The rigged Hilbert spaces approach in singular perturbation theory

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Abstract

We discuss a new approach in singular perturbation theory which is based on the method of rigged Hilbert spaces. Let A be a self-adjoint unbounded operator in a state space $\mathcal{H}_0$ and $\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+$ be the rigged Hilbert space associated with A in the sense that $\mathrm{dom} A = \mathcal{H}_+$ in the graph-norm. We propose to define the perturbed operator $\tilde{A}$ as the self-adjoint operator uniquely associated with a new rigged Hilbert space $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+$ constructed using a given perturbation of A. We show that the well-known form-sum and self-adjoint extensions methods are included in the above construction. Moreover, we show that the super singular perturbations may also be described in our framework.

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1 Introduction

Let $A = A^* \geq 1$ be an unbounded self-adjoint operator in a Hilbert space $\mathcal{H}_0$ with the inner product $(\cdot,\cdot)_0$. And let

$$\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+ \tag{1.1}$$

be the rigged Hilbert space associated with A in the sense that the domain $\mathrm{Dom}A = \mathcal{H}_+$ in the graph-norm. Here the symbol $\sqsupset$ means dense and continuous embedding. We note that a given pre-rigged pair $\mathcal{H}_0 \sqsupset \mathcal{H}_+$ the Hilbert space $\mathcal{H}_-$ is uniquely defined as the conjugate space to $\mathcal{H}_+$ with respect to $\mathcal{H}_0$ (for details see [8, 9]).

Besides the triplet (1.1) we will use also the chain of five spaces

$$\mathcal{H}_- \sqsupset \mathcal{H}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_1 \sqsupset \mathcal{H}_+, \tag{1.2}$$

where $\mathcal{H}_1 = \mathrm{Dom}A^{1/2}$, and $\mathcal{H}_{-1}$ is the completion of $\mathcal{H}_0$ in the norm $\|\cdot\|_{-1} = \|A^{-1/2}\cdot\|$.

Given $A = A^*$ another self-adjoint operator $\tilde{A}$ in $\mathcal{H}_0$ is said to be a purely singular perturbation of A if the set

$$\mathcal{D} := \{f \in \mathrm{Dom}A \cap \mathrm{Dom}\tilde{A} \ : \ Af = \tilde{A}f\} \text{ is dense in } \mathcal{H}_0 \tag{1.3}$$

(see [3, 5, 15]-[17], [20]-[30]). Under condition (1.3) we write $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$ if $\tilde{A}$ is bounded from below. We write $\tilde{A} \in \mathcal{P}_{\mathrm{ws}}(A)$ if $\mathrm{Dom}A^{1/2} = \mathrm{Dom}\tilde{A}^{1/2}$ (ws means weakly singular, i.e., a perturbation belongs to the $\mathcal{H}_{-1}$-class), and $\tilde{A} \in \mathcal{P}_{\mathrm{ss}}(A)$ if the set $\mathcal{D}$ is dense in $\mathcal{H}_1$ (ss stands for strongly singular, i.e., a perturbation belongs to the $\mathcal{H}_{-2}-$ class). Thus $\mathcal{P}_{\mathrm{s}}(A) = \mathcal{P}_{\mathrm{ws}}(A) \cup \mathcal{P}_{\mathrm{ss}}(A)$.

It is clear that for each $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$ there exists a densely defined symmetric operator

$$\mathring{A} := A|\mathcal{D} = \tilde{A}|\mathcal{D}$$

with non-trivial deficiency indices $\mathbf{n}^{\pm}(\mathring{A}) = \dim\ker(\mathring{A} \mp z)^* \neq 0$, $\mathrm{Im}z \neq 0$. Therefore each $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$ may be defined as a self-adjoint extension of $\mathring{A}$, different from A. In singular perturbation theory each $\tilde{A}$ is fixed by some abstract boundary condition, which corresponds to a singular perturbation. In turn a singular perturbation is usually presented by a singular quadratic form γ given in the rigged Hilbert space (1.1).

In the present paper we propose to use a singular quadratic form γ (corresponding to a perturbation) for the construction of a new chain of Hilbert spaces similar to (1.2),

$$\tilde{\mathcal{H}}_- \sqsupset \tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_1 \sqsupset \tilde{\mathcal{H}}_+, \tag{1.4}$$

and then to define the perturbed operator $\tilde{A}$ as an operator associated with this new rigging (1.4).

In the paper, see below Theorem 5.1, Theorem 5.2, Theorem 6.1, and Theorem 7.1 we establish a one-to-one correspondence between three families of objects: singular perturbations $\tilde{A} \in \mathcal{P}_{\rm ss}(A)$, rigged Hilbert spaces of the form (1.4), and singular quadratic forms γ with fixed properties. We extend this one-to-one correspondences to a more general set of objects involving super singular perturbations.

2 Singular quadratic forms in A-scales

Let $A \geq 1$ be a self-adjoint unbounded operator in a separable Hilbert space $\mathcal{H}_0$ which is equipped in such a way that the domain $\mathrm{Dom}A = \mathcal{H}_+$ in the norm $\|\cdot\|_+ := \|A\cdot\|$ (see (1.1)).

In the paper we discuss a new construction of singularly perturbed operator $\tilde{A}$ in $\mathcal{H}_0$. Namely, we define $\tilde{A}$ as the operator associated with a new rigged Hilbert space $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+$, where $\tilde{\mathcal{H}}_+ = \mathcal{D}(\tilde{A})$. The inner product $(\cdot,\cdot)^\sim_+$ in $\tilde{\mathcal{H}}_+$ is defined as a perturbation of the inner product $(\cdot,\cdot)_+$ in $\mathcal{H}_+$. Formally one can write $(\cdot,\cdot)^\sim_+ = (\cdot,\cdot)_+ + \gamma(\cdot,\cdot)$, where the form γ corresponds to a singular perturbation. Respectively the space $\tilde{\mathcal{H}}_-$ is the completion of $\mathcal{H}_0$ in the inner product of a form $(\cdot,\cdot)^\sim_- = (\cdot,\cdot)_- + \tau(\cdot,\cdot)$, where $(\cdot,\cdot)_-$ denotes the inner product in $\mathcal{H}_-$ and $\tau(\cdot,\cdot)$ stands for the symmetric singular quadratic form which is defined by γ (see below). The construction of $\tilde{\mathcal{H}}_-$ and $\tilde{\mathcal{H}}_+$ by a given singular perturbation γ is one of the main problem which we solve in the paper.

We show also that our method includes the usual well-known approaches in the singular perturbations theory [2, 6, 25].

We start with recalling standard constructions connected with the rigged Hilbert spaces [8, 9] (see also [1]) and some definitions concerning the singular perturbation theory [2, 6, 7, 12, 19, 30] and singular quadratic forms [3, 13]-[20, 22, 23, 25, 27].

We remind that given $A = A^* \geq 1$ the domain $\mathrm{Dom}A \equiv \mathcal{H}_+$ is a complete Hilbert space with respect to the inner product $(\cdot,\cdot)_+ := (A\cdot, A\cdot)_0$. Let $\mathcal{H}_-$ be the space conjugate to $\mathcal{H}_+$ with respect to $\mathcal{H}_0$. Then we get the triplet of continuously and densely imbedding of spaces

$$\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+, \tag{2.1}$$

called the rigged Hilbert space associated with A.

In the same way one can construct the A-scale of Hilbert spaces

$$\cdots \sqsupset \mathcal{H}_{-k} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_k \sqsupset \cdots, \quad k \geq 0, \tag{2.2}$$

where $\mathcal{H}_k \equiv \mathcal{H}_k(A) = \mathrm{Dom} A^{k/2}$ in the inner product $(\cdot,\cdot)_k := (A^{k/2}\cdot, A^{k/2}\cdot)_0$. So $(\cdot,\cdot)_2 = (\cdot,\cdot)_+$ and $(\cdot,\cdot)_{-2} = (\cdot,\cdot)_-$. Let $D_{-k,k} : \mathcal{H}_k \to \mathcal{H}_{-k}$ denote the canonical identification operator,

$$\langle D_{-k,k}\varphi, \psi\rangle_{-k,k} = (\varphi,\psi)_k, \;\; \varphi,\psi \in \mathcal{H}_k,$$

where $\langle\cdot,\cdot\rangle_{-k,k}$ stands for the dual inner product between $\mathcal{H}_{-k}$ and $\mathcal{H}_k$. Using the invariance property of the scale (2.2) with respect to the shift one can easily construct the canonical identification operator $D_{l,k} : \mathcal{H}_k \to \mathcal{H}_l$ for a couple of spaces $\mathcal{H}_k, \;\; \mathcal{H}_l, \;\; k,l \in \mathbb{R}$.

We write $I_{k,l}$ for $D^{-1}_{l,k}$. Clearly, $I_{k,l}$ is the unitary operator mapping $\mathcal{H}_l$ onto $\mathcal{H}_k$.

Theorem 2.1. *In the above notations the following mappings define the same operator $A^{k/2}$, $k > 0$ in $\mathcal{H}_0$:*

(a) $D_{0,k} \equiv A^{k/2}$,

(b) $D_{-k/2,k/2} \mid \{f \in \mathcal{H}_{k/2} \mid D_{-k/2,k/2} f \in \mathcal{H}_0\} \equiv \mathrm{Dom} A^{k/2}$,

(c) $D_{-k,0} \mid \{f \in \mathcal{H}_0 \mid D_{-k,0} f \in \mathcal{H}_0\} \equiv \mathrm{Dom} A^{k/2}$,

(d) $A^{-k/2} = I_{0,-k} \mid \{\omega \in \mathcal{H}_{-k} \mid I_{0,-k}\omega \in \mathcal{H}_k\} \equiv \mathcal{H}_0, \quad I_{0,-k} := D^{-1}_{-k,0} : \mathcal{H}_{-k} \to \mathcal{H}_0$

In particular,

$$D_{0,2} = A = D_{-1,1}|\{f \in \mathcal{H}_1 \mid D_{-1,1} f \in \mathcal{H}_0\} = D_{-2,0}|\{f \in \mathcal{H}_0 \mid D_{-2,0} f \in \mathcal{H}_0\}$$

and

$$A^{-1} = I_{0,-2}|\{\omega \in \mathcal{H}_{-2} \mid I_{0,-2}\omega \in \mathcal{H}_0\}.$$

In what follows we will use the notation $\mathbf{A} := D_{-2,0}$, which is the closure of the operator A as a mapping from $\mathcal{H}_0$ to $\mathcal{H}_{-2}$.

