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Interpolation, Schur Functions and Moment Problems

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Editorial Introduction

The present volume, entitled “Interpolation, Schur functions and moment problems”, is the second in the new subseries LOLS (*Linear Operators and Linear Systems* of the series *Operator Theory: Advances and Applications*). The main part of this volume is a selection of essays on various aspects of what is by some authors called *Schur analysis.*

To present the papers and set the volume into perspective, let us recall that a function analytic and contractive in the open unit disk is called a Schur function. In 1917, Schur associated to such a function a sequence, finite or infinite, of numbers in the open unit disk $\mathbb{D}$, called Schur coefficients. One can associate such a sequence also to a function analytic and with a positive real part in $\mathbb{D}$. Such functions are called Carathéodory functions and the associated coefficients are sometimes called Verblunsky coefficients. Carathéodory functions appear in the trigonometric moment problems via the Herglotz representation formula. Carathéodory and Schur functions have no poles in the open unit disk. Allowing functions with poles in $\mathbb{D}$ was first considered by Takagi in his 1924 paper [7]. Functions of the form $s(z) = \frac{p(z)}{z^n p(1/z^*)^*}$ (where $p(z)$ is a polynomial of degree n) play an important role in that paper, and are a particular instance of what was later known as generalized Schur functions. These are functions meromorphic in $\mathbb{D}$ and such that the kernel $\frac{1-s(z)s(w)^*}{1-zw^*}$ has a finite number of negative squares in the domain of holomorphy of s. Generalized Schur functions have been introduced independently (and in different ways) by M.G. Kreĭn and H. Langer [5] (these authors also defined in a similar way generalized Carathéodory functions) and by C. Chamfy and Dufresnoy [3], [2]. The theory of Schur and generalized Schur functions also make sense in the matrix and operator-valued cases, and are a continuous source of new problems, as is illustrated in the papers presented in this volume. We note that the translation of the papers of Schur and research papers on the Schur algorithm form the contents of volume 16 of the series OTAA, see [4] and that operator-valued generalized Schur functions have been studied in the volume 96 of the series OTAA, see [1].

Now we can say that under the word Schur analysis one encounters the variety of problems related to Schur and Carathéodory functions such as interpolation problems, moment problems, study of the relationships between the Schur coefficients and the properties of the function, study of underlying operators,... Such questions are also considered in the setting of generalized Schur and generalized Carathéodory functions, and in the “line case”, where functions analytic in a half-plane rather than in the open unit disk are considered and where Hankel operators replace Toeplitz operators.

The volume contains seven papers, and we now review their contents:

Boundary interpolation of generalized Schur functions: In the paper "*Basic boundary interpolation for generalized Schur functions and factorization of rational J-unitary matrix functions*" by **D. Alpay, A. Dijksma, H. Langer** and **G. Wanjala**, the authors develop the counterpart of the Schur algorithm for a generalized Schur function at a boundary point. This approach allows to solve the so-called *basic interpolation problem* introduced in earlier work for an inner point. In the paper "*Boundary Nevanlinna–Pick interpolation problems for generalized Schur functions*", **V. Bolotnikov** and **A. Kheifets** solve three different multipoints boundary interpolation problems. In both papers the problems take into account the particularities of the nonpositive case and have no direct analog in the positive case.

Discrete first-order systems: In a previous paper (which appeared in the first volume of the LOLS subseries), **D. Alpay** and **I. Gohberg** introduced the characteristic spectral functions associated to a discrete first order systems. The paper "*Discrete analogs of canonical systems with pseudo-exponential potential. Inverse problems*" continues this study and focuses on inverse problems. An important role is played by the solutions of an underlying Nehari interpolation problem which take unitary values on the unit circle and which admit a generalized Wiener–Hopf factorization.

Schur parameters of pseudocontinuable Schur functions: In the paper "*Shift operators contained in contractions, Schur parameters and pseudocontinuable Schur functions*", **V.K. Dubovoy** studies relationships between the maximal shift and coshift operator of a completely non unitary contraction. A main result in the paper is the characterisation of sequence of Schur coefficients for Schur functions which are not inner but admit a pseudo-analytic continuation of bounded type in the exterior of the open unit disk. The methods of the paper are an illustration of the feedback between function theory and operator theory methods.

The matrix-valued case: The matrix-valued case has difficulties of its own, in particular in the degenerate cases. In the paper "*A Truncated Matricial Moment Problem on a Finite Interval*", **A. Choque Rivero, Y. Dyukarev, B. Fritzsche** and **B. Kirstein** use Potapov's method of the Fundamental Matrix Inequality (FMI) to solve a matrix truncated moment problem on an interval. The scalar case had been considered by M.G. Kreĭn and A. Nudelman (see [6]). A complete description of the set of solutions is given in the strictly positive case. In the paper "*The Matricial Carathéodory Problem in Both Nondegenerate and Degenerate Cases*", **B. Fritzsche, B. Kirstein** and **A. Lasarow** develop a new approach to the matricial Carathéodory interpolation problem.

Inversion formula: In the paper "*A Gohberg-Heinig type inversion formula involving Hankel operators*", **G.J. Groenewald** and **M.A. Kaashoek** prove a formula for the inverse of an operator of the form $I - K_1K_2$ where K_1 and K_2 are Hankel operators between matricial L_1 spaces. The proof is given first for kernel functions of stable exponential type, and then uses an approximation argument. In the first step the state space method is used.

We note that the fourth and seventh papers are related to the line case, while the others deal with the disk case. This ends a short review of this volume.

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Basic Boundary Interpolation for Generalized Schur Functions and Factorization of Rational J-unitary Matrix Functions

Daniel Alpay, Aad Dijksma, Heinz Langer and Gerald Wanjala

Abstract. We define and solve a boundary interpolation problem for generalized Schur functions $s(z)$ on the open unit disk $\mathbb{D}$ which have preassigned asymptotics when z from $\mathbb{D}$ tends nontangentially to a boundary point $z_1 \in \mathbb{T}$. The solutions are characterized via a fractional linear parametrization formula. We also prove that a rational J-unitary 2×2-matrix function whose only pole is at z_1 has a unique minimal factorization into elementary factors and we classify these factors. The parametrization formula is then used in an algorithm for obtaining this factorization. In the proofs we use reproducing kernel space methods.

Mathematics Subject Classification (2000). Primary: 47A57, 46C20, 47B32; Secondary: 47A15.

Keywords. Generalized Schur function, Boundary interpolation, Rational J-unitary matrix function, Minimal factorization, Elementary factor, Brune section, Reproducing kernel space, Indefinite metric.

1. Introduction

Recall that $s(z)$ is a generalized Schur function with κ negative squares (for the latter we write $\mathrm{sq}_-(s) = \kappa$), if it is holomorphic in a nonempty open subset of the open unit disk $\mathbb{D}$ and if the kernel

$$K_s(z,w) = \frac{1 - s(z)s(w)^*}{1 - zw^*}, \quad z, w \in \mathcal{D}(s), \tag{1.1}$$

has κ negative squares on $\mathcal{D}(s)$, the domain of holomorphy of $s(z)$. We denote the class of generalized Schur functions $s(z)$ with $\mathrm{sq}_-(s) = \kappa$ by $\mathbf{S}_\kappa$ and set $\mathbf{S} =$

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$\cup_{\kappa\geq 0}\mathbf{S}_\kappa$. The function $s(z)\in\mathbf{S}_0$ has a holomorphic and contractive continuation to all of $\mathbb{D}$ and is called a (classical) Schur function. In fact, the following three statements are equivalent:

(a) $s(z)\in\mathbf{S}_0$.
(b) $s(z)$ is holomorphic on $\mathbb{D}$ and bounded by 1 there.
(c) $s(z)$ has the form

$$s(z)=\gamma z^n\prod_j\frac{|\alpha_j|}{\alpha_j}\frac{z-\alpha_j}{1-\alpha_j^*z}\,\exp\left(-\int_0^{2\pi}\frac{\mathrm{e}^{it}+z}{\mathrm{e}^{it}-z}\,d\mu(t)\right), \tag{1.2}$$

where n is a nonnegative integer, the α_j's are the zeros of $s(z)$ in $\mathbb{D}\setminus\{0\}$ repeated according to multiplicity, γ is a number of modulus one, and $\mu(t)$ is a nondecreasing bounded function on $[0,2\pi]$. The Blaschke product on the right-hand side of the first equality in (1.2) is finite or infinite and converges on $\mathbb{D}$, because $\sum_j(1-|\alpha_j|)<\infty$.

By a result of M.G. Krein and H. Langer [24], a function $s(z)\in\mathbf{S}_\kappa$ has a meromorphic extension to $\mathbb{D}$ and can be written as

$$s(z)=\left(\prod_{j=1}^{\kappa}\frac{z-\beta_j}{1-\beta_j^*z}\right)^{-1}s_0(z), \tag{1.3}$$

where $s_0(z)\in\mathbf{S}_0$, and the zeros β_j of the Blaschke product of order κ belong to $\mathbb{D}$ and satisfy $s_0(\beta_j)\neq 0$, $j=1,\dots,\kappa$. Conversely, every function $s(z)$ of the form (1.3) belongs to $\mathbf{S}_\kappa$. It follows from (1.3) that any function $s(z)\in\mathbf{S}$ has nontangential boundary values from $\mathbb{D}$ in almost every point of the unit circle $\mathbb{T}$. In particular, a rational function $s(z)\in\mathbf{S}$ of modulus one on $\mathbb{T}$ is holomorphic on $\mathbb{T}$, and it is the quotient of two finite Blaschke products.

A nonconstant function $s(z)\in\mathbf{S_0}$ has in $z_1\in\mathbb{T}$ a Carathéodory derivative , if the limits

$$\tau_0=\lim_{z\hat{\to}z_1}s(z)\text{ with }|\tau_0|=1,\quad \tau_1=\lim_{z\hat{\to}z_1}\frac{s(z)-\tau_0}{z-z_1} \tag{1.4}$$

exist, and then

$$\lim_{z\hat{\to}z_1}s'(z)=\tau_1.$$

Here and in the sequel $z\hat{\to}z_1$ means that z tends from $\mathbb{D}$ non-tangentially to z_1. The relation (1.4) is equivalent to the fact that the limit

$$\lim_{z\hat{\to}z_1}\frac{1-|s(z)|}{1-|z|}$$

exists and is finite and positive; in this case it equals $\tau_0^*\tau_1z_1$; see [33, p. 48]. The following basic boundary interpolation problem for Schur functions is a particular case of a multi-point interpolation problem considered by D. Sarason in [34]: Given

$z_1 \in \mathbb{T}$ and numbers τ_0, τ_1, $|\tau_0| = 1$, such that $\tau_0^* \tau_1 z_1$ is positive. Find all functions $s(z) \in \mathbf{S}_0$ such that the Carathéodory derivative of $s(z)$ in z_1 exists and

$$\lim_{z \hat{\to} z_1} s(z) = \tau_0, \qquad \lim_{z \hat{\to} z_1} \frac{s(z) - \tau_0}{z - z_1} = \tau_1.$$

The study of the Schur transformation for generalized Schur functions in [14], [1], and [3] motivates the generalization of this basic interpolation problem for generalized Schur functions, which we consider in this note.

Problem 1.1. *Let $z_1 \in \mathbb{T}$, an integer $k \geq 1$, and complex numbers τ_0, τ_k, τ_{k+1}, $\ldots$, τ_{2k-1} with $|\tau_0| = 1$, $\tau_k \neq 0$ be given. Find all functions $s(z) \in \mathbf{S}$ such that*

$$s(z) = \tau_0 + \sum_{i=k}^{2k-1} \tau_i (z - z_1)^i + \mathrm{O}((z - z_1)^{2k}), \quad z \hat{\to} z_1. \tag{1.5}$$

We solve this problem under the assumption that the matrix

$$\mathbb{P} := \tau_0^* T B \tag{1.6}$$

is Hermitian, where

$$T = \begin{pmatrix} \tau_k & 0 & \cdots & 0 & 0 \\ \tau_{k+1} & \tau_k & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tau_{2k-2} & \tau_{2k-3} & \cdots & \tau_k & 0 \\ \tau_{2k-1} & \tau_{2k-2} & \cdots & \tau_{k+1} & \tau_k \end{pmatrix} \tag{1.7}$$

and

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{k-1}\binom{k-1}{0} z_1^{2k-1} \\ 0 & 0 & \cdots & (-1)^{k-2}\binom{k-2}{0} z_1^{2k-3} & (-1)^{k-1}\binom{k-1}{1} z_1^{2k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -\binom{1}{0} z_1^3 & \cdots & (-1)^{k-2}\binom{k-2}{k-3} z_1^{k} & (-1)^{k-1}\binom{k-1}{k-2} z_1^{k+1} \\ z_1 & -\binom{1}{1} z_1^2 & \cdots & (-1)^{k-2}\binom{k-2}{k-2} z_1^{k-1} & (-1)^{k-1}\binom{k-1}{k-1} z_1^{k} \end{pmatrix}. \tag{1.8}$$

Evidently, for $k = 1$ the expression in (1.6) reduces to $\tau_0^* \tau_1 z_1$ from above. In Theorem 3.2 we describe all solutions of this problem by a parametrization formula of the form

$$s(z) \;=\; T_{\Theta(z)}(s_1(z)) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)}, \quad \Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}, \tag{1.9}$$

where the parameter $s_1(z)$ runs through a subclass of $\mathbf{S}$. The matrix function $\Theta(z)$ is rational with a single pole at $z = z_1$ and J-unitary on $\mathbb{T}$ for

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that a rational 2×2-matrix function $\Theta(z)$ is J-unitary on $\mathbb{T}$ if

$$\Theta(z)J\Theta(z)^* = J, \quad z \in \mathbb{T} \setminus \{\text{poles of } \Theta(z)\}.$$

We prove the description (1.9) of the solutions of the Problem 1.1 by making use of the theory of reproducing kernel Pontryagin spaces, see [19], [4], [5], [6] for the positive definite (Hilbert space) case and [2], [3] for the indefinite case. The essential tool is a representation theorem for reproducing kernel Pontryagin spaces which will be formulated at the end of this Introduction.

Boundary interpolation problems for classical Schur functions have been studied by I.V. Kovalishina in [23], [22], by J.A. Ball, I. Gohberg, and L. Rodman in [12, Section 21] and by D. Sarason [34], and for generalized Schur functions which are holomorphic at the interpolation points by J.A. Ball in [11]. In these papers different methods were used: the fundamental matrix inequality, realization theory and extension theory of operators.

Problem 1.1 is similar to the basic interpolation problem for generalized Schur functions at the point $z = 0$ considered in [3]. There, given an arbitrary complex number σ_0, one looks for generalized Schur functions $s(z)$ which are holomorphic in $z = 0$ and satisfy $s(0) = \sigma_0$. In the case that $|\sigma_0| = 1$ a certain number of derivatives has to be preassigned in order to find all solutions. In Problem 1.1 this additional information comes from the preassigned values τ_j, $j = k, k+1, \dots, 2k-1$, and $\tau_1 = \tau_2 = \cdots = \tau_{k-1} = 0$.

The Problem 1.1 is equivalent to a basic boundary interpolation problem for generalized Nevanlinna functions at infinity, where one looks for the set of all generalized Nevanlinna functions $N(\zeta)$ with an asymptotics of the form

$$N(\zeta) = -\frac{s_0}{\zeta} - \frac{s_1}{\zeta^2} - \cdots - \frac{s_{2k-2}}{\zeta^{2k-1}} + \mathrm{O}\left(\frac{1}{\zeta^{2k}}\right), \quad \zeta = i\eta, \ \eta \to \infty.$$

In fact, these problems can be transformed into each other via Cayley transformation, and we mention that the cases $\tau_0^*\tau_1 z_1 > 0$, $= 0$, or < 0 correspond to the cases $s_0 > 0$, $= 0$, or < 0, respectively, and the hermiticity of the matrix $\mathbb{P}$ in (1.6) corresponds to the reality of the moments s_j. On the other hand, each of these problems has special features and it seems reasonable to study them also separately. Moreover, the boundary interpolation problem for generalized Nevanlinna functions at infinity is equivalent to the indefinite power moment problem as considered in (see [25], [26], [27], [28] [17], [18]). We shall come back to the basic versions of these problems in another publication.

Basic interpolation problems are closely related to the problem of decomposing a rational J-unitary 2×2-matrix function as a minimal product of elementary factors. For the positive definite case these results go back to V.P. Potapov ([30], [31] and the joint paper [20] with A.V. Efimov); see also L. de Branges [16, Problem 110, p 116]. In the indefinite case, for a J-unitary matrix function on the circle $\mathbb{T}$ with poles in $\mathbb{D}$ this was done in [2], and for the line case in [7]. Here we prove a corresponding factorization result for a rational J-unitary 2×2-matrix function $\Theta(z)$ with a single pole on the boundary $\mathbb{T}$ of $\mathbb{D}$. In fact, with the given matrix function $\Theta(z)$ a basic boundary interpolation problem can be associated, such that the matrix function which appears in the description of its solutions is an elementary factor of $\Theta(z)$.

A short outline of the paper is as follows. In Section 2 we study the asymptotic behavior of the kernel $K_s(z,w)$ near z_1 for a generalized Schur function $s(z)$ which has an asymptotic behavior (1.5) with not necessarily vanishing coefficients $\tau_1,\dots,\tau_{k-1}$. It turns out, that an expansion of $s(z)$ up to an order $2k$ implies a corresponding expansion of the kernel up to an order $2k-1$ only if a certain matrix $\mathbb{P}$ is Hermitian. This matrix $\mathbb{P}$, in some interpolation problems called the Pick or Nevanlinna matrix, is the essential ingredient for the solution of the basic interpolation problem. It satisfies the so-called Stein equation (see (2.17)) which is a basic tool for the definition of the underlying reproducing kernel spaces.

In Section 3 the main result of the paper (Theorem 3.2) is proved, which contains the solution of Problem 1.1. In Section 4 we consider a basic boundary interpolation problem with data given in several points $z_1,z_2,\dots,z_N$ of the circle $\mathbb{T}$ and describe all its solutions via a parametrization formula. In Section 5 the existence of a minimal factorization of a J-unitary matrix function on $\mathbb{T}$ with a single pole on $\mathbb{T}$ is proved. Finally, in Section 6 we show how by means of the Schur algorithm, based on the parametrization formula of Theorem 3.2, such a minimal factorization can be obtained.

For the convenience of the reader we formulate here a basic representation theorem for reproducing kernel Pontryagin spaces, see [9], which will be essentially used in this paper. Infinite-dimensional versions of this result were proved by L. de Branges [15] and J. Rovnyak [29] for the line case, and by J.A. Ball [10] for the circle case. For a rational J-unitary 2×2-matrix function $\Theta(z)$ on $\mathbb{D}$ we denote by $\mathcal{P}(\Theta)$ the reproducing kernel Pontryagin space with reproducing kernel

$$K_\Theta(z,w)=\frac{J-\Theta(z)J\Theta(w)^*}{1-zw^*},\qquad z,w\in\mathcal{D}(\Theta).$$

Theorem 1.2. *. Let $\mathcal{M}$ be a finite-dimensional reproducing kernel Pontryagin space. Then $\mathcal{M}=\mathcal{P}(\Theta)$ for some rational J-unitary 2×2-matrix function $\Theta(z)$ which is holomorphic at $z=0$ if and only if the following three conditions hold:*

(1) *The elements of $\mathcal{M}$ are 2-vector functions holomorphic at $z=0$.*
(2) *$\mathcal{M}$ is invariant under the difference quotient operator*

$$(R_0f)(z)=\frac{f(z)-f(0)}{z},\quad f\in\mathcal{M}.$$

(3) *The following identity holds:*

$$\langle f,g\rangle_{\mathcal{M}}-\langle R_0f,R_0g\rangle_{\mathcal{M}}=g(0)^*Jf(0),\quad f,g\in\mathcal{M}. \tag{1.10}$$

In this case $\mathcal{M}$ is spanned by the elements of the form $R_0^n\Theta(z)\mathbf{c}$, where n runs through the integers ≥ 1 and $\mathbf{c}$ through $\mathbb{C}^2$.

In the sequel, for $s(z) \in \mathbf{S}$ we denote by $\mathcal{P}(s)$ the reproducing kernel Pontryagin space with reproducing kernel $K_s(z,w)$ given by (1.1). The negative index of this space equals the number of negative squares of $s(z)$.

2. Auxiliary statements

For given numbers $\tau_0, \tau_1, \dots, \tau_{2k-1}$ we introduce the following $k \times k$-matrices:

$$\widehat{T} = (t_{\ell r})_{\ell,r=0}^{k-1}, \quad t_{\ell r} = \tau_{\ell+r+1}, \tag{2.1}$$

$$\widehat{B} = (b_{rs})_{r,s=0}^{k-1}, \quad b_{rs} = z_1^{k+r-s}\binom{k-1-s}{r}(-1)^{k-1-s}, \tag{2.2}$$

and

$$Q = (c_{sm})_{s,m=0}^{k-1}, \quad c_{sm} = \tau^*_{s+m-(k-1)}. \tag{2.3}$$

Here $\widehat{B}$ is a left upper, Q is a right lower triangular matrix.

Lemma 2.1. *Suppose that the function $s(z) \in \mathbf{S}$ has the asymptotic expansion*

$$s(z) = \tau_0 + \sum_{\ell=1}^{2k-1} \tau_\ell (z-z_1)^\ell + \mathrm{O}\left((z-z_1)^{2k}\right), \quad z \hat{\to} z_1, \tag{2.4}$$

with $|\tau_0| = 1$, and that the matrix $\mathbb{P} := \widehat{T}\widehat{B}Q$ is Hermitian. Then the kernel $K_s(z,w)$ has the asymptotic expansion

$$\begin{aligned} K_s(z,w) = &\sum_{0 \le \ell+m \le 2k-2} \alpha_{\ell m}(z-z_1)^\ell (w-z_1)^{*m} \\ &+\mathrm{O}\left((\max\{|z-z_1|,|w-z_1|\})^{2k-1}\right), \quad z,w \hat{\to} z_1, \end{aligned} \tag{2.5}$$

where the coefficients $\alpha_{\ell m}$ for $0 \le \ell, m \le k-1$ are the entries of the matrix $\mathbb{P}$: $\mathbb{P} = (\alpha_{\ell m})_{\ell,m=0}^{k-1}$.

Proof. The asymptotic expansion (2.5) will follow if we show that the relation

$$\begin{aligned} 1 - s(z)s(w)^* - &\sum_{0 \le \ell+m \le 2k-2} \alpha_{\ell m}(z-z_1)^\ell (w-z_1)^{*m}(1-zw^*) \\ &= \mathrm{O}\left((\max\{|z-z_1|,|w-z_1|\})^{2k}\right) \end{aligned} \tag{2.6}$$

holds, where the symbol O refers again to the non-tangential limit $z,w \hat{\to} z_1$. To see this we consider only the radial limits of z and w and observe that then for z and w sufficiently close to z_1 the relation

$$|1 - zw^*| \ge \max\{|z-z_1|,|w-z_1|\}$$

holds. Dividing (2.6) by $1 - zw^*$ we obtain

$$K_s(z,w) - \sum_{0 \le \ell+m \le 2k-2} \alpha_{\ell m}(z-z_1)^\ell (w-z_1)^{*m} = \frac{\mathrm{O}\left((\max\{|z-z_1|,|w-z_1|\})^{2k}\right)}{\max\{|z-z_1|,|w-z_1|\}},$$

and this is (2.5).

To prove (2.6) we set $u = z - z_1$, $v = w^* - z_1^*$. Then the expression on the left-hand side of (2.6) becomes

$$
\begin{aligned}
1 - \left(\tau_0 + \tau_1 u + \tau_2 u^2 + \cdots + \mathrm{O}(u^{2k})\right) \left(\tau_0^* + \tau_1^* v + \tau_2^* v^2 + \cdots + \mathrm{O}(v^{2k})\right) \\
- \textstyle\sum_{0 \le \ell + m \le 2k-2} \alpha_{\ell m} u^\ell v^m (-u z_1^* - v z_1 - uv).
\end{aligned} \tag{2.7}
$$

Comparing coefficients we find that the following relations are equivalent for (2.6) to hold:

$$
u\colon \quad \tau_0^* \tau_1 = \alpha_{00} z_1^*, \qquad v\colon \quad \tau_0 \tau_1^* = \alpha_{00} z_1, \tag{2.8}
$$

$$
u^2\colon \; \tau_2 \tau_0^* = \alpha_{10} z_1^*, \quad uv\colon \; \tau_1^* \tau_1 = \alpha_{00} + \alpha_{01} z_1^* + \alpha_{10} z_1, \quad v^2\colon \; \tau_0 \tau_2^* = \alpha_{01} z_1, \tag{2.9}
$$

$$
\begin{aligned}
u^3\colon \quad \tau_3 \tau_0^* = \alpha_{20} z_1^*, \quad u^2 v\colon \quad \tau_2 \tau_1^* = \alpha_{10} + \alpha_{11} z_1^* + \alpha_{20} z_1, \\
uv^2\colon \quad \tau_1 \tau_2^* = \alpha_{01} + \alpha_{11} z_1 + \alpha_{02} z_1^*, \quad v^3\colon \quad \tau_0 \tau_3^* = \alpha_{02} z_1,
\end{aligned}
$$

etc. The general relation is

$$
\begin{aligned}
\tau_\ell \tau_m^* = \alpha_{\ell-1,m} z_1^* + \alpha_{\ell,m-1} z_1 + \alpha_{\ell-1,m-1}, \\
\ell, m = 0, 1, \ldots, 2k-2, \; 1 \le \ell + m \le 2k-2,
\end{aligned} \tag{2.10}
$$

where all $\alpha's$ with one index $= -1$ are set equal to zero, and we have to find solutions $\alpha_{\ell m}$ of this system (2.10). The relation (2.10) can be written as

$$
\alpha_{\ell m} = -z_1^* \alpha_{\ell-1,m} - z_1^{*2} \alpha_{\ell-1,m+1} + z_1^* \tau_\ell \tau_{m+1}^*, \quad 0 \le \ell + m \le 2k-2, \tag{2.11}
$$

and also as

$$
\alpha_{\ell m} = -z_1 \alpha_{\ell,m-1} - z_1^2 \alpha_{\ell+1,m-1} + z_1 \tau_{\ell+1} \tau_m^*, \quad 0 \le \ell + m \le 2k-2. \tag{2.12}
$$

The numbers $\alpha_{\ell m}$, $0 \le \ell + m \le 2k-2$ in (2.6) or (2.10) can be considered as the entries of a left upper triangular matrix $\widetilde{\mathbb{P}}$, which has the matrix $\mathbb{P}$ as its left upper $k \times k$ diagonal block. According to the assumption, $\mathbb{P}$ is a Hermitian matrix. The elements of the last row of $\mathbb{P}$ determine according to (2.11) the left lower $k \times k$ block of $\widetilde{\mathbb{P}}$, which is a left upper triangular matrix, and, similarly, the last column of $\mathbb{P}$ determines by the relations (2.10) the right upper $k \times k$ block of $\widetilde{\mathbb{P}}$. These relations and the hermiticity of $\mathbb{P}$ imply that also the matrix $\widetilde{\mathbb{P}}$ is Hermitian.

From (2.12) we find successively

$$
\begin{aligned}
\alpha_{\ell 0} &= \tau_0^* z_1 \tau_{\ell+1}, && \ell = 0, \ldots, 2k-2, \\
\alpha_{\ell 1} &= \tau_1^* z_1 \tau_{\ell+1} - \tau_0^* (z_1^2 \tau_{\ell+1} + z_1^3 \tau_{\ell+2}), && \ell = 0, \ldots, 2k-3, \\
\alpha_{\ell 2} &= \tau_2^* z_1 \tau_{\ell+1} - \tau_1^* (z_1^2 \tau_{\ell+1} + z_1^3 \tau_{\ell+2}) + \tau_0^* (z_1^3 \tau_{\ell+1} + 2 z_1^4 \tau_{\ell+2} + z_1^5 \tau_{\ell+3}), \\
& && \ell = 0, \ldots, 2k-4, \\
\alpha_{\ell 3} &= \tau_3^* z_1 \tau_{\ell+1} - \tau_2^* (z_1^2 \tau_{\ell+1} + z_1^3 \tau_{\ell+2}) - \tau_1^* (z_1^3 \tau_{\ell+1} + 2 z_1^4 \tau_{\ell+2} + z_1^5 \tau_{\ell+3}) \\
& \quad - \tau_0^* (z_1^3 \tau_{\ell+1} + 3 z_1^4 \tau_{\ell+2} + 3 z_1^5 \tau_{\ell+3} + z_1^6 \tau_{\ell+4}), && \ell = 0, \ldots, 2k-5,
\end{aligned} \tag{2.13}
$$

and so for $m = 0, \dots, 2k-2$, we have

$$\alpha_{\ell m} = \sum_{s=0}^{m} \tau^*_{m-s} \sum_{r=0}^{s} (-1)^s \binom{s}{r} z_1^{s+r+1} \tau_{\ell+r+1}, \quad \ell = 0, \dots, 2k-2-m.$$

With the convention that $\tau_\ell = 0$ for $\ell < 0$, observing that $\binom{s}{r} = 0$ if $r > s$, and substituting s by $k-1-s$ we find for $0 \le \ell, m \le k-1$

$$\alpha_{\ell m} = \sum_{r,s=0}^{k-1} \tau_{\ell+r+1} (-1)^{k-1-s} \binom{k-1-s}{r} z_1^{k-s+r} \tau^*_{m+s-(k-1)} = \sum_{r,s=0}^{k-1} t_{\ell r} b_{rs} c_{sm}$$

and hence (see (2.1)–(2.3))

$$(\alpha_{\ell m})_{\ell,m=0}^{k-1} = \widehat{T}\widehat{B}Q.$$

These considerations also imply that if a solution of the equations (2.10) exists, it is unique.

