$\widetilde{Q}-$ representation of real numbers and fractal probability distributions

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December 25, 2003

Abstract

A $\widetilde{Q}-$representation of real numbers is introduced as a generalization of the $p-$adic and $Q-$representations. It is shown that the $\widetilde{Q}-$representation may be used as a convenient tool for the construction and study of fractals and sets with complicated local structure. Distributions of random variables ξ with independent $\widetilde{Q}-$symbols are studied in details. Necessary and sufficient conditions for the probability measures μ_ξ associated with ξ to be either absolutely continuous or singular (resp. pure continuous, or pure point) are found in terms of the $\widetilde{Q}-$representation. In addition the metric-topological and fractal properties for the distribution of ξ are investigated. A number of examples are presented.

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AMS Subject Classifications (2000):11A67, 11K55, 26A30, 26A46, 28A80, 28D05, 60G30

Key words: $\widetilde{Q}$-representation of real numbers; random variables with independent $\widetilde{Q}$-symbols; fractals, pure point, singular continuous, absolutely continuous measures; C, P, S types of measures

1 Introduction

As well known there exist only three types of pure probability distributions: discrete, absolutely continuous and singular. During a long period mathematicians had a rather low interest in singular probability distributions, which was basically caused by the two following reasons: the absence of effective analytic tools and the widely spread point of view that such distributions do not have any applications, in particular in physics, and are interesting only for theoretical reasons. The interest in singular probability distributions increased however in 1990's due their deep connections with the theory of fractals. On the other hand, recent investigations show that singularity is generic for many classes of random variables, and absolutely continuous and discrete distributions arise only in exceptional cases (see, e.g. [8, 13]).

Usually the singular probability distributions are associated with the Cantor-like distributions. Such distributions are supported by nowhere dense sets of zero Lebesgue measures. In the sequel we shall call such distributions the distributions of the C-type. But there exist singular probability distributions with other metric-topological properties of their topological support S (the minimal closed set supported the distribution):

1) S is the closure of the union of the closed intervals (S-type);

2) S is a nowhere dense set such that for all $\varepsilon > 0$ and $x_0 \in S$ the set $S \cap (x_0 - \varepsilon; x_0 + \varepsilon)$ has positive Lebesgue measure (P-type).

In [8] it has been proved that any singular continuous function F_s can be decompose into the following sum:

$$F_s = \beta_1 F_{sc} + \beta_2 F_{ss} + \beta_3 F_{sp},$$

where $\beta_i \geq 0$, $\beta_1 + \beta_2 + \beta_3 = 1$; F_{sc}, F_{ss}, F_{sp} are distribution function of $C-$, $S-$ or $P-$type correspondingly.

It is easy to construct examples of singular continuous probability distributions of the C- or S-type (see, e.g. [1, 8]), but a construction of a simple example of singular continuous probability distributions of the P-type is more complicated.

The main goal of this paper is to introduce into consideration the so-called $\widetilde{Q}$-representation of real numbers which is a convenient tool for construction of a wide class of fractals. This class contains Cantor-like sets as well as everywhere dense noncompact fractals with any desirable Hausdorff-Besicovitch dimension $\alpha_0 \in [0; 1]$. By using the $\widetilde{Q}$ -representation we introduce a class of random variables with independent $\widetilde{Q}$-symbols. This class contains all possible above mentioned types of singular distributions.

An additional reason for the investigation of the distribution of the random variables with independent $\widetilde{Q}$-symbols is to extend the so-called Jessen-Wintner theorem to the case of sums of random variables which are not independent. In fact this theorem asserts that if a random variable is the sum of the convergent series of the independent discretely random variables, then it has a pure distribution. Necessary and sufficient conditions for probability distributions to be singular resp. absolutely continuous are still unknown.

In this paper we completely investigated the structure of the random variables with independent $\widetilde{Q}$-symbols (necessary and sufficient conditions for absolutely continuity and singularity will be proven in Section 5). Moreover we investigated in details the metric-topological properties of the above mentioned class of probability distributions.

2 $\widetilde{Q}$-representation of real numbers

We describe the notion of the so-called $\widetilde{Q}-$**representation** for real numbers $x \in [0,1]$. Let us consider a $\mathbf{N}_k \times \mathbf{N}-$matrix $\widetilde{Q} = \|q_{ik}\|$, $i \in \mathbf{N}_k$, $k \in \mathbf{N}$, where $\mathbf{N}$ stands for the set of natural numbers and $\mathbf{N}_k = \{0, 1, ..., N_k\}$, with $0 < N_k \leq \infty$. We suppose that

$$q_{ik} > 0 \quad \forall\, i \in \mathbf{N}_k,\ k \in \mathbf{N}. \tag{1}$$

Besides, we assume that for each $k \in \mathbf{N}$:

$$\sum_{i \in \mathbf{N}_k} q_{ik} = 1, \tag{2}$$

and

$$\prod_{k=1}^{\infty} \max_{i \in \mathbf{N}_k} \{q_{ik}\} = 0. \tag{3}$$

Given a $\widetilde{Q}-$matrix we consecutively perform decompositions of the segment $[0,1]$ as follows.

Step 1. We decompose $[0,1]$ (from the left to the right) into the union of closed intervals $\Delta_i \equiv \Delta_{i_1}$, $i_1 \in \mathbf{N}_1$ (without common interior points) of the length $|\Delta_{i_1}| = q_{i_1 1}$,

$$[0,1] = \bigcup_{i_1 \in \mathbf{N}_1} \Delta_{i_1}.$$

Each interval Δ_{i_1} is called a 1-rank interval.

