Decompositions of singular continuous spectra of $\mathcal{H}_{-2}$ -class rank one perturbations

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The decomposition theory for the singular continuous spectrum of rank one singular perturbations is studied. A generalization of the well-known Aronszajn-Donoghue theory to the case of decompositions with respect to α-dimensional Hausdorff measures is given and a characterization of the supports of the α-singular, α-absolutely continuous, and strongly α-continuous parts of the spectral measure of $\mathcal{H}_{-2}$ - class rank one singular perturbations is given in terms of the limiting behaviour of the regularized Borel transform.

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1 Introduction

Let $A = A^*$ be a self-adjoint unbounded operator in a Hilbert space $\mathcal{H}$ with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$. Let $\{\mathcal{H}_k(A)\}_{k\in\mathbb{R}}$ denote the associated $A-$scale of Hilbert spaces and $\langle\cdot, \cdot\rangle$ the dual inner product between $\mathcal{H}_k$ and $\mathcal{H}_{-k}$.

Here we use only a part of the $A-$scale of Hilbert spaces:

$$\mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2,$$

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where $\mathcal{H}_k \equiv \mathcal{H}_k(A) = \mathcal{D}(|A|^{k/2})$, $k = 1, 2$, in the norm $\|\varphi\|_k := \|(|A| + I)^{k/2}\varphi\|$, where I stands for identity, and $\mathcal{H}_{-k} \equiv \mathcal{H}_{-k}(A)$ is the dual space ($\mathcal{H}_{-k}$ is the completion of $\mathcal{H}$ in the norm $\|f\|_{-k} := \|(|A| + I)^{-k/2} f\|$). Obviously A is bounded as a map from $\mathcal{H}_1$ to $\mathcal{H}_{-1}$ and from $\mathcal{H}$ to $\mathcal{H}_{-2}$, and therefore the expression $\langle f, Ag\rangle$ has sense for any $f, g \in \mathcal{H}_1$.

Let $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, $\|\varphi\|_{-2} = 1$, be fixed. Define a rank one (singular) perturbation (see, e.g., [1, 2, 3, 4, 9, 14]) A_λ of A, formally written as $A_\lambda = A + \lambda \langle \varphi, \cdot\rangle \varphi$, $0 \neq \lambda \in \mathbb{R} \cup \infty$ ($\infty^{-1} := 0$) by Krein's resolvent formula

$$(A_\lambda - z)^{-1} = (A - z)^{-1} - \frac{\mathbf{1}}{\lambda^{-1} + F(z)}((A - \overline{z})^{-1}\varphi, \cdot)(A - z)^{-1}\varphi, \ \ \mathrm{Im} z \neq 0. \quad (1.1)$$

Here

$$F(z) = \left\langle \varphi, \frac{1 + zA}{A - z}(A^2 + 1)^{-1}\varphi \right\rangle \quad (1.2)$$

is the regularized Borel transform of the scalar spectral measure of A associated with $\varphi \in \mathcal{H}_{-2}$.

In this paper we study the structure of the singular continuous spectrum of A_λ. We recall that the well-known Aronszajn-Donoghue theory gives a decomposition of the spectral measure of A_λ into an absolutely continuous, a singular continuous, and a pure point part in terms of the limiting behaviour of $F(\lambda + i\varepsilon)$ as $\varepsilon \to 0$. Our aim here is to give an analogue of this result for decompositions with respect to α - dimensional Hausdorff measures ($\alpha \in [0, 1)$). In particular, our results can be considered as a development of the approach proposed in [7]. Note, that in comparison with [7] we made two new steps. First, we consider the more singular case of $\mathcal{H}_{-2}$ - perturbations. Second, we give the explicit description of the decomposition of the spectral measure of A_λ with respect to α - dimensional Hausdorff measure ($\alpha \in [0, 1)$) in terms of the limiting behaviour of F. We remark that the final formulation of our results is new even for regular rank one perturbations.