As to the existence of a solution, the first relation in (2.13) determines the elements of the first column of $\widetilde{\mathbb{P}}$, and the following columns are successively determined by the other relations of (2.13) or by (2.12). Because of the symmetry of $\widetilde{\mathbb{P}}$, the resulting elements $\alpha_{0\ell}$ are the complex conjugates of $\alpha_{\ell 0}$, $\ell = 1, 2, \dots, 2k-2$, and α_{00} is real. Thus, these $\alpha' s$ satisfy all the relations of the system (2.10) and hence are its unique solution. □

The relation (2.10) implies that

$$\alpha_{\ell-1,m} z_1^* + \alpha_{\ell,m-1} z_1 + \alpha_{\ell-1,m-1} = \tau_\ell \tau_m^*, \qquad 1 \le \ell,\, m \le k-1. \tag{2.14}$$

If we introduce the $k \times k$-matrices

$$S_k = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \qquad A = z_1^* I_k \,+\, S_k, \tag{2.15}$$

and the $2 \times k$-matrix

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tau_0^* & \tau_1^* & \cdots & \tau_{k-1}^* \end{pmatrix}, \tag{2.16}$$

then the relation (2.14) is equivalent to the relation (2.17) below, and hence we have:

Corollary 2.2. *Under the assumptions of Lemma* 2.1 *the matrix* $\mathbb{P}$ *satisfies the Stein equation*

$$\mathbb{P} - A^* \mathbb{P} A = C^* J C. \tag{2.17}$$

Remark 2.3. 1) Formula (2.8) implies a condition on τ_0 and τ_1: the number $\tau_0^*\tau_1 z_1$ has to be real. As was mentioned in the Introduction, for Schur functions this number must be nonnegative if it is finite. In (2.9) the first and the last equation determine α_{10} and α_{01}, the second equation is an additional condition. Similarly in the relations following (2.9): the first and last equation determine α_{20} and α_{02}, then there are 2 equations left for to determine α_{11}. These additional conditions are automatically satisfied since the matrix $\mathbb{P}$ is Hermitian.

2) If the equations (2.10) have a solution $\alpha_{\ell m}$, $0 \le \ell + m \le 2k-2$, then these numbers must be symmetric in the sense that $\alpha_{\ell m} = \alpha_{m\ell}^*$, $0 \le \ell + m \le 2k-2$, since they are the coefficients of the expansion of the Hermitian kernel $K_s(z,w)$.

3) For a function $s(z) \in \mathbf{S}$ with an expansion (2.4), such that the corresponding matrix $\mathbb{P}$ is not Hermitian, the kernel $K_s(z,w)$ does in general not have an expansion (2.5). An example is the function

$$s(z) = 1 + \frac{1}{2}(z-1),$$

which has at $z=1$ an expansion (2.4) with any $k \ge 1$ but for the corresponding kernel we obtain, for example, for real z, w,

$$K_s(z,w) = \frac{1}{2} + \frac{1}{4}\frac{(z-1)(w^*-1)}{1-zw^*} = \frac{1}{2} + \mathrm{O}\big(\max\{|1-z|, |1-w|\}\big),$$

and the order of the last term cannot be improved. For this example it holds

$$\mathbb{P} = \begin{cases} 1/2 & k=1, \\ \begin{pmatrix} 1/2 & -1/4 \\ 0 & 0 \end{pmatrix} & k=2. \end{cases}$$

4) For a function $s(z)$ which is analytic on an arc around z_1 and has values of modulus one on this arc the matrices $\mathbb{P}$ are Hermitian for all k and the kernel $K_s(z,w)$ is analytic in z and w^* near $z = w = z_1$. To see this we observe that the function $s(z)$ satisfies in some neighborhood of this arc the relation $s(1/z^*) = 1/s(z)^*$. Now it follows that in this neighborhood, for each fixed w the function $K_s(\,\cdot\,, w)$ and for each fixed z the function $K_s(z,\,\cdot\,)^*$ is holomorphic. According to a theorem of Hartogs [32, Theorem 16.3.1] the kernel $K_s(z,w)$ is holomorphic in z and w and the claim follows. We mention, that a function $s(z) \in \mathbf{S}_\kappa$ has the above properties if and only if in its representation (see (1.2) and (1.3))

$$s(z) = \left(\prod_{j=1}^{\kappa} \frac{z-\beta_j}{1-\beta_j^* z}\right)^{-1} \gamma z^n \prod_j \frac{|\alpha_j|}{\alpha_j}\frac{z-\alpha_j}{1-\alpha_j^* z} \;\exp\left(-\int_0^{2\pi} \frac{\mathrm{e}^{it}+z}{\mathrm{e}^{it}-z}\, d\mu(t)\right)$$

the nondecreasing function $\mu(t)$ is constant at t_1 where $z_1 = \exp(it_1)$. In particular, all rational functions in $\mathbf{S}$, which are of modulus one on $\mathbb{T}$, have these properties.

Lemma 2.4. *Under the assumptions of Lemma* 2.1 *the functions*

$$f_0(z) = \frac{1 - s(z)\tau_0^*}{1 - zz_1^*}$$

and

$$f_\ell(z) = \frac{zf_{\ell-1}(z) - s(z)\tau_\ell^*}{1 - zz_1^*}, \quad \ell = 1, 2, \ldots, k-1,$$

are elements of $\mathcal{P}(s)$ *and* $\langle f_\ell, f_m \rangle_{\mathcal{P}(s)} = \alpha_{m\ell}$.

Proof. First we note that for $z \in \mathbb{D}$ and $\ell = 0, 1, \ldots, k-1$,

$$f_\ell(z) = \lim_{w \hat{\to} z_1} \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(z, w).$$

This implies that for all $w' \in \mathbb{D}$

$$\lim_{w \hat{\to} z_1} \left\langle \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(\,\cdot\,, w), K_s(\,\cdot\,, w') \right\rangle_{\mathcal{P}(s)} = \lim_{w \hat{\to} z_1} \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(w', w) = f_\ell(w'), \tag{2.18}$$

and for $\ell, m = 0, 1, \ldots, k-1$

$$\lim_{w \hat{\to} z_1, w' \hat{\to} z_1} \left\langle \frac{1}{\ell!} \frac{\partial^\ell}{\partial w^{*\ell}} K_s(\,\cdot\,, w), \frac{1}{m!} \frac{\partial^m}{\partial w'^{*m}} K_s(\,\cdot\,, w') \right\rangle_{\mathcal{P}(s)} \tag{2.19}$$

$$= \lim_{w \hat{\to} z_1, w' \hat{\to} z_1} \frac{1}{\ell! m!} \frac{\partial^{\ell+m}}{\partial w^{*\ell} \partial w'^m} K_s(w', w) = \alpha_{m\ell}. \tag{2.20}$$

The claim follows now from [21, Theorem 2.4] and [8, Theorem 1.1.2]. In fact, (2.18) and (2.19) imply $f_\ell \in \mathcal{P}(s)$, $\ell = 1, 2, \ldots, k-1$, and (2.19) also yields the formula for the inner product between the f_ℓ's. □

In Section 4 below we also need the following generalization of Lemma 2.1. To formulate it, we suppose that at two points $z_1, z_2 \in \mathbb{T}$, $z_1 \neq z_2$, the function $s(z) \in \mathbf{S}$ has the asymptotic expansions

$$s(z) = \tau_{1;0} + \sum_{\ell=1}^{2k_1-1} \tau_{1;\ell}(z - z_1)^\ell + \mathrm{O}\left((z - z_1)^{2k_1}\right), \quad z \hat{\to} z_1, \tag{2.21}$$

$$s(z) = \tau_{2;0} + \sum_{m=1}^{2k_2-1} \tau_{2;m}(z - z_2)^m + \mathrm{O}\left((z - z_2)^{2k_2}\right), \quad z \hat{\to} z_2, \tag{2.22}$$

and we introduce for $i = 1, 2$ the $k_i \times k_i$-matrices

$$A_i = z_i^* I_{k_i} + S_{k_i}$$

and the $2 \times k_i$-matrices

$$C_i = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tau_{i;0}^* & \tau_{i;1}^* & \cdots & \tau_{i;k_i-1}^* \end{pmatrix}.$$

Lemma 2.5. *Suppose that at two points* $z_1, z_2 \in \mathbb{T}$, $z_1 \neq z_2$, *the function* $s(z) \in \mathbf{S}$ *has the asymptotic expansions* (2.21) *and* (2.22). *Then the kernel* $K_s(z,w)$ *has the asymptotic expansion*

$$K_s(z,w) = \sum_{\substack{0 \le \ell \le k_1 - 1, \\ 0 \le m \le k_2 - 1}} \alpha_{\ell m}(z-z_1)^\ell (w-z_2)^{*m} + \mathrm{O}\left((\max\{|z-z_1|^{k_1}, |w-z_2|^{k_2}\})\right), \quad z \hat{\to} z_1,\ w \hat{\to} z_2,$$

where

$$\alpha_{\ell m} = \lim_{z \hat{\to} z_1, w \hat{\to} z_2} \frac{1}{\ell!\, m!} \frac{\partial^\ell}{\partial z^\ell} \frac{\partial^m}{\partial w^{*m}} K_s(z,w).$$

Moreover, the $k_1 \times k_2$*-matrix* $\mathbb{P}_{12} = (\alpha_{\ell m})$, $0 \le \ell \le k_1 - 1, 0 \le m \le k_2 - 1$, *satisfies the relation*

$$\mathbb{P}_{12} - A_1^* \mathbb{P}_{12} A_2 = C_1^* J C_2. \tag{2.23}$$

Proof. Similar to the proof of Lemma 2.1 we set now $u = z - z_1$, $v = w^* - z_2^*$, and equate the coefficients of their powers in the analog of the expression in (2.7):

$$1 - \left(\tau_{1;0} + \tau_{1;1}u + \tau_{1;2}u^2 + \cdots + \mathrm{O}(u^{2k_1})\right)\left(\tau_{2;0}^* + \tau_{2;1}^* v + \tau_{2;2}^* v^2 + \cdots + \mathrm{O}(v^{2k_2})\right)$$
$$- \textstyle\sum_{0 \le \ell \le k_1 - 1, 0 \le m \le k_2 - 1} \alpha_{\ell m} u^\ell v^m (-u z_2^* - v z_1 - uv + 1 - z_1 z_2^*).$$

This gives

$$1 - \tau_{1;0}\tau_{2;0}^* = \alpha_{0,0}(1 - z_1 z_2^*),$$

and for $0 \le \ell \le k_1 - 1$, $0 \le m \le k_2 - 1$, and $\ell + m > 0$,

$$\tau_{1;\ell}\tau_{2;m}^* = \alpha_{\ell-1,m} z_2^* + \alpha_{\ell,m-1} z_1 + \alpha_{\ell-1,m-1} + \alpha_{\ell m}(1 - z_1 z_2^*),$$

which is easily seen to be equivalent to (2.23). □

3. The basic interpolation problem at one boundary point

With the data of the Problem 1.1 the $k \times k$-matrix T was defined in (1.7), and we recall the definition of B in (1.8). Then the matrix $\mathbb{P}$ from Lemma 2.1 can be written in the form

$$\mathbb{P} = \tau_0^* T B. \tag{3.1}$$

Observe that $\mathbb{P}$ is a right lower triangular matrix, which is invertible because of τ_0, τ_k, $z_1 \neq 0$. We define the vector function

$$R(z) = \begin{pmatrix} \dfrac{1}{1 - z z_1^*} & \dfrac{z}{(1 - z z_1^*)^2} & \cdots & \dfrac{z^{k-1}}{(1 - z z_1^*)^k} \end{pmatrix},$$

fix some $z_0 \in \mathbb{T}$, $z_0 \neq z_1$ and introduce the polynomial $p(z)$ by

$$p(z) = (1 - z z_1^*)^k\, R(z) \mathbb{P}^{-1} R(z_0)^*. \tag{3.2}$$

It has degree at most $k - 1$ and $p(z_1) \neq 0$.

Lemma 3.1. *With* $p(z)$ *from* (3.2) *we have that*

$$\tau_0 \frac{(1-zz_1^*)^k}{(1-zz_0^*)p(z)} = -\sum_{i=k}^{2k-1} \tau_i (z-z_1)^i + \mathrm{O}\left((z-z_1)^{2k}\right), \quad z \hat{\to} z_1.$$

Proof. Since $1 - zz_1^* = -z_1^*(z-z_1)$, it suffices to show that if

$$\tau_0 \frac{(-1)^{k-1} z_1^{*k}}{(1-zz_0^*)p(z)} = \sigma_k + \sigma_{k+1}(z-z_1) + \cdots + \sigma_{2k-1}(z-z_1)^{k-1} + \mathrm{O}\left((z-z_1)^k\right), \tag{3.3}$$

then $\sigma_j = \tau_j$, $j = k, k+1, \ldots, 2k-1$. An expansion of the form (3.3) exists because the quotient on the left-hand side is rational and the denominator does not vanish at $z = z_1$. Write

$$1 - zz_0^* = -z_0^*[(z-z_1) + (z_1 - z_0)],$$

$$p(z) = \sum_{j=0}^{k-1} p_j (z-z_1)^j = \begin{pmatrix} 1 & z-z_1 & \cdots & (z-z_1)^{k-1} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix},$$

and define

$$T' = \begin{pmatrix} \sigma_k & 0 & \cdots & 0 \\ \sigma_{k+1} & \sigma_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{2k-1} & \sigma_{2k-2} & \cdots & \sigma_k \end{pmatrix}.$$

From

$$\begin{aligned} &\left(\sigma_k + \sigma_{k+1}(z-z_1) + \cdots + \sigma_{2k-1}(z-z_1)^{k-1}\right) \begin{pmatrix} 1 & z-z_1 & \cdots & (z-z_1)^{k-1} \end{pmatrix} \\ &\quad = \begin{pmatrix} 1 & z-z_1 & \cdots & (z-z_1)^{k-1} \end{pmatrix} T' + \mathrm{O}\left((z-z_1)^k\right), \end{aligned}$$

the definition of the shift matrix S_k from (2.15), and (3.3) we obtain

$$\begin{aligned} &\tau_0 (-1)^{k-1} z_0 z_1^{*k} \\ &\quad = \begin{pmatrix} 1 & (z-z_1) & \cdots & (z-z_1)^{k-1} \end{pmatrix} \left((z_1 - z_0) I_k + S_k^*\right) T' \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix}, \end{aligned}$$

and it follows that

$$T' \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \tau_0 \frac{(-1)^{k-1} z_0 z_1^{*k}}{z_0 - z_1} \begin{pmatrix} 1 \\ \dfrac{1}{z_0 - z_1} \\ \vdots \\ \dfrac{1}{(z_0 - z_1)^{k-1}} \end{pmatrix}. \tag{3.4}$$

On the other hand, from the definition of $p(z)$ it follows that

$$p(z) = \tau_0 \begin{pmatrix} (1-zz_1^*)^{k-1} & z(1-zz_1^*)^{k-2} & \cdots & z^{k-1} \end{pmatrix} \times B^{-1}T^{-1}\frac{z_0}{z_0-z_1}\begin{pmatrix} 1 \\ \frac{1}{z_0-z_1} \\ \vdots \\ \frac{1}{(z_0-z_1)^{k-1}} \end{pmatrix}.$$

A straightforward calculation shows that

$$\begin{pmatrix} (1-zz_1^*)^{k-1} & z(1-zz_1^*)^{k-2} & \cdots & z^{k-1} \end{pmatrix} = \begin{pmatrix} 1 & z-z_1 & \cdots & (z-z_1)^{k-1} \end{pmatrix} B(-1)^{k-1}z_1^{*k}$$

and hence

$$T\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \tau_0\frac{(-1)^{k-1}z_0z_1^{*k}}{z_0-z_1}\begin{pmatrix} 1 \\ \frac{1}{z_0-z_1} \\ \vdots \\ \frac{1}{(z_0-z_1)^{k-1}} \end{pmatrix}.$$

This equality and (3.4) imply

$$\begin{pmatrix} \sigma_k & 0 & \cdots & 0 \\ \sigma_{k+1} & \sigma_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{2k-1} & \sigma_{2k-2} & \cdots & \sigma_k \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} = \begin{pmatrix} \tau_k & 0 & \cdots & 0 \\ \tau_{k+1} & \tau_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{2k-1} & \tau_{2k-2} & \cdots & \tau_k \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{k-1} \end{pmatrix}.$$

From this relation, because of $p_0 = p(z_1) \neq 0$, it readily follows that $\sigma_j = \tau_j$, $j = k, k+1, \ldots, 2k-1$. □

For a Hermitian matrix $\mathbb{P}$, by $\mathrm{ev}_-(\mathbb{P})$ we denote the number of negative eigenvalues of $\mathbb{P}$.

Theorem 3.2. *Given $z_1 \in \mathbb{T}$ and $\tau_0, \tau_k, \ldots, \tau_{2k-1}$ as in Problem 1.1 such that the matrix $\mathbb{P}$ in (1.6) is Hermitian, and let $\Theta(z)$ be the J-unitary rational matrix function*

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = I_2 - \frac{(1-zz_0^*)p(z)}{(1-zz_1^*)^k}\mathbf{u}\mathbf{u}^*J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ \tau_0^* \end{pmatrix}, \tag{3.5}$$

with $p(z)$ from (3.2) and fixed $z_0 \in \mathbb{T}$, $z_0 \neq z_1$. Then the fractional linear transformation

$$s(z) = T_{\Theta(z)}(s_1(z)) = \frac{a(z)s_1(z)+b(z)}{c(z)s_1(z)+d(z)}, \tag{3.6}$$

establishes a bijective correspondence between all solutions $s(z)$ *of Problem* 1.1 *and all* $s_1(z) \in \mathbf{S}$ *with the property*

$$\liminf_{z \hat{\to} z_1} |s_1(z) - \tau_0| > 0. \tag{3.7}$$

Moreover, if $s(z)$ *and* $s_1(z)$ *are related by* (3.6) *then*

$$\mathrm{sq}_-(s) = \mathrm{sq}_-(s_1) + \mathrm{ev}_-(\mathbb{P}). \tag{3.8}$$

Proof. With the given numbers $\tau_0, \tau_k, \ldots, \tau_{2k-1}$ we define the space $\mathcal{M}$ as the span of the functions

$$\mathbf{f}_\ell(z) = \frac{z^\ell}{(1 - zz_1^*)^{\ell+1}}\mathbf{u}, \quad \ell = 0, 1, \ldots, k-1. \tag{3.9}$$

Then

$$\begin{pmatrix} \mathbf{f}_0(z) & \mathbf{f}_1(z) & \ldots & \mathbf{f}_{k-1}(z) \end{pmatrix} = C(I_k - zA)^{-1}, \tag{3.10}$$

where the matrix C from (2.16) specializes now to

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \tau_0^* & 0 & \cdots & 0 \end{pmatrix}, \tag{3.11}$$

and $A = z_1^* I_k + S_k$ as in (2.15) with S_k being the $k \times k$ shift matrix. Endowing the space $\mathcal{M}$ with the inner product

$$\langle \mathbf{f}_m, \mathbf{f}_\ell \rangle_{\mathcal{M}} = (\mathbb{P})_{\ell,m} = \alpha_{\ell m} \tag{3.12}$$

we have that $\mathcal{M}$ is a reproducing kernel Pontryagin space with reproducing kernel equal to

$$C(I_k - zA)^{-1}\mathbb{P}^{-1}(I_k - wA)^{-*}C^*. \tag{3.13}$$

Evidently, the negative index of this space is equal to $\mathrm{ev}_-(\mathbb{P})$.

On the other hand, according to (2.17) the matrix $\mathbb{P}$ satisfies the Stein equation

$$\mathbb{P} - A^*\mathbb{P}A = C^*JC,$$

where now the expressions on both sides are equal to zero. Therefore for $\mathcal{M}$ all the conditions of Theorem 1.2 are satisfied, and hence there exists a J-unitary rational 2×2-matrix function

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

such that $\mathcal{M} = \mathcal{P}(\Theta)$, the reproducing kernel Pontryagin space with reproducing kernel $\dfrac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}$. By the uniqueness of the reproducing kernel it must coincide with the kernel from (3.13):

$$C(I_k - zA)^{-1}\mathbb{P}^{-1}(I_k - wA)^{-*}C^* = \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*}.$$

Thus if we normalize $\Theta(z)$ by $\Theta(z_0) = I_2$ we obtain

$$\Theta(z) = I_2 - (1 - zz_0^*)C(I_k - zA)^{-1}\mathbb{P}^{-1}(I_k - z_0A)^{-*}C^*J.$$

By (3.9) and (3.10) this matrix function can be written as

$$\Theta(z) = I_2 - (1 - zz_0^*)\,\mathbf{u}\,R(z)\,\mathbb{P}^{-1}R(z_0)^*\,\mathbf{u}^*J,$$

and this coincides with the formula for $\Theta(z)$ in the theorem.

Now we consider a solution $s(z)$ of Problem 1.1:

$$s(z) = \tau_0 + \sum_{\ell=k}^{2k-1} \tau_\ell (z - z_1)^\ell + \mathrm{O}((z - z_1)^{2k}), \quad z \hat{\to} z_1.$$

According to Lemma 2.1 the corresponding kernel $K_s(z,w)$ admits the representation (2.5):

$$K_s(z,w) = \sum_{0 \le \ell+m \le 2k-2} \alpha_{\ell m}(z - z_1)^\ell (w - z_1)^{*m}$$
$$+\mathrm{O}\left((\max\{|z - z_1|, |w - z_1|\})^{2k-1}\right), \quad z, w \hat{\to} z_1,$$

with

$$\alpha_{\ell m} = \lim_{z,w \hat{\to} z_1} \frac{1}{\ell! m!} \frac{\partial^{\ell+m}}{\partial w^{*m} \partial z^\ell} K_s(z,w) = \alpha_{m\ell}^*. \tag{3.14}$$

From

$$K_s(z,w) = \frac{1 - s(z)s(w)^*}{1 - zw^*} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{\begin{pmatrix} 1 \\ s(w)^* \end{pmatrix}}{1 - zw^*}$$

we see that

$$\lim_{w \hat{\to} z_1} \frac{1}{m!} \frac{\partial^m}{\partial w^{*m}} K_s(z,w) = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \mathbf{f}_m(z), \quad m = 0, \ldots, k-1.$$

On the other hand, according to Lemma 2.4 the elements

$$f_m(z) = \lim_{w \hat{\to} z_1} \frac{1}{m!} \frac{\partial^m}{\partial w^{*m}} K_s(z,w) = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \mathbf{f}_m(z), \quad m = 0, 1, \ldots, k-1,$$

belong to the reproducing kernel Pontryagin space $\mathcal{P}(s)$ with reproducing kernel $K_s(z,w)$ and

$$\left\langle \begin{pmatrix} 1 & -s \end{pmatrix} \mathbf{f}_m, \begin{pmatrix} 1 & -s \end{pmatrix} \mathbf{f}_\ell \right\rangle_{\mathcal{P}(s)} = \lim_{z,w \hat{\to} z_1} \frac{1}{\ell! m!} \frac{\partial^{m+\ell}}{\partial w^{*m} \partial z^\ell}\, K_S(z,w). \tag{3.15}$$

By (3.15), (3.12), and (3.14) the map $\mathcal{T}$ of multiplication by $\begin{pmatrix} 1 & -s(z) \end{pmatrix}$ is an isometry from $\mathcal{M}$ into $\mathcal{P}(s)$. Setting

$$s_1(z) = \frac{b(z) - d(z)s(z)}{c(z)s(z) - a(z)}$$

we have that $s(z)$ is of the desired form:

$$s(z) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)}. \tag{3.16}$$

From

$$K_s(z,w) = \begin{pmatrix}1 & -s(z)\end{pmatrix} \frac{J-\Theta(z)J\Theta(w)^*}{1-zw^*} \begin{pmatrix}1 & -s(w)\end{pmatrix}^* \tag{3.17}$$
$$+ (a(z)-c(z)s(z))K_{s_1}(z,w)(a(w)-c(w)s(w))^*,$$

and since $\mathcal{T}$ is an isometry, it follows that $s_1(z)$ is a generalized Schur function and

$$\mathcal{P}(s) = \mathcal{T}\mathcal{M} \oplus (a-cs)\mathcal{P}(s_1).$$

By the observations at the end of the Introduction and after formula (3.12) this implies the equality (3.8).

From the definition (3.5) of $\Theta(z)$:

$$\Theta(z) = \begin{pmatrix}1-\theta(z) & \tau_0\theta(z)\\ -\tau_0^*\theta(z) & 1+\theta(z)\end{pmatrix}, \quad \theta(z) = \frac{(1-zz_0^*)p(z)}{(1-zz_1^*)^k} = (1-zz_0^*)R(z)\mathbb{P}^{-1}R(z_0)^*, \tag{3.18}$$

and (3.16) we obtain

$$s(z) - \tau_0\left(1 - \frac{(1-zz_1^*)^k}{(1-zz_0^*)p(z)}\right)$$
$$= \frac{\tau_0(1-zz_1^*)^{2k}}{(1-zz_0^*)p(z)\left\{(1-zz_1^*)^k - \tau_0^*(1-zz_0^*)p(z)(s_1(z)-\tau_0)\right\}}. \tag{3.19}$$

By Lemma 3.1 the expression on the left is $\mathrm{O}((z-z_1)^{2k})$, $z\hat{\to}z_1$, and this can only be the case if (3.7) holds. Thus, every solution of the Problem 1.1 is of the form given in the theorem.

As to the existence of solutions, the equality (3.19) readily implies that any function $s(z)$ of the form (3.6) has the desired asymptotics and since $\Theta(z)$ is J-unitary and rational, the formula (3.17) implies that if $s_1(z)$ belongs to the class $\mathbf{S}$ then also $s(z)$ belongs to this class. $\square$

Remark 3.3. 1) The J-unitarity of $\Theta(z)$ implies that

$$p(z) = z_0(-z_1^*)^k z^{k-1} p\left(\frac{1}{z^*}\right)^*. \tag{3.20}$$

2) Note that the matrix function $\Theta(z)$ in Theorem 3.2 is normalized such that $\Theta(z_0) = I_2$. Replacing z_0 by another point $\widehat{z}_0 \in \mathbb{T}$, $\widehat{z}_0 \neq z_1$, amounts to multiplying $\Theta(z)$ from the right by a J-unitary constant matrix. This follows from the fact that the fractional linear transformations with the corresponding matrix function $\widehat{\Theta}(z)$ and with $\Theta(z)$ have the same range. It can also be shown directly using the equality (3.22) below.

3) For $\theta(z)$ as in (3.18) we have

$$\theta(z) = (1-zz_0^*)R(z)\mathbb{P}^{-1}R(z_0)^*, \quad R(z) = \begin{pmatrix}1 & 0 & \cdots & 0\end{pmatrix}(I-zA)^{-1}, \tag{3.21}$$

where $A = S_k + z_1^* I_k$. If the point z_0 is replaced by another point $\widehat{z}_0 \in \mathbb{T}$, $\widehat{z}_0 \neq z_0, z_1$, then for the corresponding function $\widehat{\theta}(z)$ the difference $\theta(z) - \widehat{\theta}(z)$ is independent of z. In fact, a direct calculation using (3.21) and (2.17) with $C^*JC = 0$ shows that

$$\theta(z) - \widehat{\theta}(z) = -\widehat{\theta}(z_0). \tag{3.22}$$

4) For rational parameters $s_1(z)$ the condition (3.7) is equivalent to the fact that the denominator in (3.6):

$$c(z)s_1(z) + d(z) = -\tau_0^*(s_1(z) - \tau_0)\theta(z) + 1$$

has a pole of order k (see (3.18)).

5) The matrix $\mathbb{P}$ in (1.6) is right lower triangular and the entries on the second main diagonal are given by

$$(\mathbb{P})_{i,k-1-i} = (-1)^{k-1-i} z_1^{2k-1-2i} \tau_0^* \tau_k, \quad i = 0, 1, \ldots, k-1. \tag{3.23}$$

If $\mathbb{P}$ is Hermitian, then by (3.23), $z_1^k \tau_0^* \tau_k$ is purely imaginary if k is even and real if k is odd, and we have

$$\mathrm{ev}_-(\mathbb{P}) = \begin{cases} k/2, & k \text{ even}, \\ (k-1)/2, & k \text{ odd}, \ (-1)^{(k-1)/2} z_1^k \tau_0^* \tau_k > 0, \\ (k+1)/2, & k \text{ odd}, \ (-1)^{(k-1)/2} z_1^k \tau_0^* \tau_k < 0. \end{cases}$$

Recall that the Schur algorithm is originally defined for a Schur function $s(z)$. Theorem 3.2 allows us to define an analog for functions $s(z)$ in the class $\mathbf{S}$ which have an asymptotics (1.5) at z_1 with a Hermitian matrix $\mathbb{P}_k$ and $\tau_k \neq 0$. The Schur transform of $s(z)$ is the function $\widehat{s}(z) := s_1(z) = T_{\Theta(z)^{-1}}(s(z))$ with $\Theta(z)$ as in Theorem 3.2. By this Schur transformation the set of functions in $\mathbf{S}$ with the above mentioned properties is mapped into $\mathbf{S}$. The Schur algorithm consists in iterating the Schur transformation. It will be considered in Sections 5 and 6.