Step 2. Each 1-rank interval Δ_{i_1} is decomposed (from the left to the right) into the union of smaller closed intervals $\Delta_{i_1 i_2}$, $i_2 \in \mathbf{N}_2$ without common interior points,

$$\Delta_{i_1} = \bigcup_{i_2 \in \mathbf{N}_2} \Delta_{i_1 i_2},$$

where the lengths $|\Delta_{i_1 i_2}|$ of $\Delta_{i_1 i_2}$ are related as follows

$$|\Delta_{i_1 0}| : |\Delta_{i_1 1}| : \cdots : |\Delta_{i_1 i_2}| : \cdots = q_{02} : q_{12} : \cdots : q_{i_2 2} : \cdots$$

Each interval $\Delta_{i_1 i_2}$ is called a 2-rank interval. It is easy to see that

$$|\Delta_{i_1 i_2}| = q_{i_1 1} \cdot q_{i_2 2}.$$

Further, we decompose each interval $\Delta_{i_1 i_2}$ by using the collection of smaller intervals $\Delta_{i_1 i_2 i_3}$, and so on.

Step $k \geq 2$. We decompose (from the left to the right) each closed $(k-1)$-rank interval $\Delta_{i_1 i_2 \ldots i_{k-1}}$ into the union of closed $k-$rank intervals $\Delta_{i_1 i_2 \ldots i_k}$,

$$\Delta_{i_1 i_2 \ldots i_{k-1}} = \bigcup_{i_k \in \mathbf{N}_k} \Delta_{i_1 i_2 \ldots i_k},$$

where their lengths

$$|\Delta_{i_1 i_2 \ldots i_k}| = q_{i_1 1} \cdot q_{i_2 2} \cdots q_{i_k k} = \prod_{s=1}^{k} q_{i_s s} \tag{4}$$

are related as follows

$$\left|\Delta_{i_1 i_2 \ldots i_{k-1} 0}\right| : \left|\Delta_{i_1 i_2 \ldots i_{k-1} 1}\right| : \cdots : \left|\Delta_{i_1 i_2 \ldots i_{k-1} i_k}\right| : \cdots = q_{0k} : q_{1k} : \cdots : q_{i_k k} : \cdots$$

Thus, for any sequence of indices $\{i_k\}$, $i_k \in \mathbf{N}_k$, there corresponds the sequence of embedded closed intervals

$$\Delta_{i_1} \supset \ \Delta_{i_1 i_2} \supset \cdots \supset \Delta_{i_1 i_2 \ldots i_k} \supset \cdots$$

such that $|\Delta_{i_1...i_k}| \to 0$, $k \to \infty$, due to (3) and (4). Therefore, there exists a unique point $x \in [0,1]$ belonging to all intervals Δ_{i_1}, $\Delta_{i_1 i_2}$, ..., $\Delta_{i_1 i_2 ... i_k}$, ... Conversely, for any point $x \in [0,1]$ there exists a sequence of embedded intervals $\Delta_{i_1} \supset \Delta_{i_1 i_2} \supset ... \supset \Delta_{i_1 i_2 ... i_k} \supset ...$ containing x, i.e.,

$$x = \bigcap_{k=1}^{\infty} \Delta_{i_1 i_2 ... i_k} = \bigcap_{k=1}^{\infty} \Delta_{i_1(x) i_2(x) ... i_k(x)} =: \Delta_{i_1(x) i_2(x) ... i_k(x) ...} \tag{5}$$

This means that every point $x \in [0,1]$ is defined by the sequence of indices $i_k = i_k(x) \in \mathbf{N}_k, k = 1, 2, ...$ Notation (5) is called the $\widetilde{Q}-$**representation** of the point $x \in [0,1]$.

Obviously a point $x \in [0,1]$ has a unique $\widetilde{Q}-$representation, if x is not an end-point of any closed interval $\Delta_{i_1 i_2 ... i_k}$.

Remark 1. The correspondence $[0,1] \in x \Leftrightarrow \{i_1(x) i_2(x) ... i_k(x) ...\}$ in (5) is one-to-one , i.e., the $\widetilde{Q}-$representation is unique for every point $x \in [0,1]$, provided that the $\widetilde{Q}$-matrix contains an infinite number of columns with an infinite number of elements. However in the case, where $N_k < \infty, \forall k > k_0$ for some k_0, there exists a countable set of points $x \in [0,1]$ having two different $\widetilde{Q}-$representations. Precisely, this is the set of all end-points of intervals $\Delta_{i_1 i_2 ... i_k}$ with $k > k_0$.

One has the formula

$$x = S_1(x) + \sum_{k=2}^{\infty} \left[S_k(x) \prod_{s=1}^{k-1} q_{i_s(x)s} \right] = \sum_{k=1}^{\infty} S_k(x) L_{k-1}(x) \tag{6}$$

where $S_k(x) := \begin{cases} 0, & \text{if} \quad i_k(x) = 0, \\ \sum\limits_{i=0}^{i_k(x)-1} q_{ik}, & \text{if} \quad i_k(x) \geq 1 \end{cases}$ and where we put (see (4))

$$L_{k-1}(x) :=| \Delta_{i_1(x)...i_{k-1}(x)} |= \prod_{s=1}^{k-1} q_{i_s(x)s}$$

for $k > 1$, and $L_{k-1}(x) = 1$, if $k = 1$.