We remind, for example, that the well-known Sobolev scale of spaces

$$W_2^{-k}(\mathbb{R}^d) \sqsupset L_2(\mathbb{R}^d) \sqsupset W_2^k(\mathbb{R}^d),\ k > 0$$

is associated with the operator $A = -\Delta + 1$, where Δ denotes the Laplacian on $\mathbb{R}^d$. In particular, for $k = 2$ the canonical identification operator $D_{0,2} : W_2^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ exactly coincides with $-\Delta + 1$ if the norm in $W_2^k(\mathbb{R}^d), k > 1$ is defined as $\|\varphi\|_k := \|(-\Delta + 1)^{k/2}\varphi\|_{L_2}$.

To develop a new point of view about the construction of singularly perturbed operators by the method of the rigged Hilbert spaces we need recall the additional definitions on singular quadratic forms and operators in the A-scale of spaces (for more details see [13, 21, 22, 24]).

A positive quadratic form γ in an abstract Hilbert space $\mathcal{H}$ is said to be singular if it is nowhere closable. Precisely this means that

$$\forall\varphi \in \mathcal{H},\ \exists\ \varphi_n\ \in \mathrm{Dom}\gamma \quad \text{such that } \varphi_n \to \varphi \text{ in } \mathcal{H} \text{ and } \gamma[\varphi_n] \to 0, \tag{2.3}$$

where $\gamma[\varphi] = \gamma(\varphi, \varphi)$. Obviously a form γ is singular in $\mathcal{H}$ if the set

$$\mathrm{Ker}\gamma := \{\varphi \in \mathrm{Dom}\gamma \mid \gamma[\varphi] = 0\} \text{ is dense in } \mathcal{H}. \tag{2.4}$$

In other words, (2.4) gives a simple sufficient condition for the singularity of a positive quadratic form in $\mathcal{H}$.

We say that a symmetric not necessarily positive quadratic form γ is singular in $\mathcal{H}$ if (2.4) holds.

In the same way one can introduce a notion of singular operator. A linear densely defined operator S is said to be singular in $\mathcal{H}$ if

$$\forall f \in \mathcal{H},\ \ \exists f_n \in \mathrm{Dom}S \quad \text{such that } \ f_n \to f \text{ and } Sf_n \to 0 \text{ in } \mathcal{H}.$$

In what follows we use operators S acting from $\mathcal{H}_k$ to $\mathcal{H}_{-k},\ k \geq 1$, such that $\mathrm{Ker}S \sqsubset \mathcal{H}_0$. Therefore these S are singular in $\mathcal{H}_0$.

We say that a Hermitian form γ is regular in $\mathcal{H}$, if it is bounded from below and closed. Each regular quadratic form is associated with a lower semi-bounded self-adjoint operator [18]. This connections may be extended to the wide class of singular quadratic forms and operators considered in the A-scale (2.2).

For a densely defined symmetric quadratic form γ in $\mathcal{H}_0$ we say that γ belongs to the $\mathcal{H}_{-k}-$class with some fixed $k \geq 1$ if two conditions are fulfilled:

(1) γ is bounded on $\mathcal{H}_k$, $\mathrm{Dom}\gamma = \mathcal{H}_k$,

(2) γ is singular in $\mathcal{H}_{k-1}$, $\mathrm{Ker}\gamma \sqsubset \mathcal{H}_{k-1}$.

Directly from this definition we obtain the following result.

Theorem 2.2. *Each quadratic form γ of the $\mathcal{H}_{-k}-$class (γ is singular in $\mathcal{H}_0$!) admits the operator representation:*

$$\gamma(\varphi,\psi) = \langle S\varphi,\psi\rangle_{-k,k}, \;\; \varphi,\psi \in \; \mathrm{Dom} S = \mathcal{H}_k, \tag{2.5}$$

where the associated operator $S : \mathcal{H}_k \to \mathcal{H}_{-k}$ may be written in the form: $S = \mathbf{A}^k \mathbf{s}$, where $\mathbf{s}$ denotes a bounded self-adjoint operator in $\mathcal{H}_k$ such that

$$\mathrm{Ker}\ \mathbf{s} = \mathrm{Ker}\ S = \mathrm{Ker}\ \gamma \sqsubset \mathcal{H}_{k-1}.$$

Example 2.1. *Rank one singular quadratic forms.*

Consider in (2.1) a fixed vector $\omega \in \mathcal{H}_-\backslash\mathcal{H}_0$ and define the operator S acting from $\mathcal{H}_+$ to $\mathcal{H}_-$ according to

$$S\varphi = \langle\varphi,\omega\rangle_{+,-}\omega, \;\;\; \varphi \in \mathcal{H}_+ = \mathrm{Dom} S.$$

Clearly S is a singular rank one operator in $\mathcal{H}_0$ since the set

$$\mathrm{Ker} S = \{\varphi \in \mathcal{H}_+ \;\; | \;\; \langle\varphi,\omega\rangle_{+,-} = 0\}$$

is dense in $\mathcal{H}_0$ due to $\omega \notin \mathcal{H}_0$. The quadratic form associated with this operator S has the form:

$$\gamma_\omega(\varphi,\psi) := \langle\varphi,\omega\rangle_{+,-}\langle\omega,\psi\rangle_{-,+} = \langle S\varphi,\psi\rangle_{-,+} = \langle \mathbf{A}^2\mathbf{s}\varphi,\psi\rangle_{-,+} = (\mathbf{s}\varphi,\psi)_+,$$

where the rank one operator $\mathbf{s}$ acts in $\mathcal{H}_+$ as follows,

$$\mathbf{s}\varphi = (\varphi,\eta_+)_+\eta_+, \text{ with } \eta_+ := \mathbf{A}^{-2}\omega.$$

Clearly, that γ_ω belongs to the $\mathcal{H}_{-2}-$class, if $\omega \in \mathcal{H}_-\backslash\mathcal{H}_{-1}$, since then $\mathrm{Ker}\gamma_\omega$ is dense in $\mathcal{H}_1$, and $\gamma_\omega \in \mathcal{H}_{-1}-$class, if $\omega \in \mathcal{H}_{-1}\backslash\mathcal{H}_0$.

In the more general case where $\omega \in \mathcal{H}_{-k}\backslash\mathcal{H}_{-k+1}, \; k > 2$ the singular quadratic form

$$\gamma_\omega(\varphi,\psi) := \langle\varphi,\omega\rangle_{k,-k}\langle\omega,\psi\rangle_{-k,k}, \;\; \varphi,\psi \in \mathcal{H}_{-k}$$

has a similar representation:

$$\gamma_\omega(\varphi,\psi) = \langle S\varphi,\psi\rangle_{-k,k} = \langle \mathbf{A}^k\mathbf{s}\varphi,\psi\rangle_{-k,k} = (\mathbf{s}\varphi,\psi)_k.$$

Here $\mathbf{s}\varphi = (\varphi,\eta_k)_k\eta_k$, with $\eta_k := \mathbf{A}^{-k}\omega$. Now the form γ_ω belongs to the $\mathcal{H}_{-k}$-class since $\omega \notin \mathcal{H}_{-k+1}$ and therefore the set $\mathrm{Ker}\gamma_\omega$ is dense in $\mathcal{H}_{k-1}$.

Example 2.2. *Finite rank singular quadratic forms.*

Let the vectors $h_i \in \mathcal{H}_0,\ i = 1, ..., n < \infty$ be orthogonal and satisfy the condition:

$$\mathrm{span}\{h_i\} \cap \mathrm{Dom}A = \{0\}.$$

Then the operator S of rank n defined as follows

$$Sf = \sum_{i=1}^{n} (Af, h_i)_0 \mathbf{A} h_i = \sum_{i=1}^{n} \langle f, \omega_i \rangle_{+,-} \omega_i,\ \ f \in \mathcal{H}_+ = \mathrm{Dom}S,\ \ \omega_i := \mathbf{A} h_i$$

is singular in $\mathcal{H}_0$ since $\mathrm{Ker}S$ is obviously dense in $\mathcal{H}_0$. The quadratic form $\gamma[f] := \langle Sf, f \rangle_{-,+}$ belongs to the $\mathcal{H}_{-2}$-class if $\mathrm{span}\{h_i\} \cap \mathrm{Dom}A^{1/2} = \{0\}$. However if all $h_i \in \mathrm{Dom}A^{1/2}$ then this form belongs to the $\mathcal{H}_{-1}$-class.

In the general case we have (cf. with [3, 13]) the following result.

Theorem 2.3. *Let γ be a Hermitian bounded quadratic form in $\mathcal{H}_k, k > 1$. Set*

$$\mathcal{M}_k := \mathrm{Ker}\gamma \text{ and } \mathcal{N}_k = \mathcal{H}_k \ominus \mathcal{M}_k.$$

Then $\gamma \in \mathcal{H}_{-k}-$class iff

$$\mathcal{N}_{-k} \cap \mathcal{H}_{-k+1} = \{0\}, \quad \text{where} \ \ \mathcal{N}_{-k} := \mathbf{A}^k \mathcal{N}_k.$$

Proof. This follows from Theorem A1 (see [3]) since

$$\mathcal{M}_k \sqsubset \mathcal{H}_{k-1} \Leftrightarrow \mathcal{N}_{-k} \cap \mathcal{H}_{-k+1} = \{0\}. \qquad \square$$

3 On rigged Hilbert spaces associated with singular perturbations

Let

$$\mathcal{H}_- \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_+ \tag{3.1}$$

be the rigged Hilbert spaces associated with a self-adjoint operator $A \geq 1$ in $\mathcal{H}_0$. We recall that $\mathcal{H}_+ = \mathrm{Dom}A$ in the graph-norm of A. Let $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$ be a singular perturbation of A. We will assume that $\tilde{A} \geq 1$. In other case, i.e., if $\tilde{A} \geq m > -\infty,\ m := \inf\sigma(\tilde{A}) < 1$, we take the operator $\tilde{A}_{m-1} := \tilde{A} + (m-1)\mathbf{1} \geq 1$

to play the role of $\tilde{A}$, where $\mathbf{1}$ stands for the identical operator. With each operator $\tilde{A}$ there is associated a new rigged Hilbert space

$$\tilde{\mathcal{H}}_{-} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_{+} \tag{3.2}$$

constructed by the standard methods using $\tilde{A}$ (see [8, 9]).