4. Multipoint boundary interpolation

We generalize Problem 1.1 to an interpolation problem with N distinct points $z_1, \ldots, z_N$ on the unit circle.

Problem 4.1. *Let $N \geq 1$ be an integer, let $z_1, \ldots, z_N$ be N distinct points on $\mathbb{T}$, let $k_1, \ldots, k_N$ be integers ≥ 1, and let $\tau_{i;0}, \tau_{i;k_i}, \tau_{i;k_i+1}, \ldots, \tau_{i;2k_i-1}$ be complex numbers such that $|\tau_{i;0}| = 1$ and $\tau_{i;k_i} \neq 0$, $i = 1, \ldots, N$. Find all generalized Schur functions $s(z) \in \mathbf{S}$ such that*

$$s(z) = \tau_{i;0} + \sum_{\ell=k_i}^{2k_i-1} \tau_{i;\ell}(z - z_i)^\ell + \mathrm{O}((z - z_i)^{2k_i}), \quad z \hat{\to} z_i, \quad i = 1, \ldots, N.$$

Let $\mathbb{P}_i$, C_i, A_i, and $\Theta_i(z)$ be related to z_i as in Section 3 the matrices $\mathbb{P}$, C, A, and $\Theta(z)$ in formulas (3.1), (3.11), (2.15) and (3.5) are related to z_1. Set

$$C = \begin{pmatrix} C_1 & C_2 & \cdots & C_N \end{pmatrix}, \quad A = \mathrm{diag}\,(A_1, A_2, \ldots, A_N),$$

and denote by $\mathbb{P} = (\mathbb{P}_{ij})_{i,j=1}^N$ the $N \times N$ block matrix with $\mathbb{P}_{ii} = \mathbb{P}_i$ and $\mathbb{P}_{ij} \in \mathbb{C}^{k_i \times k_j}$ being the matrix given by (2.23) for $z_1 = z_i$ and $z_2 = z_j$, $i, j = 1, 2, \ldots, N$. Then, according to (2.17) and (2.23) the matrix $\mathbb{P}$ satisfies the Stein equation

$$\mathbb{P} - A^* \mathbb{P} A = C^* J C. \tag{4.1}$$

We note that the relation (2.23) in the situation of this section reads as

$$\mathbb{P}_{ij} - A_i^* \mathbb{P}_{ij} A_j = C_i^* J C_j = \begin{pmatrix} 1 - \tau_{i;0}\tau_{j;0}^* & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If no derivatives are involved, $\mathbb{P}_{ij}$ is a complex number and equal to $\dfrac{1 - \tau_{i;0}\tau_{j;0}^*}{1 - z_i^* z_j}$.

Theorem 4.2. *Assume that the matrix $\mathbb{P}$ is invertible and Hermitian and define the J-unitary matrix function $\Theta(z)$ by*

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} = I_2 - (1 - z z_0^*) C (I - zA)^{-1} \mathbb{P}^{-1} (I - z_0 A)^{-*} C^* J,$$

where z_0 is any point in $\mathbb{T}$ different from the interpolation points. Then the fractional linear transformation

$$s(z) = T_{\Theta(z)}(s_1(z)) = \frac{a(z)s_1(z) + b(z)}{c(z)s_1(z) + d(z)} \tag{4.2}$$

establishes a bijective correspondence between all solutions $s(z)$ of Problem 4.1 and all $s_1(z) \in \mathbf{S}$ with the properties

$$\liminf_{z \hat{\to} z_1} \left| \frac{\widehat{a}_i(z)s_1(z) + \widehat{b}_i(z)}{\widehat{c}_i(z)s_1(z) + \widehat{d}_i(z)} - \tau_{i,0} \right| > 0, \quad i = 1, \ldots, N, \tag{4.3}$$

where

$$\begin{pmatrix} \widehat{a}_i(z) & \widehat{b}_i(z) \\ \widehat{c}_i(z) & \widehat{d}_i(z) \end{pmatrix} = \widehat{\Theta}_i(z) := \Theta_i^{-1}(z)\Theta(z).$$

In the correspondence (4.2),

$$\mathrm{sq}_-(s) = \mathrm{ev}_-(\mathbb{P}) + \mathrm{sq}_-(s_1). \tag{4.4}$$

Proof. As in the proof of Theorem 3.2, to each of the interpolation points z_i is associated the finite-dimensional resolvent invariant space $\mathcal{M}_i$ of $\mathbb{C}^2$-valued rational functions spanned by the columns of the matrix function $C_i(I - zA_i)^{-1}$. Then the space $\mathcal{M} = \oplus_{i=1}^N \mathcal{M}_i$ is spanned by the columns of the matrix function $C(I - zA)^{-1}$. We endow $\mathcal{M}$ with the inner product defined by $\mathbb{P}$. It follows from Theorem 1.2 that $\mathcal{M} = \mathcal{P}(\Theta)$ with $\Theta(z)$ as in the theorem.

Assume that $s(z)$ is a solution of the interpolation problem. We claim that the map $\mathcal{T} : f(z) \mapsto \begin{pmatrix} 1 & -s(z) \end{pmatrix} f(z)$ is an isometry from $\mathcal{P}(\Theta)$ into $\mathcal{P}(s)$. Indeed, because of the Stein equation (4.1) and the relations

$$\mathcal{T}C_i(I-zA_i)^{-1} = \left(\lim_{w \hat{\to} z_i} K_s(z,w) \quad \lim_{w \hat{\to} z_i} \frac{\partial}{\partial w^*} K_s(z,w) \quad \cdots \quad \lim_{w \hat{\to} z_i} \frac{1}{(k_i-1)!} \frac{\partial^{k_i-1}}{\partial w^{*(ki-1)}} K_s(z,w)\right),$$

where $i = 1, 2, \ldots, N$, the entries of the Gram matrix associated with the basis of the space $\mathcal{M}$, which is the union of the bases of the spaces $\mathcal{M}_i$, coincides with the Gram matrix of the images under $\mathcal{T}$. Hence

$$\mathcal{P}(s) = \mathcal{T}\mathcal{P}(\Theta) \oplus (a - cs)\mathcal{P}(s_1)$$

and $s(z) = T_{\Theta(z)}(s_1(z))$ for some generalized Schur function $s_1(z)$ satisfying (4.4). Since $\mathcal{M}_i$ is a non-degenerate R_0-invariant subspace of $\mathcal{M}$, $\Theta(z)$ admits the factorization $\Theta(z) = \Theta_i(z)\widehat{\Theta}_i(z)$, see [9]. Hence $s(z) = T_{\Theta(z)}(s_1(z)) = T_{\Theta_i(z)}(\widehat{s}_1(z))$ with

$$\widehat{s}_1(z) = T_{\widehat{\Theta}_i(z)}(s_1(z)) = \frac{\widehat{a}_i(z)s_1(z) + \widehat{b}_i(z)}{\widehat{c}_i(z)s_1(z) + \widehat{d}_i(z)}.$$

This shows that $s(z)$ is a solution of the interpolation problem at z_i with parameter $\widehat{s}_1(z)$, therefore, according to (3.7), $\widehat{s}_1(z)$ satisfies (4.3).

Conversely, let $s(z) = T_{\Theta(z)}(s_1(z))$ be given with a function $s_1(z)$ as in the theorem. If we write $s(z) = T_{\Theta_i(z)}(\widehat{s}_1(z))$, then, since $\widehat{\Theta}_i(z) = \Theta_i^{-1}(z)\Theta(z)$ is J-unitary, $\widehat{s}_1(z)$ is a generalized Schur function and by (3.7) it has all the properties of the parameters in Theorem 3.2 and hence $s(z)$ is a solution of Problem 4.1 □

Remark 4.3. 1) There exist rational parameters $s_1(z)$ satisfying the conditions (4.3) for $i=1,\ldots,N$. Indeed for each i there is a unique constant $s_i = T_{\Theta(z_i)^{-1}}(\tau_{i;0})$ such that in (4.3) there is equality rather than inequality. It suffices to take for $s_1(z)$ any constant of modulus 1 which is different from these s_i, $i = 1, 2, \ldots, N$.

2) If $k_i = 1$, $i = 1, 2, \ldots, N$, a description of all rational Schur functions which satisfy the given interpolation conditions was given by J.A. Ball, I. Gohberg, and L. Rodman [12, Theorem 21.1.2]: in this case the conditions (4.3) reduce to the fact that $c(z)s_1(z) + d(z)$ has poles of order 1 at $z = z_i$, $i = 1, 2, \ldots, N$. Indeed, with

$$\Theta_i(z) = \begin{pmatrix} a_i(z) & b_i(z) \\ c_i(z) & d_i(z) \end{pmatrix}$$

and the relations in the proof of the theorem we have

$$c(z)s_1(z) + d(z) = (c_i(z)\widehat{s}_1(z) + d_i(z))(\widehat{c}_i(z)s_1(z) + \widehat{d}_i(z)).$$

According to Remark 3.3, 4) the first factor on the right-hand side has a pole of order 1 at z_i and the second factor is rational and nonzero at z_i.

3) We give an example where $\mathbb{P}$ is not invertible while its diagonal entries are invertible. For such matrices the assumptions of Theorem 4.2 are not satisfied. Take $N = 2$, two distinct points z_1 and z_2 on $\mathbb{T}$, $k_1 = k_2 = 1$, $\tau_{1;0} = 1$, $\tau_{2;0} = -1$, and numbers $\tau_{1;1}, \tau_{2;1}$ such that $z_1\tau_{1;1}, z_2\tau_{2;1} \in \mathbb{R}$ and $z_1z_2\tau_{1;1}\tau_{2;1} = 4/|1-z_1z_2^*|^2$. Then $\mathbb{P}_1$ and $\mathbb{P}_2$ are invertible, $\mathbb{P}$ satisfies the Stein equation (4.1) but is not invertible.

5. J-unitary factorization

In this section z_0 and z_1 are two distinct points in $\mathbb{T}$. By $\mathcal{U}_{z_1}$ we denote the set of all rational J-unitary 2×2-matrix functions $\Theta(z)$ with a pole only at $z = z_1$, and by $\mathcal{U}_{z_1}^{z_0}$ the set of all matrix functions $\Theta(z) \in \mathcal{U}_{z_1}$ which are normalized such that $\Theta(z_0) = I_2$. In particular, the matrix functions of $\mathcal{U}_{z_1}$ are bounded at ∞.

Lemma 5.1. *If* $\Theta(z) \in \mathcal{U}_{z_1}$ *then* $\det \Theta(z) \equiv c$ *for some* $c \in \mathbb{T}$, *and* $\Theta(z)^{-1} \in \mathcal{U}_{z_1}$.

Proof. The J-unitarity of $\Theta(z)$ on $\mathbb{T}$ and the analyticity outside $z = z_1$ imply the identity

$$\Theta(z) J \Theta(1/z^*)^* = J, \quad z \in \mathbb{C} \setminus \{0, z_1\}.$$

For the rational function $f(z) = \det\Theta(z)$ it follows that $|f(z)| = 1$, $z \in \mathbb{T}$. Therefore f cannot have a pole at z_1, and since it is also bounded at ∞ it must be constant. □

By the degree of a rational J-unitary matrix function $\Theta(z)$ we mean the McMillan degree (see [13]) and we write it as $\deg \Theta(z)$. If $\Theta(z) \in \mathcal{U}_{z_1}$ and

$$\Theta(z) = \sum_{i=0}^{n} T_i (z - z_1)^{-i},$$

where the T_i's are constant 2×2-matrices and $T_n \neq 0$, then

$$\deg \Theta = \operatorname{rank} \begin{pmatrix} T_n & 0 & \cdots & 0 \\ T_{n-1} & T_n & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_1 & T_2 & \cdots & T_n \end{pmatrix}.$$

A product $\Theta_1(z)\Theta_2(z) \cdots \Theta_n(z) = \Theta(z)$ of rational J-unitary matrix functions is called minimal if the degrees add up, that is,

$$\deg \Theta_1(z) + \deg \Theta_2(z) + \cdots + \deg \Theta_n(z) = \deg \Theta(z).$$

In this case the product on the left-hand side is also called a minimal factorization of $\Theta(z)$. An example of a nonminimal product is given by the equality $\Theta(z)\Theta(z)^{-1} = I_2$ for any nonconstant $\Theta(z) \in \mathcal{U}_{z_1}$, since, because of Lemma 5.1, the inverse $\Theta(z)^{-1}$ also belongs to $\mathcal{U}_{z_1}$.

A matrix function $\Theta(z) \in \mathcal{U}_{z_1}$ is called elementary if in any minimal factorization $\Theta(z) = \Theta_1(z)\Theta_2(z)$ at least one of the factors is a J-unitary constant.

Theorem 5.2. *Assume* $z_0, z_1 \in \mathbb{T}$ *and* $z_0 \neq z_1$. *Then:*

(i) *The matrix function* $\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$ *is elementary if and only if it is of the form*

$$\Theta(z) = I_2 - \frac{(1 - z z_0^*) p(z)}{(1 - z z_1^*)^k} \mathbf{u}\mathbf{u}^* J, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 \\ \zeta \end{pmatrix}, \tag{5.1}$$

where k *is an integer* ≥ 1, $\zeta \in \mathbb{T}$, $p(z)$ *is a polynomial of degree* $\leq k - 1$ *satisfying* (3.20) *and* $p(z_1) \neq 0$.

(ii) *Every* $\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$ *admits a unique minimal factorization*

$$\Theta(z) = \Theta_1(z) \cdots \Theta_n(z), \tag{5.2}$$

in which each $\Theta_j(z)$ *is an elementary normalized factor of the form* (5.1).

The theorem implies that the matrix function $\Theta(z)$ in (3.5) belongs to the class $\mathcal{U}_{z_1}^{z_0}$ and is elementary. The proof of Theorem 5.2 hinges on the fact that the reproducing kernel space $\mathcal{P}(\Theta)$ consists of one Jordan chain for the difference quotient operator R_0, which makes the elementary factors unique. In case of higher dimensions this uniqueness does not hold.

Proof of Theorem 5.2. Let $\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$. We claim that $\mathcal{P}(\Theta)$ is spanned by a single chain for R_0 at the eigenvalue $\lambda = z_1^*$. To see this, let λ be an eigenvalue of R_0 with eigenelement $\mathbf{f}_0(z)$: $R_0\mathbf{f}_0(z) = \lambda\mathbf{f}_0(z)$. Then

$$\mathbf{f}_0(z) = \frac{\mathbf{c}_0}{1-\lambda z}, \quad \mathbf{c}_0 = \mathbf{f}_0(0) \neq 0,$$

and since the elements of $\mathcal{P}(\Theta)$ have a pole only at $z = z_1$, we conclude that $\lambda = z_1^*$. The identity (1.10) and $|z_1| = 1$ imply that $\mathbf{c}_0$ is J-neutral:

$$\mathbf{c}_0^* J\mathbf{c}_0 = \langle \mathbf{f}_0, \mathbf{f}_0\rangle_{\mathcal{P}(\Theta)} - \langle z_1^*\mathbf{f}_0, z_1^*\mathbf{f}_0\rangle_{\mathcal{P}(\Theta)} = 0.$$

If

$$\mathbf{g}_0(z) = \frac{\mathbf{d}_0}{1-zz_1^*}, \quad \mathbf{d}_0 \in \mathbb{C}^2,$$

is another eigenfunction, then also $\mathbf{d}_0$ is J-neutral and (1.10) yields $\mathbf{c}_0^*J\mathbf{d}_0 = 0$. Since J is invertible, this implies that $\mathbf{d}_0$ is a multiple of $\mathbf{c}_0$ and hence the geometric multiplicity of the eigenvalue $\lambda = z_1^*$ is 1. This proves the claim. It follows that there are vectors $\mathbf{c}_j \in \mathbb{C}^2$, $\mathbf{c}_0$ being J-neutral, such that $\mathcal{P}(\Theta)$ is spanned by

$$\mathbf{f}_j(z) = \frac{z\mathbf{f}_{j-1}(z) + \mathbf{c}_j}{1-zz_1^*}, \quad j = 0, \dots, N-1, \quad \mathbf{f}_{-1}(z) \equiv 0.$$

Since $\mathbf{c}_0$ is nonzero and J-neutral, its components have the same nonzero absolute value and hence we may suppose without loss of generality that for some unimodular number ζ_0,

$$\mathbf{c}_0 = \begin{pmatrix} 1 \\ \zeta_0 \end{pmatrix}.$$

Let k be the smallest integer ≥ 1 such that $\langle \mathbf{f}_0, \mathbf{f}_{k-1}\rangle_{\mathcal{P}(\Theta)} \neq 0$, hence, if $k \geq 2$,

$$\langle \mathbf{f}_0, \mathbf{f}_j\rangle_{\mathcal{P}(\Theta)} = 0, \quad j = 0, \dots, k-2.$$

Then the subspace

$$\mathcal{M} = \operatorname{span}\{\mathbf{f}_0, \mathbf{f}_1, \dots, \mathbf{f}_{k-1}\}$$

is the smallest R_0-invariant subspace of $\mathcal{P}(\Theta)$ which is non-degenerate and hence, by Theorem 1.2, it is a $\mathcal{P}(\Theta_1)$-space for some rational J-unitary 2×2-matrix function $\Theta_1(z)$. We prove that $\Theta_1(z)$ is of the form described by (5.1).

To this end we first show that without loss of generality we may assume that

$$\mathbf{c}_1 = \cdots = \mathbf{c}_{k-1} = 0. \tag{5.3}$$

By the identity (1.10) we have

$$\mathbf{c}_0^* J \mathbf{c}_j = \langle \mathbf{f}_j, \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} - \langle z_1^* \mathbf{f}_j + \mathbf{f}_{j-1}, z_1^* \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} = 0$$

and, since $\mathbf{c}_0^* J \mathbf{c}_0 = 0$ and J is invertible, $\mathbf{c}_j$ is a multiple of $\mathbf{c}_0$. Successively for $j = 1, \ldots, k-1$, we may replace $\mathbf{c}_j$ in $\mathbf{f}_j(z)$ by zero by subtracting from $\mathbf{f}_j(z)$ a suitable multiple of the eigenfunction $\mathbf{f}_0(z)$. Thus we obtain a chain which satisfies (5.3) and still spans $\mathcal{M}$. By (5.3), this new chain coincides with the columns of the matrix $C(I_k - zA)^{-1}$ with C and A as in (3.11) and $\tau_0^* = \zeta$. Denote by $\mathbb{P}$ the corresponding Gram matrix:

$$\mathbb{P} = (p_{ij})_{i,j=0}^{k-1}, \quad p_{ij} = \langle \mathbf{f}_j, \mathbf{f}_i \rangle_{\mathcal{P}(\Theta)}, \ i, j = 0, 1, \ldots, k-1.$$

For the reproducing kernel $\Theta_1(z)$ of the space $\mathcal{M}$ we obtain

$$\frac{J - \Theta_1(z) J \Theta_1(w)^*}{1 - zw^*} = C(I_k - zA)^{-1} \mathbb{P}^{-1} (I_k - wA)^{-*} C^*,$$

and hence

$$\Theta_1(z) = I_2 - (1 - z_0^* z) C (I_k - zA)^{-1} \mathbb{P}^{-1} (I_k - z_0 A)^{-*} C^* J.$$

As in the proof of Theorem 3.2 one can show that $\Theta_1(z)$ is of the form (5.1). From its construction it follows that $\Theta_1(z)$ is elementary: Assume on the contrary, that $\Theta_1(z) = \Theta'(z)\Theta''(z)$ is a minimal factorization with nonconstant factors. Then $\mathcal{P}(\Theta_1) = \mathcal{P}(\Theta') \oplus \Theta' \mathcal{P}(\Theta'')$ and $\mathcal{P}(\Theta')$ is a proper non-degenerate R_0-invariant subspace of $\mathcal{P}(\Theta_1)$ and hence also a subspace of $\mathcal{P}(\Theta)$. The construction above and the minimality of k imply that $\mathcal{P}(\Theta')$ is spanned by the same chain as $\mathcal{P}(\Theta_1)$, that is, $\mathcal{P}(\Theta') = \mathcal{P}(\Theta_1)$. The normalization implies $\Theta'(z) = \Theta_1(z)$ and $\Theta''(z) = I_2$.

Now we prove (i) and (ii).

(i) The arguments above imply that if $\Theta(z)$ is elementary, then $\Theta(z) = \Theta_1(z)$. We now prove that if $\Theta(z)$ is given by (5.1), then it is elementary. The formula (5.1) implies that $\Theta(z)$ is J-unitary, rational with only one pole of order k at $z = z_1$ and normalized by $\Theta(z_0) = I_2$. The space $\mathcal{P}(\Theta)$ is spanned by the elements $R_0^n \Theta(z)\mathbf{c}$, $n = 0, 1, \ldots$, and these are 2-vector functions of the form $x(z)\mathbf{u}$, where $x(z)$ is a rational function with at most one pole at $z = z_1$. The chain argument above shows that the space $\mathcal{P}(\Theta)$ is spanned by the following chain of R_0 at z_1

$$\mathbf{g}_0(z) = \frac{1}{(1 - zz_1^*)^1}\mathbf{u}, \quad \mathbf{g}_1(z) = \frac{z}{(1 - zz_1^*)^2}\mathbf{u}, \quad \ldots \quad , \quad \mathbf{g}_{k-1}(z) = \frac{z^{k-1}}{(1 - zz_1^*)^k}\mathbf{u}.$$

We claim that the Gram matrix G associated with this chain is right lower triangular. Then, since the space $\mathcal{P}(\Theta)$ is non-degenerate, the entries on the second diagonal of G are nonzero. The triangular form of G implies that the span of any sub-chain of the given chain is degenerate and hence $\Theta(z)$ is elementary.

It remains to prove the claim. For this we use the matrix representation of the operator R_0 relative to the basis $\mathbf{g}_j(z)$: it is the matrix $A = z_1^* I_k + S_k$ from (2.15). From (1.10) and since $\mathbf{u}$ is J-neutral, we have that

$$G - (z_1^* I_k + S_k)^* G (z_1^* I_k + S_k) = 0,$$

and hence

$$S_k^* G = G\left(-z_1^2 S_k + z_1^3 S_k + \cdots (-1)^{k-1} z_1^k S_k^{k-1}\right).$$

The triangular form of G can be deduced from this equality by comparing the entries of the matrices on both sides.

(ii) If $\Theta(z)$ and $\Theta_1(z)$ are as in the beginning of this proof, then by Lemma 5.1, $\Theta_2(z) = \Theta_1(z)^{-1}\Theta(z) \in \mathcal{U}_{z_1}^{z_0}$. From the orthogonal decomposition

$$\mathcal{P}(\Theta) = \mathcal{P}(\Theta_1) \oplus \Theta_1 \mathcal{P}(\Theta_2)$$

it follows that $\deg \Theta_2 = \deg \Theta - k$. The minimal factorization mentioned in part (ii) of the theorem now follows by repeating the foregoing arguments. $\square$

Since $\operatorname{rank} \mathbf{u}\mathbf{u}^* J = 1$, the elementary factor $\Theta(z)$ in Theorem 5.2 (i) has McMillan degree k, which, evidently, is the order of the pole of $\Theta(z)$ at $z = z_1$. The function $\Theta(z)$ in (5.1) is a generalization of a Brune section in the positive definite case where it is of the form

$$(I + \frac{1}{\gamma}\frac{z+a}{z-a}\mathbf{u}\mathbf{u}^* J)V,$$

with a normalizing constant J-unitary factor V, $a \in \mathbb{T}$, $\mathbf{u} \in \mathbb{C}^2$ with $\mathbf{u}^* J \mathbf{u} = 0$, and $\gamma > 0$.

6. A factorization algorithm

In this section we show how the factorization of a matrix function

$$\Theta(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix} \in \mathcal{U}_{z_1}^{z_0}$$

with z_1, $z_0 \in \mathbb{T}$, $z_0 \neq z_1$, can be derived from the Schur algorithm described at the end of Section 3. Similar arguments were presented in our previous papers [2] and [7] for polynomial matrix functions which are J-unitary on the unit circle or on the real line. We proceed in a number of steps.

Step 1: *Choose a number* $\tau \in \mathbb{T}$ *such that* (i)

$$s(z) = s_\tau(z) = \frac{a(z)\tau + b(z)}{c(z)\tau + d(z)} \tag{6.1}$$

is not a constant, (ii) $c(0)\tau + d(0) \neq 0$, *and* (iii)

$$O_{a\tau+b} = \max\{O_a, O_b\}, \quad O_{c\tau+d} = \max\{O_c, O_d\},$$

where, for example, O_a *stands for the order of the pole of the function* $a(z)$ *at* $z = z_1$. *Then* $s(z) \in \mathbf{S}$, *it is a rational function holomorphic and of modulus one on* $\mathbb{T}$ *and hence the quotient of two Blaschke factors.*

There are at most five distinct points $\tau \in \mathbb{T}$ for which (i)–(iii) do not hold: Assume that for three distinct points $\tau_1, \tau_2, \tau_3 \in \mathbb{T}$ the function $s(z)$ is a constant. Then, since $\Theta(z_0) = I_2$,

$$\frac{a(z)\tau_j + b(z)}{c(z)\tau_j + d(z)} = \tau_j, \quad j = 1, 2, 3, \ z \in \mathbb{C},$$

and we obtain that $c(z) \equiv 0$, $b(z) \equiv 0$, $a(z) \equiv d(z)$. Hence $\Theta(z) = a(z)I_2$. Since $\det \Theta(z)$ is a constant, we have that $a(z)$ is a constant, and so that $\Theta(z)$ is a constant matrix, which is a contradiction. Hence (i) holds with the exception of at most two different values of $\tau \in \mathbb{T}$. The condition in (ii) holds with the exception of at most one $\tau \in \mathbb{T}$, since $|\det \Theta(0)| = 1$. Finally, the conditions in (iii) hold, each with the exception at most one point $\tau \in \mathbb{T}$.

Step 2: *Let $s_1(z) = \widehat{s}(z)$ be the Schur transform of $s(z)$ (see the end of Section 3). Then $s_1(z) = T_{\Theta_1(z)^{-1}}(s(z))$ and $\Theta_1(z)$ is an elementary factor of $\Theta(z)$.*

From the proof of Theorem 3.2 we know that the map $\mathcal{T} : \mathbf{f}(z) \mapsto \begin{pmatrix} 1 & -s(z) \end{pmatrix} \mathbf{f}(z)$ is an isometry from $\mathcal{P}(\Theta_1)$ into $\mathcal{P}(s)$. We first show that $\mathcal{T}$ is a unitary mapping from $\mathcal{P}(\Theta)$ onto $\mathcal{P}(s)$. The fact that τ in (6.1) is a constant of modulus one implies the identity

$$\frac{1 - s(z)s(w)^*}{1 - zw^*} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{J - \Theta(z)J\Theta(w)^*}{1 - zw^*} \begin{pmatrix} 1 \\ -s(w)^* \end{pmatrix}. \tag{6.2}$$

This in turn implies that $\mathcal{T}$ is a partial isometry from $\mathcal{P}(\Theta)$ onto $\mathcal{P}(s)$, which is unitary if its kernel $\ker \mathcal{T}$ is trivial, see [8, Theorem 1.5.7]. Suppose

$$0 \neq \mathbf{f} = \begin{pmatrix} f \\ g \end{pmatrix} \in \ker \mathcal{T},$$

that is, $\begin{pmatrix} 1 & -s \end{pmatrix} \mathbf{f} = 0$, then

$$\mathbf{f} = \begin{pmatrix} s \\ 1 \end{pmatrix} g = \Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} x \in \mathcal{P}(\Theta), \quad x = \frac{g}{c\tau + d}.$$

Note that since $\det \Theta \neq 0$, we have that $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \neq 0$. Apply R_0 to $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} x$ to obtain

$$(R_0\Theta) \begin{pmatrix} \tau \\ 1 \end{pmatrix} x(0) + \Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} R_0 x \in \mathcal{P}(\Theta).$$

The first summand belongs to $\mathcal{P}(\Theta)$ and hence the second summand also belongs to $\mathcal{P}(\Theta)$. By repeatedly applying R_0, we find that

$$\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} R_0^j x \in \mathcal{P}(\Theta), \quad j = 0, 1, 2, \ldots.$$

Since x is a rational function there is an integer $n \geq 0$ such that the span of the functions $R_0^j x$, $j = 0, 1, \ldots, n$, is finite-dimensional and R_0-invariant. It follows that R_0 has an eigenvector v which has one of three possible forms: either $v \equiv 1$

or $v(z) = 1/(z - z_2)$ with $z_2 \neq z_1$ or $v(z) = 1/(1 - zz_1^*)$. All three possibilities lead to a contradiction:

$v \equiv 1$: This implies that $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \in \mathcal{P}(\Theta)$, and hence, since the elements in $\mathcal{P}(\Theta)$ all tend to 0 as $z \to \infty$, we see that $\Theta(\infty) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0$, but this cannot hold since $\det \Theta(\infty) \neq 0$.