We note that (6) follows from (5) since the common length of all intervals lying on the left side of a point $x = \Delta_{i_1(x)...i_k(x)...}$ can be calculate as the sum of all $1-$rank intervals lying on the left from x (it is the first term $S_1(x)$ in (5)), plus the sum of all $2-$rank intervals from $\Delta_{i_1(x)}$, lying on the left side from x (the second term $S_2(x) \cdot q_{i_1(x)1}$ in (5)), and so on.

Remark 2. If $q_{ik} = q_i$,$\forall k \in \mathbf{N}$, then the $\widetilde{Q}-$representation coincides with the $Q-$representation (see [10]); moreover, if $q_{ik} = \frac{1}{s}$, for some natural number $s > 1$, then the $\widetilde{Q}-$representation coincides with the classical $s-$adic representation of real numbers.

3 $\widetilde{Q}(\mathbf{V})-$representation for fractals

The $\widetilde{Q}$-representation allows to construct in a convenient way a wide class of fractals on R^1 and other mathematical objects with fractal properties.

Firstly we consider compact fractals from R^1.

Let $\mathbf{V} := \{\mathbf{V}_k\}_{k=1}^{\infty}$, $\mathbf{V}_k \subseteq \mathbf{N}_k$. If in the $\widetilde{Q}-$representation the symbols i_k run not along all set $\mathbf{N}_k$ but only along some of its subsets $\mathbf{V}_k$, then we say that we have the $\widetilde{Q}(\mathbf{V})-$representation.

Let us consider the set

$$\Gamma_{\widetilde{Q}(\mathbf{V})} \equiv \Gamma := \{x \in [0,1] : \quad x = \Delta_{i_1 i_2 \ldots i_k \ldots},\ i_k \in \mathbf{V}_k\} . \tag{7}$$

This subset of $[0,1]$ consists of points, which can be $\widetilde{Q}-$represented by using only symbols i_k from the set $\mathbf{V}_k$ on each k-th position of their $\widetilde{Q}-$representation.

Of course, if $\mathbf{V}_k = \mathbf{N}_k$ for all k, then $\Gamma = [0,1]$.

If $\mathbf{V}_k \neq \mathbf{N}_k$ at least for one $k < k_0$, and $\mathbf{V}_k = \mathbf{N}_k$ for all $k \geq k_0$ with some fixed $k_0 > 1$, then Γ is *a union of closed intervals*. In this case one can get Γ removing from $[0,1]$ all open intervals $\dot{\Delta}_{i_1 \ldots i_k}$, $k < k_0$ with $i_k \notin \mathbf{V}_k$ (where a point over Δ means that an interval is open).

If the condition $\mathbf{V}_k \neq \mathbf{N}_k$ holds for infinitely many values of k, then obviously Γ is *a nowhere dense set*. All fractals from the unit segment have zero Lebesgue measure. Firstly, we shall study the metric properties of the sets $\Gamma_{\widetilde{Q}(\mathbf{V})}$.

Let $S_k(\mathbf{V})$ denote the sum of all elements q_{ik} such that $i_k \in \mathbf{V}_k$, i.e.,

$$S_k(\mathbf{V}) := \sum_{i \in \mathbf{V}_k} q_{ik}.$$

We note that $0 < S_k(\mathbf{V}) \leq 1$ due to (1), (2).

Lemma 1. *The Lebesgue measure $\lambda(\Gamma)$ of the set Γ defined by (7) is equal to*

$$\lambda(\Gamma) = \prod_{k=1}^{\infty} S_k(\mathbf{V}). \tag{8}$$

Proof. Let $\Gamma_1 := \bigcup\limits_{i_1 \in \mathbf{V}_1} \Delta_{i_1}$. Then $\lambda(\Gamma_1) = S_1(\mathbf{V})$. Let further $\Gamma_2 := \bigcup\limits_{i_1 \in \mathbf{V}_1, i_2 \in \mathbf{V}_2} \Delta_{i_1 i_2} \subset \Gamma_1$. Then

$$\lambda(\Gamma_2) = \sum_{i_1 \in \mathbf{V}_1, i_2 \in \mathbf{V}_2} q_{i_1 1} q_{i_2 2} = S_1(\mathbf{V}) S_2(\mathbf{V}).$$

Similarly, for any $n \in \mathbf{N}$, let $\Gamma_n := \bigcup\limits_{i_1 \in \mathbf{V}_1 \ldots i_n \in \mathbf{V}_n} \Delta_{i_1 \ldots i_n}$, $\Gamma_n \subseteq \Gamma_{n-1}$. Then

$$\lambda(\Gamma_n) = \sum_{i_1 \in \mathbf{V}_1, \ldots, i_k \in \mathbf{V}_k} q_{i_1 1} \ldots q_{i_n n} = \prod_{k=1}^{n} S_k(\mathbf{V}).$$

It is easy to see that $\Gamma = \bigcap\limits_{n=1}^{\infty} \Gamma_n$ and $\Gamma_{k-1} \supset \Gamma_k$. Therefore,

$$\lambda(\Gamma) = \lim_{n \longrightarrow \infty} \lambda(\Gamma_n)$$

which coincides with (8). ■

Let $W_k(\mathbf{V}) = 1 - S_k(\mathbf{V}) \geq 0$.