2 Hausdorff measures and decomposition theory

Let us recall some basic facts of the decomposition theory, due to Rodgers and Taylor, with respect to Hausdorff measures. The detailed presentation of this theory can be found in the book [11] or the original papers [12, 13].

For any subset S of $\mathbb{R}$ and $\alpha \in [0, 1]$, the α-dimensional Hausdorff measure, h^α, is defined by

$$h^\alpha(S) := \lim_{\delta \downarrow 0} \inf_{\delta - covers} \sum_{\nu=1}^{\infty} |b_\nu|^\alpha,$$

where a δ-cover is a cover of S by a countably collection of intervals, $S \subset \bigcup_{\nu=1}^{\infty} b_\nu$, such that for each ν the length of b_ν is at most δ. Then h^α is an outer measure on $\mathbb{R}$ and its restriction to Borel subsets is a Borel measure. h^1 coincides with Lebesgue measure and h^0 is the counting measure (assigning to each set the number of points in it). Given any $\emptyset \neq S \subset \mathbb{R}$, there exist a unique $\alpha = \alpha(S) \in [0,1]$ such that $h^\alpha(S) = 0$ for any $\alpha > \alpha(S)$, and $h^\alpha(S) = \infty$ for any $\alpha < \alpha(S)$. This unique $\alpha(S)$ is called the Hausdorff dimension of S.

Definition 2.1. *Let μ be a Borel measure on $\mathbb{R}$ and let $\alpha \in [0,1]$.*

(i) μ is called α-continuous (denoted αc) if $\mu(S) = 0$ for any set S with $h^\alpha(S) = 0$.

(ii) μ is called strongly α-continuous (denoted $s\alpha c$) if $\mu(S) = 0$ for any set S which has σ-finite h^α measure.

(iii) μ is called α-singular (denoted αs) if it is supported on a set S with $h^\alpha(S) = 0$.

(iv) μ is called absolutely continuous with respect to h^α (denoted αac) if $d\mu = f(x)dh^\alpha$ for some Borel function f.

Remark 2.2. If μ is σ-finite, then it follows from the Radon-Nikodym theorem that μ is absolutely continuous with respect to h^α if and only if it is α-continuous and supported on a set of σ-finite h^α measure.

We say that a Borel measure on $\mathbb{R}$ is locally finite if it is finite on any bounded Borel set. The following unique decomposition of a locally finite Borel measure μ on $\mathbb{R}$ holds [11]

$$\mu = \mu_{\alpha s} + \mu_{\alpha ac} + \mu_{s\alpha c}, \tag{2.1}$$

where $\mu_{\alpha s}$ is α-singular, $\mu_{\alpha ac}$ is absolutely continuous with respect to h^α (on a set of σ-finite h^α measure), and $\mu_{s\alpha c}$ is strongly α-continuous. This decomposition extends the usual Lebesgue decomposition into pure point, singular continuous, and absolutely continuous parts. In particular if $\alpha = 0$, one has $\mu_{\alpha s} = 0$, and the decomposition $\mu = \mu_{\alpha ac} + \mu_{s\alpha c}$ coincides with the decomposition of μ into a pure point part and a continuous part. If $\alpha = 1$, then $\mu_{s\alpha c} = 0$, and the decomposition $\mu = \mu_{\alpha s} + \mu_{\alpha ac}$ coincides with the decomposition of μ into a singular part and an absolutely continuous part (with respect to Lebesgue measure).