$v(z) = 1/(z - z_2)$: This implies that $\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \dfrac{1}{z - z_2} \in \mathcal{P}(\Theta)$, and hence, since the elements in $\mathcal{P}(\Theta)$ are all holomorphic at $z = z_2$, we see that $\Theta(z_2) \begin{pmatrix} \tau \\ 1 \end{pmatrix} = 0$, and again this cannot hold since $\det \Theta(z_2) \neq 0$.

$v(z) = 1/(1 - zz_1^*)$: This implies that

$$\Theta \begin{pmatrix} \tau \\ 1 \end{pmatrix} \frac{1}{1 - zz_1^*} = \begin{pmatrix} \dfrac{a(z)\tau + b(z)}{1 - zz_1^*} \\ \dfrac{c(z)\tau + d(z)}{1 - zz_1^*} \end{pmatrix} \in \mathcal{P}(\Theta),$$

but this cannot hold because of conditions (iii) in Step 1 and because, according to the last statement in Theorem 1.2, if $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{P}(\Theta)$ then $O_f \leq \max\{O_a, O_b\}$ and $O_g \leq \max\{O_c, O_d\}$.

These contradictions imply that $\mathcal{T}$ has a trivial kernel and hence $\mathcal{T}$ is unitary.

We now claim that $\mathcal{P}(\Theta_1) \subset \mathcal{P}(\Theta)$ and that the inclusion map is isometric. Let $N_1 = \dim \mathcal{P}(\Theta_1)$ and $\mathbf{g}_0, \ldots, \mathbf{g}_{N_1-1}$ be a basis of $\mathcal{P}(\Theta_1)$ such that $R_0 \mathbf{g}_j = z_1 \mathbf{g}_j + \mathbf{g}_{j-1}$. One can choose $\mathbf{g}_j = \mathbf{f}_j$ for $j = 1, \ldots, N_1 - 1$. Indeed, let

$$\mathbf{g}_0(z) = \frac{1}{1 - zz_1^*} \begin{pmatrix} 1 \\ \eta \end{pmatrix},$$

then the function

$$\begin{pmatrix} 1 & -s(z) \end{pmatrix} (\mathbf{f}_0(z) - \mathbf{g}_0(z)) = -\frac{s(z)(\zeta_0 - \eta)}{1 - zz_1^*}$$

belongs to $\mathcal{P}(s)$, and thus $\zeta_0 = \eta$ since the elements of $\mathcal{P}(s)$ are holomorphic in z_1. Hence $\mathbf{f}_0(z) = \mathbf{g}_0(z)$. Moreover,

$$\langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta)} = \langle \mathcal{T}\mathbf{f}_0, \mathcal{T}\mathbf{f}_0 \rangle_{\mathcal{P}(s)} = \langle \mathbf{f}_0, \mathbf{f}_0 \rangle_{\mathcal{P}(\Theta_1)}.$$

In the same way it follows that $\mathbf{f}_\ell(z) = \mathbf{g}_\ell(z)$, $\ell = 1, \ldots, N_1 - 1$, and that for $i, j = 0, \ldots, N_1 - 1$ the inner products satisfy

$$\langle \mathbf{f}_i, \mathbf{f}_j \rangle_{\mathcal{P}(\Theta)} = \langle \mathcal{T}\mathbf{f}_i, \mathcal{T}\mathbf{f}_j \rangle_{\mathcal{P}(s)} = \langle \mathbf{f}_i, \mathbf{f}_j \rangle_{\mathcal{P}(\Theta_1)}.$$

We conclude that $\mathcal{P}(\Theta_1)$ is isometrically included in $\mathcal{P}(\Theta)$, and the claim is proved. According to [9], $\Theta_1(z)$ is an elementary factor of $\Theta(z)$.

Step 3: *If $s_1(z)$ is a constant, then $\Theta(z) = \Theta_1(z)$. If $s_1(z)$ is not a constant, let $s_2(z) = \widehat{s}_1(z)$ be the Schur transform of $s_1(z)$ and denote the corresponding coefficient matrix by $\Theta_2(z)$. Then $\Theta_2(z)$ is an elementary factor of $\Theta_1(z)^{-1}\Theta(z)$. We iterate n times until $s_n(z) = \widehat{s}_{n-1}(z)$ is a unitary constant and conclude that $\Theta(z) = \Theta_1(z)\cdots\Theta_n(z)$.*

Because of (6.2) and the relation

$$\frac{1-s(z)s(w)^*}{1-zw^*} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \frac{J-\Theta(z)J\Theta(w)^*}{1-zw^*} \begin{pmatrix} 1 \\ -s(w)^* \end{pmatrix}$$
$$+(a_1(z)-c_1(z)s(z))\frac{1-s_1(z)s(w)^*}{1-zw^*}(a_1(w)-c_1(w)s(w))^*$$

we have the following equalities:

$$\begin{aligned} \mathcal{P}(s) &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta), \\ \mathcal{P}(s) &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta_1) \oplus (a_1 - c_1 s)\mathcal{P}(s_1). \end{aligned} \tag{6.3}$$

In particular, the map

$$\mathbf{f} \mapsto (a_1 - c_1 s)\,\mathbf{f} \tag{6.4}$$

is an isometry from $\mathcal{P}(s_1)$ into $\mathcal{P}(s)$.

If $s_1(z)$ is a constant then $\mathcal{P}(s_1) = \{0\}$ and (6.3) implies that $\mathcal{P}(\Theta) = \mathcal{P}(\Theta_1)$. Since $\Theta(z)$ and $\Theta_1(z)$ are normalized they must be equal.

If $s_1(z)$ is not a constant, we define $\Theta_2(z)$ via $s_1(z) = T_{\Theta_2(z)}(s_2(z))$. Then $\Theta_2(z) \in \mathcal{U}_{z_1}^{z_0}$ and we have the decomposition

$$\mathcal{P}(s_1) = \begin{pmatrix} 1 & -s_1 \end{pmatrix} \mathcal{P}(\Theta_2) \oplus (a_2 - c_2 s_1)\mathcal{P}(s_2).$$

Since (6.4) is an isometry and

$$(a_1(z) - c_1(z)s(z)) \begin{pmatrix} 1 & -s_1(z) \end{pmatrix} = \begin{pmatrix} 1 & -s(z) \end{pmatrix} \Theta_1(z)$$

we obtain that

$$(a_1 - c_1 s)\mathcal{P}(s_1) = \begin{pmatrix} 1 & -s \end{pmatrix} \Theta_1 \mathcal{P}(\Theta_2) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2).$$

Thus

$$\begin{aligned} \mathcal{P}(s) &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta_1) \oplus \begin{pmatrix} 1 & -s \end{pmatrix} \Theta_1 \mathcal{P}(\Theta_2) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2) \\ &= \begin{pmatrix} 1 & -s \end{pmatrix} (\mathcal{P}(\Theta_1) \oplus \Theta_1 \mathcal{P}(\Theta_2)) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2) \\ &= \begin{pmatrix} 1 & -s \end{pmatrix} \mathcal{P}(\Theta_1\Theta_2) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2). \end{aligned}$$

It follows as above that $\mathcal{P}(\Theta_1\Theta_2)$ is isometrically included in $\mathcal{P}(\Theta)$, and, if $s_2(z)$ is constant, that $\Theta(z) = \Theta_1(z)\Theta_2(z)$. If $s_2(z)$ is not constant, we observe that

$$(a_1 - c_1 s)(a_2 - c_2 s_1) \begin{pmatrix} 1 & -s_2 \end{pmatrix} = (a_1 - c_1 s) \begin{pmatrix} 1 & -s_1 \end{pmatrix} \Theta_2 = \begin{pmatrix} 1 & -s \end{pmatrix} \Theta_1\Theta_2.$$

and define $\Theta_3(z)$ via $s_2(z) = T_{\Theta_3(z)}(s_3(z))$. Then we have

$$\begin{aligned} &(a_1 - c_1 s)(a_2 - c_2 s_1)\mathcal{P}(s_2) \\ &\quad = \begin{pmatrix} 1 - s \end{pmatrix} \Theta_1\Theta_2\mathcal{P}(\Theta_3) \oplus (a_1 - c_1 s)(a_2 - c_2 s_1)(a_3 - c_3 s_2)\mathcal{P}(s_3). \end{aligned}$$

and the factorization (5.2) follows by repeating the arguments.

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Discrete Analogs of Canonical Systems with Pseudo-exponential Potential. Inverse Problems

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Abstract. We study the inverse problems associated to the characteristic spectral functions of first-order discrete systems. We focus on the case where the coefficients defining the discrete system are strictly pseudo-exponential. The arguments use methods from system theory. An important role is played by the description of the unitary solutions of a related Nehari interpolation problem and by Hankel operators with unimodular symbols. An application to inverse problems for Jacobi matrices is also given.

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1. Introduction

In the present work we continue our study of first-order discrete systems. In [4, Section 3.1] we defined one-sided first-order discrete systems to be expressions of the form

$$X_{n+1}(z) = \begin{pmatrix} 1 & -\rho_n \\ -\rho_n^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} X_n(z), \quad n = 0, 1, \ldots \tag{1.1}$$

where the ρ_n are in the open unit disk. Two-sided first-order discrete systems are given by the formula

$$X_{n+1}(z) = \begin{pmatrix} 1 & -\rho_n \\ -\rho_n^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} X_n(z) \quad n = 0, 1, \ldots \tag{1.2}$$

Such systems arise from the discretization of the telegrapher's equation and in the theory of orthogonal polynomials; see, e.g., [13].

As in [4] we focus on the case where the coefficients ρ_n are of the form

$$\rho_n = -ca^n(I_p - \Delta a^{*(n+1)}\Omega a^{n+1})^{-1}b. \tag{1.3}$$

In this equation $(a, b, c) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times 1} \times \mathbb{C}^{1\times p}$ is a minimal triple of matrices (see Section 2.3 for the definition), the spectrum of a is in the open unit disk and Δ and Ω are the solutions of the Stein equations

$$\Delta - a\Delta a^* = bb^* \quad \text{and} \quad \Omega - a^*\Omega a = c^*c. \tag{1.4}$$

Furthermore, one requires that:

$$\Omega^{-1} > \Delta. \tag{1.5}$$

Sequences of the form (1.3) with condition (1.5) are called *strictly pseudo-exponential sequences* and have been introduced in [7, Theorem 4.3]. There we studied the connections between the Carathéodory–Fejér and the Nehari extension problems. In the process, we proved recursions formulas for analogs of orthogonal polynomials associated to the Hankel operator

$$\Gamma = \begin{pmatrix} \gamma_0 & \gamma_{-1} & \cdots \\ \gamma_{-1} & \gamma_{-2} & \cdots \\ \vdots & \vdots & \\ \vdots & \vdots & \end{pmatrix}, \quad \ell_2 \to \ell_2, \quad \text{where} \quad \gamma_{-j} = ca^jb, \quad j = 0, 1, 2, \ldots \tag{1.6}$$

The analysis in [7, Section 4] allows to find explicit forms for the solutions of the systems (1.1) in terms of a, b and c. See Theorem 2.1.

In [4] we associated to such systems a number of functions of z, which we called the *characteristic spectral functions* of the system. The main problem in this paper is to find the sequence ρ_n when one of the characteristic spectral functions is given. Classically, when no hypothesis of rationality is made, there are three approaches to solve such a problem (starting from the spectral function), namely

1. The Gelfand–Levitan approach.
2. Kreĭn's approach.
3. Marchenko's approach.

The connections between these approaches are explained in [17] and discussed in the rational case in our previous paper [5].

In the present paper, we present a different approach, based on realization theory. A key tool in the arguments is the description of all unitary solutions of the Nehari extension problem which admit a generalized Wiener–Hopf factorization. First recall that the Wiener algebra $\mathcal{W}$ of the unit circle consists of complex-valued functions $f(z)$ of the form

$$f(z) = \sum_{\mathbb{Z}} f_\ell z^\ell \tag{1.7}$$

for which

$$\|f\|_{\mathcal{W}} \stackrel{\text{def.}}{=} \sum_{\mathbb{Z}} |f_\ell| < \infty.$$

The Nehari extension problem is defined as follows:

Definition 1.1. *Given $\gamma_{-j} = ca^j b$, $j = 0, 1, 2, \ldots$, find all elements $f \in \mathcal{W}$ for which*

$$f_{-j} = \gamma_{-j}, \qquad j = 0, 1, 2, \ldots$$

and such that $\sup_{|z|=1} |f(z)| < 1$.

Recall (see for instance [20, p. 956–961]) that a necessary and sufficient condition for the Nehari problem to be solvable is that the Hankel operator Γ defined by (1.6) has a norm strictly less than 1.

Condition (1.5) insures that the Hankel operator $\Gamma = (ca^{(\ell+k)}b)_{\ell,k=0,\ldots,}$ is a strict contraction from ℓ_2 into itself. Indeed, let

$$C = \begin{pmatrix} c \\ ca \\ ca^2 \\ \vdots \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b & ab & a^2b & \cdots & \end{pmatrix}.$$

Then C and B are bounded operators from ℓ_2 into $\mathbb{C}$ and $\mathbb{C}$ into ℓ_2 respectively. We have that $C^*C = \Omega$ and $BB^* = \Delta$. Furthermore, $\Gamma = CB$ and

$$\begin{aligned} \|\Gamma\| < 1 &\iff \Gamma\Gamma^* < I \\ &\iff CBB^*C^* < I \\ &\iff C^*CBB^*C^*C < C^*C \\ &\iff BB^* < (C^*C)^{-1}, \end{aligned}$$

which is (1.5).

We note that a different kind of discrete systems has been recently studied in [25], also using the state space method.

The paper consists of nine sections including the introduction. Section two is of a preliminary nature. We review the definitions of the characteristic spectral functions and the description of all unitary solutions to the Nehari interpolation problem. A new result in this section is Theorem 2.15, which states that the strictly pseudo-exponential sequence ρ_n determines uniquely (up to a similarity matrix) the minimal triple (a, b, c). The inverse scattering problem is considered in Section 3. Inverse problems associated to the other spectral functions are considered in Section 4. One of the main results of this paper, Theorem 4.1, states that rational functions strictly contractive in the closed unit disk are exactly the functions with sequence of Schur coefficients of the form $-\rho_n$. Section 5 deals with the inverse problem associated to the asymptotic equivalence matrix function. In Section 6, we consider the case of two-sided systems. In Section 7 we present a numerical example. In Section 8 we compute explicitly an example of a rational Schur function which is not the reflection coefficient function of a first-order system with strictly pseudo-exponential sequence. In the last section we present an application to Jacobi matrices.

We conclude this introduction with some notation: we denote by $f^\sharp$ the function

$$f^\sharp(z) = f(1/z^*)^*.$$

We denote by $\mathbb{D}$ the open unit disk and by $\overline{\mathbb{D}}$ the closed unit disk. The symbol $\mathbb{E}$ denotes the exterior of the closed unit disk, and we set

$$\overline{\mathbb{E}} = \{z \in \mathbb{C} \ : \ |z| \geq 1\} \cup \{\infty\}\,.$$

We already defined the Wiener algebra $\mathcal{W}$. The subalgebra of functions for which in (1.7) $f_\ell = 0$ for $\ell < 0$ (resp. for $\ell > 0$) will be denoted by $\mathcal{W}_+$ (resp. $\mathcal{W}_-$).

2. Preliminaries

2.1. The characteristic spectral functions

In this section we review the definitions of the characteristic spectral functions associated to a one-sided first-order discrete system given in our previous paper [4]. We begin with a result, which is proved in [4] and uses [7, Theorem 4.5], and which explains how solutions to the system (1.1) can be expressed explicitly in terms of a, b and c.

Theorem 2.1. *Let $\rho_0, \rho_1, \ldots$ be a strictly pseudo-exponential sequence of the form*

$$\rho_n = -ca^n(I_p - \Delta a^{*(n+1)}\Omega a^{n+1})^{-1}b.$$

Every solution of the first-order discrete system (1.1) *is of the form*

$$X_n(z) = \prod_{\ell=0}^{n-1}(1-|\rho_\ell|^2)\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix} H_n(z)^{-1}\begin{pmatrix}z^n & 0\\ 0 & 1\end{pmatrix} H_0(z)\begin{pmatrix}1 & 0\\ 0 & z^{-1}\end{pmatrix} X_0(z), \quad (2.1)$$

where

$$H_n(z) = \begin{pmatrix}\alpha_n(z) & \beta_n(z)\\ \gamma_n(z) & \delta_n(z)\end{pmatrix}$$

and, for $n = 0, 1, \ldots$

$$\begin{aligned}
\alpha_n(z) &= 1 + ca^n z(zI_p - a)^{-1}(I_p - \Delta\Omega_n)^{-1}\Delta a^{*n}c^* && (2.2)\\
\beta_n(z) &= ca^n z(zI_p - a)^{-1}(I_p - \Delta\Omega_n)^{-1}b && (2.3)\\
\gamma_n(z) &= b^*(I_p - za^*)^{-1}(I_p - \Omega_n\Delta)^{-1}a^{*n}c^* && (2.4)\\
\delta_n(z) &= 1 + b^*(I_p - za^*)^{-1}(I_p - \Omega_n\Delta)^{-1}\Omega_n b, && (2.5)
\end{aligned}$$

with $\Omega_n = a^{*n}\Omega a^n$.

The function

$$X_n(z) = \frac{\prod_{\ell=0}^{n-1}(1-|\rho_\ell|^2)}{\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2)}\begin{pmatrix}1 & 0\\ 0 & z\end{pmatrix} H_n(z)^{-1}\begin{pmatrix}z^n & 0\\ 0 & z^{-1}\end{pmatrix},$$

is a solution to (1.1). *It corresponds to*

$$X_0(z) = \frac{1}{\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2)} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} H_0(z)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix},$$

and it has the asymptotic

$$\lim_{n\to\infty} \begin{pmatrix} z^{-n} & 0 \\ 0 & 1 \end{pmatrix} X_n(z) = I_2.$$

The function

$$\prod_{\ell=0}^{\overset{\curvearrowleft}{\ell=n-1}} \begin{pmatrix} 1 & -\rho_\ell \\ -\rho_\ell^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

is also a solution to (1.1). *It corresponds to* $X_0(z) = I_2$.

Finally, we have:

$$\begin{aligned} &\prod_{\ell=0}^{\overset{\curvearrowleft}{\ell=n-1}} \begin{pmatrix} 1 & -\rho_\ell \\ -\rho_\ell^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \\ &\qquad = \prod_{\ell=0}^{n-1}(1-|\rho_\ell|^2) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} H_n(z)^{-1} \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix} H_0(z) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}, \end{aligned}$$

where we denote

$$\prod_{\ell=0}^{\overset{\curvearrowleft}{\ell=n-1}} A_\ell = A_{n-1}\cdots A_0.$$

The proof of this result is based on the following recurrence formulas, proved in [7, Theorem 4.5],

$$\alpha_{n+1}(z) = \alpha_n(z) + \rho_n^*\beta_n(z) \tag{2.6}$$
$$\beta_{n+1}(z) = z(\rho_n\alpha_n(z) + \beta_n(z)) \tag{2.7}$$
$$z\gamma_{n+1}(z) = \gamma_n(z) + \rho_n^*\delta_n(z) \tag{2.8}$$
$$\delta_{n+1}(z) = \delta_n(z) + \rho_n\gamma_n(z), \tag{2.9}$$

and which force the recurrence relationship

$$H_{n+1}(z) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{z} \end{pmatrix} H_n(z) \begin{pmatrix} 1 & \rho_n \\ \rho_n^* & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix},$$

between $H_n(z)$ and $H_{n+1}(z)$. Such recursions were developed in a general setting in [18].

We also recall that it holds that

$$\delta_n(z) = \alpha_n^\sharp(z) \quad \text{and} \quad \gamma_n(z) = \beta_n^\sharp(z). \tag{2.10}$$

The first characteristic spectral function which we introduce is the scattering function. To define it, we first look for the $\mathbb{C}^2$-valued solution of the system (1.1), with the boundary conditions

$$\begin{aligned} \begin{pmatrix} 1 & -1 \end{pmatrix} Y_0(z) &= 0, \\ \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n(z) &= 1 + o(n). \end{aligned} \tag{2.11}$$

In view of (2.1) the first condition implies that the solution is of the form

$$Y_n(z) = (\prod_{\ell=0}^{n-1}(1-|\rho_\ell|^2)) \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} H_n(z)^{-1} \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix} H_0(z) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} x(z) \\ x(z) \end{pmatrix}$$

where $x(z)$ is to be determined via the second boundary condition. We compute

$$\begin{pmatrix} 0 & 1 \end{pmatrix} Y_n(z) = (\prod_{\ell=0}^{n-1}(1-|\rho_\ell|^2)) \begin{pmatrix} 0 & z \end{pmatrix} H_n(z)^{-1} \begin{pmatrix} z^n & 0 \\ 0 & 1 \end{pmatrix} H_0(z) \begin{pmatrix} x(z) \\ \frac{x(z)}{z} \end{pmatrix}.$$

Taking into account that $\lim_{n\to\infty} H_n(z) = I_2$ we get that

$$\lim_{n\to\infty} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_n(z) = (\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2)) \begin{pmatrix} 0 & z \end{pmatrix} H_0(z) \begin{pmatrix} x(z) \\ \frac{x(z)}{z} \end{pmatrix}$$

and hence $1 = (\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2))(z\gamma_0(z)+\delta_0(z))x(z)$, that is

$$x(z) = \frac{1}{(\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2))(z\gamma_0(z)+\delta_0(z))}.$$

Furthermore,

$$\begin{aligned} \lim_{n\to\infty} \begin{pmatrix} 1 & 0 \end{pmatrix} Y_n(z)z^{-n} &= (\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2)) \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} H_0(z) \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} x(z) \\ x(z) \end{pmatrix} \\ &= (\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2)) \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_0(z) + \frac{\beta_0(z)}{z} \\ \gamma_0(z) + \frac{\delta_0(z)}{z} \end{pmatrix} x(z) \\ &= \frac{\alpha_0(z) + \frac{\beta_0(z)}{z}}{z\gamma_0(z)+\delta_0(z)}. \end{aligned}$$

Definition 2.2. *Let $\rho_0, \rho_1, \ldots$ be a strictly pseudo-exponential sequence of the form*

$$\rho_n = -ca^n(I_p - \Delta a^{*(n+1)}\Omega a^{n+1})^{-1}b,$$

and let $\alpha_0(z), \beta_0(z), \gamma_0(z)$ and $\delta_0(z)$ be the functions given by (2.2)–(2.5) *with $n = 0$. The function*

$$S(z) = \frac{\alpha_0(z) + \frac{\beta_0(z)}{z}}{z\gamma_0(z)+\delta_0(z)} = \frac{1}{z}\frac{\alpha_0(z)z+\beta_0(z)}{\gamma_0(z)z+\delta_0(z)}, \tag{2.12}$$

is called the scattering function associated to the discrete system (1.1) *with the boundary conditions* (2.11).

We note that

$$(z\gamma_0(z)+\delta_0(z))^\sharp(z)=\frac{1}{z}(\alpha_0(z)z+\beta_0(z)),$$
$$\left(\alpha_0(z)+\frac{\beta_0(z)}{z}\right)^\sharp(z)=\delta_0(z)+z\gamma_0(z),$$

and in particular $S(z)S(z)^\sharp=1$.

From the preceding analysis we obtain the following result (see [4, Theorem 3.14]):

Theorem 2.3. *The scattering function is of the form* $S(z)=\frac{S_-(z)}{S_-^\sharp(z)}$ *where* $S_-(z)$ *and its inverse are analytic in* $\overline{\mathbb{E}}$. *Equivalently, the scattering function can be represented as* $\frac{B_1(z)}{B_2(z)}$, *where* $B_1(z)$ *and* $B_2(z)$ *are two Blaschke products of same degree.*

The factor $S_-(z)$ is defined up to a multiplicative constant, and we will normalize it by the condition

$$\int_0^{2\pi}\frac{dt}{|S_-(e^{-it})|^2}=2\pi\quad\text{and}\quad S_-(\infty)>0. \tag{2.13}$$

Let

$$d=\sqrt{\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|\alpha_0(e^{-it})+e^{it}\beta_0(e^{-it})|^2}}.$$

The choice

$$S_-(z)=d\left(\alpha_0(z)+\frac{\beta_0(z)}{z}\right) \tag{2.14}$$

satisfies the normalization (2.13) since

$$\lim_{z\longrightarrow\infty}\left(\alpha_0(z)+\frac{\beta_0(z)}{z}\right)=1+ca(I_p-\Delta\Omega)^{-1}\Delta c^*>0.$$

We remark that we have two factorizations for the scattering function, which are of different kinds. The first one, $S(z)=\frac{S_-(z)}{S_-^\sharp(z)}=\frac{\alpha_0(z)+\frac{\beta_0(z)}{z}}{z\gamma_0(z)+\delta_0(z)}$, is a Wiener–Hopf factorization (recall that the function $w\in\mathcal{W}$ is said to have a Wiener–Hopf factorization if it can be written as $w=w_+w_-$, where w_+ and its inverse are in $\mathcal{W}_+$ while w_- and its inverse are in $\mathcal{W}_-$). The second one, $S(z)=\frac{B_1(z)}{B_2(z)}$, is a quotient of two finite Blaschke products of same degree. In the first case, the spectral factor $S_-(z)$ uniquely determines (up to a unitary constant factor) the function $S(z)$ since this latter is unitary on the unit circle. In the second case, starting from any finite Blaschke $B_1(z)$, any other Blaschke factor $B_2(z)$ of same degree and without common zero with $B_1(z)$ will lead (once more, up to a unitary constant factor) to a scattering function $S(z)=\frac{B_1(z)}{B_2(z)}$. See Theorem 3.2. The second factorization is a special case of factorizations considered by Kreĭn and Langer for generalized Schur functions. See [26], [12], [1], and see [8] for a discussion of similar factorizations for scattering functions associated to canonical differential systems.

We now turn to the definition of the reflection coefficient function. We set

$$C(\rho) = \begin{pmatrix} 1 & -\rho \\ -\rho^* & 1 \end{pmatrix}$$

and

$$M_n(z) = C(\rho_0)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} C(\rho_1)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots C(\rho_n)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.15}$$

In the statement we use the following notation for linear fractional transformations:

$$T_\Theta(x) = \frac{ax+b}{cx+d}, \quad \text{where} \quad \Theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Theorem 2.4. *Let $\rho_n, n = 1, 2, \ldots$ be a strictly pseudo-exponential sequence of the form*

$$\rho_n = -ca^n(I_p - \Delta a^{*(n+1)}\Omega a^{n+1})^{-1}b$$

and let $M_n(z)$ be defined by (2.15). *The limit*

$$R(z) = \lim_{n\to\infty} T_{M_n(z)}(0) \tag{2.16}$$

exists and is equal to

$$R(z) = \frac{\beta_0}{\alpha_0}(1/z).$$

It is a function analytic and contractive in the open unit disk, called the reflection coefficient function. It takes strictly contractive values on the unit circle.

It follows from Theorem 2.4 that the ρ_n are in $\mathbb{D}$. Indeed, the proof that $R(z)$ is analytic and contractive in $\mathbb{D}$ depends only on the fact that (1.5) holds and on the properties of $H_0(z)$. By (2.16), the sequence $-\rho_0, -\rho_1, \ldots$ is the sequence of Schur coefficients of $R(z)$ and hence the ρ_n are in $\mathbb{D}$.

The proof of Theorem 2.4 (see [4, Theorem 3.9]) is based on the equation

$$M_n(z) = \left(\prod_{\ell=0}^{n}(1-|\rho_\ell|^2)\right) H_0(z^*)^* \begin{pmatrix} z^{n+1} & 0 \\ 0 & 1 \end{pmatrix} H_{n+1}(z^*)^*$$

which relates M_n and H_{n+1}, and on the asymptotic property of H_{n+1}.

We note that a finite Blaschke product is not the reflection coefficient function of a first-order one-sided discrete system with strictly pseudo-exponential sequence. We also note that a function such as

$$R(z) = \frac{z}{2-z}$$

is not appropriate either, since $R(1) = 1$. The Schur coefficients of this function are computed in Section 8.

In [4, Theorem 3.10] we proved the following realization result for the reflection coefficient function.

Theorem 2.5. *Let $(a,b,c) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times 1} \times \mathbb{C}^{1\times p}$ is a minimal triple of matrices such that* (1.5) *holds. Then the function*

$$R(z) = c\left\{(I - \Delta a^*\Omega a) - z(I - \Delta\Omega)a\right\}^{-1} b \tag{2.17}$$

is analytic and strictly contractive in the closed unit disk.

In Section 4.1 we show that every rational function strictly contractive in the closed unit disk admits a realization of the type (2.17). See Theorem 4.1.