Lemma 2. *The set Γ defined by (7) has zero Lebesgue measure if and only if*

$$\sum_{k=1}^{\infty} W_k(\mathbf{V}) = \infty, \tag{9}$$

Proof. This assertion is a direct consequence of the previous lemma and the well known relation between infinite products and infinite series. Namely, for a sequence $0 \leq a_k < 1$, the product $\prod\limits_{k=1}^{\infty} (1 - a_k) = 0$ if and only if the sum $\sum\limits_{k=1}^{\infty} a_k = \infty$. In our case $a_k = 1 - S_k(\mathbf{V})$. ■

The above mentioned procedure allows to construct nowhere dense compact fractal sets E with desirable Hausdorff-Besicovitch dimension (including the anomalously fractal case ($\alpha_0(E) = 0$) and the superfractal case ($\alpha_0(E) = 1$)) in a very compact way.

Theorem 1. *Let* $\mathbf{N}_k = N^0_{s-1} := \{0, 1, ..., s-1\}$ $\forall k \in N$ *and let the matrix* $\widetilde{Q}$ *have the following asymptotic property:*

$$\lim_{k\to\infty} q_{ik} = q_i, \quad i \in N^0_{s-1}.$$

Then:

1) the Hausdorff-Besicovitch dimension of the set $\Gamma_{\widetilde{Q}(\mathbf{V})}$ *is the root of the following equation*

$$\sum_{i\in\mathbf{V}} q_i^x = 1, \quad \mathbf{V} = \{v_1, v_2, ..., v_m\};$$

2) if

$$M[Q, (\nu_0, ..., \nu_{s-1})] = \left\{ x : \Delta^{\tilde{Q}}_{\alpha_1(x)...\alpha_k(x)...}, \lim_{k\to\infty} \frac{N_i(x,k)}{k} = \nu_i, \; i \in N^0_{s-1} \right\},$$

where $N_i(x, k)$ *is the amount of symbols "i" in the* $\widetilde{Q}$*-representation of* x *until the* k*-th position, then*

$$\alpha_0(M[Q, (\nu_0, ..., \nu_{s-1})]) = \frac{\sum\limits_{i=0}^{s-1} \nu_i \ln \nu_i}{\sum\limits_{i=0}^{s-1} \nu_i \ln q_i}. \tag{10}$$

Proof. Let the matrix $\widetilde{Q}$ has the exactly s rows and assume all columns are the same: $(q_0, q_1, ..., q_{s-1})$. In such a case the $\widetilde{Q}$-representation reduces to the Q-representation studied in [8].

It is easy to prove (see, e.g., [10]), that to calculate the Hausdorff-Besicovitch dimension of any subset $E \subset [0; 1]$ it is sufficient to consider a class of cylinder sets of different ranks generated by Q-partitions of the unit interval.

The Billingsley theorem (see, e.g., [3], p.141) admits a generalization to the class of Q-cylinders, and, therefore, the set

$$E = \left\{ x : \lim_{k\to\infty} \frac{\ln \prod\limits_{i=1}^{k} \nu_{c_i}}{\ln |\Delta^Q_{c_1...c_k}|} = \delta \right\}$$

has the Hausdorff-Besicovitch dimension $\alpha_0(E) = \delta$.

It is well known (see, e.g., [6, 8]) that the set $\Gamma_{Q,\mathbf{V}}$ is a self-similar fractal with the Hausdorff-Besicovitch dimension satisfying the condition (10).

It is also not hard to prove (see, e.g., [2]), that the following transformation f of $[0;1]$

$$f(x) = f(\Delta^{\widetilde{Q}}_{\alpha_1(x)\ldots\alpha_k(x)\ldots}) = \Delta^{Q}_{\alpha_1(x)\ldots\alpha_k(x)\ldots}$$

preserves the Hausdorff-Besicovitch dimension of any subset of $[0;1]$. Therefore, $\alpha_0(\Gamma_{\widetilde{Q}(\mathbf{V})}) = \alpha_0(\Gamma_{Q(\mathbf{V})})$, which proves the second part the theorem. ■

By using theorem 1 it is easy to construct compact fractals as well as everywhere dense noncompact fractals with any desirable Hausdorff-Besicovitch dimension $\alpha_0 \in [0;1]$

Examples.

1. If again $\mathbf{N}_k = \{0,1,2\}, \mathbf{V}_k = \{0,2\}, q_{1k} \to 0$, but $\sum\limits_{k=1}^{\infty} q_{1k} = \infty$ with $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, then Γ is a nowhere dense set of zero Lebesgue measure. One can check that the Hausdorff dimension of this set is equal 1. In the terminology of [8] a set of this kind is called a superfractal set.

2. If $\mathbf{N}_k = \{0,1,2\}, \mathbf{V}_k = \{0,2\}$, $q_{1k} \to 1$ (but $\prod\limits_{k=1}^{\infty} q_{1k} = 0$), and $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, then Γ is a nowhere dense set of zero Lebesgue measure and of zero Hausdorff dimension, i.e., Γ is an anomalously fractal set (see [8]).

4 Random variables with independent $\widetilde{Q}-$symbols

Let $\{\xi_k\}$, $k \in \mathbf{N}$, be a sequence of independent random variables with the following distributions

$$P(\xi_k = i) := p_{ik}, \quad \forall i \in \mathbf{N}_k, \ \ \forall k \in \mathbf{N}.$$

We have, of course,

$$\sum_{i \in \mathbf{N}_k} p_{ik} = 1, \quad \forall k \in \mathbf{N}. \tag{11}$$

By using ξ_k and the $\widetilde{Q}$-representation we construct a random variable ξ as follows:

$$\xi := \Delta_{\xi_1\xi_2\ldots\xi_k\ldots} \ . \tag{12}$$

Thus, the distribution of ξ is completely fixed by two matrices: $\widetilde{Q}$ and $\widetilde{P} = ||p_{ik}||$, $i \in \mathbf{N}_k$, $k \in \mathbf{N}$, where some elements of the matrix $\widetilde{P}$ possibly are equal to zero. Of course, all sets $\mathbf{N}_k$ are the same as those in the $\widetilde{Q}$-matrix.