In this section we study the structure of (3.2) in terms of singular perturbations.

By the assumption that $\tilde{A} \geq 1$ the space $\tilde{\mathcal{H}}_{+}$ coincides with $\mathrm{Dom}\tilde{A}$ endowed by the inner product $(f,g)_{+}^{\sim} = (\tilde{A}f, \tilde{A}g)_0$. Thanks to $\tilde{A} \in \mathcal{P}_s(A)$ there exist a liner set $\mathcal{D}$ dense in $\mathcal{H}_0$ and such that

$$(f,g)_{+} = (f,g)_{+}^{\sim}, \quad f,g \in \mathcal{D}. \tag{3.3}$$

Thus, the set $\mathcal{D}$ consists a proper subspace in each of the spaces $\mathcal{H}_{+}$, $\tilde{\mathcal{H}}_{+}$:

$$\mathcal{H}_{+} = \mathcal{M}_{+} \oplus \mathcal{N}_{+}, \quad \tilde{\mathcal{H}}_{+} = \tilde{\mathcal{M}}_{+} \oplus \tilde{\mathcal{N}}_{+}, \tag{3.4}$$

where just due to (3.3) we can write

$$\mathcal{M}_{+} = \tilde{\mathcal{M}}_{+} = \mathcal{D} \sqsubset \mathcal{H}_0. \tag{3.5}$$

From (3.4) (3.5) it follows that

$$\mathcal{H}_0 = \mathcal{M}_0 \oplus \mathcal{N}_0, \ \text{ where } \ \mathcal{M}_0 = A\mathcal{M}_{+} = \tilde{A}\mathcal{M}_{+},\ \mathcal{N}_0 = A\mathcal{N}_{+} = \tilde{A}\tilde{\mathcal{N}}_{+}. \tag{3.6}$$

Now we establish some more complete connections between (3.1) and (3.2).

Proposition 3.1. *Given two rigged triplets (3.1) and (3.2) assume that (3.4) (3.5) hold. Then the spaces $\mathcal{H}_{-}, \tilde{\mathcal{H}}_{-}$ admit the orthogonal decompositions:*

$$\mathcal{H}_{-} = \mathcal{M}_{-} \oplus \mathcal{N}_{-}, \quad \tilde{\mathcal{H}}_{-} = \tilde{\mathcal{M}}_{-} \oplus \tilde{\mathcal{N}}_{-}, \tag{3.7}$$

such that

$$\mathcal{M}_{-} = \tilde{\mathcal{M}}_{-}, \tag{3.8}$$

and

$$\mathcal{N}_{-} \cap \mathcal{H}_0 = \{0\} = \tilde{\mathcal{N}}_{-} \cap \mathcal{H}_0. \tag{3.9}$$

Proof. Let $D_{-,+}, \tilde{D}_{-,+}$ denote the standard canonical identification operators in (3.1), (3.2) resp. Applying $D_{-,+}, \tilde{D}_{-,+}$ to (3.4) we get (3.7). For $\omega = D_{-,+}\varphi$ and $\tilde{\omega} = \tilde{D}_{-,+}\varphi,\ \varphi \in \mathcal{D}$ due to (3.5) we have:

$$\langle \omega, \psi \rangle_{-,+} = \langle \tilde{\omega}, \psi \rangle_{-,+}^{\sim}, \quad \psi \in \mathcal{D}. \tag{3.10}$$

Therefore due to density of $\mathcal{D}$ we get:

$$\|\omega\|_- = \|D_{-,+}\varphi\|_- = \|\varphi\|_+ = \|\tilde{D}_{-,+}\varphi\|_-^{\sim} = \|\tilde{\omega}\|_-^{\sim}.$$

Moreover, by this construction we also have

$$\langle \omega, \eta \rangle_{-,+} = 0 = \langle \tilde{\omega}, \tilde{\eta} \rangle_{-,+}^{\sim}, \quad \eta \in \mathcal{N}_+, \quad \tilde{\eta} \in \tilde{\mathcal{N}}_+. \tag{3.11}$$

Therefore (3.8) is proved. The relation (3.9) follows from density $\mathcal{D}$ in $\mathcal{H}_0$. $\square$

Since $\tilde{A} \geq 1$ we can use Krein's formula for this operator:

$$\tilde{A}^{-1} = A^{-1} + B, \tag{3.12}$$

where B is a bounded and positive operator in $\mathcal{H}_0$ with $\mathrm{Ker}B = \mathcal{M}_0$, where $\mathcal{M}_0 := A\mathcal{D}$. We recall that in terms of B the domain of $\tilde{A}$ has the description:

$$\mathrm{Dom}\tilde{A} = \{g \in \mathcal{H}_0 \ : \ g = f + BAf, \ f \in \mathcal{H}_+ = \mathrm{Dom}A\}. \tag{3.13}$$

Proposition 3.2. *For each operator* $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$, $\tilde{A} \geq 1$, *the space* $\tilde{\mathcal{H}}_+ = \mathrm{Dom}\tilde{A}$ *has the following structure*

$$\tilde{\mathcal{H}}_+ = \tilde{\mathcal{M}}_+ \oplus \tilde{\mathcal{N}}_+ = \mathcal{M}_+ \oplus \tilde{\mathcal{N}}_+, \ \text{where } \tilde{\mathcal{M}}_+ = \mathcal{M}_+ = \mathcal{D} \sqsubset \mathcal{H}_0, \tag{3.14}$$

where the subspace $\tilde{\mathcal{N}}_+$ *is connected with* $\mathcal{N}_+$ *in the following way:*

$$\tilde{\mathcal{N}}_+ = \{\theta_+ \in \mathcal{H}_0 \ : \ \theta_+ = \eta_+ + BA\eta_+, \ \eta_+ \in \mathcal{N}_+\}, \quad \| \theta_+ \|_+^{\sim} = \| \eta_+ \|_+ . \tag{3.15}$$

Proof. (3.14) holds due to (3.4)) (3.5)). Since $\mathcal{H}_+ = \mathcal{M}_+ \oplus \mathcal{N}_+, \ \mathcal{M}_+ = \mathcal{D}$, for each $f \in \mathcal{H}_+$ we can write:

$$f = \varphi_+ \oplus \eta_+, \ \varphi_+ = P_{\mathcal{M}_+} f, \quad \eta_+ = P_{\mathcal{N}_+} f,$$

where $P_{\mathcal{L}}$ stands for the orthogonal projector onto the subspace $\mathcal{L}$. Using that $\tilde{\mathcal{H}}_+ = \mathrm{Dom}\tilde{A}$ for $g \in \mathrm{Dom}\tilde{A}$ by (3.13) we have:

$$g = \varphi_+ + \eta_+ + BA(\varphi_+ + \eta_+) = \varphi_+ + \theta_+, \ \theta_+ := \eta_+ + BA\eta_+.$$

Here $BA\varphi_+ = 0$ thanks to $A\varphi_+ \in \mathrm{Ker}B$. By $Af = \tilde{A}g$, we get $A\eta_+ = \tilde{A}\theta_+$ that proves (3.15). $\square$

Now we able to formulate the important new result.

Theorem 3.1. *For each $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$, $\tilde{A} \geq 1$, the inner product $(\cdot,\cdot)^{\sim}_{-}$ in the space $\tilde{\mathcal{H}}_{-}$ is the form-sum perturbation of the inner product in $\mathcal{H}_{-}$. It means that:*

$$(\cdot,\cdot)^{\sim}_{-} = (\cdot,\cdot)_{-} + \tau(\cdot,\cdot), \tag{3.16}$$

where the Hermitian quadratic form τ is singular in $\mathcal{H}_{-}$.