To introduce the Weyl coefficient function we consider the matrix function

$$U_n(z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \prod_{\ell=0}^{\overset{\curvearrowleft}{\ell=n-1}} \begin{pmatrix} 1 & -\rho_\ell \\ -\rho_\ell^* & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Definition 2.6. *The Weyl coefficient function $N(z)$ is defined for $z \in \mathbb{D}$ by the following property: The sequence $n \mapsto U_n(z)\begin{pmatrix} iN(z^*)^* \\ 1 \end{pmatrix}$ belongs to ℓ_2^2, that is:*

$$\sum_{n=0}^{\infty} \begin{pmatrix} -iN(z^*) & 1 \end{pmatrix} U_n(z)^* U_n(z) \begin{pmatrix} iN(z^*)^* \\ 1 \end{pmatrix} < \infty, \quad z \in \mathbb{D}.$$

A similar definition appears in [27, Theorem 1, p. 231]. See also [25, equation (0.6)].

For the next result, see also [31, equation (3.7) p. 416].

Theorem 2.7. *The relation between the Weyl coefficient function and the reflection coefficient function is given by:*

$$N(z) = i\frac{1 - zR(z)}{1 + zR(z)}. \tag{2.18}$$

The following realization result for the Weyl coefficient function was proved in [4].

Theorem 2.8. *Let $\rho_n, n = 1, 2, \ldots$ be a strictly pseudo-exponential sequence of the form*

$$\rho_n = -ca^n(I_p - \Delta a^{*(n+1)}\Omega a^{n+1})^{-1} b.$$

The Weyl coefficient function associated to the corresponding one-sided first-order discrete system is given by:

$$N(z) = i\left(1 + 2zc\left\{I - \Delta a^*\Omega a + zbc - z(I - \Delta\Omega)a\right\}^{-1} b\right). \tag{2.19}$$

The function

$$W(z) = \frac{c_0}{|\alpha_0(1/z) + z\beta_0(1/z)|^2}, \quad c_0 = \frac{1}{\prod_{\ell=0}^{\infty}(1 - |\rho_\ell|^2)}, \qquad |z| = 1. \tag{2.20}$$

is called the *spectral function*, and plays an important role, in particular in the theory of orthogonal polynomials associated to the system (1.1).

Theorem 2.9. *The Weyl coefficient function* $N(z)$ *is such that* $\text{Im } N(z) = W(z)$ *on the unit circle, and it holds that*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{|\alpha_0(e^{-it}) + e^{it}\beta_0(e^{-it})|^2} = \prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2). \tag{2.21}$$

Proof. We have

$$\text{Im } N(z) = \frac{|\alpha_0(1/z)|^2 - |\beta_0(1/z)|^2}{|\alpha_0(1/z) + z\beta_0(1/z)|^2} = \frac{\det H_0(1/z)}{|\alpha_0(1/z) + z\beta_0(1/z)|^2}$$

and

$$\det H_0(z) \equiv \frac{1}{\prod_{\ell=0}^{\infty}(1-|\rho_\ell|^2)}$$

See [7] for the latter. Comparing with (2.14) one obtains (2.21). □

Definition 2.10. *The function*

$$V(z) = \begin{pmatrix} \delta_0(z) & -\frac{\beta_0(z)}{z} \\ -z\gamma_0(z) & \alpha_0(z) \end{pmatrix} = \begin{pmatrix} \alpha_0^\sharp(z) & -\frac{\beta_0(z)}{z} \\ -z\beta_0^\sharp(z) & \alpha_0(z) \end{pmatrix} \tag{2.22}$$

is called the asymptotic equivalence matrix function *of the one-sided first-order discrete system* (1.1).

The second equality stems from (2.10). The terminology is explained in the next theorem:

Theorem 2.11. *Let* c_1 *and* c_2 *be in* $\mathbb{C}^2$, *and let* $X_n^{(1)}$ *and* $X_n^{(2)}$ *be the* $\mathbb{C}^2$*-valued solutions of* (1.1), *corresponding to the case of zero potential and to a potential* ρ_n *respectively and with initial conditions* $X_0^{(1)}(z) = c_1$ *and* $X_0^{(2)}(z) = c_2$. *Then, for every* z *on the unit circle,*

$$\lim_{n\to\infty} \|X_n^{(1)}(z) - X_n^{(2)}(z)\| = 0 \qquad \Longleftrightarrow \qquad c_2 = V(z)c_1.$$

See [4, Theorem 3.2].

2.2. Unitary solutions of the Nehari problem

We follow here [20, p. 956–961] specialized to the scalar case for the solution of this problem when the γ_j are of the form

$$\gamma_{-j} = ca^jb, \quad j = 0, -1, \dots$$

where (a, b, c) is a minimal triple. We already remarked that $\|\Gamma\| < 1$ is equivalent to (1.5).

The Nehari extension problem associated with this sequence is then solved as follows. In the statement, $\alpha_0, \beta_0, \gamma_0$ and δ_0 are defined by (2.6)–(2.9) with $n = 0$.

Theorem 2.12. *All solutions of the Nehari extension problem which are strictly contractive on the unit circle are given by the linear fractional transformation*

$$\frac{\alpha_0(z)z\epsilon(z)+\beta_0(z)}{\gamma_0(z)z\epsilon(z)+\delta_0(z)}, \qquad |z|=1,$$

where $\epsilon(z)$ *varies in* $\mathcal{W}_+$ *and is strictly contractive on the unit circle.*

We are interested in solutions of the Nehari interpolation problem which are unitary rather than strictly contractive on $\mathbb{T}$. We focus on the case where the solution can be written as $w_+(z)z^\ell w_-(z)$ where w_+ and its inverse are in $\mathcal{W}_+$ and w_- and its inverse are in $\mathcal{W}_-$, and $\ell \in \mathbb{Z}$. Such factorizations are called generalized Wiener–Hopf factorizations.

Theorem 2.13. *All solutions of the Nehari extension problem which take unitary values on the unit circle and which admit a generalized Wiener–Hopf factorization. are given by the linear fractional transformation*

$$\frac{\alpha_0(z)z\epsilon(z)+\beta_0(z)}{\gamma_0(z)z\epsilon(z)+\delta_0(z)}, \qquad |z|=1,$$

where $\epsilon(z)$ *varies among finite Blaschke products.*

See [9, Theorem 4.3 p. 33]. We refer also to [16] and [15] for more information on unitary solutions of the Nehari problem.

We note that, in particular, the function $zS(z)$, where $S(z)$ is the scattering function defined by (2.12), is a solution of the Nehari interpolation problem associated to $\gamma_{-j} = ca^jb, \quad j=0,1,\ldots$ which is unitary and admits a generalized Wiener–Hopf factorization.

2.3. Uniqueness theorem

A priori, a pseudo-exponential sequence may have different representations of the form (1.3). The purpose of this section is to show that in fact the ρ_n determines uniquely the minimal triple (a,b,c) (up to a similarity matrix). Recall that minimality means the following:

$$\cap_{\ell=0}^m \ker ca^\ell = \{0\} \quad \text{and} \quad \cup_{\ell=0}^m \operatorname{Im} a^\ell b = \mathbb{C}^p$$

for m large enough. The first condition means that the pair (c,a) is observable while the second means that the pair (a,b) is controllable. When both conditions are in force, the triple is called minimal. Two minimal triples are unique up to a uniquely defined similarity matrix, that is, there exists an invertible matrix σ such that:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}. \tag{2.23}$$

See [10] for more information.

We begin with a preliminary lemma. In the statement the letters N, R, W, S and S_- denote the rational functions previously introduced, i.e., functions with the following properties:

1. Im $N(z) > 0$ in $\overline{\mathbb{D}}$ and $N(0) = i$.
2. The function $W(z)$ has no pole on the unit circle, at the origin and at infinity and moreover $W(0)$ and $W(\infty)$ are different from 0, $W(e^{it}) > 0$ for all $t \in [0, 2\pi]$ and it holds that
$$\frac{1}{2\pi}\int_0^{2\pi} W(e^{it})dt = 1. \tag{2.24}$$
3. R is strictly contractive in $\overline{\mathbb{D}}$.
4. S_- is analytic and invertible in $\overline{\mathbb{E}}$, with $S_-(\infty) > 0$, and
$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{|S_-(e^{-it})|^2} = 1.$$
5. $S = \frac{S_-}{S_-^\sharp}$ where S_- is as in the previous item.

Lemma 2.14. *Any of the characteristic spectral functions N, W, R, S and S_- determines uniquely the other four via the formulas*
$$\begin{aligned} W(z) &= \text{Im } N(z), \quad |z| = 1, \\ W(z) &= \frac{1}{|S_-(1/z)|^2}, \quad |z| = 1, \\ N(z) &= i\frac{1 - zR(z)}{1 + zR(z)} \\ S(z) &= \frac{S_-(z)}{S_-(1/z^*)^*}. \end{aligned} \tag{2.25}$$

Proof. We start with a rational function W without poles and strictly positive on $\mathbb{T}$, and satisfying (2.24). We can write $W(z) = q(z)q(1/z^*)^*$, where q is rational and moreover, q and its inverse have no poles in $\overline{\mathbb{D}}$. The function q is defined up to a constant of modulus one. To define it in a unique way, we require $q(0) > 0$. It suffices then to define $S_-(z) = \frac{1}{q(1/z)}$.

Since $W(z)$ is analytic in a neighborhood of the closed unit disk, the Herglotz formula
$$N(z) = \frac{i}{2\pi}\int_0^{2\pi} W(e^{it})\frac{e^{it} + z}{e^{it} - z}dt, \qquad z \in \mathbb{D}$$
defines a rational function with positive real part in $\mathbb{D}$ and such that $W(z) =$ Im $N(z)$ for $|z| = 1$. The function $R(z)$ is in turn uniquely determined by
$$R(z) = \frac{1}{z}\frac{1 + iN(z)}{1 - iN(z)}.$$
The arguments when one starts from one of the other functions are similar. $\square$

Before proving Theorem 2.15, we recall the following. The Schur algorithm (see [29], [13]) associates to a function R analytic and contractive in the open unit disk (that is, a Schur function) a sequence of numbers k_n and a sequence of Schur functions R_n via the formulas $R_0(z) = R(z)$, $k_0 = R(0)$ and

$$\begin{aligned} R_{n+1}(z) &= \frac{R_n(z) - R_n(0)}{z(1 - R_n(z)R_n(0)^*)}, \\ k_n &= R_n(0). \end{aligned} \tag{2.26}$$

The recursion stops if at some stage $|k_n| = 1$. Moreover, when the recursion is infinite (that is, when $|k_n| < 1$ for all n), (2.16) holds with M_n defined as in (2.15). The numbers k_n are called Schur coefficients or reflection coefficients.

Theorem 2.15. *A strictly pseudo-exponential sequence ρ_n determines uniquely* (*up to a similarity matrix*) *the minimal triple* (a, b, c) *subject to* (1.5).

Proof. For the purpose of the proof, let us use the notation $\rho_n = \rho_n(a, b, c)$ to denote the dependence on (a, b, c). We assume that for two minimal triples (a_1, b_1, c_1) and (a_2, b_2, c_2) we have

$$\rho_n(a_1, b_1, c_1) = \rho_n(a_2, b_2, c_2), \quad n = 0, 1, \ldots$$

A priori, a_1 and a_2 may be of different sizes (say, $n_1 \times n_1$ and $n_2 \times n_2$ respectively). The reflection coefficient function $R(z)$ does not depend on the given representation $\rho_n(a, b, c)$. Indeed, by (2.6)–(2.9),

$$\frac{\beta_{n+1}(z)}{\alpha_{n+1}(z)} = \frac{\frac{\beta_n(z)}{\alpha_n(z)} + \rho_n}{z(1 + \rho_n^* \frac{\beta_n(z)}{\alpha_n(z)})}, \quad n = 0, 1, \ldots$$

and thus the reflection coefficients of $R(z)$ are the $-\rho_n$. Thus the ρ_n determine uniquely $R(z)$. Furthermore, from Lemma 2.14 we note that the scattering function is uniquely determined by $R(z)$. Finally, there exists an invertible matrix σ such that (2.23) holds. Indeed, let (a, b, c) be a minimal triple subject to (1.5) and let $\rho_n(a, b, c)$ the associated strictly pseudo-exponential sequence. The function $zS(z)$ is a solution to the Nehari interpolation problem associated to the series of negative Fourier coefficients $\gamma_{-j} = ca^jb$ ($j = 0, 1, 2, \ldots$). Hence, if $\rho_n(a_1, b_1, c_1) = \rho_n(a_2, b_2, c_2)$, we have

$$\gamma_{-j} = c_1 a_1^j b_1 = c_2 a_2^j b_2, \quad j = 0, 1, \ldots,$$

and hence in a neighborhood of the origin we have that:

$$c_1(I_{n_1} - za_1)^{-1} b_1 = c_2(I_{n_2} - za_2)^{-1} b_2.$$

The above equality expresses two different minimal realizations of a common rational function. The result follows. □

3. Inverse scattering problem

From the characterization of the scattering function (see Theorem 2.3) there are two possible starting points for studying inverse problems: the first is a rational function S_- such that both S_- and S_-^{-1} are analytic in the exterior of the open unit disk, including the point at infinity, and the second is a finite Blaschke product.

3.1. Inverse scattering problem associated to the spectral factor

Theorem 3.1. *Let S_- be a rational function such that both S_- and S_-^{-1} are analytic in the exterior of the open unit disk, including the point at infinity, and $S_-(\infty) > 0$. Then the function*

$$S(z) = \frac{S_-(z)}{S_-(1/z^*)^*}$$

is the scattering matrix of a discrete first-order system with strictly pseudo-exponential sequence.

Proof. The function $S(z)$ is rational and has no pole on the unit circle and its negative coefficients are of the form

$$s_{-j} = ca^{j-1}b, \quad j = 1, 2, \ldots, \tag{3.1}$$

where the spectrum of a is in the open unit disk. See [21, Corollary 3.2, p. 397], [20, (11), p. 593] (in particular $S(z)$ belongs to the Wiener algebra). By considering the function

$$\sum_{j=0}^{\infty} s_{-j} z^j = c(I - za)^{-1} b$$

one obtains a minimal triple (which we still call (a, b, c)) such that (3.1) holds. Let

$$zS(z) = \sum_{j \in \mathbb{Z}} \gamma_j z^j, \quad |z| = 1.$$

Then,

$$\gamma_{-j} = ca^j b, \quad j = 0, 1, 2, \ldots \tag{3.2}$$

We claim that the corresponding Hankel operator (1.6) is a strict contraction, i.e.,

$$\mathrm{dist}_{\mathbf{L}_\infty(\mathbb{T})}(zS(z), \mathbf{H}_\infty(\mathbb{T})) < 1.$$

It is enough to show that the Hankel operator with symbol S is a strict contraction since

$$\begin{aligned}\mathrm{dist}_{\mathbf{L}_\infty(\mathbb{T})}(zS(z), \mathbf{H}_\infty(\mathbb{T})) &\le \mathrm{dist}_{\mathbf{L}_\infty(\mathbb{T})}(zS(z), z\mathbf{H}_\infty(\mathbb{T})) \\ &= \mathrm{dist}_{\mathbf{L}_\infty(\mathbb{T})}(S(z), \mathbf{H}_\infty(\mathbb{T})).\end{aligned}$$

To that purpose we let $\mathtt{p}$ denote the orthogonal projection from the Lebesgue space $\mathbf{L}_2(\mathbb{T})$ onto the Hardy space $\mathbf{H}_2(\mathbb{T})$, and set $\mathtt{q} = I - \mathtt{p}$. Viewing Γ as an operator from $\mathbf{H}_2(\mathbb{T})$ onto $\mathbf{H}_2(\mathbb{T})^\perp$ we have $\Gamma = H_S = \mathtt{q}S\mathtt{p}$ and

$$T_S^* T_S + H_S^* H_S = I. \tag{3.3}$$

Since S admits a Wiener–Hopf factorization it follows from that T_S is boundedly invertible and its inverse is given by $\mathtt{p}S_-^{-1}\mathtt{p}S_-^\sharp\mathtt{P}$ (see [20, Theorem 4.1, p. 588] for more information in the matrix-valued case). It follows from (3.3) that H_S is a strict contraction.

One can also get to the same conclusion as follows: the function $\frac{S_-(z)}{S_-(1/z^*)^*}$ is unimodular, and $S_-(1/z^*)^*$ is outer (it belongs as well as its inverse to $\mathbf{H}_\infty(\mathbb{T})$). In particular the function $w(z) = |S_-(1/z^*)|^2$, being bounded from above and below, satisfies in a trivial way the Muckenhoupt condition

$$\sup_{I \text{ interval of } \mathbb{T}} \frac{\int_I w(e^{it})dt}{\int_I w^{-1}(e^{it})dt} < \infty$$

(or the equivalent Helson–Szëgo condition; [24, Theorem 2 p. 229]). By [23, Theorem 5 p. 259],

$$\mathrm{dist}_{\mathbf{L}_\infty(\mathbb{T})}(S(z), \mathbf{H}_\infty(\mathbb{T})) < 1.$$

We note that this last argument is valid only in the scalar case, while the first argument is true in the matrix-valued case as well.

We now build the functions $\alpha_0(z), \beta_0(z), \gamma_0(z)$ and $\delta_0(z)$ as in (2.2)–(2.5) with $n = 0$ and a, b, c as in the present proof. From (3.2) we see that the function $zS(z)$ is a unitary solution to the Nehari interpolation problem (1.1). By Theorem 2.13 we have

$$zS(z) = \frac{\alpha_0(z)z\epsilon(z) + \beta_0(z)}{\gamma_0(z)z\epsilon(z) + \delta_0(z)}$$

for some finite Blaschke product $\epsilon(z)$. It follows that $\epsilon(z) \equiv 1$. Indeed, assume that ϵ is not a constant: there exist then a constant $c = e^{i\theta} \in \mathbb{T}$ and numbers $a_1, \ldots, a_p \in \mathbb{D}$ such that

$$\epsilon(z) = \prod_1^p \frac{z - a_i}{1 - za_i^*} = z^p \frac{\prod_1^p (1 - \frac{a_i}{z})}{\prod_1^p (1 - za_i^*)}.$$

In particular, the positive factorization index of $z\epsilon(z)$ is equal to $p + 1$. By [9, Theorem 4. 3 p. 33] the positive factorization index of $zS(z)$ will then be also equal to $p+1$. Hence $p = 0$ and ϵ is a constant. We show that $\epsilon = 1$. Write $\epsilon = \frac{u}{u^*}$ with $u \in \mathbb{T}$. Then,

$$S(z) = \frac{Y_-(z)}{Y_-^\sharp(z)}, \quad \text{with} \quad Y_-(z) = u^{-1}\left(\alpha_0(z)\epsilon + \frac{\beta_0(z)}{z}\right).$$

The function Y_- is analytic and invertible in $|z| \geq 1$ and thus there exists a complex number k such that

$$S_-(z) = kY_-(z). \tag{3.4}$$

Since

$$\frac{Y_-}{Y_-^\sharp} = S = \frac{S_-}{S_-^\sharp} = \frac{k}{k^*}\frac{Y_-}{Y_-^\sharp}$$

we have that $\frac{k}{k^*} = 1$, and so k is real. Let $z \to \infty$ in (3.4) we obtain

$$S_-(\infty) = ku^{-1}\alpha_0(\infty)\epsilon.$$

Since k is real, $S_-(\infty) > 0$ and $\alpha_0(\infty) > 0$ we obtain that $u^{-1}\epsilon \in \mathbb{R}$, and so $u^{-1}\epsilon = 1$, and hence $\epsilon = 1$. Thus,

$$S(z) = \frac{\alpha_0(z) + \frac{\beta_0(z)}{z}}{z\gamma_0(z) + \delta_0(z)},$$

that is, S is the scattering function of the first-order discrete system (1.1) with boundary conditions (2.11) and strictly pseudo-exponential sequence ρ_n, $n = 0, 1, 2, \ldots$ This concludes the proof. $\square$

In the proof, one could also use the representation of S as a quotient of two Blaschke products of same degree, and in the case of simple poles, use [28, Corollary 1 p. 205].

Note that the triple (a, b, c) in (3.2) is unique up to a similarity matrix. See [10].

3.2. Inverse scattering problem associated to a Blaschke product

In this section the starting point is a Blaschke product B_1. We ask the following question. When is there a Blaschke product B_2 such that $S = \frac{B_1}{B_2}$ is the scattering function of a one-sided discrete first-order system?

Theorem 3.2. *Let $B_1(z) = \prod_1^n \frac{z-b_i}{1-zb_i^*}$ be a finite Blaschke product, with $b_1, \ldots, b_n$ not necessarily distinct points in $\mathbb{D}$. Assume that all the $b_i \neq 0$. Then, for every points $a_1, \ldots, a_n$ different from 0 and from the b_i, the function $S = B_1B_2^{-1}$ with $B_2(z) = \prod_1^n \frac{z-a_i}{1-za_i^*}$ is the scattering function of a one-sided discrete first-order system. The corresponding spectral factor S_- is given by*

$$S_-(z) = \prod_1^n \frac{z - b_i}{z - a_i}, \tag{3.5}$$

up to the normalization (2.13) *and*

$$S(z) = \prod_1^n \frac{z - b_i}{1 - zb_i^*} \frac{1 - za_i^*}{z - a_i}.$$

Indeed, with $B_2(z)$ as in the theorem we have:

$$B_1(z)B_2(z)^{-1} = \prod_1^n \frac{z - b_i}{1 - zb_i^*} \prod_1^n \frac{1 - za_i^*}{z - a_i} = \frac{S_-(z)}{S_-^\sharp(z)}$$

where $S_-(z)$ as in (3.5). The function $w(z) = |S_-(1/z^*)|^2$ satisfies the Muckenhoupt condition, and this ends the proof.

We can compute the corresponding sequence of Schur coefficients as follows: write

$$S(z) = \sum_1^n \frac{c_i}{z - a_i} + \sum_1^n \frac{d_i}{1 - zb_i^*} + \prod_1^n \frac{a_i^*}{b_i^*}.$$

The coefficients c_i are equal to

$$c_i = \left(\prod_{\ell=1}^{n} \frac{a_i - b_\ell}{1 - a_i b_\ell^*} \right) \frac{\prod_{\ell=1}^{n}(1 - a_i a_\ell^*)}{\prod_{\substack{\ell=1,\ldots,n,\\ \ell \neq i}} (a_i - a_\ell)}, \tag{3.6}$$

and the d_i need not be computed. We have

$$\begin{aligned} zS(z) &= \sum_1^n \frac{zc_i}{z - a_i} + \sum_1^n \frac{zd_i}{1 - zb_i^*} + z\prod_1^n \frac{a_i^*}{b_i^*} \\ &= \sum_1^n c_i \left(\sum_{j=0}^{\infty} z^{-j} a_i^j \right) + \sum_1^n \frac{zd_i}{1 - zb_i^*} + z \prod_1^n \frac{a_i^*}{b_i^*}. \end{aligned}$$

Thus

$$\gamma_{-j} = \sum_{i=1}^{n} c_i a_i^j = ca^j b \quad j = 0, 1, 2, \ldots$$

where

$$a = \text{diag}\ (a_1, a_2, \ldots, a_n), \quad b = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}.$$

Finally, the matrices Δ and Ω solutions of the Stein equations (1.4) are equal to

$$\Delta = \left(\frac{1}{1 - a_i a_j^*} \right)_{i,j=1,\ldots,n} \quad \text{and} \quad \Omega = \left(\frac{c_i^* c_j}{1 - a_i^* a_j} \right)_{i,j=1,\ldots,n}.$$

Plugging these various expressions in (1.3) one obtains a formula for the Schur coefficients in terms of the a_i and b_i.

We note that when deg $B_1 \neq$ deg B_2, the above theorem of [23] cannot be used (or more precisely, the theorem insures that the norm of the Hankel operator will be 1). For instance if $B_1(z) = 1$ and $B_2(z) = \frac{z-a}{1-za^*}$ with $a \in \mathbb{D}$,

$$\|\Gamma\| = \inf_{\mathbf{H}_\infty} \|B_2^{-1} - h\|_\infty = \inf_{\mathbf{H}_\infty} \|1 - B_2 h\|_\infty = 1$$

since for every $h \in \mathbf{H}_\infty$,

$$\|1 - B_2 h\|_\infty = \sup_{z \in \mathbb{D}} |1 - B_2(z)h(z)\| \geq |1 - B_2(a)h(a)| = 1.$$

4. Other inverse problems

The three inverse problems which we now present are solved via the same principle: from either of the chosen characteristic spectral functions, compute the spectral factor S_-. Then apply Theorem 3.1. The case of the asymptotic equivalence matrix function is treated in a separate section.

4.1. Inverse problem associated to the reflection coefficient function

The Schur algorithm solves the inverse problem associated to the reflection coefficient function of a first-order one-sided discrete system, but the question which we ask here is a bit different. Is any rational function with no poles in $\mathbb{D}$ and strictly contractive on the unit circle of the form (2.17)?

Theorem 4.1. *Let $R(z)$ be a rational function strictly contractive in the closed unit disk. Then, $R(z)$ is the reflection coefficient function of a first-order discrete system of the form* (1.1) *with strictly pseudo-exponential potential. In particular it admits a minimal realization of the form* (2.17).

Proof. We set $N(z) = i\frac{1-zR(z)}{1+zR(z)}$ and $W(z) = \text{Im } N(z)$, and factorize $W(z)$ as

$$W(z) = \frac{1}{|S_-(1/z)|^2}, \quad |z| = 1$$

with S_- and its inverse analytic in $\overline{\mathbb{E}}$ and $S_-(\infty) > 0$. This last condition insures that the function S_- is uniquely determined by R. Forming $S = \frac{S_-}{S_-^\sharp}$ we associate a unique minimal pair (a, b, c) such that (3.2) holds. At this stage we have the formulas (2.2)–(2.5) (with $n = 0$) for the entries of $H_0(z)$ and we know from Section 2.1 (see formula (2.12)) that $S(z) = \frac{\alpha_0(z)+\frac{\beta_0(z)}{z}}{\gamma_0(z)z+\delta_0(z)}$.

From the uniqueness of the normalized spectral factor, we see that the function $S_-(z)$ is given by (2.14). Define now the functions N_0 and R_0 by:

$$N_0(z) = i\frac{\alpha_0(1/z) - z\beta_0(1/z)}{\alpha_0(1/z) + z\beta_0(1/z)}$$
$$R_0(z) = \frac{\beta_0(1/z)}{\alpha_0(1/z)}.$$

We show that $N_0(z) = N(z)$ and $R_0(z) = R(z)$. We have the relationships (with $|z| = 1$):

$$\begin{aligned}
\text{Im } \frac{1 - zR_0(z)}{1 + zR_0(z)} &= \text{Im } N_0(z) \\
&= \frac{c_0}{|\alpha_0(1/z) + z\beta_0(1/z)|^2} \quad \text{(see (2.20))} \\
&= \frac{1}{|S_-(1/z)|^2} \\
&= \text{Im } N(z) \\
&= \text{Im } \frac{1 - zR(z)}{1 + zR(z)}.
\end{aligned}$$

and so $N(z) = N_0(z)$ (because of the common normalization at $z = 0$) and this forces $R_0(z) = R(z)$. $\square$

As a corollary we have the following partial realization result:

Theorem 4.2. *Let $p \in \mathbb{N}$ and let $\rho_0, \ldots, \rho_p$ be $(p+1)$ numbers in the open unit disk. Then, there exists a minimal triple $(a, b, c) \in \mathbb{C}^{(p+1)\times(p+1)} \times \mathbb{C}^{(p+1)\times 1} \times \mathbb{C}^{1\times(p+1)}$ such that*

$$\rho_n = -ca^n(I_p - \Delta a^{*(n+1)}\Omega a^{n+1})^{-1}b, \quad n = 0, \ldots, p.$$

Indeed, let $R(z) = T_{M_p(z)}(0)$, where $M_p(z)$ is built from the sequence $\rho_0, \ldots, \rho_p$ as in (2.15). Let $J_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The function $M_p(z)$ is J_0-inner:

$$M_p(z)^* J_0 M_p(z) \begin{cases} \leq J_0, & z \in \mathbb{D} \\ = J_0, & |z| = 1. \end{cases}.$$

Set $M_p(z) = \begin{pmatrix} a_p(z) & b_p(z) \\ c_p(z) & d_p(z) \end{pmatrix}$. Then

$$|b_p(z)|^2 - |d_p(z)|^2 = -1, \quad |z| = 1$$

and it follows (see also [14]) that $T_{M_p(z)}(0) = \frac{b_p(z)}{d_p(z)}$ takes strictly contractive values on the unit circle. Therefore one can apply to it Theorem 4.1. The result follows since the first $p+1$ Schur coefficients of $T_{M_p(z)}$ are exactly $-\rho_0, \ldots, -\rho_p$.