One can obtain the decomposition (2.1) in the following way. Given a (locally) finite Borel measure μ and $\alpha \in [0,1]$, we define the upper α - derivative of μ by

$$D_\mu^\alpha(x) := \limsup_{\varepsilon \to 0} \frac{\mu(x-\varepsilon, x+\varepsilon)}{(2\varepsilon)^\alpha}.$$

Set

$$\begin{aligned} T_0 := T_0(\alpha,\mu) &:= \{x : D_\mu^\alpha(x) = 0\}, \\ T_+ := T_+(\alpha,\mu) &:= \{x : 0 < D_\mu^\alpha(x) < \infty\}, \\ T_\infty := T_\infty(\alpha,\mu) &:= \{x : D_\mu^\alpha(x) = \infty\}. \end{aligned}$$

Theorem 2.3. *(Rodgers and Taylor [11, 12])* T_0, T_+, T_∞ *are disjoint Borel sets, and*
(i) $h^\alpha(T_\infty) = 0$,
(ii) T_+ *has* σ*-finite* h^α *measure,*
(iii) $\mu(S \cap T_+) = 0$ *for any* S *with* $h^\alpha(S) = 0$,
(iv) $\mu(S \cap T_0) = 0$ *for any* S *which has* σ*-finite* h^α *measure.*

Corollary 2.4. *Set*

$$\mu_{\alpha s}(\Delta) = \mu(\Delta \cap T_\infty), \mu_{\alpha ac}(\Delta) = \mu(\Delta \cap T_+), \mu_{s\alpha c}(\Delta) = \mu(\Delta \cap T_0). \tag{2.2}$$

Then the formula (2.2) gives the decomposition (2.1) of the measure μ.

We shall say that a Borel measure μ is supported by a Borel set T (or T is a (not necessarily closed) support of μ) if $\mu(\mathbb{R} \setminus T) = 0$. It follows that T_∞, T_+, T_0 are the supports of $\mu_{\alpha s}, \mu_{\alpha ac}, \mu_{s\alpha c}$, respectively.

3 Regularized Borel Transform

In this section we generalize the well-known connection between the decomposition (2.1) of measures μ and the limiting behaviour of their Borel transforms. For a measure μ with

$$\int (1 + |x|)^{-1} d\mu(x) < \infty, \tag{3.1}$$

one can define its Borel transform by

$$\Phi_\mu(z) = \int \frac{1}{x - z} d\mu(x), \ \mathrm{Im} z > 0, \tag{3.2}$$

The properties of $\Phi_\mu(z)$ discussed in details in [14]. Here we consider the more general situation of a Borel measure μ on $\mathbb{R}$ satisfying

$$\int (1 + x^2)^{-1} d\mu(x) < \infty. \tag{3.3}$$

In this case we define its regularized Borel transform by

$$F_\mu(z) = \int (\frac{1}{x - z} - \frac{x}{1 + x^2}) d\mu(x), \ \mathrm{Im} z > 0. \tag{3.4}$$

Note that the generalized Borel transform plays a crucial role in the $\mathcal{H}_{-2}$ rank one perturbation theory (see [3, 4, 1]). Fix $\gamma \leq 1$ and $x \in \mathbb{R}$. Let

$$Q_\mu^\gamma(x) := \limsup_{\varepsilon \downarrow 0} \varepsilon^\gamma \mathrm{Im} F_\mu(x + i\varepsilon),$$

$$R_\mu^\gamma(x) := \limsup_{\varepsilon \downarrow 0} \varepsilon^\gamma |F_\mu(x + i\varepsilon)|,$$

Theorem 3.1. *Fix $x_0 \in \mathbb{R}$ and $\alpha \in [0,1)$ and let $\gamma := 1-\alpha$. Then $D^{\alpha}_{\mu}(x_0)$, $Q^{\gamma}_{\mu}(x_0)$, and $R^{\gamma}_{\mu}(x_0)$ are either all infinite, all zero, or all belonging to $(0,\infty)$.*

Remark 3.2. In the case of the measure μ satisfying (3.1) Theorem 3.1 is proved in [7].

Remark 3.3. The relation between $D^{\alpha}_{\mu}(x_0)$ and $Q^{\gamma}_{\mu}(x_0)$ extends to the range $\alpha \in [1,2)$.

