain subsequence n_k, the ratio between $1 - p_j^{(n_k)}$ and $1 - \varepsilon_n - \sum_{j \in \Omega_0} (p_j^{(n_k)})^2$ tends to 1, which is possible only if $p_j^{(n_k)} \to 0$. Consequently, $(\mathbf{p}^n, \mathbf{q}^n) \to 0$ as $n \to \infty$ in the general case, which guarantees the validity of the statements of the theorem.

Now assume that $\mathbf{p}^0 = \mathbf{q}^0$ and $p_i^{(0)} = q_i^{(0)} \neq 0$ for all $i = 1, 2, \ldots, d$. Then it obviously follows from (1) that $p_i^{(n)} = q_i^{(n)} \neq 0$ for all n. Without loss of generality, we can assume that the coordinates of the vector $\mathbf{p}^0$ are ordered as follows:

$$0 < p_1^{(0)} \leq p_2^{(0)} \leq \ldots \leq p_d^{(0)} < 1. \tag{10}$$

Since $\mathbf{p}^0 = \mathbf{q}^0$, the inverse ordering

$$1 > q_1^{(0),c} \geq q_2^{(0),c} \geq \ldots \geq q_d^{(0),c} > 0 \tag{11}$$

follows from (10) for the numbers $q_i^{(0),c} := 1 - q_i^{(0)}$.

It follows from (11) that

$$q_1^{(0),c} > z_1 > q_d^{(0),c}. \tag{12}$$

Indeed, by definition, we have

$$z_1 = 1 - (\mathbf{p}^0, \mathbf{q}^0) = \sum_i p_i^{(0)} q_i^{(0),c}.$$

Therefore, replacing all numbers $q_i^{(0),c}$ in the last sum by $q_1^{(0),c}$ [or by $q_d^{(0),c}$] and using the equality

$$\sum_i p_i^{(0)} = 1,$$

we obtain inequality (12) from (11).

We now show that the coordinates of the vector $\mathbf{p}^1$ satisfy the inequalities

$$p_1^{(0)} \le p_1^{(1)} \le p_i^{(1)} \le p_d^{(1)} \le p_d^{(0)}, \qquad i = 1, 2, \ldots, d. \tag{13}$$

Indeed, it directly follows from (12) that $p_1^{(0)} \le p_1^{(1)}$ and $p_d^{(1)} \le p_d^{(0)}$. To prove the inequalities $p_1^{(1)} \le p_i^{(1)} \le p_d^{(1)}$, we note that, by virtue of the fact that $p_i^{(0)} = q_i^{(0)} \ne 0$, they can be rewritten in the following equivalent form:

$$p_1^{(0)}(1 - p_1^{(0)}) \le p_i^{(0)}(1 - p_i^{(0)}) \le p_d^{(0)}(1 - p_d^{(0)}).$$

These relations are true because the function $y = x(1 - x)$, $x \in [0, 1]$, is symmetric with respect to the point $x = 1/2$, at which it has a maximum. Therefore,

$$y(p_1^{(0)}) \le y(x) \le y(1 - p_d^{(0)})$$

for any point $x = p_i^{(0)} \in (0, 1)$ under the conditions $d \ge 3$, $\sum_i p_i^{(0)} = 1$, and $\min_i p_i^{(0)} \le x \le \max_i p_i^{(0)}$. Inequalities (13) can now be extended by induction to the coordinates of the vector $\mathbf{p}^n$ with arbitrary n:

$$p_1^{(0)} \le p_1^{(1)} \le \ldots \le p_1^{(n)} \le p_i^{(n)} \le p_d^{(n)} \le \ldots \le p_d^{(1)} \le p_d^{(0)}.$$

Since the vectors $\mathbf{p}^n$ and $\mathbf{q}^n$ are stochastic, we necessarily obtain

$$\lim_{n \to \infty} p_i^{(n)} = 1/d = \lim_{n \to \infty} q_i^{(n)}, \qquad i = 1, 2, \ldots, d,$$

which proves (7). It is clear that, in the case where $p_i^{(0)} = q_i^{(0)} \ne 0$ only for $i = 1, 2, \ldots, m < d$, the limits in (14) are equal to $1/m$.

5. Discussion

According to the theorem on conflict proved above, the composition $\ast$ [see (1)] has a purely repulsive effect. As a result of an infinite (in the general case) conflict, different opponents from a nontrivial conflict system [the vectors $\mathbf{p}^n$ and $\mathbf{q}^n$ are not identical and $(\mathbf{p}^n, \mathbf{q}^n) \ne 0$] occupy different positions (the vectors $\mathbf{p}^\infty$ and $\mathbf{q}^\infty$ are orthogonal). In this case, questionable positions are absent at the border: at least, one of the coordinates $p_i^{(\infty)}$ and $q_i^{(\infty)}$ is equal to zero. A uniform (parity) distribution is realized only for identical sides.

It is clear that, for the construction of a more perfect model of conflicts, an attractive effect should also be taken into account. In fact, this manifests itself in the increase in the coefficient of claims (the coordinates of the vectors $\mathbf{p}^n$ and $\mathbf{q}^n$) to occupy certain positions simultaneously by each of the opposing sides as $n \to \infty$. Mathematically, this can be realized by introducing a functional (controlled) dependence of these coordinates on time, i.e., $p_i^{(n)}(t)$ and $q_i^{(n)}(t)$. In particular, at each step of the composition $\ast$, the coordinates $p_i^{(n)}$ and $q_i^{(n)}$ can additionally have a power or an exponential dependence on the argument t. This means a passage to the construction of conflict models with control (see, e.g., [1]). The law of transformation of distribution functions of the corresponding states, as well as the coordinates $p_i^{(n)}(t)$ and $q_i^{(n)}(t)$, can be given by stochastic square matrices.

Note that, in the proposed model of conflict system, we do not use the notion of payoff function, which is the main object of the ordinary theory of games [2–7]. However, in a more perfect version of conflict composition, this function necessarily appears (see [8]).

In conclusion, note that the principle of indestructibility of opponents is introduced in the present paper. Neither side can win or lose, which results in a conflict-free state. In the subsequent paper, we shall show that the conflict composition $\ast$ introduced in the present paper generates a vector dynamical system in the space $R^d \times R^d$, which is much more complicated than the one-dimensional one [9].

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