As an example, let us take

$$\rho_0 = \cdots = \rho_{p-1} = 0 \quad \text{and} \quad \rho_p \in \mathbb{D}.$$

This sequence is of the form (1.3) with $c = -\begin{pmatrix} \rho_p & 0 & 0 & \cdot & \end{pmatrix} \in \mathbb{C}^{1\times(p+1)}$ and

$$a = \begin{pmatrix} 0 & 1 & 0 & \cdot & \\ 0 & 0 & 1 & 0 & \cdot \\ & & & & \\ 0 & 0 & \cdot & 0 & 1 \\ 0 & 0 & \cdot & 0 & 0 \end{pmatrix} \in \mathbb{C}^{(p+1)\times(p+1)}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ 0 \end{pmatrix} \in \mathbb{C}^{(p+1)}.$$

Indeed, the matrices Δ and Ω are equal to

$$\Delta = bb^* + abb^*a^* + \cdots = I_4 \quad \text{and} \quad \Omega = c^*c + a^*c^*ca + \cdots = |\rho_p|^2 I_4.$$

Condition (1.5) is thus in force and to check that (1.3) holds with this choice of a, b and c is a straightforward computation. Let us find back this result by the method described above. We have $R(z) = -\rho_p z^p$. Computations are easier since $R(z)$ has constant norm on the unit circle. We have $N(z) = i\frac{1+\rho_p z^{p+1}}{1-\rho_p z^{p+1}}$ and for $|z| = 1$,

$$\text{Im } N(z) = \frac{1 - |\rho_p|^2}{(1 - \rho_p z^{p+1})(1 - \frac{\rho_p{}^*}{z^{p+1}})} = \frac{1 - |\rho_p|^2}{S_-(1/z)S_-(1/z)^*}$$

with $S_-(z) = \frac{1 - \frac{\rho_p}{z^{p+1}}}{\sqrt{1-|\rho_p|^2}}$, and

$$zS(z) = z\frac{S_-(z)}{S_-(1/z^*)^*} = -\frac{\rho_p}{z^p} + \varphi(z)$$

where $\varphi(z) \in z\mathcal{W}_+$. We thus have to look for a minimal triple (a, b, c) such that

$$ca^j b = \begin{cases} 0, & j = 0, \dots, p-1, \\ -\rho_p, & j = p. \end{cases}$$

We are thus back to the direct computation just done above.

4.2. Inverse problem associated to the Weyl coefficient function

The proofs of the next Theorem as well as of Theorem 4.5 are similar to the proof of Theorem 4.1 and will be outlined for completeness.

Theorem 4.3. *Necessary and sufficient conditions for a rational function to be the Weyl coefficient function of a discrete first-order system* (1.1) *with pseudo-exponential sequence ρ_n are:*

(a) $N(0) = i$,
(b) $\mathrm{Im}\ N(z) > 0$ *for* $z \in \overline{\mathbb{D}}$.

When these conditions are in force, the inverse problem associated to N is solved as follows:

(1) *Compute $S_-(z)$ invertible and analytic in $\overline{\mathbb{E}}$ such that $S_-(\infty) > 0$ and* $\mathrm{Im}\ N(z) = \frac{1}{|S_-(1/z)|^2}$.
(2) *Set $S = \frac{S_-}{S_-^\sharp}$. The function S is in the Wiener algebra and its negative coefficients are of the form $\gamma_{-j} = ca^j b$ $(j = 0, 1, 2, \dots)$ for a unique (up to similarity) minimal triple of matrices (a, b, c).*

The coefficients ρ_n are then computed from (a, b, c) as in (1.3).

This problem is solved by reduction to the solution of the associated inverse scattering problem. Indeed, conditions (a) and (b) are necessary from the analysis in [4]. See (2.18). To prove that these conditions are also necessary we remark (Lemma 2.14) that N determines uniquely the normalized spectral factor $S_-(z)$. Steps (1) and (2) solve the inverse scattering problem associated to $S_-(z)$ and gives the series of reflection coefficients ρ_n in terms of a unique (up to similarity) minimal triple of matrices (a, b, c). By uniqueness of the Weyl coefficient function, N is the Weyl coefficient function of the corresponding system.

As a corollary we have:

Corollary 4.4. *Let $N(z)$ be a rational function. The following are equivalent:*

(1) *The function $N(z)$ has no pole, has a strictly positive imaginary part in the closed unit disk, and $N(0) = i$.*
(2) *$N(z)$ can be written as* (2.19).

4.3. Inverse spectral problem

Theorem 4.5. *Necessary and sufficient conditions for a rational function W to be the spectral function of a discrete first-order system* (1.1) *with pseudo-exponential sequence ρ_n are:*

(a) *$W(z)$ has no pole on the unit circle, at the origin and at infinity and moreover $W(0)$ and $W(\infty)$ are different from 0, $W(e^{it}) > 0$ for all $t \in [0, 2\pi]$.*

(b) *We have $\frac{1}{2\pi}\int_0^{2\pi} W(e^{it})dt = 1$.*

When these conditions are in force, the inverse spectral problem associated to W is solved as follows:

(1) *Compute $S_-(z)$ invertible and analytic in $\overline{\mathbb{E}}$ to be such that $S_-(\infty) > 0$ and $W(z) = \frac{1}{|S_-(1/z)|^2}$ for $z \in \mathbb{T}$.*

(2) *Set $S = \frac{S_-}{S_-^\sharp}$. The function S is in the Wiener algebra and its negative coefficients are of the form $\gamma_j = ca^{-j}b$ ($j = 0, -1, -2, \ldots$) for a unique (up to similarity) minimal triple of matrices (a, b, c).*

The coefficients ρ_n are then computed from (a, b, c) as in (1.3).

As in the previous section, this problem is also solved by reduction to the solution of the associated inverse scattering problem. Conditions (a) and (b) are necessary from the analysis in [4]. See (2.20). To prove that these conditions are also necessary we remark (Lemma 2.14) that W determines uniquely the normalized spectral factor $S_-(z)$. Steps (1) and (2) solve the inverse scattering problem associated to $S_-(z)$ and gives the series of reflection coefficients ρ_n in terms of a unique (up to similarity) minimal triple of matrices (a, b, c). By uniqueness of the spectral function, W is the spectral function of the corresponding system.

We mention that another approach to these two inverse problems (when the coefficients ρ_n do not necessarily form a strictly pseudo-exponential sequence) uses the theory of reproducing kernel spaces of the kind introduced by de Branges and Rovnyak. See [2], [3]. Yet another approach uses a realization

$$W(z) = D + zC(I - zA)^{-1}B,$$

for the weight function and formulas for the inverse of the Toeplitz matrix

$$\mathbb{T}_n = \begin{pmatrix} w_0 & w_1^* & \cdots & w_n^* \\ w_1 & w_0 & \cdots & w_{n-1}^* \\ \vdots & \vdots & & \vdots \\ w_n & w_{n-1} & \cdots & w_0 \end{pmatrix}, \tag{4.7}$$

when the entries are of the form

$$w_k = \begin{cases} CA^{k-1}(I-P)B & \text{if } k = 1, 2, \ldots \\ D - CPB & \text{if } k = 0 \\ -CA^{k-1}PB & \text{if } k = -1, -2, \ldots \end{cases}$$

where P is the Riesz projection defined by

$$P = -\frac{1}{2\pi i}\int_{\mathbb{T}} (\lambda I - A)^{-1} \mathrm{d}\lambda.$$

Theorem 4.6. *Let W be a rational function strictly positive on the unit circle and without pole at the origin, and let*

$$W(z) = D + zC(I - zA)^{-1}B$$

be a minimal realization of $W(z)$. Then the associated Schur coefficients are given by

$$k_n = \frac{D^{-1}CA^{\times}V_{n+1}^{-1}PA^{-(n+1)}A^{\times}BD^{-1}}{D^{-1} + D^{-1}CA^{\times}V_{n+1}^{-1}PA^{-(n+1)}(A^{\times})^nBD^{-1}}.$$

In this expression, $A^{\times} = A - BD^{-1}C$ and $V_n = (I - P + PA)^{-n}(I - P + P(A^{\times})^n)$.

To prove this theorem we first recall the following result (see [17, pp. 235–236]).

Theorem 4.7. *Let $R(z)$ be a Schur function and let*

$$\phi(z) = \frac{1 - R(z)}{1 + R(z)} = w_0 + 2\sum_{\ell=1}^{\infty} w_\ell z^\ell,$$

and assume that the matrix $\mathbb{T}_n$ (given by (4.7)) is invertible. Set

$$\mathbb{T}_n^{-1} = \begin{pmatrix} \gamma_{00}^{(n)} & \gamma_{01}^{(n)} & \cdots & \gamma_{0n}^{(n)} \\ \gamma_{10}^{(n)} & \gamma_{11}^{(n)} & \cdots & \gamma_{1n}^{(n)} \\ \vdots & \vdots & & \vdots \\ \gamma_{n0}^{(n)} & \gamma_{n1}^{(n)} & \cdots & \gamma_{nn}^{(n)} \end{pmatrix}.$$

Then the nth Schur coefficient of R is equal to

$$k_n = \frac{\gamma_{0n}^{(n)}}{\gamma_{00}^{(n)}}. \tag{4.8}$$

Note that in [17] the function $\frac{1+R(z)}{1-R(z)}$ is considered, and this introduces a minus sign in the Schur coefficients.

In [22, p. 36] formulas are given for the entries of the inverse of $\mathbb{T}_n$. More precisely, it is proved that for every n the matrix $V_n = (I - P + PA)^{-n}(I - P + P(A^{\times})^n)$ is invertible, and,

(a) for $0 \le j < i \le n$.

$$\gamma_{ij}^{(n)} = (D^{-1}C(A^{\times})^iV_{n+1}^{-1}PA^{-(n+1)}(A^{\times})^{n-j}B - D^{-1}C(A^{\times})^{i-j-1}BD^{-1}).$$

(b) for $0 \le i \le j \le n$

$$\gamma_{ij}^{(n)} = \delta_{ij}D^{-1} + D^{-1}C(A^{\times})^iV_{n+1}^{-1}PA^{-n}(A^{\times})^{n-j}BD^{-1}.$$

In particular,

$$\begin{aligned} k_n &= \frac{\gamma_{0n}^{(n)}}{\gamma_{00}^{(n)}} \\ &= \frac{D^{-1}C(A^\times)V_{n+1}^{-1}PA^{-(n+1)}(A^\times)BD^{-1}}{D^{-1}+D^{-1}C(A^\times)V_{n+1}^{-1}PA^{-(n+1)}(A^\times)^nBD^{-1}} \\ &= \frac{C\pi_{n+1}(A^\times)^{-n-1}BD^{-1}}{1+D^{-1}C\pi_{n+1}A^{\times-1}B}, \end{aligned}$$

where $\pi_{n+1} = V_{n+1}^{-1}PA^{-n-1}(A^\times)^{n+1}$. See [6] for more details.

5. Inverse problem associated to the asymptotic equivalence matrix function

The asymptotic equivalence matrix can be expressed in the following way (see [4]):

$$V(z) = \frac{1}{2}\begin{pmatrix} (1+iN(z^*)^*)S_+(z)^{-1} & -(1+iN(1/z))S_-(1/z) \\ -(1-iN(z^*)^*)S_+(z)^{-1} & (1-iN(1/z))S_-(1/z) \end{pmatrix}.$$

To tackle inverse problems associated to V it is easier to consider the expression (2.22) for $V(z)$, that is,

$$V(z) = \begin{pmatrix} \alpha_0^\sharp(z) & -\frac{\beta_0(z)}{z} \\ -z\beta_0^\sharp(z) & \alpha_0(z) \end{pmatrix}. \tag{5.1}$$

Theorem 5.1. *A $\mathbb{C}^{2\times 2}$-valued rational matrix function $V = (v_{\ell,j})$ is the asymptotic equivalence matrix function of a one-sided discrete first-order system with strictly pseudo-exponential potential if and only if the following conditions hold:*

1. *V is of the form (5.1) with α_0 and β_0 without poles in $|z| \geq 1$, and moreover α_0 does not vanish in $|z| \geq 1$.*
2. *It holds that*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{|v_{22}(e^{-it}) - v_{12}(e^{-it})|^2} \equiv (|v_{22}(z)|^2 - |v_{11}(z)|^2)^{-1}, \quad |z| = 1. \tag{5.2}$$

When these conditions are in force, the solution to the inverse problem associated to V is obtained by solving the inverse scattering problem associated to

$$S(z) = \frac{(v_{22}-v_{12})(z)}{(v_{22}-v_{12})^\sharp(z)}.$$

Indeed, (5.2) follows from (2.21), and the conditions in the theorem insure that $\frac{(v_{22}-v_{12})(z)}{(v_{22}-v_{12})^\sharp(z)}$ is a scattering function. Solving the corresponding inverse scattering problem gives us the first-order discrete system with asymptotic equivalence matrix function v since this function is uniquely determined from either of the other characteristic spectral functions.

6. The case of two-sided first-order systems

The relationships between the systems (1.1) and (1.2) has been studied in [4, Section 4]. There we proved that the solutions of the system (1.2) are of the form

$$\begin{aligned} Y_n(z) &= \prod_{\ell=0}^{n-1}(1-|\rho_\ell|^2)\begin{pmatrix}1 & 0\\ 0 & z^2\end{pmatrix} H_n(z^2)^{-1}\begin{pmatrix}z^n & 0\\ 0 & z^{-n}\end{pmatrix} H_0(z^2)\begin{pmatrix}1 & 0\\ 0 & \frac{1}{z^2}\end{pmatrix} Y_0(z)\\ &= z^{-n}\prod_{\ell=0}^{n-1}(1-|\rho_\ell|^2)\begin{pmatrix}1 & 0\\ 0 & z^2\end{pmatrix} H_n(z^2)^{-1}\begin{pmatrix}(z^2)^n & 0\\ 0 & 1\end{pmatrix} H_0(z^2)\begin{pmatrix}1 & 0\\ 0 & \frac{1}{z^2}\end{pmatrix} Y_0(z). \end{aligned}$$

In view of formula (2.1), and since the scalar factor z^{-n} does not affect the various linear transformations, this suggests that the set of characteristic spectral functions of both systems (for a given sequence ρ_n) are related by the map $z \mapsto z^2$. This is indeed the case, as explained in [4, Section 4].

The following result is proved in [4, Section 4].

Theorem 6.1. *Let ρ_n, $n = 0, 1, \ldots$ be a strictly pseudo-exponential sequence. The system* (1.2) *has a solution uniquely defined by the conditions*

$$\begin{aligned} \begin{pmatrix}1 & -1\end{pmatrix} Y_0(z) &= 0,\\ \begin{pmatrix}0 & 1\end{pmatrix} Y_n(z) &= z^{-n} + o(n). \end{aligned}$$

Then the limit

$$\lim_{n\to\infty}\begin{pmatrix}1 & 0\end{pmatrix} Y_n(z) z^{-n}$$

exists and is called the scattering function of the system (1.2). *It is related to the scattering function of the system* (1.1) *by the map $z \mapsto z^2$.*

The counterpart of Theorem 2.3 is now:

Theorem 6.2. *A rational function S is the scattering function of a two-sided first-order discrete system* (1.2) *if and only if it can be written as $S(z) = \frac{S_-(z^2)}{S_-(1/z^{*2})^*}$ where S_- is analytic and invertible in $\overline{\mathbb{E}}$. When this is the case, the inverse scattering problem is solved by solving the inverse scattering problem for the system* (1.1) *associated to the function $\frac{S_-(z)}{S_-(1/z^*)^*}$.*

To define the reflection coefficient function we now set

$$\begin{aligned} Q_n(z) &= C(\rho_0)\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix} C(\rho_1)\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}\cdots C(\rho_n)\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}\\ &= z^{-n-1}M_n(z^2). \end{aligned} \tag{6.1}$$

Theorem 6.3. *Let $\rho_n, n = 1, 2, \ldots$ be a strictly pseudo-exponential sequence and let $Q_n(z)$ be defined by* (6.1). *The limit*

$$R(z) = \lim_{n\to\infty} T_{Q_n(z)}(0) \tag{6.2}$$

exists and is equal to

$$R(z) = \frac{\beta_0}{\alpha_0}(1/z^2).$$

It is a function analytic and contractive in the open unit disk, called the reflection coefficient function. It takes strictly contractive values on the unit circle, and is related to the reflection coefficient function of the system (1.1) *by the map* $z \mapsto z^2$.

Indeed, in view of (6.1), we note that

$$T_{Q_n(z)}(0) = T_{M_n(z^2)}(0).$$

The arguments follow then those of the one-sided case.

As in Theorem 6.2 the inverse problem associated to R is solved by considering the corresponding problem for $\frac{\beta_0}{\alpha_0}(1/z)$.

One can also introduce the Weyl coefficient function

$$N(z) = i\frac{1 + z^2 R(z)}{1 - z^2 R(z)}$$

and the spectral function

$$W(z) = \text{Im } N(z).$$

Theorem 6.4. *The characteristic spectral functions of a two-sided first-order discrete system are even functions of z. They can be all expressed in term of a rational even function $\sigma_-(z)$, which is analytic and invertible in $\overline{\mathbb{E}}$ and normalized by*

$$\int_0^{2\pi} \frac{dt}{|\sigma_-(e^{-it})|^2} = 2\pi, \quad \textit{and} \quad \sigma_-(\infty) > 0$$

via:

$$\begin{aligned} S(z) &= \frac{\sigma_-(z)}{\sigma_-^\sharp(z)} \\ W(z) &= \frac{1}{|\sigma_-(1/z)|^2} \\ W(z) &= \text{Im } N(z) \\ N(z) &= i\frac{1 + z^2 R(z)}{1 - z^2 R(z)}. \end{aligned}$$

7. A numerical example

In this section we consider a numerical example. We take $S_-(z) = \frac{1-3z}{1-2z}$ (that is, the normalization (2.13) is not taken into account at this stage). Then

$$\begin{aligned}
z\frac{S_-(z)}{S_-^\sharp(z)} &= z\frac{z-2}{1-2z}\frac{1-3z}{z-3}\\
&= z\left(-\frac{3}{10}\frac{1}{(1-2z)} + \frac{8}{5}\frac{1}{(z-3)} + \frac{3}{2}\right)\\
&= \frac{3}{20}\frac{1}{(1-\frac{1}{2z})} + \frac{8}{5}\frac{z}{(-3)(1-\frac{z}{3})} + \frac{3}{2}z.
\end{aligned}$$

Hence, the negative Fourier coefficients of S are

$$\gamma_j = \frac{3}{20}\frac{1}{2^{|j|}} = ca^{|j|}b, \quad j = 0, -1, -2, \ldots$$

with

$$a = \frac{1}{2}, \quad c = \frac{3}{20}, \quad \text{and} \quad b = 1.$$

We compute the solutions of the Stein equations in (1.4):

$$\Delta = \frac{4}{3}, \quad \Omega = \frac{3}{100}.$$

Condition (1.5) is in force and we get

$$\rho_n = -\frac{3}{20}\frac{1}{2^n}\frac{1}{(1-\frac{1}{25}\frac{1}{2^{2n+2}})} = -\frac{15.2^n}{(5.2^{n+1}-1)(5.2^{n+1}+1)}. \tag{7.1}$$

The corresponding Schur function is given by formula (2.17) and we obtain

$$R(z) = \frac{5}{33-16z},$$

which is strictly contractive in the closed unit disk.

We now check directly that the Schur coefficients of this function are indeed given by (7.1). We proceed by induction. We first prove that for every positive integer n, the nth iteration R_n of the Schur algorithm is of the form

$$R_n(z) = \frac{1}{p_n - zq_n},$$

with $p_n \neq 0$ (and thus, $\rho_n = -\frac{1}{p_n}$), and that we have the recursion relation

$$\begin{aligned}
p_{n+1} &= \frac{|p_n|^2-1}{q_n}\\
q_{n+1} &= p_n^*.
\end{aligned} \tag{7.2}$$

We first remark that $|p_n| \neq 1$. Indeed, if $|p_n| = 1$ we would have $|R_n(0)| = 1$, and by the maximum modulus principle, $R_n(z)$ is a unitary constant. It would follow that R is a finite Blaschke product, which it is not. A direct computation shows that $R(0) = -\rho_0$. Applying twice the Schur algorithm leads to

$$R_1(z) = \frac{10}{133 - 66z} \quad \text{and} \quad R_2(z) = \frac{660}{(143.123) - 133.66.z}$$

and to Schur coefficients equal respectively to $\frac{10}{133}$ and $\frac{20}{13.41}$, which are in turn respectively equal to $-\rho_1$ and $-\rho_2$.

Assume now that the hypothesis is true at rank n. Then,

$$\begin{aligned} R_{n+1}(z) &= \frac{\frac{1}{p_n - zq_n} - \frac{1}{p_n}}{1 - \frac{1}{p_n - zq_n}\frac{1}{p_n^*}} \\ &= \frac{q_n}{|p_n|^2 - 1 - zp_n^* q_n} \\ &= \frac{1}{\frac{|p_n|^2-1}{q_n} - zq_n}. \end{aligned}$$

The division by $q_n = p_{n-1}^*$ is legitimate since the induction hypothesis holds at rank $n-1$, and hence $p_{n-1} \neq 0$. Furthermore, we already remarked that $|p_n| \neq 1$, and so $p_{n+1} = \frac{|p_n|^2-1}{q_n} \neq 0$, and hence the induction hypothesis is proved at rank $n+1$.

We now check that $\frac{1}{p_n} = -\rho_n$, where the sequence ρ_n is given by (7.1). We also prove this claim by induction. The result is true for $n = 0$ and $n = 1$, as mentioned above. From (7.2) we see that we have to show that for every n:

$$\rho_{n+1} = \frac{1}{\rho_{n-1}\left(\frac{1}{\rho_n^2} - 1\right)},$$

that is,

$$1 = \left(\frac{1}{\rho_n^2} - 1\right)\rho_{n-1}\rho_{n+1}.$$

This amounts to check that

$$1 = \left(\frac{(100 \cdot 2^{2n} - 1)^2}{225 \cdot 2^{2n}} - 1\right) \cdot \frac{225 \cdot 2^{2n}}{(25 \cdot 2^{2n} - 1)(400 \cdot 2^{2n} - 1)},$$

that is,

$$(25 \cdot 2^{2n} - 1)(400 \cdot 2^{2n} - 1) = (100 \cdot 2^{2n} - 1)^2 - 225 \cdot 2^{2n}.$$

This in turn is readily verified.

The Weyl coefficient function is equal to

$$N(z) = i\frac{1 - zR(z)}{1 + zR(z)} = i\frac{3}{11}\frac{11 - 7z}{3 - z},$$

and the spectral function is given by:

$$\begin{aligned}W(z) &= \text{Im } N(z)\\ &= \frac{24}{11}\left(\frac{5-4\text{Re } z}{|3-z|^2}\right)\\ &= \frac{24}{11}\frac{1}{|S_-(1/z)|^2}, \quad |z|=1.\end{aligned}$$

Thus the normalized spectral factor is equal to $\sqrt{\frac{11}{24}\frac{1-3z}{1-2z}}$. A direct computation using Cauchy's theorem shows that

$$\begin{aligned}\int_0^{2\pi}\frac{dt}{|S_-(e^{-it})|^2} &= \frac{1}{2\pi i}\int_{|z|=1}\frac{1}{S_-(z)S_-(1/z)}\frac{dz}{z}\\ &= \frac{1}{2\pi i}\int_{|z|=1}\frac{(1-2z)(2-z)}{(1-3z)(3-z)z}dz = \frac{11}{24}.\end{aligned}$$

8. An example of a non-strictly pseudo-exponential sequence

As already mentioned, the Schur function $R(z)=\frac{z}{2-z}$ takes value 1 for $z=1$ and in particular is not strictly contractive in the closed unit disk. By Theorem 4.1 we know that its sequence of Schur coefficients is not strictly pseudo-exponential. We check this directly here. To that purpose we use a formula for the Schur coefficients in terms of the Taylor series of the function $\phi(z)=\frac{1-R(z)}{1+R(z)}$ recalled in Theorem 4.7. For $R(z)=\frac{z}{2-z}$ we have that $\phi(z)=1-z$ and hence

$$c_0 = 1 \quad \text{and} \quad c_1 = -\frac{1}{2}.$$

Since the coefficients are real $\gamma_{0n}^{(n)}=\gamma_{n0}^{(n)}$, and we have to compute the entries of the first column of $\mathbb{T}_n^{-1}$. To ease the notation we denote the entries of this column by $a_0,\ldots,a_n$. One has to solve

$$\begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 & \cdots & 0\\ -\frac{1}{2} & 1 & \frac{1}{2} & 0 & \cdots & 0\\ 0 & -\frac{1}{2} & 1 & \frac{1}{2} & 0 & \cdots\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\ 0 & 0 & \cdot & & -\frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}a_0\\ a_1\\ \vdots\\ a_n\end{pmatrix} = \begin{pmatrix}1\\0\\ \vdots\\ 0\end{pmatrix},$$

that is, the system of equations

$$\begin{aligned}a_0 - \frac{a_1}{2} &= 1\\ a_0 + a_2 &= 2a_1\\ &\vdots\\ a_{n-2}+a_n &= 2a_{n-1}\\ a_{n-1} &= 2a_n.\end{aligned}$$

This system of equations has a unique solution, which is found as follows: we set

$$a_j = (j+1)a_0 - 2j, \quad j = 0, \dots, n.$$

These a_j satisfy the above equations, at the exception of the last one,

$$na_0 - (2n-2) = 2((n+1)a_0 - 2n)$$

which gives the value of a_0:

$$a_0 = \frac{2n+2}{n+2}.$$

Hence we obtain the value of the coefficient k_n:

$$k_n = \frac{a_n}{a_0} = n + 1 - \frac{2n}{a_0} = \frac{1}{n+1}$$

for $n \geq 1$. This sequence does not decrease exponentially fast to 0 and hence is not a strictly pseudo-exponential sequence.

Finally, we note that the Schur coefficients can also be computed by proving by induction that the nth iteration R_n in (2.26) is equal to $R_n(z) = \frac{1}{(n+1)-nz}$ for $n \geq 1$. Indeed, the claim is true for $n = 1$. Assume it holds at rank n. Then,

$$\begin{aligned} R_{n+1}(z) &= \frac{\frac{1}{(n+1)-nz} - \frac{1}{n+1}}{1 - \frac{1}{n+1}\frac{1}{(n+1)-nz}} \\ &= \frac{n}{(n+1)^2 - 1 - n(n+1)z} \\ &= \frac{1}{(n+2) - (n+1)z}. \end{aligned}$$

The Schur coefficients of the function $\frac{z}{2-z}$ were already computed in [30, Section 14].

9. Jacobi matrices

We now give an application to Jacobi matrices. We begin with a brief review of these matrices and of the associated inverse problem. We use extensively the papers [17], [19] and our previous paper [5]. For the general theory of Jacobi matrices we refer to [11, Chapter VII].

Jacobi matrices are infinite matrices of the form

$$J = \begin{pmatrix} b_0 & a_0 & 0 & 0 & & \cdot & \cdot & \cdot \\ a_0 & b_1 & a_1 & 0 & & \cdot & \cdot & \cdot \\ 0 & a_1 & b_2 & a_2 & 0 & \cdot & \cdot & \cdot \\ \vdots & & & & & & & \end{pmatrix},$$

where the numbers a_n are strictly positive and the b_n are real numbers.

One associates to such an infinite matrix a sequence of polynomials $P_0, P_1, \ldots$ via $P_0(\lambda) \equiv k_0 > 0$ and the recursion formulas

$$b_0 P_0(\lambda) + a_0 P_1(\lambda) = \lambda P_0(\lambda)$$
$$a_{n-1}P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda P_n(\lambda).$$

Favard proved in 1935 that there exist positive measures on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} P_n(\lambda) d\sigma(\lambda) P_m(\lambda) = \delta_{n,m}. \tag{9.1}$$

The inverse problem associated to $d\sigma(\lambda)$ consists in recovering the a_n and b_n from $d\sigma$. Of course, these can be obtained directly from $d\sigma(\lambda)$ via the formula

$$a_n = \frac{\kappa_n}{\kappa_{n+1}}, \quad \kappa_n \text{ being the coefficient of } \lambda^n \text{ in } P_n(\lambda),$$
$$b_n = \int_{\mathbb{R}} \lambda P_n(\lambda)^2 d\sigma(\lambda).$$

This is the analog of computing the coefficients ρ_n in the discrete system (1.1) directly via (4.8) (see the discussion at the end of page 236 of [17]), and does not take into account possible special properties of $d\sigma(\lambda)$.

When the sequences a_n and b_n are bounded, J defines a bounded self-adjoint operator (see [11, Theorem 1.2 p. 504]), and the measure is unique. Under the assumption that both limits

$$\lim_{n\to\infty} a_n = a \quad \text{and} \quad \lim_{n\to\infty} b_n = b$$

exist, the first limit being strictly positive, and that, moreover

$$\sum_{n=1}^{\infty} n \left\{ \left| 1 - \frac{a_n^2}{a^2} \right| + \left| \frac{b_n - b}{a} \right| \right\} < \infty, \tag{9.2}$$

one can say more on the measure; see [19, Theorem 3, p. 474]; $d\sigma(\lambda)$ has then a simple form and one can relate the inverse problem to the inverse problem for a related discrete first-order one-sided system. Following [17] we will assume

$$\lim_{n\to\infty} a_n = 1 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

One then has:

Theorem 9.1. *Assume that* (9.2) *holds with* $a = 1$ *and* $b = 0$. *There exists a measure* $d\sigma$ *with support in* $[-2, 2]$, *with a finite number of jumps outside* $[-2, 2]$ *such that* (9.1) *holds. Furthermore,* $d\sigma(\lambda)$ *is absolutely continuous with respect to Lebesgue measure on* $[-2, 2]$ *and*

$$\frac{d\sigma(\lambda)}{d\lambda} = \frac{\sin\theta}{|f_+(e^{i\theta})|^2}, \quad \lambda = 2\cos\theta, \tag{9.3}$$

where the function $zf_+(z)$ *belongs to* $\mathcal{W}_+$.