As a rule, the distribution of the r.v. ξ is concentrated on fractals. Our main aim in this paper is to study the structure of the r.v. ξ, and its metric, topological and fractal properties.

Let $F_\xi(x),\ x \in [0,1]$ be the distribution function of the r. v. ξ given by (12).

Theorem 2. *The values of $F_\xi(x)$ can be calculated according to the formula*

$$F_\xi(x) = P_1(x) + \sum_{k=2}^{\infty}\left[P_k(x)\prod_{s=1}^{k-1} p_{i_s(x)s}\right] = \sum_{k=1}^{\infty} P_k(x)T_{k-1}(x) \qquad (13)$$

where we put $T_0(x)$=1, $T_{k-1}(x) := \prod\limits_{s=1}^{k-1} p_{i_s(x),s}$ and $P_k(x) := \begin{cases} 0, & if \quad i_k(x) = 0, \\ \sum\limits_{j=0}^{i_k(x)-1} p_{jk}, & if \quad i_k(x) \geq 1. \end{cases}$

Proof. By the definition of the r.v. ξ, the event $\{\xi < x\}$ is equivalent to

$$\{\xi_1 < i_1(x)\}\bigcup\{\xi_1 = i_1(x), \xi_2 < i_2(x)\}\bigcup ...$$

$$\bigcup\{\xi_1 = i_1(x), \xi_2 = i_2(x), ..., \xi_{k-1} = i_{k-1}(x), \xi_k < i_k(x)\}\bigcup ...$$

Since all $\xi_1, \xi_2, ..., \xi_k, ...$ are independent and the events in the brackets $\{\cdot\}$ are disjoint, we have

$$F_\xi(x) = P\{\xi_1 < i_1(x)\} + P\{\xi_1 = i_1(x)\} \cdot P\{\xi_2 < i_2(x)\} + ...$$

$$+P\{\xi_1 = i_1(x)\} \cdot P\{\xi_2 = i_2(x)\} \cdot ... \cdot P\{\xi_{k-1} = i_{k-1}(x)\} \cdot P\{\xi_k < i_k(x)\} + ...$$

$$= P_1(x) + \sum_{k=2}^{\infty}\left[P_k(x)\prod_{s=1}^{k-1} p_{i_s(x),s}\right] = P_1(x) + \sum_{k=2}^{\infty} P_k(x)T_{k-1}(x),$$

where we recall that $P_k(x) = P\{\xi_k < i_k(x)\}$ and $T_{k-1}(x) = P\{\xi_1 = i_1(x)\} \cdot ... \cdot P\{\xi_{k-1} = i_{k-1}(x)\}$. This proves (13). ■

5 Structure of the distributions of random variables with independent $\widetilde{Q}$-symbols

Let μ_ξ be the measure corresponding the distribution of the random variable ξ with independent $\widetilde{Q}$-symbols.

The goal of this section is to prove the purity of the distributions of random variables ξ with independent $\widetilde{Q}-$symbols. We shall establish necessary and sufficient conditions for the measure μ_ξ to be pure point, resp., absolute continuous, or singular continuous.

Theorem 3. *The measure μ_ξ is of pure type, i.e., it is either purely absolutely continuous, resp., purely point, resp., purely singular continuous. Precisely,*

1) μ_ξ is purely absolutely continuous, $\mu_\xi = (\mu_\xi)_{ac}$, if and only if

$$\rho := \prod_{k=1}^{\infty} \left\{ \sum_{i \in \mathbf{N}_k} \sqrt{p_{ik} \cdot q_{ik}} \right\} > 0; \tag{14}$$

2) μ_ξ is purely point, $\mu_\xi = (\mu_\xi)_{pp}$, if and only if

$$P_{max} := \prod_{k=1}^{\infty} \max_{i \in \mathbf{N}_k} \{p_{ik}\} > 0; \tag{15}$$

3) μ_ξ is purely singular continuous, $\mu_\xi = (\mu_\xi)_{sc}$, if and only if

$$\rho = 0 = P_{max}. \tag{16}$$

Proof. Let $\Omega_k = \mathbb{N}_k = \{0, 1, ..., N_k\}$, $\mathcal{A}_k = 2^{\Omega_k}$. We define measures μ_k and ν_k in the following way:

$$\forall i \in \Omega_k : \quad \mu_k(i) = p_{ik}; \;\; \nu_k(i) = q_{ik}.$$

Let

$$(\Omega, \mathcal{A}, \mu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, \mu_k), \quad (\Omega, \mathcal{A}, \nu) = \prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, \nu_k)$$

be the infinite products of probability spaces.

Let us consider the measurable mapping $f : \Omega \to [0; 1]$ defined as follows:

$$\forall \omega = (\omega_1, \omega_2, ..., \omega_k, ...) \in \Omega, \quad f(\omega) = x = \Delta_{i_1(x) i_2(x) ... i_k(x) ...}$$

with $\omega_k = i_k(x)$ $k \in N$. Here we used the $\widetilde{Q}$-representation of $x \in [0;1]$.

We define the measures μ^* and ν^* as the image measure of μ resp. ν under f:

$$\forall B \in \mathcal{B}, \ \ \mu^*(B) := \mu(f^{-1}(B)); \ \ \nu^*(B) = \nu(f^{-1}(B)).$$

It is easy to see that ν^* coincides with Lebesgue measure λ on $[0;1]$, and $\mu^* \equiv \mu_\xi$.