Proof. By construction $\tilde{\mathcal{H}}_{-}$ is the completion of $\mathcal{H}_0$ with respect to the inner product

$$(h_1, h_2)^{\sim}_{-} := (\tilde{A}^{-1}h_1, \tilde{A}^{-1}h_2)_0, \quad h_1, h_2 \in \mathcal{H}_0.$$

By Krein's formula (3.12) we get:

$$(h_1, h_2)^{\sim}_{-} = (A^{-1}h_1, A^{-1}h_2)_0 + \tau(h_1, h_2),$$

where

$$\tau(\cdot,\cdot) := (A^{-1}\cdot, B\cdot)_0 + (B\cdot, A^{-1}\cdot)_0 + (B\cdot, B\cdot)_0. \tag{3.17}$$

Obviously the form τ is Hermitian but non-positive. From(3.17) it follows that

$$\mathrm{Ker}\tau = \mathrm{Ker}B = \mathcal{M}_0.$$

We recall that $\mathcal{M}_0 = A\mathcal{D}$. Therefore the inner product in $\tilde{\mathcal{H}}_{-}$ on vectors from $\mathcal{M}_0$ is the same as in $\mathcal{H}_{-}$:

$$(\cdot,\cdot)_{-} \upharpoonright \mathcal{M}_0 = (\cdot,\cdot)^{\sim}_{-} \upharpoonright \mathcal{M}_0. \tag{3.18}$$

This means that τ is singular in $\mathcal{H}_{-}$ since the set $\mathrm{Ker}\tau = \mathcal{M}_0$ is dense in $\mathcal{H}_{-}$. The latter fact is true due to the general criterion (see for example [1]): $\mathcal{M}_0 \sqsubset \mathcal{H}_{-} \Longleftrightarrow \mathcal{N}_{-} \cap \mathcal{H}_0 = \{0\}$, where $\mathcal{N}_{-} := \mathbf{A}\mathcal{N}_0$. $\square$

(3.18) implies that in $\tilde{\mathcal{H}}_{-} = \tilde{\mathcal{M}}_{-} \oplus \tilde{\mathcal{N}}_{-}$ the subspace $\tilde{\mathcal{M}}_{-} = \mathcal{M}_{-}$ and is the completion of $\mathcal{M}_0$ in the norm

$$\|\mu\|^{\sim}_{-} = \|A^{-1}\mu\|_0 = \|\tilde{A}^{-1}\mu\|_0, \; \mu \in \mathcal{M}_0,$$

but the subspace $\tilde{\mathcal{N}}_{-} \neq \mathcal{N}_{-}$ and is the completion of $\mathcal{N}_0$ in the norm

$$\|\eta\|^{\sim,2}_{-} = \|\eta\|^2_{-} + \tau[\eta], \; \eta \in \mathcal{N}_0. \tag{3.19}$$

Moreover, (3.18) means that the operators $\mathbf{A} : \mathcal{H}_0 \to \mathcal{H}_{-}$ and $\tilde{\mathbf{A}} : \mathcal{H}_0 \to \tilde{\mathcal{H}}_{-}$ coincide not only on $\mathcal{D}$ but on $\mathcal{M}_0$ too:

$$\mathbf{A}\mathcal{M}_0 = \mathcal{M}_{-} = \tilde{\mathbf{A}}\mathcal{M}_0. \tag{3.20}$$

Remark 3.1. It is well-known that for $\tilde{A} \in \mathcal{P}_{\rm ws}(A)$ the space $\tilde{\mathcal{H}}_1$ may be produced by the form-sum method, i.e., the inner product $(\cdot,\cdot)_1^{\sim} = (\cdot,\cdot)_1 + \gamma(\cdot,\cdot)$, where the singular perturbation is given by a quadratic form γ of the $\mathcal{H}_{-1}-$class. Above Theorem 3.1 shows that in the more general case where $\gamma \in \mathcal{H}_{-2}-$class and $\tilde{A}$ is defined by the method of self-adjoint extensions, we can use the form-sum method also, but for construction of the space $\tilde{\mathcal{H}}_{-}$. In this way the operator $\tilde{A}$ is produced as an operator associated with the rigged Hilbert space (3.2). In other words Theorem 3.1 has the following consequence.

Theorem 3.2. *For each $\tilde{A} \in \mathcal{P}_{\rm s}(A)$, $\tilde{A} \geq 1$, the inverse operator $\tilde{A}^{-1}$ is uniquely associated (in the sense of the second representation theorem (see [18])) with the positive quadratic form $\chi_{-}^{\sim}[\cdot] := (\cdot,\cdot)_{-}^{\sim}$:*

$$\chi_{-}^{\sim}(h_1,h_2) = (Th_1,Th_2)_0,\ T \equiv \tilde{A}^{-1},\ h_1,h_2 \in \mathcal{H}_0.$$

Proof. By the above constructions the form $\chi_{-}^{\sim}[\cdot] = \chi_{-}[\cdot] + \tau[\cdot]$ is positive. Here $\chi_{-}[\cdot] := \|\cdot\|_{-}^2$ and τ has a view (3.17) and is defined by a positive operator B in $\mathcal{H}_{-}$. From $\tilde{A} \geq 1$ it follows that $\chi_{-}^{\sim} \leq \chi_0$, where $\chi_0[\cdot] := (\cdot,\cdot)_0$. Therefore $\chi_{-}^{\sim}(\cdot,\cdot) = (T\cdot,T\cdot)_0$, and $T = \tilde{A}^{-1}$ due to uniqueness of the operator representation. Conversely if we assumed that the quadratic form $\gamma_B[\cdot] := (B\cdot,\cdot)_0$ of a bounded operator B satisfies the inequality

$$\chi_{-1} \leq \gamma_B \leq \chi_0 - \chi_{-1}, \tag{3.21}$$

and the set $\mathcal{M}_0 :=$Ker$B$ is dense in $\mathcal{H}_0$ then it is easy to see that the operator T associated with $\chi_{-}^{\sim}$ coincides with $\tilde{A}^{-1}$ for some $\tilde{A} \in \mathcal{P}_{\rm s}(A)$, $\tilde{A} \geq 1$. □

Example 3.1. *Construction of rank one singular perturbations by the rigged Hilbert spaces method.*

Consider a rank one singular perturbation $\tilde{A}$ formally given as $\tilde{A} = A + \gamma_\omega$, where $\gamma_\omega(\cdot,\cdot) = \langle\cdot,\omega\rangle\langle\omega,\cdot\rangle$, $\omega \in \mathcal{H}_{-}\backslash\mathcal{H}_0$, $\|\omega\|_{-} = 1$ stands for the singular quadratic form. Precisely $\tilde{A} \in \mathcal{P}_s(A)$ is defined by Krein's formula:

$$\tilde{A}^{-1} = A^{-1} + \beta(\cdot,\eta)_0\eta,\ \eta = A^{-1}\omega,\quad \beta \in \mathbb{R}. \tag{3.22}$$

For $\tilde{A} \geq 1$ the parameter β should satisfy the condition

$$0 < \beta \leq 1 - (A^{-1}\eta,\eta)_0.$$

It is known that

$$\tilde{A}g = Af,\quad g \in {\rm Dom}\tilde{A},\ f \in {\rm Dom}A, \tag{3.23}$$

where

$$\mathrm{Dom}\tilde{A} = \{g \in \mathcal{H} \ : g = f + \beta(Af, \eta)_0 \eta = f + \beta\langle f, \omega\rangle \eta, \quad f \in \mathrm{Dom}A.\} \tag{3.24}$$

At first we introduce $\tilde{\mathcal{H}}_1$ as $\mathrm{Dom}\tilde{A}$ equipped by the inner product

$$(g_1, g_2)_1^{\sim} := (\tilde{A}g_1, g_2)_0 = (Af_1, g_2)_0 = (Af_1, f_2)_0 + \beta(Af_1, \eta)_0(\eta, Af_2)_0$$

$$= (f_1, f_2)_1 + \beta\langle f_1, \omega\rangle\langle \omega, f_2\rangle = (f_1, f_2)_1 + \beta\gamma_\omega(f_1, f_2).$$

Thus, if we assume that $\gamma_\omega \in \mathcal{H}_{-2}-$class, i.e., if $\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, then

$$\tilde{\mathcal{H}}_1 = \mathcal{H}_1 \oplus \tilde{\mathcal{N}}_1, \tag{3.25}$$

where $\tilde{\mathcal{N}}_1$ is the one-dimensional space constructed by the form γ_ω. Clearly γ_ω is singular in $\mathcal{H}_1$ since $\mathrm{Ker}\gamma_\omega$ is dense in $\mathcal{H}_1$.

In turn, the conjugate space $\tilde{\mathcal{H}}_{-1}$ is the completion of $\mathcal{H}_0$ in the inner product

$$(h, l)_{-1}^{\sim} := (\tilde{A}^{-1}h, l)_0 = (A^{-1}h, l)_0 + \beta(h, \eta)_0(\eta, l)_0 = (h, l)_{-1} + \beta\langle A^{-1}h, \omega\rangle\langle \omega, A^{-1}l\rangle,$$

i.e.,

$$(\cdot, \cdot)_{-1}^{\sim} = (\cdot, \cdot)_{-1} + \beta\gamma_\eta(\cdot, \cdot) = (\cdot, \cdot)_{-1} + \beta\gamma_\omega(A^{-1}\cdot, A^{-1}\cdot),$$

where $\gamma_\eta(\cdot, \cdot) := (\cdot, \eta)(\eta, \cdot)$. Obviously, the quadratic form γ_η is singular in $\mathcal{H}_{-1}$ since $\mathcal{M}_0 := \{h \in \mathcal{H}_0 : (h, \eta)_0 = 0\}$ is dense in $\tilde{\mathcal{H}}_{-1}$. Consequently we have

$$\tilde{\mathcal{H}}_{-1} = \mathcal{H}_{-1} \oplus \tilde{\mathcal{N}}_{-1}, \tag{3.26}$$

where $\tilde{\mathcal{N}}_{-1}$ is a one-dimensional space constructed by the form γ_η.