See [19, Theorem 1, p. 473] for the above result. We also note that the function f_+ has real Fourier coefficients. Formulas are also available for the jumps of $d\sigma(\lambda)$. We will not recall them here.

We assume that $d\sigma(\lambda)$ is of the form

$$d\sigma(\lambda)=\begin{cases}\frac{2}{\pi}\left(\sin^2\theta\right)W(e^{i\theta})d\theta, & |\lambda|\le 2\\ 0, \quad |\lambda|>2\end{cases} \tag{9.4}$$

where $W(z)=\sum_{\mathbb{Z}} w_n z^n$ is in $\mathcal{W}$, has real Fourier coefficients and is strictly positive on the unit circle. In [17] H. Dym and A. Iacob wrote explicitly the relationships between the Schur coefficients ϵ_n of the function

$$r(z)=\frac{(w_0+2\sum_{n=1}^{\infty}w_k z^k)-1}{(w_0+2\sum_{n=1}^{\infty}w_k z^k)+1} \tag{9.5}$$

and the sequences a_n and b_n.

Theorem 9.2. *Assume that $d\sigma(\lambda)$ is of the form* (9.4), *and let $\epsilon_0,\epsilon_1,\ldots$ be the Schur coefficients of the function* (9.5). *Then,*

$$\begin{aligned}a_n &= \left\{(1+\epsilon_{2n+2})(1-|\epsilon_{2n+3}|^2)(1-\epsilon_{2n+4})\right\}^{1/2}\\ b_n &= \epsilon_{2n+1}(1-\epsilon_{2n+2})-\epsilon_{2n+3}(1+\epsilon_{2n+2}), \quad n=0,1,\ldots\end{aligned} \tag{9.6}$$

In the next theorem we specialize (9.3) to the case where $W(z)$ is moreover rational. In [5, pp. 165–166] we computed the coefficients a_n and b_n in terms of a minimal realization of $W(z)$. In the present section we chose a different route. We remark that $W(z)$ is the spectral function of a first-order one-sided discrete system with strictly pseudo-exponential potential (see Theorem 4.3), and we can use the results proved earlier in the paper to compute the sequences a_n and b_n.

Theorem 9.3. *Let $d\sigma(\lambda)$ be of the form*

$$d\sigma(\lambda)=\begin{cases}\frac{2}{\pi}\left(\sin^2\theta\right)W(e^{i\theta})d\theta, & |\lambda|\le 2\\ 0, \quad |\lambda|>2\end{cases}$$

where $W(z)$ is a real rational function without poles and positive on the unit circle and such that $\int_0^{2\pi}W(e^{i\theta})d\theta=1$. Write $W(z)=\frac{1}{|S_-(1/z)|^2}$ where S_- and its inverse are invertible in $\overline{\mathbb{E}}$. Let

$$z\frac{S_-(z)}{S_-(1/z)}=\sum_{\mathbb{Z}}\gamma_j z^j, \quad |z|=1,$$

and let (a,b,c) be a minimal triple such that

$$\gamma_{-j}=ca^jb, \quad j=0,1,2,\ldots$$

holds. Finally, let ρ_n be built from (a,b,c) via (1.3) *and set $\epsilon_0=0$ and $\epsilon_n=\rho_{n-1}$ for $n\ge 1$. Then, the coefficients a_n and b_n are given by* (9.2) *with this choice of ϵ_n.*

Indeed, the function $r(z)$ given by (9.5) vanishes at the origin because of the normalization $\int_0^{2\pi} W(e^{i\theta})d\theta = 1$. Write $r(z) = -zR(z)$ and set

$$N(z) = i\left(w_0 + 2\sum_{n=1}^{\infty} w_n z^n\right).$$

Then, $\text{Im } N(z) = W(z), \quad |z| = 1$ and

$$N(z) = i\frac{1 - zR(z)}{1 + zR(z)},$$

that is, R is the reflection coefficient function associated to the discrete first-order system with spectral function $W(z)$. Furthermore the sequence of Schur coefficients of $r(z)$ is

$$0, \rho_0, \rho_1, \ldots$$

since the $-\rho_j$ are the Schur coefficients of $R(z)$.

We conclude with an example:

Example 9.4. Let $\epsilon \in (-1, 1)$ and

$$W(z) = \frac{1 - \epsilon^2}{(1 + \epsilon z)(1 + \frac{\epsilon}{z})}.$$

Then, $\frac{1}{2\pi}\int_0^{2\pi} W(e^{it})dt = 1$ and

$$S_-(z) = \frac{\sqrt{1 - \epsilon^2}}{1 + \frac{\epsilon}{z}}.$$

Then, the negative Fourier coefficients of

$$z\frac{S_-(z)}{S_-(1/z)} = (z + \epsilon z^2)\left\{1 - \frac{\epsilon}{z} + \frac{\epsilon^2}{z^2} - \frac{\epsilon^3}{z^3} + \cdots\right\}$$

are equal to $\gamma_{-j} = ca^j b$ $(j = 0, 1, \ldots)$ with

$$c = \epsilon, \quad a = -\epsilon \quad \text{and} \quad b = \epsilon^2 - 1.$$

The Stein equations (1.4) have solutions

$$\Delta = 1 - \epsilon^2 \quad \text{and} \quad \Omega = \frac{\epsilon^2}{1 - \epsilon^2}.$$

In particular, inequality (1.5) holds.

The Schur coefficients of the associated first-order system are thus equal to

$$k_n = \frac{(-\epsilon)^n(\epsilon^2 - 1)}{1 - \epsilon^{2n+2}},$$

and we have:

$$
\begin{aligned}
a_n &= \left\{ \left(1 + \frac{\epsilon^{2n+2}(\epsilon^2-1)}{1-\epsilon^{4n+6}}\right)\left(1 - \left(\frac{\epsilon^{2n+3}(\epsilon^2-1)}{1-\epsilon^{4n+8}}\right)^2\right)\right. \\
&\qquad \left. \times \left(1 - \frac{\epsilon^{2n+4}(\epsilon^2-1)}{1-\epsilon^{4n+10}}\right)\right\}^{1/2} \\
b_n &= \left\{ -\frac{\epsilon^{2n+1}(\epsilon^2-1)}{1-\epsilon^{4n+4}}\left(1 - \frac{\epsilon^{2n+2}(\epsilon^2-1)}{1-\epsilon^{4n+6}}\right)\right. \\
&\qquad \left. + \frac{\epsilon^{2n+3}(\epsilon^2-1)}{1-\epsilon^{4n+8}}\left(1 + \frac{\epsilon^{2n+2}(\epsilon^2-1)}{1-\epsilon^{4n+6}}\right)\right\}.
\end{aligned}
$$

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Boundary Nevanlinna–Pick Interpolation Problems for Generalized Schur Functions

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Abstract. Three boundary multipoint Nevanlinna-Pick interpolation problems are formulated for generalized Schur functions. For each problem, the set of all solutions is parametrized in terms of a linear fractional transformation with a Schur class parameter.

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1. Introduction

The Schur class $\mathcal{S}$ of complex-valued analytic functions mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$ can be characterized in terms of positive kernels as follows: a function w belongs to $\mathcal{S}$ if and only if the kernel

$$K_w(z,\zeta) := \frac{1-\overline{w(\zeta)}w(z)}{1-\bar{\zeta}z} \tag{1.1}$$

is positive definite on $\mathbb{D}$ (in formulas: $K_w \succeq 0$), i.e., if and only if the Hermitian matrix

$$[K_w(z_j,z_i)]_{i,j=1}^n = \left[\frac{1-\overline{w(z_i)}w(z_j)}{1-\bar{z}_i z_j}\right]_{i,j=1}^n \tag{1.2}$$

is positive semidefinite for every choice of an integer n and of n points $z_1,\ldots,z_n \in \mathbb{D}$. The significance of this characterization for interpolation theory is that it gives the necessity part in the Nevanlinna-Pick interpolation theorem: *given points $z_1,\ldots,z_n \in \mathbb{D}$ and $w_1,\ldots,w_n \in \mathbb{C}$, there exists $w \in \mathcal{S}$ with $w(z_j) = w_j$ for $j = 1,\ldots,n$ if and only if the associated Pick matrix $P = \left[\frac{1-\overline{w}_i w_j}{1-\overline{z}_i z_j}\right]$ is positive semidefinite.*

There are at least two obstacles to get an immediate boundary analogue of the latter result just upon sending the points $z_1, \ldots, z_n$ in (1.2) to the unit circle $\mathbb{T}$. Firstly, the boundary nontangential (equivalently, radial) limits

$$w(t) := \lim_{z \to t} w(z) \tag{1.3}$$

exist at almost every (but not every) point t on $\mathbb{T}$. Secondly, although the nontangential limits

$$d_w(t) := \lim_{z \to t} \frac{1 - |w(z)|^2}{1 - |z|^2} \geq 0 \quad (t \in \mathbb{T}) \tag{1.4}$$

exist at every $t \in \mathbb{T}$, they can be infinite. However, if $d_w(t) < \infty$, then it is readily seen that the limit (1.3) exists and is unimodular. Then we can pass to limits in (1.2) to get the necessity part of the following interpolation result:

Given points $t_1, \ldots, t_n \in \mathbb{T}$ and numbers $w_1, \ldots, w_n$ and $\gamma_1, \ldots, \gamma_n$ such that

$$|w_i| = 1 \quad \text{and} \quad \gamma_i \geq 0 \quad \text{for} \quad i = 1, \ldots, n, \tag{1.5}$$

there exists $w \in \mathcal{S}$ with

$$w(t_i) = w_i \quad \text{and} \quad d_w(t_i) \leq \gamma_i \quad \text{for} \quad i = 1, \ldots, n \tag{1.6}$$

if and only if the associated Pick matrix

$$P = [P_{ij}]_{i,j=1}^n \quad \text{with the entries} \quad P_{ij} = \begin{cases} \dfrac{1 - \overline{w}_i w_j}{1 - \overline{t}_i t_j} & \text{for} \quad i \neq j \\ \gamma_i & \text{for} \quad i = j \end{cases} \tag{1.7}$$

is positive semidefinite.

This result in turn, suggests the following well-known boundary Nevanlinna–Pick interpolation problem.

Problem 1.1. *Given points $t_1, \ldots, t_n \in \mathbb{T}$ and numbers $w_1, \ldots, w_n$, $\gamma_1, \ldots, \gamma_n$ as in* (1.5) *and such that the Pick matrix P defined in* (1.7) *is positive semidefinite, find all functions $w \in \mathcal{S}$ satisfying interpolation conditions* (1.6).

Note that assumptions (1.5) and $P \geq 0$ are not restrictive since they are necessary for the problem to have a solution.

The boundary Nevanlinna–Pick interpolation problem was worked out using quite different approaches: the method of fundamental matrix inequalities [12], the recursive Schur algorithm [7], the Grassmannian approach [3], via realization theory [2], and via unitary extensions of partially defined isometries [1, 11]. If P is singular, then Problem 1.1 has a unique solution which is a finite Blaschke product of degree $r = \operatorname{rank} P$. If P is positive definite, Problem 1.1 has infinitely many solutions that can be described in terms of a linear fractional transformation with a free Schur class parameter.

Note that a similar problem with equality sign in the second series of conditions in (1.6) was considered in [19, 9, 6]:

Problem 1.2. *Given the data as in Problem* 1.1, *find all functions* $w \in \mathcal{S}$ *such that*

$$w(t_i) = w_i \quad \textit{and} \quad d_w(t_i) = \gamma_i \quad \textit{for} \quad i = 1, \dots, n \tag{1.8}$$

The solvability criteria for this modified problem is also given in terms of the Pick matrix (1.7) but it is more subtle: condition $P \geq 0$ is necessary (not sufficient, in general) for the Problem 1.2 to have a solution while the condition $P > 0$ is sufficient.

The objective of this paper is to study the above problems in the setting of *generalized Schur functions.* A function w is called *a generalized Schur function* if it is of the form

$$w(z) = \frac{S(z)}{B(z)}, \tag{1.9}$$

for some Schur function $S \in \mathcal{S}$ and a finite Blaschke product B. Without loss of generality we can (and will) assume that S and B in representation (1.9) have no common zeroes. For a fixed integer $\kappa \geq 0$, we denote by $\mathcal{S}_\kappa$ the class of generalized Schur functions with κ poles inside $\mathbb{D}$, i.e., the class of functions of the form (1.9) with a Blaschke product B of degree κ. Thus, $\mathcal{S}_\kappa$ is a class of functions w such that

1. w is meromorphic in $\mathbb{D}$ and has κ poles inside $\mathbb{D}$ counted with multiplicities.
2. w is bounded on an annulus $\{z : \ \rho < |z| < 1\}$ for some $\rho \in (0, 1)$.
3. Boundary nontangential limits $w(t) := \lim\limits_{z \to t} w(z)$ exist and satisfy $|w(t)| \leq 1$ for almost all $t \in \mathbb{T}$.

It is clear that the class $\mathcal{S}_0$ coincides with the classical Schur class.

The class $\mathcal{S}_\kappa$ can be characterized alternatively (and sometimes this characterization is taken as the definition of the class) as the set of functions w meromorphic on $\mathbb{D}$ and such that the kernel $K_w(z, \zeta)$ defined in (1.1) has κ negative squares on $\mathbb{D} \cap \rho(w)$ ($\rho(w)$ stands for the domain of analyticity of w); in formulas: $\mathrm{sq}_-(K_w) = \kappa$. The last equality means that for every choice of an integer n and of n points $z_1, \dots, z_n \in \mathbb{D} \cap \rho(w)$, the Hermitian matrix (1.9) has at most κ negative eigenvalues:

$$\mathrm{sq}_- \left[\frac{1 - \overline{w(z_i)} w(z_j)}{1 - \bar{z}_i z_j} \right]_{i,j=1}^n \leq \kappa, \tag{1.10}$$

and for at least one such choice it has exactly κ negative eigenvalues counted with multiplicities. In what follows, we will say "w has κ negative squares" rather than "the kernel K_w has κ negative squares".

Due to representation (1.9) and in view of the quite simple structure of finite Blaschke products, most of the results concerning the boundary behavior of generalized Schur functions can be derived from the corresponding classical results for the Schur class functions. For example, the nontangential boundary limit $d_w(t)$ (defined in (1.4)) exists for every $t \in \mathbb{T}$ and satisfies $d_w(t) > -\infty$ (not necessarily nonnegative, in contrast to the definite case). Indeed, if w is of the form (1.9), then

$$\frac{1-|w(z)|^2}{1-|z|^2} = \frac{1}{|B(z)|^2}\left(\frac{1-|S(z)|^2}{1-|z|^2} - \frac{1-|B(z)|^2}{1-|z|^2}\right). \tag{1.11}$$

Passing to the limits as z tends to $t \in \mathbb{T}$ in the latter equality and taking into account that $|B(t)| = 1$, we get

$$d_w(t) = d_S(t) - d_B(t) > -\infty,$$

since $d_w(t_0) \geq 0$ and $d_B(t) < \infty$. Furthermore, as in the definite case, if $d_w(t) < \infty$, then the nontangential limit (1.3) exists and is unimodular.

Now we formulate indefinite analogues of Problems 1.1 and 1.2. The data set for these problems will consist of n points $t_1, \dots, t_n$ on $\mathbb{T}$, n unimodular numbers $w_1, \dots, w_n$ and n real numbers $\gamma_1, \dots, \gamma_n$:

$$t_i \in \mathbb{T}, \quad |w_i| = 1, \quad \gamma_i \in \mathbb{R} \qquad (i = 1, \dots, n). \tag{1.12}$$

As in the definite case, we associate to the interpolation data (1.12) the Pick matrix P via the formula (1.7) which is still Hermitian (since $\gamma_j \in \mathbb{R}$), but not positive semidefinite, in general. Let κ be the number of its negative eigenvalues:

$$\kappa := \text{sq}_- P, \tag{1.13}$$

where

$$P = [P_{ij}]_{i,j=1}^n \quad \text{and} \quad P_{ij} = \begin{cases} \dfrac{1-\overline{w}_i w_j}{1-\bar{t}_i t_j} & \text{for} \quad i \neq j, \\ \gamma_j & \text{for} \quad i = j. \end{cases} \tag{1.14}$$

The next problem is an indefinite analogue of Problem 1.2 and it coincides with Problem 1.2 if $\kappa = 0$.

Problem 1.3. *Given the data set* (1.12), *find all functions* $w \in \mathcal{S}_\kappa$ *(with* κ *defined in* (1.13)) *such that*

$$d_w(t_i) := \lim_{z \to t_i} \frac{1-|w(z)|^2}{1-|z|^2} = \gamma_i \qquad (i = 1, \dots, n) \tag{1.15}$$

and

$$w(t_i) := \lim_{z \to t_i} w(z) = w_i \qquad (i = 1, \dots, n). \tag{1.16}$$

The analogue of Problem 1.1 is:

Problem 1.4. *Given the data set* (1.12), *find all functions* $w \in \mathcal{S}_\kappa$ *(with* κ *defined in* (1.13)) *such that*

$$d_w(t_i) \leq \gamma_i \quad \text{and} \quad w(t_i) = w_i \qquad (i = 1, \dots, n). \tag{1.17}$$

Interpolation conditions for the two above problems are clear: existence of the nontangential limits $d_w(t_i)$'s implies existence of the nontangential limits $w(t_i)$'s; upon prescribing the values of these limits (or upon prescribing upper bounds for $d_w(t_i)$'s) we come up with interpolation conditions (1.15)–(1.17). The choice (1.13) for the index of $\mathcal{S}_\kappa$ should be explained in some more detail.

Remark 1.5. *If a generalized Schur function w satisfies interpolation conditions* (1.17)*, then it has at least $\kappa = \mathrm{sq}_{-}P$ negative squares.*

Indeed, if w is a generalized Schur function of the class $\mathcal{S}_{\widetilde{\kappa}}$ and $t_1, \ldots, t_n$ are distinct points on $\mathbb{T}$ such that

$$d_w(t_i) < \infty \quad \text{for} \quad i = 1, \ldots, n,$$

then the nontangential boundary limits $w(t_i)$'s exist (and are unimodular) and one can pass to the limit in (1.10) (as $t_i \to z_i$ for $i = 1, \ldots, n$) to conclude that the Hermitian matrix

$$P^w(t_1, \ldots, t_n) = \left[P^w_{ij}\right]_{i,j=1}^n, \ P^w_{ij} = \begin{cases} \dfrac{1 - \overline{w(t_i)}w(t_j)}{1 - \bar{t}_i t_j} & \text{for} \quad i \neq j \\ d_w(t_i) & \text{for} \quad i = j \end{cases} \tag{1.18}$$

satisfies

$$\mathrm{sq}_{-}P^w(t_1, \ldots, t_n) \leq \widetilde{\kappa}. \tag{1.19}$$

If w meets conditions (1.16), then the nondiagonal entries in the matrices

$$P^w(t_1, \ldots, t_n) \quad \text{and} \quad P$$

coincide which clearly follows from the definitions (1.14) and (1.18). It follows from the same definitions that

$$P - P^w(t_1, \ldots, t_n) = \begin{bmatrix} \gamma_1 - d_w(t_1) & & 0 \\ & \ddots & \\ 0 & & \gamma_n - d_w(t_n) \end{bmatrix}$$

and thus, conditions (1.15) and the first series of conditions in (1.17) can be written equivalently in the matrix form as

$$P^w(t_1, \ldots, t_n) = P \quad \text{and} \quad P^w(t_1, \ldots, t_n) \leq P, \tag{1.20}$$

respectively. Each one of the two last relations implies, in view of (1.19) that

$$\mathrm{sq}_{-}P \leq \widetilde{\kappa}.$$

Thus, the latter condition is necessary for existence of a function w of the class $\mathcal{S}_{\widetilde{\kappa}}$ satisfying interpolation conditions (1.17) (or (1.15) and (1.16)). The choice (1.13) means that we are concerned about generalized Schur functions with the minimally possible negative index.

Problems 1.3 and 1.4 are indefinite analogues of Problems 1.2 and 1.1, respectively. Now we introduce another boundary interpolation problem that does not appear in the context of classical Schur functions.

Problem 1.6. *Given the data set* (1.12)*, find all functions $w \in \mathcal{S}_{\kappa'}$ for some $\kappa' \leq \kappa = \mathrm{sq}_{-}P$ such that conditions* (1.17) *are satisfied at all but $\kappa - \kappa'$ points $t_1, \ldots, t_n$.*

In other words, a solution w to the last problem is allowed to have less then κ negative squares and to omit some of interpolation conditions (but not too many of them). The significance of Problem 1.6 will be explained in the next section.

2. Main results

The purpose of the paper is to obtain parametrizations of solution sets $\mathbb{S}_{13}$, $\mathbb{S}_{14}$ and $\mathbb{S}_{16}$ for Problems 1.3, 1.4 and 1.6, respectively. First we note that

$$\mathbb{S}_{13} \subseteq \mathbb{S}_{14} \subseteq \mathbb{S}_{16} \quad \text{and} \quad \mathbb{S}_{14} = \mathbb{S}_{16} \cap \mathcal{S}_\kappa. \tag{2.1}$$

Inclusions in (2.1) are self-evident. If w is a solution of Problems 1.6 with $\kappa' = \kappa$, then $\kappa - \kappa' = 0$ which means that conditions (1.17) are satisfied at all points $t_1, \dots, t_n$ and thus, $w \in \mathbb{S}_{14}$. Thus, $\mathbb{S}_{14} \subseteq \mathbb{S}_{16} \cap \mathcal{S}_\kappa$. The reverse inclusion is evident, since $\mathbb{S}_{14} \subseteq \mathcal{S}_\kappa$. Note also that if $\kappa = 0$, then Problems 1.4 and 1.6 are equivalent: $\mathbb{S}_{14} = \mathbb{S}_{16}$.

It turns out that in the indefinite setting (i.e., when $\kappa > 0$), Problem 1.6 plays the same role as Problem 1.4 does in the classical setting: it always has a solution and, in the indeterminate case, the solution set $\mathbb{S}_{16}$ admits a linear fractional parametrization with the free Schur class parameter. The case when P is singular, is relatively simple:

Theorem 2.1. *Let P be singular. Then Problem* 1.6 *has a unique solution w which is the ratio of two finite Blaschke products*

$$w(z) = \frac{B_1(z)}{B_2(z)}$$

with no common zeroes and such that

$$\deg B_1 + \deg B_2 = \operatorname{rank} P.$$

Furthermore, if $\deg B_2 = \kappa$, then w is also a solution of Problem 1.4.

The proof will be given in Section 7. Now we turn to a more interesting case when P is not singular. In this case, we pick an arbitrary point $\mu \in \mathbb{T} \setminus \{t_1, \dots, t_n\}$ and introduce the 2×2 matrix-valued function

$$\begin{aligned} \Theta(z) &= \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix} \\ &= I_2 + (z-\mu) \begin{bmatrix} C \\ E \end{bmatrix} (zI_n - T)^{-1} P^{-1} (I_n - \mu T^*)^{-1} \begin{bmatrix} C^* & -E^* \end{bmatrix} \end{aligned} \tag{2.2}$$

where

$$T = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{bmatrix}, \quad E = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}, \quad C = \begin{bmatrix} w_1 & \dots & w_n \end{bmatrix}. \tag{2.3}$$

Note that the Pick matrix P defined in (1.14) satisfies the following identity

$$P - T^*PT = E^*E - C^*C. \tag{2.4}$$

Indeed, equality of nondiagonal entries in (2.4) follows from the definition (1.18) of P, whereas diagonal entries in both sides of (2.4) are zeroes. Identity (2.4) and all its ingredients will play an important role in the subsequent analysis.

The function Θ defined in (2.2) is rational and has simple poles at $t_1, \dots, t_n$. Note some extra properties of Θ. Let J be a signature matrix defined as

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.5}$$

It turns out that Θ is J-unitary on the unit circle, i.e., that

$$\Theta(t)J\Theta(t)^* = J \quad \text{for every } \ t \in \mathbb{T} \cap \rho(\Theta) \tag{2.6}$$

and the kernel

$$K_{\Theta,J}(z,\zeta) := \frac{J - \Theta(z)J\Theta(\zeta)^*}{1 - z\bar{\zeta}} \tag{2.7}$$

has $\kappa = \mathrm{sq}_- P$ negative squares on $\mathbb{D}$:

$$\mathrm{sq}_- K_{\Theta,J} = \kappa. \tag{2.8}$$

We shall use the symbol $\mathcal{W}_\kappa$ for the class of 2×2 meromorphic functions satisfying conditions (2.6) and (2.8). It is well known that for *every* function $\Theta \in \mathcal{W}_\kappa$, the linear fractional transformation

$$\mathbf{T}_\Theta : \ \mathcal{E} \longrightarrow \frac{\Theta_{11}\mathcal{E} + \Theta_{12}}{\Theta_{21}\mathcal{E} + \Theta_{22}} \tag{2.9}$$

is well defined for every Schur class function $\mathcal{E}$ and maps $\mathcal{S}_0$ into $\bigcup_{\kappa' \le \kappa} \mathcal{S}_{\kappa'}$. This map is not onto and the question about its range is of certain interest. If Θ is of the form (2.2), the range of the transformation (2.9) is $\mathbb{S}_{16}$:

Theorem 2.2. *Let P, T, E and C be defined as in* (1.14) *and* (2.3) *and let w be a function meromorphic on $\mathbb{D}$. If P is invertible, then w is a solution of Problem* 1.6 *if and only if it is of the form*

$$w(z) = \mathbf{T}_\Theta[\mathcal{E}](z) := \frac{\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z)}{\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z)}, \tag{2.10}$$

for some Schur function $\mathcal{E} \in \mathcal{S}_0$.

It is not difficult to show that every rational function Θ from the class $\mathcal{W}_\kappa$ with simple poles at $t_1, \dots, t_n \in \mathbb{T}$ and normalized to I_2 at $\mu \in \mathbb{T}$, is necessarily of the form (2.2) for some row vector $C \in \mathbb{C}^{1\times n}$ with unimodular entries, with E as in (2.3) and with a Hermitian invertible matrix P having κ negative squares and being subject to the Stein identity (2.4). Thus, Theorem 2.2 clarifies the interpolation meaning of the range of a linear fractional transformation based on a rational function Θ of the class $\mathcal{W}_\kappa$ with simple poles on the boundary of the unit disk.

The necessity part in Theorem 2.2 will be obtained in Section 3 using an appropriate adaptation of the V.P. Potapov's method of the Fundamental Matrix Inequality (FMI) to the context of generalized Schur functions. The proof of the sufficiency part rests on Theorems 2.3 and 2.5 which are of certain independent interest. To formulate these theorems, let us introduce the numbers $\widetilde{c}_1,\dots,\widetilde{c}_n$ and $\widetilde{e}_1,\dots,\widetilde{e}_n$ by

$$\widetilde{c}_i^* := -\lim_{z\to t_i}(z-t_i)\Theta_{21}(z) \quad \text{and} \quad \widetilde{e}_i^* := \lim_{z\to t_i}(z-t_i)\Theta_{22}(z) \quad (i=1,\dots,n) \tag{2.11}$$

(for notational convenience we will write sometimes a^* rather than $\overline{a}$ for $a\in\mathbb{C}$). It turns out $|\widetilde{c}_i| = |\widetilde{e}_i| \neq 0$ (see Lemma 3.1 below for the proof) and therefore the following numbers

$$\eta_i := \frac{\widetilde{c}_i}{\widetilde{e}_i} = \frac{\widetilde{e}_i^*}{\widetilde{c}_i^*} = -\lim_{z\to t_i}\frac{\Theta_{22}(z)}{\Theta_{21}(z)} \quad (i=1,\dots,n) \tag{2.12}$$

are unimodular:

$$|\eta_i| = 1 \quad (i=1,\dots,n). \tag{2.13}$$

Furthermore let $\widetilde{p}_{ii}$ stand for the ith diagonal entry of the matrix P^{-1}, the inverse of the Pick matrix. It is self-evident that for a fixed i, any function $\mathcal{E}\in\mathcal{S}_0$ satisfies exactly one of the following six conditions:

$\mathbf{C}_1$: The function $\mathcal{E}$ fails to have a nontangential boundary limit η_i at t_i.

$$\mathbf{C}_2: \quad \mathcal{E}(t_i) := \lim_{z\to t_i}\mathcal{E}(z) = \eta_i \quad \text{and} \quad d_{\mathcal{E}}(t_i) := \frac{1-|\mathcal{E}(z)|^2}{1-|z|^2} = \infty. \tag{2.14}$$

$$\mathbf{C}_3: \quad \mathcal{E}(t_i) = \eta_i \quad \text{and} \quad -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2} < d_{\mathcal{E}}(t_i) < \infty. \tag{2.15}$$

$$\mathbf{C}_4: \quad \mathcal{E}(t_i) = \eta_i \quad \text{and} \quad 0 \le d_{\mathcal{E}}(t_i) < -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2}. \tag{2.16}$$

$$\mathbf{C}_5: \quad \mathcal{E}(t_i) = \eta_i \quad \text{and} \quad d_{\mathcal{E}}(t_i) = -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2} > 0. \tag{2.17}$$

$$\mathbf{C}_6: \quad \mathcal{E}(t_i) = \eta_i \quad \text{and} \quad d_{\mathcal{E}}(t_i) = \widetilde{p}_{ii} = 0. \tag{2.18}$$

Note that condition $\mathbf{C}_1$ means that either the nontangential boundary limit

$$\mathcal{E}(t_i) := \lim_{z\to t_i}\mathcal{E}(z)$$

fails to exist or it exists and is not equal to η_i. Let us denote by $\mathbf{C}_{4-6}$ the disjunction of conditions $\mathbf{C}_4$, $\mathbf{C}_5$ and $\mathbf{C}_6$:

$$\mathbf{C}_{4-6}: \quad \mathcal{E}(t_i) = \eta_i \quad \text{and} \quad d_{\mathcal{E}}(t_i) \le -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2}. \tag{2.19}$$

The next theorem gives a classification of interpolation conditions that are or are not satisfied by a function w of the form (2.10) in terms of the corresponding parameter $\mathcal{E}$.