If the matrix $\widetilde{Q}$ contains an infinite number of columns with an infinite number of elements, then the mapping f is bijective. If $N_k < +\infty$ $\forall k > k_0$ for some $k_0 \in N$, then there exists a set Ω_0 such that $\nu(\Omega_0) = \mu(\Omega_0) = 0$ and the mapping $f : (\Omega \setminus \Omega_0) \to [0;1]$ is bijective. If $\prod\limits_{k=k_0}^{\infty} p_{0k} = 0$, then $\Omega_0 = \{\omega : \omega_j = 0 \ \forall j > j_0(\omega)\}$. If $\prod\limits_{k=k_0}^{\infty} p_{N_k k} = 0$, then $\Omega_0 = \{\omega : \omega_j = N_j, \ \forall j > j_0(\omega)\}$. Therefore, the measure $\mu_\xi = \mu^*$ is absolutely continuous (singular) with respect to Lebesgue measure if and only if the measure μ is absolutely continuous (singular) with respect to the measure ν.

Since, $q_{ik} > 0$, we conclude that $\mu_k \ll \nu_k$ $\forall k \in N$. By using Kakutani is theorem [7], we have

$$\mu_\xi \ll \lambda \ \Leftrightarrow \ \prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k > 0 \ \Leftrightarrow \ \prod_{k=1}^{\infty} \left(\sum_{i \in \mathbb{N}_k} \sqrt{p_{ik} q_{ik}} \right) > 0, \tag{17}$$

$$\mu_\xi \perp \lambda \ \Leftrightarrow \ \prod_{k=1}^{\infty} \int_{\Omega_k} \sqrt{\frac{d\mu_k}{d\nu_k}} d\nu_k = 0 \ \Leftrightarrow \ \prod_{k=1}^{\infty} \left(\sum_{i \in \mathbb{N}_k} \sqrt{p_{ik} q_{ik}} \right) = 0. \tag{18}$$

Of course, a singularly distributed random variable ξ can also be distributed discretely. For any point $x \in [0;1]$ the set $f^{-1}(x)$ consists of at most two points from Ω. Therefore, the measure μ_ξ is an atomic measure if and only if the measure μ is atomic. If $\prod\limits_{k=1}^{\infty} \max\limits_i p_{ik} = 0$, then

$$\mu(\omega) = \prod_{k=1}^{\infty} p_{\omega_k k} \leq \prod_{k=1}^{\infty} \max_i p_{ik} = 0 \ \text{ for any } \ \omega \in \Omega,$$

and μ is continuous. Therefore, conditions (15) is necessary for the measure μ to be pure point. If $\prod\limits_{k=1}^{\infty} \max\limits_i p_{ik} > 0$, then we consider the subset $A_+ = \{\omega : \ \mu(\omega) > 0\}$.

The set A_+ contains the point ω^* such that $p_{\omega_k^* k} = \max\limits_i p_{ik}$, It is easy to see that for all $\omega \in A_+$ the condition $p_{\omega_k k} \neq \max\limits_i p_{ik}$ holds only for a finite amount of values k. This means that A_+ is a countable set and the event "$\omega \in A_+$" does not depend on any finite coordinates of ω. Therefore, by using Kolmogorov's "0 and 1" theorem, we conclude that $\mu(A_+) = 0$ or $\nu(A_+) = 1$. Since $\mu(A_+) \geq \mu(\omega^*) > 0$, we have $\mu(A_+) = 1$, which proves the equality $\mu = \mu_{pp}$. $\blacksquare$

Remark 3. If there exists a positive number q^+ such that $q_{ik} \geq q^+, \forall k \in \mathbf{N}$, $\forall i \in \mathbf{N}_k$, then condition (17) is equivalent to the convergence of the following series:

$$\sum_{k=1}^{\infty}\{\sum_{i \in \mathbf{N}_k}(1 - \frac{p_{ik}}{q_{ik}})^2\} < \infty. \tag{19}$$

If $\varliminf\limits_{k \to \infty} q_{ik} = 0$, then, generally speaking, conditions (17) and (19) are not equivalent. For example, let us consider the matrices $\widetilde{Q}$ and $\widetilde{P}$ as follows:$\mathbf{N}_k = \{0,1,2\}, q_{1k} = \frac{1}{2^k}$, $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}$, $p_{1k} = 0$, $p_{0k} = p_{2k} = \frac{1}{2}$. In this case condition (17) holds, but (19) does not hold.

6 Metric-topological classification and fractal properties of the distributions of the random variables with independent $\widetilde{Q}-$symbols

For any probability distribution there exist sets which essentially characterize the properties of the distribution. We would like to stress the role of the following sets.

a) Topological support $S_\psi = \{x : F(x+\varepsilon) - F(x-\varepsilon) > \varepsilon, \ \forall \varepsilon > 0\}$. S_ψ is the smallest closed support of the distribution of ψ.

b) Essential support $N_\psi^\infty = \left\{x : \varliminf\limits_{\varepsilon \to 0} \frac{F(x+\varepsilon)-F(x-\varepsilon)}{2\varepsilon} = +\infty\right\}$.

All singular probability distributions are concentrated on sets of zero Lebesgue measure and they have fractal properties. If the topological support of a distribution is a fractal, then the corresponding distribution is said to be externally fractal.

The probability distribution of a random variable ψ is said to be internally fractal if the essential support of the distribution is a fractal set.