Further, the space $\tilde{\mathcal{H}}_+ = \mathrm{Dom}\tilde{A}$ in the inner product:

$$(g_1, g_2)_+^{\sim} = (\tilde{A}g_1, \tilde{A}g_2)_0 = (Af_1, Af_2)_0 = (f_1, f_2)_+, \tag{3.27}$$

where vectors $f_1, f_2 \in \mathrm{Dom}A$ are connected with $g_1, g_2 \in \mathrm{Dom}\tilde{A}$ according to (3.24). In particular, $g_1 = f_1, g_2 = f_2$ if the vectors f_1, f_2 are orthogonal to ω in the sense of the dual inner product. Then they belong to the set $\mathcal{M}_+ :=\mathrm{Ker}\gamma_\omega$, and we have

$$(\cdot, \cdot)_+^{\sim} \mid \mathcal{D} = (\cdot, \cdot)_+ \mid \mathcal{D}.$$

This means that $\tilde{\mathcal{M}}_+$ coincides with $\mathcal{M}_+$, and therefore we have

$$\tilde{\mathcal{H}}_+ = \mathcal{M}_+ \oplus \tilde{\mathcal{N}}_+, \tag{3.28}$$

where $\tilde{\mathcal{N}}_+$ is a one-dimensional space unitary equivalent to $\mathcal{N}_0$. Finally, the conjugate space $\tilde{\mathcal{H}}_-$ is the completion of $\mathcal{H}_0$ in the inner product:

$$(h_1,h_2)_-^{\sim} := (\tilde{A}^{-1}h_1, \tilde{A}^{-1}h_2)_0, \ \ h_1,h_2 \in \mathcal{H}_0.$$

By Krein's formula (3.22) we get

$$\begin{aligned}(h_1,h_2)_-^{\sim} &= (A^{-1}h_1 + \beta(h_1,\eta)_0\eta, A^{-1}h_2 + \beta(h_2,\eta)_0\eta)_0 \\ &= (A^{-1}h_1, A^{-1}h_2)_0 + \tau_\omega(h_1,h_2) = (h_1,h_2)_- + \tau_\omega(h_1,h_2),\end{aligned}$$

where the Hermitian quadratic form τ_ω has the form

$$\begin{aligned}\tau_\omega(\cdot,\cdot) &= \beta(A^{-1}\cdot,\eta)_0(\eta,\cdot)_0 + \beta(\cdot,\eta)_0(\eta,A^{-1}\cdot)_0 + \beta^2(\cdot,\eta)_0(\eta,\cdot)_0 \\ &= \beta(\cdot,\eta_+)_0(\eta,\cdot)_0 + \beta(\cdot,\eta)_0(\eta_+,\cdot)_0 + \beta^2(\cdot,\eta)_0(\eta,\cdot)_0, \qquad (3.29)\end{aligned}$$

where $\eta_+ := A^{-1}\eta$ and where we used $\|\eta\|_0^2 = 1$. Thus (cf. with (3.16))

$$(\cdot,\cdot)_-^{\sim} = (\cdot,\cdot)_- + \tau_\omega(\cdot,\cdot). \qquad (3.30)$$

The quadratic forms τ_ω is obviously singular in $\mathcal{H}_-$ since vectors $\eta,\eta_+ \notin \mathcal{H}_+$, but it is non-positive. By the latter reason it is impossible to present the space $\tilde{\mathcal{H}}_-$ as a sum $\mathcal{H}_- \oplus \tilde{\mathcal{N}}_-$. However we have

$$\tilde{\mathcal{H}}_- = \tilde{\mathcal{M}}_- \oplus \tilde{\mathcal{N}}_-, \ \ \tilde{\mathcal{M}}_+ = \mathcal{M}_+ \qquad (3.31)$$

where $\tilde{\mathcal{N}}_-$ is conjugate to $\tilde{\mathcal{N}}_+$.

As a general result of the above analysis we conclude that for $\gamma_\omega \in \mathcal{H}_{-2}-$class a singular rank one perturbation admits a construction by the form-sum method along two ways: (**1**) to define $\tilde{A}$ as the operator associated with a new triplet $\tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_1$, where the inner products in $\tilde{\mathcal{H}}_{-1}, \tilde{\mathcal{H}}_1$ have the form-sum representations:

$$(\cdot,\cdot)_{-1}^{\sim} = (\cdot,\cdot)_{-1} + \beta\gamma_\omega(\cdot,\cdot), \ (\cdot,\cdot)_1^{\sim} = (\cdot,\cdot)_1 + \beta\gamma_\omega(A^{-1}\cdot, A^{-1}\cdot),$$

(**2**) to define $\tilde{A}^{-1}$ as the operator associated by the second representation theorem (see [18]) with the quadratic form $\tilde{\chi}_-(\cdot,\cdot) := (\cdot,\cdot)_-^{\sim}$ which is a singular form-sum perturbation of $(\cdot,\cdot)_-$ (see (3.30)).

4 The singularity phenomenon

Let $\mathcal{S} \subset \mathcal{G}$ be a pair of linear sets and $(\cdot,\cdot)$, $(\cdot,\cdot)^{\sim}$ be two inner products on $\mathcal{G}$. Let $\mathcal{H}$, $\tilde{\mathcal{H}}$ denote the corresponding Hilbert spaces constructed in a standard way. Assume that

(**1**) the above inner products coincide on $\mathcal{S}$, i.e.,

$$(\cdot,\cdot) \mid \mathcal{S} = (\cdot,\cdot)^{\sim} \mid \mathcal{S},$$

(**2**) the set $\mathcal{S}$ is dense both in $\mathcal{H}$ and $\tilde{\mathcal{H}}$.

Then one can naively think that the spaces $\mathcal{H}$, $\tilde{\mathcal{H}}$ are identical. However this is not true. In general $\mathcal{H} \neq \tilde{\mathcal{H}}$ in the sense that $\| g \| \neq \| g \|^{\sim}$ for $g \in \mathcal{G} \setminus \mathcal{S}$. In other words, the quadratic form $\tau[\cdot] := (\cdot,\cdot)^{\sim} - (\cdot,\cdot)$ is non-trivial and singular both in $\mathcal{H}$ and $\tilde{\mathcal{H}}$ in the sense that $\mathrm{Ker}\tau \sqsubset \mathcal{H}, \tilde{\mathcal{H}}$. However the Hermitian form τ is not positive. Indeed, if we assume that $\tau \geq 0$ then the space $\tilde{\mathcal{H}}$ should have the structure of an orthogonal sum: $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}_{\tau}$ (see [24]) that is impossible under (**1**) and (**2**).

We will call the above described situation with conditions (**1**), (**2**) as a *singularity phenomenon.*

In fact we already met this phenomenon in the previous section. Namely, for each $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$, $\tilde{A} \geq 1$ the corresponding Hilbert space $\tilde{\mathcal{H}}_{-}$ contains the same linear set $\mathcal{M}_0$ with two properties: (**1**) the inner products in $\tilde{\mathcal{H}}_{-}$ and $\mathcal{H}_{-}$ restricted to this set are identical: $(\cdot,\cdot)_{-}^{\sim} \mid \mathcal{M}_0 = (\cdot,\cdot)_{-} \mid \mathcal{M}_0$, (**2**) $\mathcal{M}_0$ is dense both in $\tilde{\mathcal{H}}_{-}$ and $\mathcal{H}_{-}$. Indeed, we recall that $\tilde{\mathcal{H}}_{-}$ is constructed as the completion of $\mathcal{H}_0$ with respect to the inner product $(\cdot,\cdot)_{-}^{\sim} = (\tilde{A}^{-1}\,\cdot, \tilde{A}^{-1}\,\cdot)_0$, where $\tilde{A}^{-1}$ is defined by Krein's formula (3.12) with a positive operator B which is non-zero only on $\mathcal{N}_0 := \mathcal{H}_0 \ominus \mathcal{M}_0$. So (**2**) is fulfilled. The condition (**1**) is evident due to (3.18). We remark that the density of $\mathcal{M}_0$ in each $\tilde{\mathcal{H}}_{-}$ can be proven independently in the following way.

Lemma 4.1 *Let $\tilde{A} \in \mathcal{P}_{\mathrm{s}}(A)$, $\tilde{A} \geq 1$ and let $\tilde{\mathcal{H}}_{-}$ be the completion of $\mathcal{H}_0$ in the inner product (3.16) where the quadratic form τ is defined by (3.17). Then the subspace $\mathcal{M}_0 := \mathrm{Ker}\tau$ is dense in $\tilde{\mathcal{H}}_{-}$:*

$$\mathcal{M}_0 \sqsubset \tilde{\mathcal{H}}_{-}. \tag{4.1}$$

Therefore the quadratic form τ is singular not only in $\mathcal{H}_{-}$ but in each $\tilde{\mathcal{H}}_{-}$ too.

Proof. By construction $\mathcal{M}_0 \sqsubset \mathcal{H}_{-}$ since $\tilde{A}$ is defined by a singular form. So, we need to prove only $\mathcal{M}_0 \sqsubset \tilde{\mathcal{H}}_{-}$. Let $h \in \mathcal{H}_0 = \mathrm{Ran}\tilde{A}$. Then $h = \tilde{A}g$ with some $g \in \mathrm{Dom}\tilde{A}$. Thanks to the density of the set $\mathcal{D} = \mathcal{M}_{+} := A^{-1}\mathcal{M}_0$ in $\mathcal{H}_0$, there exists a sequence $\varphi_n \in \mathcal{M}_+$ such that $\| \varphi_n - g \|_0 \to 0$. Set $f_n := A\varphi_n = \tilde{A}\varphi_n$. Obviously $f_n \in \mathcal{M}_0$. Let us check that the sequence f_n converges to the vector h in $\tilde{\mathcal{H}}_-$. Indeed, using that $\tilde{A}^{-1}A\varphi_n = \varphi_n$ we have

$$\| h - f_n \|_-^{\sim} = \| \tilde{A}^{-1}(h - f_n) \|_0 = \| \tilde{A}^{-1}(\tilde{A}g - A\varphi_n) \|_0 = \| g - \varphi_n \|_0 \to 0. \ \square$$

We can face the singularity phenomenon in a slightly other form. Let $\overset{\circ}{A}$ be the symmetric densely defined restriction of $A = A^* \geq 1$ in $\mathcal{H}_0$. So, $\mathcal{M}_+ := \mathrm{Dom}\overset{\circ}{A} \sqsubset \mathcal{H}_0$. Let $\tilde{A}$ be a strongly positive self-adjoint extension of $\overset{\circ}{A}$ and $\tilde{\mathcal{H}}_-$ be the corresponding space constructed by the inner product $(\cdot,\cdot)_-^{\sim} := (\tilde{A}^{-1}\cdot, \tilde{A}^{-1}\cdot)_0$. Then the subspace $\mathcal{M}_0 := \overset{\circ}{A}\mathcal{M}_+ = A\mathcal{M}_+ = \tilde{A}\mathcal{M}_+$ has two properties: (**1**) it is dense both in $\mathcal{H}_-$ and $\tilde{\mathcal{H}}_-$ and (**2**) the norms $\| \cdot \|_-^{\sim}$ and $\| \cdot \|_-$ coincide on $\mathcal{M}_0$ due to $\mathcal{M}_- := \mathbf{A}\mathcal{M}_0 = \tilde{\mathbf{A}}\mathcal{M}_0$.