Proof. Consider

$$\mathrm{Im} F(x_0 + i\varepsilon) = \varepsilon \int \frac{d\mu(x)}{(x-x_0)^2+\varepsilon^2}.$$

For any $\delta > 0$ we have

$$\int\limits_{|x-x_0|\geq\delta} \frac{d\mu(x)}{(x-x_0)^2+\varepsilon^2} \leq C(\delta) < \infty$$

where $C(\delta)$ does not depend on ε. Therefore for $\gamma > -1$

$$Q^{\gamma}_{\mu}(x) = \limsup_{\varepsilon\downarrow 0} \varepsilon^{\gamma+1} \int\limits_{|x-x_0|<\delta} \frac{d\mu(x)}{(x-x_0)^2+\varepsilon^2}. \tag{3.5}$$

Let $I \subset \mathbb{R}$ be an open bounded interval containing x_0. By (3.5) one can replace in the definition of $Q^{\gamma}_{\mu}(x_0)$ the measure μ by its restriction $\mu_I := \mu \restriction I$. The same arguments show that $R^{\gamma}_{\mu}(x_0) = R^{\gamma}_{\mu_I}(x_0)$ for any $\gamma > 0$. Therefore it is sufficient to consider the case of compactly supported measures μ and apply the results of [7] (see Remark 3.2). □

4 Decompositions of singular spectra

In this section we prove our main result (see Theorem 4.1 below). It generalizes the well-known Aronszajn-Donoghue theory to the case of decompositions (2.1) with respect to Hausdorff measures.

Let $E(\cdot)$ be the operator spectral measure (the resolution of the identity) of a self-adjoint operator A, and let $\mu(\Delta) \equiv \mu_{\varphi}(\Delta) = (\varphi, E(\Delta)\varphi)$ denote the scalar spectral measure of A associated with $\varphi \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, $\|\varphi\|_{-2} = 1$. This measure is not finite as $\varphi \notin \mathcal{H}$. One can introduce a regularization of this measure by

$$d\mu^{\mathrm{reg}}(x) := \frac{d\mu(x)}{1+x^2},$$

so that $\mu^{\mathrm{reg}}(\mathbb{R}) = \int d\mu^{\mathrm{reg}}(x) = 1$. Clearly the measures μ and μ^{reg} are equivalent.

Let a rank one (singular) perturbation A_λ of A, formally written as $A_\lambda = A + \lambda \langle \varphi, \cdot \rangle \varphi$, defined by (1.1). Consider the operator spectral measure $E_{\lambda}(\cdot)$ for A_λ. Similarly to above constructions one can introduce

$$\mu_\lambda^{\text{reg}}(\Delta) := ((A+i)^{-1}\varphi, E_{\lambda}(\Delta)(A+i)^{-1}\varphi),$$

and

$$d\mu_\lambda(x) := (1+x^2)d\mu_\lambda^{\text{reg}}(x).$$

Henceforth, we shall assume that φ is a cyclic vector for A, i.e. $\{(A-z)^{-1}\varphi \mid \text{Im}z \neq 0\}$ is a total set for $\mathcal{H}$. (In general, if $\mathcal{H}_\varphi$ denotes the closed subspace in $\mathcal{H}$ generated by the vectors from this set, then $\mathcal{H}_\varphi$ is an invariant subspace for each A_λ and $A_\lambda = A$ on the orthogonal complement to $\mathcal{H}_\varphi$. Thus the extension from the cyclic to the general case is trivial.) It is easy to see that $(A+i)^{-1}\varphi$ is a cyclic vector for A_λ (cf. [8]) and μ_λ is equivalent to the spectral measure $E_\lambda(\cdot)$. In the following we shall say that μ_λ is the scalar spectral measure of A_λ associated with φ.