First of all we shall analyze the metric and topological properties of the topological support of the random variable with independent $\widetilde{Q}$-symbol. We establish the metric-topological classification of the support of a probability measure μ_ψ by introducing three disjoint types of closed sets.

We say that a set S is of the pure C-type if it is a perfect nowhere dense set of zero Lebesgue measure. By definition a probability measure μ_ψ is said to be of the pure C-type if its support S_ψ is a set of the pure C-type.

We say that S is a set of the pure P-type, if for any interval (a, b) the set $(a, b) \cap S$ is either empty or a perfect nowhere dense set of positive Lebesgue measure. By definition a probability measure μ_ψ is said to be of the pure P-type, if its support S_ψ is a set of the pure P-type.

We say that S is a set of the pure S-type if it is the closure of a union of an at most countable family of closed intervals, i.e.,

$$\S_\psi = \left(\cup_i [a_i, b_i]\right)^{cl}, \quad a_i < b_i.$$

By definition a probability measure μ_ψ is said to be of the pure S-type if its support S_ψ is a set of the pure S-type.

In [1, 8] it was proven that arbitrary singular continuous probability measures can be decomposed into linear combinations of singular probability measures of S-, C- and P-types.

We shall prove now that the above considered probability measures μ_ξ are of the pure above mentioned metric-topological types. Moreover we give necessary and sufficient conditions for a probability measure to belong to each of these types.

Theorem 4. *The distribution of the random variable ξ with independent $\widetilde{Q}-$symbols has pure metric-topological type. Namely, the support of the corresponding measure μ_ξ is one of following three type:*

1) it is of the pure S-type, if and only if the matrix $\widetilde{P}$ contains only a finite number of zero elements;

2) it is of the pure C-type, if and only if the matrix $\widetilde{P}$ contains infinitely many columns having some elements $p_{ik} = 0$, and besides

$$\sum_{k=1}^{\infty} \Big(\sum_{i: p_{ik}=0} q_{ik} \Big) = \infty; \tag{20}$$

3) it is of the pure P-type, if and only if the matrix $\widetilde{P}$ contains infinitely many columns having zero elements and besides

$$\sum_{k=1}^{\infty}\left(\sum_{i:p_{ik}=0} q_{ik}\right) < \infty; \tag{21}$$

Proof. Let us consider the set $\Gamma \equiv \Gamma_{\widetilde{Q}(\mathbf{V})}$ (see Sect. 3) with $\mathbf{V} = \{\mathbf{V}_k\}_{k=1}^{\infty}$ defined by the $\widetilde{P}-$matrix as follows: $\mathbf{V}_k = \{i \in \mathbf{N}_k : \; p_{ik} \neq 0\}$. Then it is not hard to understand (see [8]) that the usual support of the measure μ_ξ coincides with a set Γ or its closure, i.e.,

$$S_\xi = \Gamma^{cl}_{\widetilde{Q}(\mathbf{V})}. \tag{22}$$

Therefore to examine the metric-topological structure of the set S_ξ we may apply the results of section 3.

So, if the matrix $\widetilde{P}$ contains only finite number of zero elements, then $\mathbf{V}_k = \mathbf{N}_k, \; k > k_0$ for some $k_0 > 0$. In such a case (see Sect. 3) Γ is the union of at most of an countable family of closed intervals. Hence (22) implies that the measure μ_ξ is of the pure S-type. Of course, in this case $\lambda(\Gamma) > 0$.

In the opposite case where the matrix $\widetilde{P}$ contains an infinite number of columns where some elements $p_{ik} = 0$, then obviously Γ is a nowhere dense set (see Sect. 3). The Lebesgue measure of the set Γ by Lemma 1 is equal to

$$\lambda(\Gamma) = \prod_{k=1}^{\infty} S_k(\mathbf{V}) = \prod_{k=1}^{\infty}\left(\sum_{i\in\mathbf{V}_k} q_{ik}\right) = \prod_{k=1}^{\infty}\left(1 - \sum_{i:p_{ik}=0} q_{ik}\right).$$

Let us set $W_k(\mathbf{V}) = 1 - S_k(\mathbf{V}) = \sum_{i=1,p_{ik}=0}^{\infty} q_{ik}$. Then, by Theorem 1, either $\lambda(\Gamma) = 0$, provided that condition (20) fulfilled, or $\lambda(\Gamma) > 0$, if condition (21) holds. Thus the measure μ_ξ either is of the C-type, or it is of the P-type.

Since the conditions 1), 2) and 3) of this theorem are mutually exclusive and one of them always holds, we conclude that the distribution of the random variable ξ with independent $\widetilde{Q}$-symbols always has a pure metric-topological type. ■

By using the latter theorems we can construct measures of **8** kinds: pure point as well as pure singular continuous of any S-, C-, or P-types, and pure absolutely continuous but only of the S- and P-types.

We illustrate this statement by examples.

Example 3.

Let $\mathbf{N}_k = \{0, 1, 2\}$ and let the $\widetilde{Q}-$matrix be given by $q_{0k} = q_{1k} = q_{2k} = \frac{1}{3},\ k = 1, 2, ...$

S$_{pp}$: If $p_{0k} = \frac{1-p_{1k}}{2}, p_{1k} = 1 - \frac{1}{2^k}, p_{2k} = \frac{1-p_{1k}}{2}$, then μ_ξ is a discrete measure of the pure S-type. In this case $S_\xi = [0, 1]$ and N_ξ^∞ is a countable set which is dense on $[0, 1]$.