5 Construction of the $\tilde{A}$-scale by a singular quadratic form

In this section we discuss connections of the new rigged Hilbert space (3.2) with a quadratic form $\gamma \in \mathcal{H}_{-2}(A)$-class associated to a singular perturbation.

We start with the rigged triplet (3.1) associated to the free operator $A = A^* \geq 1$ in $\mathcal{H}_0$ and take in the consideration a chain of five spaces

$$\mathcal{H}_- \equiv \mathcal{H}_{-2} \sqsupset \mathcal{H}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_1 \sqsupset \mathcal{H}_2 \equiv \mathcal{H}_+ (= \mathrm{Dom}A), \tag{5.1}$$

which consists of a part of the A-scale (2.2). We remind that both (3.1) and the whole scale (2.2) can be reconstructed by any couples of spaces: $\mathcal{H}_0 \sqsupset \mathcal{H}_k$ or $\mathcal{H}_{-k} \sqsupset \mathcal{H}_0$, $k > 0$ from the A-scale (see [8]).

Given a positive quadratic form $\gamma \in \mathcal{H}_{-2}$-class define a new inner product on $\mathcal{H}_0$:

$$(h_1, h_2)_{-1}^{\sim} := (A^{-1}h_1, h_2)_0 + \gamma(A^{-1}h_1, A^{-1}h_2), \quad h_1, h_2 \in \mathcal{H}_0. \tag{5.2}$$

We note that (5.2) is well defined since the operator A^{-1} maps $\mathcal{H}_0$ onto $\mathcal{H}_+$ and therefore vectors $A^{-1}h_1, A^{-1}h_2 \in \mathcal{H}_+ = \mathrm{Dom}\gamma$. Let $\tilde{\mathcal{H}}_{-1}$ be the Hilbert space corresponding to the inner product (5.2), i.e., $\tilde{\mathcal{H}}_{-1}$ is the completion of $\mathcal{H}_0$ in the norm

$$\| \cdot \|_{-1}^{\sim} := (\| A^{-1/2} \cdot \|_0^2 + \gamma[A^{-1}\cdot])^{1/2}. \tag{5.3}$$

Assume that γ is such that

$$\| \cdot \|_{-1}^{\sim} \leq \| \cdot \|_0 \, . \tag{5.4}$$

Then

$$\tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0, \tag{5.5}$$

and one can extend this couple of spaces to the rigged triplet

$$\tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_1. \tag{5.6}$$

and construct the associated operator

$$\tilde{A} := \tilde{D}_{-1,1} \mid \{f \in \tilde{\mathcal{H}}_1 : \tilde{D}_{-1,1} f \in \mathcal{H}_0\}, \tag{5.7}$$

where $\tilde{D}_{-1,1} : \tilde{\mathcal{H}}_1 \to \tilde{\mathcal{H}}_{-1}$ is the standard canonical isomorphism. Clearly that $\tilde{A} \geq 1$ since by (5.4),

$$\| \cdot \|_0 \leq \| \cdot \|_1^{\sim} = \| \tilde{A} \cdot \|_0 \, . \tag{5.8}$$

Further, by $\tilde{A}$ we can introduce the chain of five spaces similar to (5.1),

$$\tilde{\mathcal{H}}_- \equiv \tilde{\mathcal{H}}_{-2} \sqsupset \tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_1 \sqsupset \tilde{\mathcal{H}}_2 \equiv \tilde{\mathcal{H}}_+ (= \mathrm{Dom}\tilde{A}). \tag{5.9}$$

Proposition 5.1. *Let a quadratic form $\gamma \in \mathcal{H}_{-2}$-class satisfies the condition:*

$$- \| f \|_1^2 \leq \gamma[f] \leq \| f \|_2^2 - \| f \|_1^2, \quad f \in \mathcal{H}_2 = \mathrm{Dom}A. \tag{5.10}$$

Then the associated operator $\tilde{A} \in \mathcal{P}_{\rm ss}(A)$.

Proof. From (5.10) we have

$$-(Af, f)_0 \leq \gamma[f] \leq \| Af \|_0^2 - (Af, f)_0, \; f \in \mathcal{H}_+.$$

This implies

$$-(A^{-1}h, h)_0 \leq \gamma[A^{-1}h] \leq \| h \|_0^2 - (A^{-1}h, h)_0, h \in \mathcal{H}_0$$

Since each $f = A^{-1}h$ for some $h \in \mathcal{H}_0$. In other terms

$$- \| h \|_{-1}^2 \leq \gamma[A^{-1}h] \leq \| h \|_0^2 - \| h \|_{-1}^2$$

that is equivalent to

$$0 \leq \gamma[A^{-1}h] + \| h \|_{-1}^2 \leq \| h \|_0^2$$

Therefore condition (5.4) is fulfilled and by the construction before Proposition 5.1 we get the operator $\tilde{A} \geq 1$. We need check now that $\tilde{A} \in \mathcal{P}_{\rm ss}(A)$. To this aim we remark that $\gamma[A^{-1}h] = 0,\ h \in \mathcal{M}_0,$ where $\mathcal{M}_0 := A\mathrm{Ker}\gamma$. Therefore $\tilde{A}^{-1}h = A^{-1}h,\ h \in \mathcal{M}_0$ and $\tilde{A}f = Af,\ f \in \mathrm{Ker}\gamma := \mathcal{D}$. Thus $\tilde{A} \in \mathcal{P}_{\mathrm{ss}}(A)$ since $\mathrm{Ker}\gamma \sqsubset \mathcal{H}_1$. $\Box$

The chain (5.9) may be constructed using the operator $S : \mathcal{H}_+ \to \mathcal{H}_-$ associated with γ (see (2.5)). So, let $S = \mathbf{A}^2\mathbf{s}$, where $\mathbf{s}$ is a bounded operator in $\mathcal{H}_+$ such that $\gamma[\cdot] = (\mathbf{s}\cdot, \cdot)_+$.

Introduce the bounded operator $\mathbf{T} : \mathcal{H}_0 \to \mathcal{H}_-$ acting as follows,

$$\mathbf{T}h = (\mathbf{1} + SA^{-1})h,\ h \in \mathcal{H}_0,$$

where $\mathbf{1}$ stands for an identical mapping.

Using $\mathbf{T}$ one can define a new inner product on $\mathcal{H}_0$,

$$(h, l)_-^{\sim} := (\mathbf{T}h, \mathbf{T}l)_- \ \forall\ h, l \in \mathcal{H}_0. \tag{5.11}$$

Assume

$$\| h \|_-^{\sim} \leq \| h \|_0,\ h \in \mathcal{H}_0, \tag{5.12}$$

and define $\tilde{\mathcal{H}}_-$ as the completion of $\mathcal{H}_0$ in the norm $\| h \|_-^{\sim}$. Due to (5.12) we get $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0$. By the standard procedure one can construct $\tilde{\mathcal{H}}_+$ and define $\tilde{A}$ as the operator associated with the triplet : $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+$.

Proposition 5.2 *Let* $\mathbf{s}$ *be a positive bounded operator in* $\mathcal{H}_+$. *Assume the inequality*

$$-(Af, f)_0 \leq (\mathbf{s}f, f)_+ \leq \| f \|_+^2 - (Af, f)_0,\ f \in \mathcal{H}_+ \tag{5.13}$$

holds and

$$\mathrm{Ker}\mathbf{s} = \mathcal{M}_+ \sqsubset \mathcal{H}_1.$$

Let the rigged Hilbert space $\tilde{\mathcal{H}}_- \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_+$ *be constructed by* $S = \mathbf{A}^2\mathbf{s}$ *and* $\mathbf{T}$ *in according with the above described way. Then the associated with this rigged Hilbert space operator* $\tilde{A} \in \mathcal{P}_{\mathrm{ss}}(A)$ *and* $\tilde{A} \geq 1$.

Proof. From (5.11) it follows that the associated with the new rigged Hilbert space operator has the representation:

$$\tilde{A}^{-1} := A^{-1}\mathbf{T} = A^{-1} + A\mathbf{s}A^{-1} = A^{-1} + B.$$

By this construction $\mathrm{Ker}B = \mathcal{M}_0 := A\mathcal{M}_+$ and therefore

$$\tilde{A}|\mathcal{D} = A|\mathcal{D},\quad \mathcal{D} \equiv \mathcal{M}_+.$$

Thus $\tilde{A} \in \mathcal{P}_{\text{ss}}(A)$ since the set $\mathcal{D}$ is dense in $\mathcal{H}_1$. Further, the inequality $\tilde{A} \geq 1$ is equivalent to (5.12) which follows from (5.13). □

We shall say that the chains (5.1) and (5.9) are *s−similar* (=singularly similar) if

$$\mathcal{H}_+ \cap \tilde{\mathcal{H}}_+ =: \mathcal{D} \sqsubset \mathcal{H}_1 \tag{5.14}$$

and

$$\| f \|_1^{\sim} = \| f \|_1, \quad f \in \mathcal{D}. \tag{5.15}$$

We get an important result.

Theorem 5.1. *The associated with (5.9) operator $\tilde{A} \in \mathcal{P}_{\text{ss}}(A)$, $\tilde{A} \geq 1$ if and only if the chains (5.1) and (5.9) are s−similar.*

Proof. By (5.15) we have

$$(\tilde{A}f, l)_0 = (Af, l)_0, \quad f, l \in \mathcal{D}.$$

Since $\mathcal{D}$ is dense in $\mathcal{H}_0$ we can introduce the symmetric operator $\overset{\circ}{A}$,

$$\overset{\circ}{A} := \tilde{A} \mid \mathcal{D} = A \mid \mathcal{D}.$$

Thus, both A and $\tilde{A}$ are different self-adjoint extensions of $\overset{\circ}{A}$. In particular, $\tilde{A} \in \mathcal{P}_{\text{ss}}(A)$ since in fact the set $\mathcal{D}$ is dense in $\mathcal{H}_1$. □

We emphasize that one can not change condition (5.15) into the condition $\| f \|_+^{\sim} = \| f \|_+, \quad f \in \mathcal{D}$.