Let F_λ be the regularized Borel transform of μ_λ (cf. (3.4))

$$F_\lambda(z) := \int (\frac{1}{x-z} - \frac{x}{1+x^2})d\mu_\lambda(x) = \int (\frac{1+zx}{x-z})d\mu_\lambda^{\text{reg}}(x)$$

and $F(z)$ be the regularized Borel transform of μ. By the Aronszajn-Krein formula (see, [1], [8], [9])

$$F_\lambda(z) = \frac{F(z)-\lambda}{1+\lambda F(z)},\ \text{Im}z > 0,\ 0 \neq \lambda \in \mathbb{R}, \tag{4.1}$$

and

$$F_\infty(z) = -\frac{1}{F(z)},\ \text{Im}z > 0. \tag{4.2}$$

In particular

$$\text{Im}F_\lambda(z) = (1+\lambda^{-2})\frac{\text{Im}F(z)}{|\lambda^{-1}+F(z)|^2},\ \text{Im}z > 0. \tag{4.3}$$

Recall that the absolutely continuous part of μ_λ is supported by

$$L := \{x \in \mathbb{R} : 0 < \text{Im}F(x+i0) < \infty\}, \tag{4.4}$$

(see, [1], [8], [14]) and the singular part $\mu_{\lambda,s}$ ($\lambda = \infty$ allowed with $\infty^{-1} = 0$) is supported by

$$S_\lambda := \left\{x \in \mathbb{R} : F(x+i0) = -\lambda^{-1}\right\}. \tag{4.5}$$

Moreover the set of eigenvalues of A_λ coincides with the set

$$P_\lambda := \left\{x \in \mathbb{R} : F(x+i0) = -\lambda^{-1}, H(x) < \infty\right\}. \tag{4.6}$$

Here

$$H(x) := \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}\text{Im}F(x+i\varepsilon) = \int \frac{d\mu(y)}{(x-y)^2}. \tag{4.7}$$

Note that if $H(x) < \infty$, then there exist the real limits $\lim_{\varepsilon\downarrow 0} F(x+i\varepsilon)$ and

$$\lim_{\varepsilon\downarrow 0}(i\varepsilon)^{-1}[F(x+i\varepsilon) - F(x+i0)] = H(x) \tag{4.8}$$

exist ([8, 14]). Let $\alpha \in [0,1]$. For $\lambda \neq 0$ define the following sets

$$S_{\alpha,\lambda} := \{x \in \mathbb{R} : \liminf_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)}|F(x+i\varepsilon) + \lambda^{-1}| = 0,\ F(x+i0) = -\lambda^{-1}\}, \tag{4.9}$$

$$P_{\alpha,\lambda} := \{x \in \mathbb{R} : 0 < \liminf_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)}|F(x+i\varepsilon) + \lambda^{-1}| < \infty,\ F(x+i0) = -\lambda^{-1}\}, \tag{4.10}$$

$$J_{\alpha,\lambda} := \{x \in \mathbb{R} : \lim_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)}|F(x+i\varepsilon) + \lambda^{-1}| = \infty\}, \tag{4.11}$$

Theorem 4.1. *Let $\alpha \in [0,1)$, $0 \neq \lambda \in \mathbb{R}\cup\infty$. Suppose that $\varphi \in \mathcal{H}_{-2}\setminus\mathcal{H}_{-1}$ and let μ_λ be the scalar spectral measure of A_λ associated with φ . Then the decomposition (2.1) for the measure μ_λ has the form*

$$\mu_\lambda = \mu_{\lambda,\alpha s} + \mu_{\lambda,\alpha ac} + \mu_{\lambda,s\alpha c}, \tag{4.12}$$

where for any Borel set $\Delta \subset \mathbb{R}$

$$\mu_{\lambda,\alpha s}(\Delta) = \mu(\Delta\cap S_{\alpha,\lambda}),\ \mu_{\lambda,\alpha ac}(\Delta) = \mu(\Delta\cap P_{\alpha,\lambda}),\ \mu_{\lambda,s\alpha c}(\Delta) = \mu(\Delta\cap J_{\alpha,\lambda}). \tag{4.13}$$