S$_{sc}$: If $p_{0k} = \frac{1}{4}, p_{1k} = \frac{1}{2}, p_{2k} = \frac{1}{4}$, then μ_ξ is a singular continuous measure of pure S-type. In this case again $S_\xi = [0, 1]$ but N_ξ^∞ is now a fractal set which is also dense on $[0, 1]$.

S$_{ac}$: If $p_{0k} = p_{1k} = p_{2k} = \frac{1}{3}$, then μ_ξ coincides with the Lebesgue measure on $[0, 1]$.

Example 4.

Let again $\mathbf{N}_k = \{0, 1, 2\}$ and let the $\widetilde{Q}-$matrix be given by $q_{0k} = q_{1k} = q_{2k} = \frac{1}{3},\ k = 1, 2, ...$ Then

C$_{pp}$: If $p_{0k} = 1 - \frac{1}{2^k}, p_{1k} = 0, p_{2k} = \frac{1}{2^k}$, then μ_ξ is a pure point measure of the pure C-type. In this case $S_\xi \equiv C_0$ coincides with the classical Cantor set C_0 and its essential support is a countable set which is dense on C_0.

C$_{sc}$: If $p_{0k} = \frac{1}{2},\ p_{1k} = 0, p_{2k} = \frac{1}{2}$, then μ_ξ is a singular continuous measure of the pure C-type. In this case again $S_\xi = C_0$ and the difference $S_\xi \setminus N_\xi^\infty$ is a countable set.

Example 5.

Let as above $\mathbf{N}_k = \{0, 1, 2\}$ and let the $\widetilde{Q}-$matrix be given by $q_{0k} = q_{2k} = \frac{1-q_{1k}}{2}, q_{1k} = \frac{1}{2^k},\ k = 1, 2, ...$ Then

P$_{pp}$: If $p_{0k} = 1 - \frac{1}{2^k}, p_{1k} = 0, p_{2k} = \frac{1}{2^k}$, then μ_ξ is a pure point measure of the pure P-type.

P$_{sc}$: If $p_{0k} = \frac{1}{4}, p_{1k} = 0, p_{2k} = \frac{3}{4}$, then μ_ξ is a singular continuous measure of the pure P-type.

P$_{ac}$: If $p_{0k} = p_{2k} = \frac{1-p_{1k}}{2}, p_{1k} = \frac{1}{2^k}$, then μ_ξ coincides with Lebesgue measure.

We would like to stress that the essential support is more suitable to describe the properties of distributions with complicated local structure.

As we saw above, a discrete probability distribution may be of C-, P- resp. S-type, and the topological support of discrete distribution can be of any Hausdorff-Besicovitch dimension $\alpha_0 \in [0; 1]$. But the essential support of a discrete distributions is always at most a countable set.

The essential support is especially suitable for singular distributions because of the following fact: a random variable ψ is singularly distributed iff $P_\psi(N_\psi^\infty)=1$.

For an absolutely continuous distribution the topological support is always of positive Lebesgue measure. But the essential support may be of very complicated local structure. In [2] we constructed an example of an absolutely continuous distribution function such that the essential support is an everywhere dense superfractal set ($\alpha_0(N_\xi^\infty)=1$). Therefore, the condition $\alpha_0(N_\xi^\infty)>0$ does not imply the singularity of the distribution.

The following notion is very important for describing the fractal properties of probability distributions. Let A_ξ be the set of all possible supports of the distribution of the r.v. ξ, i.e.,

$$A_\xi=\{E: \ E\in\mathcal{B}, \ P_\xi(E)=1\}.$$

The number $\alpha_0(\xi)=\inf\limits_{E\in A_\xi}\{\alpha_0(E)\}$ is said to be the Hausdorff-Besicovitch dimension of the distribution of the r.v. ξ.

It is obvious that $\alpha_0(\xi)=0$ for any discrete distribution; on the other hand, $\alpha_0(\xi)=1$ for any absolutely continuous distribution. $\alpha_0(\xi)$ can be an arbitrary number from $[0;1]$ for a singular continuous distribution.

The problem of determination of the Hausdorff-Besicovitch dimension of the distribution of the random variable ξ with independent $\widetilde{Q}$-symbols is still open.

In [11] this problem is solved for some partial cases. In particular, we have the following theorem which is proven in [11].

Theorem 5. *If $p_{ik}=p_i$, $q_{ik}=q_i$ $\forall k\in N$, $i\in N_{s-1}^0$, then*

$$\alpha_0(\xi)=\frac{\sum\limits_{i=0}^{s-1}p_i\ln p_i}{\sum\limits_{i=0}^{s-1}p_i\ln q_i}. \qquad \blacksquare$$

Remark 4. Let us consider the set $M[Q,(p_0,...,p_{s-1})]$ which consists of the points whose Q-representation contains the digit "i" with the relative frequency p_i. It is known (see, e.g., [10]) that $\alpha_0(M[Q,(p_0,...,p_{s-1})])=\frac{\sum\limits_{i=0}^{s-1}p_i\ln p_i}{\sum\limits_{i=0}^{s-1}p_i\ln q_i}$. Therefore, the set $M[Q,(p_0,...,p_{s-1})]$ can be considered as the "dimensionally minimal" support of the distribution of the random variable with independent identically distributed $\widetilde{Q}$-symbols.

Acknowledgement

This work was supported by DFG 436 UKR 113/43 and DFG 436 UKR 113/53, INTAS 00-257, and SFB-611 projects. The last three named authors gratefully acknowledged the hospitality of the Institute of Applied Mathematics and of the IZKS of the University of Bonn.

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