The following theorem is the main result of this section.

Theorem 5.2. *There exists a one-to-one correspondence between three families of objects: the operators $\tilde{A} \in \mathcal{P}_{\text{ss}}(A)$, $\tilde{A} \geq 1$, the quadratic forms $\gamma \in \mathcal{H}_{-2}$−class with condition (5.10), and the chains of spaces (5.9) which are s−similar to (5.1). These correspondences are fixed by the formulas*

$$\gamma[f] = (\tilde{A}^{-1}h, h)_0 - (Af, f)_0, \;\; h = Af, f \in \mathcal{H}_+, \tag{5.16}$$

$$(h, l)_{-1}^{\sim} = (\tilde{A}^{-1}h, l)_0 = (h, l)_{-1} + \gamma(A^{-1}h, A^{-1}l), h, l \in \mathcal{H}_0. \tag{5.17}$$

Proof. By an operator $\tilde{A} \in \mathcal{P}_{\text{ss}}(A)$, $\tilde{A} \geq 1$ we can define a form $\gamma \in \mathcal{H}_{-2}$−class according to (5.16). This form satisfies condition (5.10) since $\tilde{A} \geq 1$. By using the form γ one can introduce the space $\tilde{\mathcal{H}}_{-1}$ completing the space $\mathcal{H}_0$ with respect to the inner product $(h,l)^{\sim}_{-1} := (h,l)_{-1} + \gamma(A^{-1}h, A^{-1}l), h, l \in \mathcal{H}_0$. Then starting with the so-called pre-rigged pair $\tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0$ one can construct the chain of spaces (5.9). Clearly, we get the chain which is $s-$similar to (5.1) by Theorem 5.1. Finally, starting from (5.9) we can reconstruct $\tilde{A}$ as the operator associated with this chain. □

Of course, the same result is true in the general case where $\tilde{A}$ is not necessarily strongly positive but only bounded from below.

Indeed, let

$$\tilde{A} \in \mathcal{P}_{\rm ss}(A),\ \ \tilde{A} \geq m, \quad m := \inf\sigma(\tilde{A}) < 1.$$

Then the quadratic form γ is defined by a formula of the form (5.16) with the operators $\tilde{A},\ A$ replaced by $\tilde{A}_a = \tilde{A} + a,\ \ A_a = A + a$, resp., where $a = 1 - m > 0$:

$$\gamma[f] = (\tilde{A}_a^{-1}h, h)_0 - (A_a f, f)_0,\ \ h = Af,\ \ f \in \mathcal{H}_+.$$

Obviously $\gamma \in \mathcal{H}_{-2}-$ class and satisfies the inequalities

$$-(A_a f, f)_0 \leq \gamma[f] \leq (A_a f, A_a f)_0 - (A_a f, f)_0,\ \ f \in \mathcal{H}_+.$$

Having γ one can introduce the space $\tilde{\mathcal{H}}_{-1}$ as the completion of $\mathcal{H}_0$ in the norm corresponding to the inner product

$$(\cdot,\cdot)^{\sim}_{-1} := (A_a^{-1}\cdot,\cdot)_0 + \gamma[A_a^{-1}\cdot].$$

Then, starting with the pre-rigged couple $\tilde{\mathcal{H}}_{-1} \sqsupset \mathcal{H}_0$ we construct whole chain of spaces of type (5.9) by the standard methods. Surely, this chain will be $s-$similar to the chain of the form (5.1) which is constructed by A_a. Finally, we may reconstruct the operator $\tilde{A}_a$ as associated with the latter chain and return to $\tilde{A} = \tilde{A}_a - a$. Of course, in the above round of implications one can start with any object: an operator $\tilde{A} \in \mathcal{P}_{\rm ss}(A)$, a quadratic form $\gamma \in \mathcal{H}_{-2}-$class, or, finally, a chain of $s-$similar to (5.1) spaces of form (5.9).

6 Singular rank one perturbation of a higher order

In this section we show that the method of rigged Hilbert spaces may be applied in the singular perturbation theory of a higher order (the so-called super singular perturbation theory, see e.g., [10] and references wherein). However our method differs from the approach developed in [10], where the state space is changed by the procedure of the orthogonal extension.

We will consider simplest case of rank-one perturbations. Let

$$\mathcal{H}_- \equiv \mathcal{H}_{-k} \sqsupset \mathcal{H}_{-k/2} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_{k/2} \sqsupset \mathcal{H}_k \equiv \mathcal{H}_+, \ k > 2 \tag{6.1}$$

be the A-scale of Hilbert spaces associated with an operator $A = A^* \geq 1$ in $\mathcal{H}_0$.

Fix a vector $\omega \in \mathcal{H}_{-k} \backslash \mathcal{H}_{-k+1}, \ k > 2$. Then the quadratic form

$$\gamma_\omega(\varphi, \psi) := \langle \varphi, \omega \rangle_{k,-k} \langle \omega, \psi \rangle_{-k,k}, \ \varphi, \psi \in \mathcal{H}_k \tag{6.2}$$

obviously belongs to $\mathcal{H}_{-k}-$class since the set

$$\mathrm{Ker}\gamma = \mathcal{M}_+ \equiv \mathcal{M}_k := \{\varphi \in \mathcal{H}_k \ : \ \langle \varphi, \omega \rangle_{k,-k} = 0\}$$

is dense in $\mathcal{H}_{k-1}$ just due to $\omega \notin \mathcal{H}_{-k+1}$. For example, if $k = 3$, then $\mathcal{M}_k$ is dense in $\mathcal{H}_2$ =Dom$A$, and it is impossible to define the perturbed operator $\tilde{A}$ by any standard method. Here we will define the operator $\tilde{A}$ in $\mathcal{H}_0$ by the method of rigged Hilbert spaces.

With this aim we at first construct by A and γ_ω a new scale of Hilbert spaces

$$\tilde{\mathcal{H}}_- \equiv \tilde{\mathcal{H}}_{-k} \sqsupset \tilde{\mathcal{H}}_{-k/2} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_{k/2} \sqsupset \tilde{\mathcal{H}}_k \equiv \tilde{\mathcal{H}}_+, \ k > 2. \tag{6.3}$$

and then introduce $\tilde{A}$ as the associated operator. We recall that the chain (6.3) is fixed by every couple of spaces of the form $\mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_j$ or $\tilde{\mathcal{H}}_{-j} \sqsupset \mathcal{H}_0, \ j > 0$, where $\tilde{\mathcal{H}}_j$ or $\tilde{\mathcal{H}}_{-j}$ may be chosen from the infinite scale of spaces (6.3). We choose the space $\tilde{\mathcal{H}}_{-k/2}$ which is defined by A and γ_ω as the completion of $\mathcal{H}_0$ with respect to the inner product

$$(h_1, h_2)^{\sim}_{-k/2} := (A^{-k/2}h_1, h_2) + \beta\gamma(A^{-k/2}h_1, A^{-k/2}h_2), \ \ h_1, h_2 \in \mathcal{H}_0, \ \beta \in \mathbb{R}. \tag{6.4}$$

We recall that the operator $A^{-k/2}$ is isometric as a map from $\mathcal{H}_0$ onto $\mathcal{H}_k$. So, we get the inequality

$$\| \cdot \|^{\sim}_{-k/2} \leq \| \cdot \|_0$$

only if the coupling constant β satisfies the condition:

$$0 \leq \beta + \|\eta_k\|^2_{k/2} \leq 1, \ \ \eta_{k/2} := \mathbf{A}^{-k}\omega \in \mathcal{H}_k. \tag{6.5}$$

Hence, under (6.5) the embedding $\tilde{\mathcal{H}}_{-k/2} \sqsupset \mathcal{H}_0$ holds. Now one can extend this pre-rigged couple to the whole scale (6.3). By using (6.3) we define the operator $\tilde{A}^{k/2}$ by:

$$\tilde{A}^{k/2} := \tilde{D}_{0,k} = \tilde{D}_{-k/2,k/2} \upharpoonright \{\varphi \in \mathcal{H}_{k/2} : \tilde{D}_{-k/2,k/2}\varphi \in \mathcal{H}_0\}$$

where $\tilde{D}_{0,k}$, $\tilde{D}_{-k/2,k/2}$ denote the canonical identification operators in the scale (6.3). Of course, the operator $\tilde{A}^{k/2}$ is a strongly singular perturbation of $A^{k/2}$, i.e., $\tilde{A}^{k/2} \in \mathcal{P}_{\rm ss}(A^{k/2})$. Finally, by the spectral theorem we can define $\tilde{A} := (\tilde{A}^{k/2})^{2/k} \equiv \tilde{D}_{0,2}$. This operator we call the super singular perturbation of A corresponding to a singular rank-one quadratic form $\gamma_\omega \in \mathcal{H}_{-k}-$class, $k > 2$, where $\omega \in \mathcal{H}_{-k} \backslash \mathcal{H}_{-k+1}$.

Theorem 6.1. *Given two chains of Hilbert spaces (6.1) and (6.3) assume that for some $k > 2$ the difference of the inner products in $\tilde{\mathcal{H}}_{-k/2}$ and $\mathcal{H}_{-k/2}$ defines a rank-one positive quadratic form on $\mathcal{H}_0$:*

$$\beta\gamma_\omega(\cdot,\cdot) := (\cdot,\cdot)^{\sim}_{-k/2} - (\cdot,\cdot)_{-k/2},\ \omega \in \mathcal{H}_{-k} \setminus \mathcal{H}_{-k+1},$$

where a constant β satisfies inequality (6.5). Then this form admits the interpretation as a super singular $\mathcal{H}_{-k}-$class perturbation of A, and define the uniquely associated with the rigged triplet $\tilde{\mathcal{H}}_{-k/2} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_{k/2}$ self-adjoint in $\mathcal{H}_0$ operator $\tilde{A}^{k/2} \in \mathcal{P}_{\rm ss}(A^{k/2})$ as well as the super singularly perturbed operator $\tilde{A}$ associated with the scale (6.3).