Remark 4.2. If $\alpha = 0$, $\mu_{\lambda,\alpha s} = 0$, and the decomposition $\mu_\lambda = \mu_{\lambda,\alpha ac} + \mu_{\lambda,s\alpha c}$ coincides with the decomposition of μ_λ into a pure point part and a continuous part. In particular, the set of eigenvalues of A_λ coincides (see (4.6) - (4.8)) with the set

$$P_{0,\lambda} = \{x \in \mathbb{R} : 0 < \liminf_{\varepsilon\downarrow 0} \varepsilon^{-1}|F(x+i\varepsilon) + \lambda^{-1}| < \infty,\ F(x+i0) = -\lambda^{-1}\} = P_\lambda.$$

Moreover, $S_{0,\lambda} = \emptyset$ and $(L\cup S_\lambda)\setminus P_\lambda \subset J_{0,\lambda}$. In particular, $J_{0,\lambda}$ supports the continuous part of μ_λ. If $\alpha = 1$, then $\mu_{\lambda,s\alpha c} = 0$, and the decomposition $\mu_\lambda = \mu_{\lambda,\alpha s}+\mu_{\lambda,\alpha ac}$ coincides with the decomposition of μ_λ into a singular part and an absolutely continuous part (with respect to Lebesgue measure). In particular (see (4.5)) the singular part of μ_λ is supported by $S_{1,\lambda} = S_\lambda$ and one has $\mu_\lambda(J_{1,\lambda}) = 0$. Note that in this case the absolutely continuous part of μ_λ is supported by $L \neq P_{1,\lambda}$ and the result of the theorem is not valid for $\alpha = 1$.

Remark 4.3. The result of Theorem 4.1 remains true for rank one regular and $\mathcal{H}_{-1}$ perturbations. One need only to replace the regularized Borel transform $F = F_\mu$ by the standard Borel transform Φ_μ (3.2) (cf. [1]).

Proof. We remark that the sets $S_{\alpha,\lambda}$, $P_{\alpha,\lambda}$, $J_{\alpha,\lambda}$, are mutually disjoint (for fixed α) and

$$L \cup S_\lambda \subset S_{\alpha,\lambda} \cup P_{\alpha,\lambda} \cup J_{\alpha,\lambda}, \ \alpha \in [0,1).$$

By Aronszajn-Donoghue theory the measure μ_λ is supported by $L \cup S_\lambda$, and therefore it is supported by $S_{\alpha,\lambda} \cup P_{\alpha,\lambda} \cup J_{\alpha,\lambda}$. Suppose that $x \in S_{\alpha,\lambda}$. Then

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{(1-\alpha)}|F_\lambda(x+i\varepsilon)| = \limsup_{\varepsilon\downarrow 0} \frac{|\lambda^{-1}F(x+i\varepsilon)-1|}{\varepsilon^{-(1-\alpha)}|F(x+i\varepsilon)+\lambda^{-1}|} = \frac{(1+\lambda^{-2})}{\liminf_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)}|F(x+i\varepsilon)+\lambda^{-1}|} = \infty.$$

By Theorem 3.1 it follows that $D^{\alpha}_{\mu_\lambda}(x) = \infty$ and $x \in T_\infty(\alpha, \mu_\lambda)$. Therefore we can apply Theorem 2.3 and Corollary 2.4 to prove that the restriction of μ_λ on $S_{\alpha,\lambda}$ is α - singular with respect to h^α. Analogously for $x \in P_{\alpha,\lambda}$

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{(1-\alpha)}|F_\lambda(x+i\varepsilon)| \in (0,\infty)$$

and therefore $\mu_\lambda \restriction P_{\alpha,\lambda}$ is absolutely continuous with respect to h^α. At last for $x \in J_{\alpha,\lambda}$ we have (see (4.3))

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{(1-\alpha)}\mathrm{Im}F_\lambda(x+i\varepsilon) = \limsup_{\varepsilon\downarrow 0} \frac{(1+\lambda^{-2})\mathrm{Im}F(x+i\varepsilon)}{\varepsilon^{-(1-\alpha)}|F(x+i\varepsilon)+\lambda^{-1}|^2} \leq \frac{(1+\lambda^{-2})}{\liminf_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)}|F(x+i\varepsilon)+\lambda^{-1}|} = 0$$

and $\mu_\lambda \restriction J_{\alpha,\lambda}$ is strongly α-continuous. □

We will end by formulating several corollaries, which contain, as particular cases, the known results for regular rank one perturbations (see [7]).