Proof. The result is true due to the arguments based on Theorem 5.2 (see also Example 3.1). □

Example 6.1. *A model $\frac{d^4}{dx^4} + \delta - \delta''$.*

Here we consider a rank one singular perturbation $\omega := \delta - \delta'' \in W_2^{-3}(\mathbb{R})$ of the operator d^4/dx^4 in $L_2(\mathbb{R})$. Formally this perturbation is given by the expression: $d^4/dx^4 + \beta(\delta - \delta'') = d^4/dx^4 + \beta\gamma_\omega$, where $\gamma_\omega(\cdot,\cdot) := \langle\cdot,\omega\rangle\langle\omega,\cdot\rangle$ and $\beta \in \mathbb{R}$. Precisely we construct $d^4/dx^4 + \beta(\delta - \delta'')$ using the method of rigged Hilbert spaces as follows.

We associate $d^4/dx^4 + \beta(\delta - \delta'')$ with the perturbed scale of Sobolev spaces

$$\tilde{W}_2^{-4}(\mathbb{R}) \sqsupset \tilde{W}_2^{-2}(\mathbb{R}) \sqsupset L_2(\mathbb{R}) \sqsupset \tilde{W}_2^{2}(\mathbb{R}) \sqsupset \tilde{W}_2^{4}(\mathbb{R}),\ k > 0. \tag{6.6}$$

By definition,

$$d^4/dx^4 + \beta(\delta - \delta'') := \tilde{D}_{-2,2} \upharpoonright \{\varphi \in \tilde{W}_2^2 | \varphi^{(4)}(x) + \beta(\varphi(0)\delta(x) - \varphi''(0)\delta''(x)) \in L_2\},$$

where $\tilde{D}_{-2,2} : \tilde{W}_2^2 \to \tilde{W}_2^{-2}$ stands for the unitary identification operator and all derivatives are taken in the generalized sense. The chain (6.6) may be constructed starting from the pre-rigged pair $\tilde{W}_2^{-2}(\mathbb{R}) \sqsupset L_2(\mathbb{R})$, where $\tilde{W}_2^{-2}(\mathbb{R})$ is the completion of $L_2(\mathbb{R})$ endowed by the inner product

$$(f,g)_{-2}^{\sim} := (f,g)_{W_2^{-2}} + \beta\gamma_\omega((1-d^2/dx^2)^{-2}f, (1-d^2/dx^2)^{-2}g).$$

We observe that $\omega = \delta - \delta''$ as a vector in $W_2^{-4}(\mathbb{R})$ admits the representation

$$\omega = (1-d^2/dx^2)\delta = (1-d^2/dx^2)^2\eta, \text{ where } \eta(x) = \frac{1}{2}e^{-|x|},$$

and therefore we can derive the positive operator

$$d^4/dx^4 + \beta(\delta - \delta'') = (1-d^2/dx^2)^2 + \beta(\delta-\delta'') + 2d^2/dx^2 - 1$$

using Krein's formula:

$$[(1-d^2/dx^2)^2 + \beta(\delta-\delta'')]^{-1} = (1-d^2/dx^2)^{-2} - \frac{\beta}{2(2+\beta)}(\cdot,\eta)_0\eta, \tag{6.7}$$

where β should satisfy the condition

$$0 < \beta \leq 1 - ((1-d^2/dx^2)^{-1}\eta, \eta)_{L_2}.$$

The corresponding integral kernels in (6.7) have the explicit representations. The domain of the operator $d^4/dx^4 + \beta(\delta - \delta'')$ has the following description

$$\mathrm{Dom}(d^4/dx^4+\beta(\delta-\delta'')) = \{g \in L_2 | g(x) = \varphi(x) + \frac{\beta}{2}(\varphi(0)-\varphi''(0))e^{-|x|}\},\ \varphi \in W_2^4\}.$$

7 On the s-similarity of Hilbert scales

Let $A, \tilde{A} \geq 1$ be a pair of self-adjoint operators in $\mathcal{H}_0$ and let

$$\mathcal{H}_{-k} \sqsupset \mathcal{H}_0 \sqsupset \mathcal{H}_k, \tag{7.1}$$

$$\tilde{\mathcal{H}}_{-k} \sqsupset \mathcal{H}_0 \sqsupset \tilde{\mathcal{H}}_k, \quad k > 0 \tag{7.2}$$

be the scales of Hilbert spaces associated with the operators A, $\tilde{A}$, resp.

We say that scales (7.1), (7.2) are *s-similar* in the generalized sense and write $\{\mathcal{H}_k\} \sim \{\tilde{\mathcal{H}}_k\}$ if there exists $k \geq 1$ such that the set

$$\mathcal{D}_k := \mathcal{H}_{2k} \cap \tilde{\mathcal{H}}_{2k} \tag{7.3}$$

is dense in $\mathcal{H}_k$,

$$\mathcal{H}_k \sqsupset \mathcal{D}_k, \tag{7.4}$$

and

$$\|\varphi\|_k = \|\varphi\|_{\tilde{k}}, \ \varphi \in \mathcal{D}_k. \tag{7.5}$$

Conditions (7.3), (7.4) imply that both spaces $\mathcal{H}_{2k}, \tilde{\mathcal{H}}_{2k}$ admit the orthogonal decompositions

$$\mathcal{H}_{2k} = \mathcal{M}_{2k} \oplus \mathcal{N}_{2k} \ \ \tilde{\mathcal{H}}_{2k} = \tilde{\mathcal{M}}_{2k} \oplus \tilde{\mathcal{N}}_{2k}, \tag{7.6}$$

such that the subspaces $\mathcal{M}_{2k}, \ \tilde{\mathcal{M}}_{2k}$ are identical and dense in $\mathcal{H}_k$:

$$\mathcal{M}_{2k} = \tilde{\mathcal{M}}_{2k} \equiv \mathcal{D}_k \sqsubset \mathcal{H}_k. \tag{7.7}$$

Theorem 7.1 *The scales of Hilbert spaces (7.1),(7.2) are s-similar in the generalized sense,* $\{\mathcal{H}_k\} \sim \{\tilde{\mathcal{H}}_k\}$*, iff for some* $k \geq 1$ *the operator* $\tilde{A}^k$ *is a strongly singular perturbation of* A^k*, i.e.,* $\tilde{A}^k \in \mathcal{P}_{\rm ss}(A^k)$.

Proof. By the construction of scales (7.1), (7.2) the inner products in $\mathcal{H}_k, \tilde{\mathcal{H}}_k$ are defined by the quadratic forms

$$\gamma_k(\varphi, \psi) := (A^k\varphi, \psi)_0, \quad \tilde{\gamma}_k(\varphi, \psi) := (\tilde{A}^k\varphi, \psi)_0.$$

Due to (7.5) we have:

$$(\varphi, \psi)_k = (A^k\varphi, \psi)_0 = (\varphi, \psi)_{\tilde{k}} = (\tilde{A}^k\varphi, \psi)_0 \quad \varphi, \psi \in \mathcal{D}_k.$$

Since $\mathcal{D}_k$ is dense in $\mathcal{H}_k$ (see (7.4)) the restrictions of $A^k, \tilde{A}^k$ onto $\mathcal{D}_k$ coincide:

$$A^k \mid \mathcal{D}_k = \tilde{A}^k \mid \mathcal{D}_k. \tag{7.8}$$

Therefore these restrictions produce the same densely defined symmetric operator $(A^k)^\circ$ in $\mathcal{H}_0$. This operator is closed since $\mathcal{D}_k = \mathcal{M}_{2k} = \tilde{\mathcal{M}}_{2k}$ is the closed subspace both in $\mathcal{H}_{2k}$ and $\tilde{\mathcal{H}}_{2k}$. Clearly each of the operators $A^k, \tilde{A}^k$ is the self-adjoint extension of $(A^k)^\circ$. We recall that since the set $\mathcal{D}_k$ is dense in $\mathcal{H}_k$, the Friedrichs extension of $(A^k)^\circ$ coincides with A^k. So, by definition any other self-adjoint extension of $(A^k)^\circ$ belongs to $\mathcal{P}_{\rm ss}(A^k)$. Thus $\tilde{A}^k \in \mathcal{P}_{\rm ss}(A^k)$. The inverse assertion evidently is also true. $\Box$

We remark that due to (7.7) similarly as for (7.5) we have

$$\|\varphi\|_{2k} = \|\varphi\|_{\widetilde{2k}},\ \varphi \in \mathcal{D}_k. \tag{7.9}$$

However in general the set $\mathcal{D}_k$ does not belong to $\mathcal{H}_{4k} \cap \tilde{\mathcal{H}}_{4k}$. For this reason (7.9) does not imply that $\tilde{A}^{2k}$ is a singular perturbation of A^{2k}.

We note else that according to Theorem 5.2, see (5.16),(5.17), the singular quadratic form defined by

$$\gamma(A^{-k}\cdot, A^{-k}\cdot) := (\tilde{A}^{-k}\cdot, \cdot)_0 - (A^{-k}\cdot, \cdot)_0$$

belongs to the $\mathcal{H}_{-2}-$class with respect to the operator A^k since the set $\mathcal{D}_k$ is dense in $\mathcal{H}_k$. In the case where $\tilde{A}^{-k} \in \mathcal{P}_{\rm ws}(A^k)$, the spaces $\tilde{\mathcal{H}}_k$, $\mathcal{H}_k$ coincide as sets in $\mathcal{H}_0$ however have different norms. Thus, the quadratic form $\gamma[\varphi] := (\tilde{A}^k\varphi, \varphi)_0 - (A^k\varphi, \varphi)_0$ is bounded in $\mathcal{H}_k$ although belongs to the $\mathcal{H}_{-1}-$class with respect to the operator A^k.

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