Corollary 4.4. *Let $\alpha \in [0,1]$, $\lambda \neq 0$. Set $R_\alpha = \{x \in \mathbb{R} : \liminf_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)}\mathrm{Im}F(x+i\varepsilon) > 0\}$. Then $\mu_\lambda \restriction R_\alpha$ is α-continuous (i.e., gives zero weight to sets of zero h^α-measure).*

Proof. First of all we note that for $\alpha = 1$ the result directly follows from the Aronszajn-Donoghue theory (see (4.4)). For $\alpha \in [0,1)$ it is clear that $R_\alpha \subset P_{\alpha,\lambda} \cup J_{\alpha,\lambda}$, ($\lambda \neq 0$). This inclusion proves the assertion of the corollary. □

Corollary 4.5. *Let $\alpha \in [0,1]$, $\lambda \neq 0$. Set $R_{\alpha,\infty} = \{x \in \mathbb{R} : \lim_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)}\mathrm{Im}F(x+i\varepsilon) = \infty\}$. Then $\mu_\lambda \restriction R_{\alpha,\infty}$ is strongly α-continuous (i.e., gives zero weight to h^α-sigma-finite sets).*

Proof. The case $\alpha = 1$ is clear and for $\alpha \in [0, 1)$ we have $R_{\alpha,\infty} \subset J_{\alpha,\lambda}$. □

Corollary 4.6. *Let $\alpha \in [0, 1)$, $\lambda \neq 0$. Suppose that μ is purely singular. Set $Q_\alpha = \{x \in \mathbb{R} : \limsup_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)} \mathrm{Im}F(x + i\varepsilon) < \infty\}$. Then $\mu_\lambda \restriction Q_\alpha$ is supported on an h^α-sigma-finite set.*

Proof. As μ is pure singular, μ_λ is supported by S_λ (see (4.5)). Suppose that $x \in Q_\alpha \cap S_\lambda$. Then (see the proof of Theorem 4.2 of [7]) for some $C > 0$

$$|1 + \lambda \mathrm{Re}F(x + i\varepsilon)| \leq C\varepsilon^{(1-\alpha)}$$

and

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)} |F(x + i\varepsilon) + \lambda^{-1}| < \infty.$$

In particular $Q_\alpha \cap S_\lambda \subset S_{\alpha,\lambda} \cup P_{\alpha,\lambda}$. It follows that $\mu_{\lambda,s\alpha c}(Q_\alpha) = 0$ and the assertion of the corollary is proven. □

Corollary 4.7. *Let $\alpha \in [0, 1)$, $\lambda \neq 0$. Suppose that μ is purely singular. Set $Q_{\alpha,0} = \{x \in \mathbb{R} : \lim_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)} \mathrm{Im}F(x + i\varepsilon) = 0\}$. Then $\mu_\lambda \restriction Q_{\alpha,0}$ is α-singular.*

Proof. By a variant of the proof of Corollary 4.6 one shows that that for $x \in Q_{\alpha,0} \cap S_\lambda$

$$\limsup_{\varepsilon\downarrow 0} \varepsilon^{-(1-\alpha)} |F(x + i\varepsilon) + \lambda^{-1}| = 0.$$

In particular one has $Q_{\alpha,0} \cap S_\lambda \subset S_{\alpha,\lambda}$ and the result directly follows from Theorem 4.1. □

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