Theorem 2.3. *Let the Pick matrix P be invertible, let $\mathcal{E}$ be a Schur class function, let Θ be given by (2.2), let $w = \mathbf{T}_\Theta[\mathcal{E}]$ and let t_i be an interpolation node.*

1. *The nontangential boundary limits $d_w(t_i)$ and $w(t_i)$ exist and are subject to*

$$d_w(t_i) = \gamma_i \quad \textit{and} \quad w(t_i) = w_i$$

if and only if the parameter $\mathcal{E}$ meets either condition $\mathbf{C}_1$ or $\mathbf{C}_2$.

2. *The nontangential boundary limits* $d_w(t_i)$ *and* $w(t_i)$ *exist and are subject to*

$$d_w(t_i) < \gamma_i \quad \textit{and} \quad w(t_i) = w_i$$

if and only if the parameter $\mathcal{E}$ *meets condition* $\mathbf{C}_3$.

3. *The nontangential boundary limits* $d_w(t_i)$ *and* $w(t_i)$ *exist and are subject to*

$$\gamma_i < d_w(t_i) < \infty \quad \textit{and} \quad w(t_i) = w_i.$$

if and only if the parameter $\mathcal{E}$ *meets condition* $\mathbf{C}_4$.

4. *If* $\mathcal{E}$ *meets* $\mathbf{C}_5$, *then* w *is subject to one of the following:*
   (a) *The limit* $w(t_i)$ *fails to exist.*
   (b) *The limit* $w(t_i)$ *exists and* $w(t_i) \neq w_i$.
   (c) $w(t_i) = w_i$ *and* $d_w(t_i) = \infty$.
5. *If* $\mathcal{E}$ *meets* $\mathbf{C}_6$, *then* w *is the ratio of two finite Blaschke products,*

$$d_w(t_i) < \infty \quad \textit{and} \quad w(t_i) \neq w_i.$$

We note an immediate consequence of the last theorem.

Corollary 2.4. *A function* $w = \mathbf{T}_\Theta[\mathcal{E}]$ *meets the ith interpolation conditions for Problem* 1.4*:*

$$d_w(t_i) \leq \gamma_i \quad \textit{and} \quad w(t_i) = w_i$$

if and only if the corresponding parameter $\mathcal{E} \in \mathcal{S}_0$ *meets the condition* $\mathbf{C}_{1-3} := \mathbf{C}_1 \vee \mathbf{C}_2 \vee \mathbf{C}_3$ *at* t_i.

Note that Problem 1.3 was considered in [2] for rational generalized Schur functions. It was shown ([2, Theorem 21.1.2]) that all rational solutions of Problem 1.3 are parametrized by the formula (2.10) when $\mathcal{E}$ varies over the set of all rational Schur functions such that (in the current terminology)

$$\mathcal{E}(t_i) \neq \eta_i \quad \text{for} \quad i = 1, \ldots, n.$$

Note that if $\mathcal{E}$ is a *rational* Schur function admitting a unimodular value $\mathcal{E}(t_0)$ at a boundary point $t_0 \in \mathbb{T}$, then the limit $d_w(t_0)$ always exists and equals $t_0\mathcal{E}'(t_0)\mathcal{E}(t_0)^*$. The latter follows from the converse Carathéodory-Julia theorem (see, e.g., [18, 20]):

$$\begin{aligned} d_w(t_0) := \lim_{z \to t_0} \frac{1 - |\mathcal{E}(z)|^2}{1 - |z|^2} &= \lim_{z \to t_0} \frac{1 - \mathcal{E}(z)\mathcal{E}(t_0)^*}{1 - z\bar{t}_0} \\ &= \lim_{z \to t_0} \frac{\mathcal{E}(t_0) - \mathcal{E}(z)}{t_0 - z} \cdot \frac{\mathcal{E}(t_0)^*}{\bar{t}_0} \\ &= t_0\mathcal{E}'(t_0)\mathcal{E}(t_0)^* < \infty. \end{aligned}$$

Thus, a Schur function $\mathcal{E}$ cannot satisfy condition $\mathbf{C}_2$ at a boundary point t_i therefore, Statement (1) in Theorem 2.3 recovers Theorem 21.1.2 in [2]. The same conclusion can be done when $\mathcal{E}$ is not rational but still analytic at t_i. In the case when $\mathcal{E}$ is not rational and admits the nontangential boundary limit $\mathcal{E}(t_i) = \eta_i$, the situation is more subtle: Statement (1) shows that even in this case (if the convergence of $\mathcal{E}(z)$ to $\mathcal{E}(t_i)$ is not too fast), the function $w = \mathbf{T}[\mathcal{E}]$ may satisfy interpolation conditions (1.15), (1.16).

The next theorem concerns the number of negative squares of the function $w = \mathbf{T}_\Theta[\mathcal{E}]$.

Theorem 2.5. *If the Pick matrix P is invertible and has κ negative eigenvalues, then a Schur function $\mathcal{E} \in \mathcal{S}_0$ may satisfy conditions $\mathbf{C}_{4-6}$ at at most κ interpolation nodes. Furthermore, if $\mathcal{E}$ meets conditions $\mathbf{C}_{4-6}$ at exactly ℓ $(\le \kappa)$ interpolation nodes, then the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ belongs to the class $\mathcal{S}_{\kappa-\ell}$.*

Corollary 2.4 and Theorem 2.5 imply the sufficiency part in Theorem 2.2. Indeed, any Schur function $\mathcal{E}$ satisfies either conditions $\mathbf{C}_{4-6}$ or $\mathbf{C}_{1-3}$ at every interpolation node t_i $(i = 1, \dots, n)$. Let $\mathcal{E}$ meet conditions $\mathbf{C}_{4-6}$ at $t_{i_1}, \dots, t_{i_\ell}$ and $\mathbf{C}_{1-3}$ at other $n-\ell$ interpolation nodes $t_{j_1}, \dots, t_{j_{n-\ell}}$. Then, by Corollary 2.4, the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ satisfies interpolation conditions (1.17) for $i \in \{j_1, \dots, j_{n-\ell}\}$ and fails to satisfy at least one of these conditions at the remaining ℓ interpolation nodes. On the other hand, w has exactly $\kappa - \ell$ negative squares, by Theorem 2.5. Thus, for every $\mathcal{E} \in \mathcal{S}_0$, the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ solves Problem 1.6.

Note also that Theorems 2.2 and 2.5 lead to parametrizations of solution sets for Problems 1.3 and 1.4. Indeed, by inclusions (2.1), every solution w to Problem 1.3 (or to Problem 1.4) is also of the form (2.10) for some $\mathcal{E} \in \mathcal{S}_0$. Thus, there is a chance to describe the solution sets $\mathbb{S}_{13}$ and $\mathbb{S}_{14}$ by appropriate selections of the parameter $\mathcal{E}$ in (2.10). Theorem 2.5 indicates how these selections have to be made.

Theorem 2.6. *A function w of the form* (2.10) *is a solution to Problem* 1.3 *if and only if the corresponding parameter $\mathcal{E} \in \mathcal{S}_0$ satisfies either condition $\mathbf{C}_1$ or $\mathbf{C}_2$ for every $i \in \{1, \dots, n\}$.*

Theorem 2.7. *A function w of the form* (2.10) *is a solution to Problem* 1.4 *if and only if the corresponding parameter $\mathcal{E} \in \mathcal{S}_0$ either fails to have a nontangential boundary limit η_i at t_i or*

$$\mathcal{E}(t_i) = \eta_i \quad \text{and} \quad d_{\mathcal{E}}(t_i) > -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2}$$

for every $i = 1, \dots, n$ (in other words, $\mathcal{E}$ meets one of conditions $\mathbf{C}_1$, $\mathbf{C}_2$, $\mathbf{C}_3$ at each interpolation node t_i).

As a consequence of Theorems 2.2 and 2.7 we get curious necessary and sufficient conditions (in terms of the interpolation data (1.12)) for Problems 1.4 and 1.6 to be equivalent (that is, to have the same solution sets).

Corollary 2.8. *Problems* 1.4 *and* 1.6 *are equivalent if and only if all the diagonal entries of the inverse P^{-1} of the Pick matrix are positive.*

Indeed, in this case, all the conditions in Theorem 2.7 are fulfilled for every $\mathcal{E} \in \mathcal{S}_0$ and every $i \in \{1, \dots, n\}$ and formula (2.6) gives a free Schur class parameter description of all solutions w of Problem 1.4.

In the course of the proof of Theorem 2.5 we will discuss the following related question: given indices $i_1, \dots, i_\ell \in \{1, \dots, n\}$, does there exist a parameter $\mathcal{E} \in \mathcal{S}_0$ satisfying conditions $\mathbf{C}_{4-6}$ at $t_{i_1}, \dots, t_{i_\ell}$? Due to Theorems 2.2 and 2.3, this question can be posed equivalently: does there exist a solution w to Problem 1.6 that misses interpolation conditions at $t_{i_1}, \dots, t_{i_\ell}$ (Theorem 2.5 claims that if such a function exists, it belongs to the class $\mathcal{S}_{\kappa-\ell}$). The question admits a simple answer in terms of a certain submatrix of $P^{-1} = [\widetilde{p}_{ij}]_{i,j=1}^n$, the inverse of the Pick matrix.

Theorem 2.9. *There exists a parameter $\mathcal{E}$ satisfying conditions $\mathbf{C}_{4-6}$ at $t_{i_1}, \dots, t_{i_\ell}$ if and only if the $\ell \times \ell$ matrix*

$$\mathcal{P} := \left[\widetilde{p}_{i_\alpha, i_\beta}\right]_{\alpha,\beta=1}^{\ell}$$

is negative semidefinite. Moreover, if $\mathcal{P}$ is negative definite, then there are infinitely many such parameters. If $\mathcal{P}$ is negative semidefinite (singular), then there is only one such parameter, which is a Blaschke product of degree $r = \operatorname{rank} \mathcal{P}$.

Note that all the results announced above have their counterparts in the context of the regular Nevanlinna-Pick problem with all the interpolation nodes inside the unit disk [5]

The paper is organized as follows: Section 3 contains some needed auxiliary results which can be found (probably in a different form) in many sources and are included for the sake of completeness. In Section 4 we prove the necessity part in Theorem 2.2 (see Remark 4.4). In Section 5 we prove Theorem 2.3. In Section 6 we present the proofs of Theorems 2.9 and 2.5 and complete the proof of Theorem 2.2 (see Remark 6.2). The proof of Theorem 2.1 is contained in Section 7; some illustrative numerical examples are presented in Section 8.

3. Some preliminaries

In this section we present some auxiliary results needed in the sequel. We have already mentioned the Stein identity

$$P - T^*PT = E^*E - C^*C \tag{3.1}$$

satisfied by the Pick matrix P constructed in (1.14) from the interpolation data. Most of the facts recalled in this section rely on this identity rather than on the special form (2.3) of matrices T, E and C.

Lemma 3.1. *Let T, E and C be defined as in* (2.3), *let P defined in* (1.14) *be invertible and let μ be a point on $\mathbb{T} \setminus \{t_1, \dots, t_n\}$. Then*

1. *The row vectors*
$$\widetilde{E} = \begin{bmatrix}\widetilde{e}_1 & \dots & \widetilde{e}_n\end{bmatrix} \quad \textit{and} \quad \widetilde{C} = \begin{bmatrix}\widetilde{c}_1 & \dots & \widetilde{c}_n\end{bmatrix} \tag{3.2}$$
defined by
$$\begin{bmatrix}\widetilde{C} \\ \widetilde{E}\end{bmatrix} = \begin{bmatrix} C \\ E \end{bmatrix}(\mu I - T)^{-1}P^{-1}(I - \mu T^*) \tag{3.3}$$
satisfy the Stein identity
$$P^{-1} - TP^{-1}T^* = \widetilde{E}^*\widetilde{E} - \widetilde{C}^*\widetilde{C}. \tag{3.4}$$
2. *The numbers* $\widetilde{c}_i$ *and* $\widetilde{e}_i$ *are subject to*
$$|\widetilde{e}_i| = |\widetilde{c}_i| \neq 0 \quad \textit{for } i = 1, \dots, n. \tag{3.5}$$
3. *The nondiagonal entries* $\widetilde{p}_{ij}$ *of* P^{-1} *are given by*
$$\widetilde{p}_{ij} = \frac{\widetilde{e}_i^*\widetilde{e}_j - \widetilde{c}_i^*\widetilde{c}_j}{1 - t_i\bar{t}_j} \qquad (i \neq j). \tag{3.6}$$

Proof. Under the assumption that P is invertible, identity (3.4) turns out to be equivalent to (3.1). Indeed, by (3.3) and (3.1),
$$\begin{aligned}
&\widetilde{E}^*\widetilde{E} - \widetilde{C}^*\widetilde{C} \\
&= (I - \bar{\mu}T)P^{-1}(\bar{\mu}I - T^*)^{-1}\left[E^*E - C^*C\right](\mu I - T)^{-1}P^{-1}(I - \mu T^*) \\
&= (I - \bar{\mu}T)P^{-1}(\bar{\mu}I - T^*)^{-1}\left[P - T^*PT\right](\mu I - T)^{-1}P^{-1}(I - \mu T^*) \\
&= (I - \bar{\mu}T)P^{-1}\left[(I - \mu T^*)^{-1}P + PT(\mu I - T)^{-1}\right]P^{-1}(I - \mu T^*) \\
&= (I - \bar{\mu}T)P^{-1} + \bar{\mu}TP^{-1}(I - \mu T^*) \\
&= P^{-1} - TP^{-1}T^*.
\end{aligned}$$
Let $P^{-1} = [\widetilde{p}_{ij}]_{i,j=1}^n$. Due to (3.2) and (2.3), equality of the ijth entries in (3.4) can be displayed as
$$\widetilde{p}_{ij} - t_i\bar{t}_j\widetilde{p}_{ij} = \widetilde{e}_i^*\widetilde{e}_j - \widetilde{c}_i^*\widetilde{c}_j \tag{3.7}$$
and implies (3.6) if $i \neq j$. Letting $i = j$ in (3.7) and taking into account that $|t_i| = 1$, we get $|\widetilde{e}_i| = |\widetilde{c}_i|$ for $i = 1, \dots, n$. It remains to show that $\widetilde{e}_i$ and $\widetilde{c}_i$ do not vanish. To this end let us assume that
$$\widetilde{e}_i = \widetilde{c}_i = 0. \tag{3.8}$$
Let $\mathbf{e}_i$ be the ith column of the identity matrix I_n. Multiplying (3.4) by $\mathbf{e}_i$ on the right we get
$$P^{-1}\mathbf{e}_i - TP^{-1}T^*\mathbf{e}_i = \widetilde{E}^*\widetilde{e}_i - \widetilde{C}^*\widetilde{c}_i = 0$$
or equivalently, since $T^*\mathbf{e}_i = \bar{t}_i\mathbf{e}_i$,
$$(I - \bar{t}_iT)P^{-1}\mathbf{e}_i = 0.$$

Since the points $t_1, \dots, t_n$ are distinct, all the diagonal entries but the ith in the diagonal matrix $I - \bar{t}_i T$ are not zeroes; therefore, it follows from the last equality that all the entries in the vector $P^{-1}\mathbf{e}_i$ but the ith entry are zeroes. Thus,

$$P^{-1}\mathbf{e}_i = \alpha \mathbf{e}_i \tag{3.9}$$

for some $\alpha \in \mathbb{C}$ and, since P is not singular, it follows that $\alpha \neq 0$. Now we compare the ith columns in the equality (3.3) (i.e., we multiply both parts in (3.3) by $\mathbf{e}_i$ on the right). For the left-hand side we have, due to assumption (3.8),

$$\begin{bmatrix} \widetilde{C} \\ \widetilde{E} \end{bmatrix} \mathbf{e}_i = \begin{bmatrix} \widetilde{c}_i \\ \widetilde{e}_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For the right-hand side, we have, due to (3.9) and (2.3),

$$\begin{bmatrix} C \\ E \end{bmatrix} (\mu I - T)^{-1} P^{-1} (I - \mu T^*) \mathbf{e}_i = \alpha \frac{1 - \mu \bar{t}_i}{\mu - t_i} \begin{bmatrix} C \\ E \end{bmatrix} \mathbf{e}_i = -\alpha \bar{t}_i \begin{bmatrix} w_i \\ 1 \end{bmatrix}.$$

By (3.3), the right-hand side expressions in the two last equalities must be the same, which is not the case. The obtained contradiction completes the proof of (3.5). □

Remark 3.2. *The numbers* $\widetilde{e}_i$ *and* $\widetilde{c}_i$ *introduced in* (3.2), (3.3) *coincide with those in* (2.11).

For the proof we first note that the formula (2.2) for Θ can be written, on account of (3.3), as

$$\Theta(z) = I_2 + (z - \mu) \begin{bmatrix} C \\ E \end{bmatrix} (zI_n - T)^{-1} (\mu I_n - T)^{-1} \begin{bmatrix} \widetilde{C}^* & -\widetilde{E}^* \end{bmatrix} \tag{3.10}$$

and then, since

$$\lim_{z \to t_i} (z - t_i)(zI - T)^{-1} = \mathbf{e}_i \mathbf{e}_i^* \quad \text{and} \quad \mathbf{e}_i^* (\mu I - T)^{-1} = (\mu - t_i)^{-1} \mathbf{e}_i^*$$

(recall that $\mathbf{e}_i$ is the ith column of the identity matrix I_n), we have

$$\begin{aligned} \lim_{z \to t_i} (z - t_i)\Theta(z) &= \lim_{z \to t_i} (z - \mu) \begin{bmatrix} C \\ E \end{bmatrix} \mathbf{e}_i \mathbf{e}_i^* (\mu I - T)^{-1} \begin{bmatrix} \widetilde{C}^* & -\widetilde{E}^* \end{bmatrix} \\ &= -\begin{bmatrix} C \\ E \end{bmatrix} \mathbf{e}_i \mathbf{e}_i^* \begin{bmatrix} \widetilde{C}^* & -\widetilde{E}^* \end{bmatrix} \\ &= -\begin{bmatrix} w_i \\ 1 \end{bmatrix} \begin{bmatrix} \widetilde{c}_i^* & -\widetilde{e}_i^* \end{bmatrix}. \end{aligned} \tag{3.11}$$

Comparing the bottom entries in the latter equality we get (2.11). □

In the rest of the section we recall some needed results concerning the function Θ introduced in (2.2). These results are well known in a more general situation when T, C and E are matrices such that the pair $\left(\begin{bmatrix} C \\ E \end{bmatrix}, T\right)$ is observable:

$$\bigcap_{j\geq 0} \operatorname{Ker} \begin{bmatrix} C \\ E \end{bmatrix} T^j = \{0\}, \tag{3.12}$$

and P is an invertible Hermitian matrix satisfying the Stein identity (3.1) (see, e.g., [2]). Note that the matrices defined in (2.3) satisfy a stronger condition:

$$\bigcap_{j\geq 0} \operatorname{Ker} CT^j = \bigcap_{j\geq 0} \operatorname{Ker} ET^j = \{0\}. \tag{3.13}$$

Remark 3.3. *Under the above assumptions, the function* Θ *defined via formula* (2.2) *belongs to the class* $\mathcal{W}_\kappa$ *with* $\kappa = \mathrm{sq}_- P$.

Proof. The desired membership follows from the formula

$$K_{\Theta,J}(z,\zeta) = \begin{bmatrix} C \\ E \end{bmatrix} (zI-T)^{-1}P^{-1}(\bar{\zeta}I - T^*)^{-1} \begin{bmatrix} C^* & E^* \end{bmatrix} \tag{3.14}$$

for the kernel $K_{\Theta,J}$ defined in (2.7). The calculation is straightforward and relies on the Stein identity (3.1) only (see, e.g., [2]). It follows from (3.14) that Θ is J-unitary on $\mathbb{T}$ (that is, satisfies condition (2.6)) and that

$$\mathrm{sq}_- K_{\Theta,J} \leq \mathrm{sq}_- P = \kappa.$$

Condition (3.12) guarantees that in fact $\mathrm{sq}_- K_{\Theta,J} = \kappa$ (see [2]). $\square$

Remark 3.4. Since Θ is J-unitary on $\mathbb{T}$ it holds, by the symmetry principle, that $\Theta(z)^{-1} = J\Theta(1/\bar{z})^*J$, which together with formula (2.2) leads us to

$$\Theta(z)^{-1} = I_2 - (z-\mu) \begin{bmatrix} C \\ E \end{bmatrix} (\mu I - T)^{-1}P^{-1}(I - zT^*)^{-1} \begin{bmatrix} C^* & -E^* \end{bmatrix}. \tag{3.15}$$

Besides (3.14) we will need realization formulas for two related kernels. Verification of these formulas (3.16) and (3.17) is also straightforward and is based on the Stein identities (3.1) and (3.4), respectively.

Remark 3.5. *Let* Θ *be defined as in* (2.2)*. The following identities hold for every choice of* $z, \zeta \notin \{t_1, \dots, t_n\}$*:*

$$\frac{\Theta(\zeta)^{-*}J\Theta(z)^{-1} - J}{1 - z\bar{\zeta}} = \begin{bmatrix} C \\ -E \end{bmatrix} (I - \bar{\zeta}T)^{-1}P^{-1}(I - zT^*)^{-1} \begin{bmatrix} C^* & -E^* \end{bmatrix}, \tag{3.16}$$

$$\frac{J - \Theta(\zeta)^*J\Theta(z)}{1 - z\bar{\zeta}} = \begin{bmatrix} \widetilde{C} \\ -\widetilde{E} \end{bmatrix} (\bar{\zeta}I - T^*)^{-1}P(zI - T)^{-1} \begin{bmatrix} \widetilde{C}^* & -\widetilde{E}^* \end{bmatrix}. \tag{3.17}$$

Let us consider conformal partitioning

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{bmatrix}, \quad T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \tag{3.18}$$

$$E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad \widetilde{E} = \begin{bmatrix} \widetilde{E}_1 & \widetilde{E}_2 \end{bmatrix}, \quad \widetilde{C} = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 \end{bmatrix} \tag{3.19}$$

where P_{22}, $\widetilde{P}_{22}$, $T_2 \in \mathbb{C}^{\ell\times\ell}$ and E_2, C_2, $\widetilde{E}_2$, $\widetilde{C}_2 \in \mathbb{C}^{1\times\ell}$. Note that these decompositions contain one restrictive assumption: it is assumed that the matrix T is block diagonal.

Lemma 3.6. *Let us assume that P_{11} is invertible and let* $\mathrm{sq}_-P_{11} = \kappa_1 \le \kappa$*. Then $\widetilde{P}_{22}$ is invertible,* $\mathrm{sq}_-\widetilde{P}_{22} = \kappa - \kappa_1$ *and the functions*

$$\Theta^{(1)}(z) = I_2 + (z-\mu)\begin{bmatrix} C_1 \\ E_1 \end{bmatrix}(zI - T_1)^{-1}P_{11}^{-1}(I - \mu T_1^*)^{-1}\begin{bmatrix} C_1^* & -E_1^* \end{bmatrix} \tag{3.20}$$

and

$$\widetilde{\Theta}^{(2)}(z) = I_2 + (z-\mu)\begin{bmatrix} \widetilde{C}_2 \\ \widetilde{E}_2 \end{bmatrix}(I - \mu T_2^*)^{-1}\widetilde{P}_{22}^{-1}(zI - T_2)^{-1}\begin{bmatrix} \widetilde{C}_2^* & -\widetilde{E}_2^* \end{bmatrix} \tag{3.21}$$

belong to $\mathcal{W}_{\kappa_1}$ and $\mathcal{W}_{\kappa-\kappa_1}$, respectively. Furthermore, the function Θ defined in (2.2) *admits a factorization*

$$\Theta(z) = \Theta^{(1)}(z)\widetilde{\Theta}^{(2)}(z). \tag{3.22}$$

Proof. The first statement follows by standard Schur complement arguments: since P and P_{11} are invertible, the matrix $P_{22}-P_{21}P_{11}^{-1}P_{12}$ (the Schur complement of P_{11} in P) is invertible and has $\kappa-\kappa_1$ negative eigenvalues. Since the block $\widetilde{P}_{22}$ in P^{-1} equals $(P_{22} - P_{21}P_{11}^{-1}P_{12})^{-1}$, it also has $\kappa - \kappa_1$ negative eigenvalues. Realization formulas

$$K_{\Theta^{(1)},J}(z,\zeta) = R(z)P_{11}^{-1}R(\zeta)^* \quad \text{and} \quad K_{\widetilde{\Theta}^{(2)},J}(z,\zeta) = \widetilde{R}(z)\widetilde{P}_{22}\widetilde{R}(\zeta)^*, \tag{3.23}$$

where we have set for short

$$R(z) = \begin{bmatrix} C_1 \\ E_1 \end{bmatrix}(zI - T_1)^{-1}, \quad \widetilde{R}(z) = \begin{bmatrix} \widetilde{C}_2 \\ \widetilde{E}_2 \end{bmatrix}(I - \mu T_2^*)^{-1}\widetilde{P}_{22}^{-1}(zI - T_2)^{-1},$$

are established exactly as in Remark 3.3 and rely on the Stein identities

$$P_{11} - T_1^*P_{11}T_1 = E_1^*E_1 - C_1^*C_1 \quad \text{and} \quad \widetilde{P}_{22}^{-1} - T_2\widetilde{P}_{22}T_2^* = \widetilde{E}_2^*\widetilde{E}_2 - \widetilde{C}_2^*\widetilde{C}_2 \tag{3.24}$$

which hold true, being parts of identities (3.1) and (3.4). Formulas (3.23) guarantee that the rational functions $\Theta^{(1)}$ and $\widetilde{\Theta}^{(2)}$ are J-unitary on $\mathbb{T}$ and moreover, that

$$\mathrm{sq}_-K_{\Theta^{(1)},J} \le \mathrm{sq}_-P_{11} = \kappa_1 \quad \text{and} \quad \mathrm{sq}_-K_{\widetilde{\Theta}^{(2)},J} \le \mathrm{sq}_-\widetilde{P}_{22} = \kappa - \kappa_1. \tag{3.25}$$

Assuming that the factorization formula (3.22) is already proved, we have

$$K_{\Theta,J}(z,\zeta) = K_{\Theta^{(1)},J}(z,\zeta) + \Theta^{(1)}(z)K_{\widetilde{\Theta}^{(2)},J}(z,\zeta)\Theta^{(1)}(\zeta)^*$$

and thus,

$$\kappa = \mathrm{sq}_- K_{\Theta,J} \le \mathrm{sq}_- K_{\Theta^{(1)},J} + \mathrm{sq}_- K_{\widetilde{\Theta}^{(2)},J}$$

which together with inequalities (3.25) imply

$$\mathrm{sq}_- K_{\Theta^{(1)},J} = \kappa_1 \quad \text{and} \quad \mathrm{sq}_- K_{\widetilde{\Theta}^{(2)},J} = \kappa - \kappa_1.$$

It remains to prove (3.22). Making use of the well-known equality

$$P^{-1} = \begin{bmatrix} P_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \widetilde{P}_{22} \begin{bmatrix} -P_{21}P_{11}^{-1} & 1 \end{bmatrix} \tag{3.26}$$

we conclude from (3.3) that

$$\begin{aligned} \begin{bmatrix} \widetilde{C}_2 \\ \widetilde{E}_2 \end{bmatrix} &= \begin{bmatrix} C \\ E \end{bmatrix} (\mu I_n - T)^{-1} P^{-1} (I_n - \mu T^*) \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \\ &= \begin{bmatrix} C \\ E \end{bmatrix} (\mu I_n - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \widetilde{P}_{22}(I_\ell - \mu T_2^*). \end{aligned} \tag{3.27}$$

This last relation allows us to rewrite (3.21) as

$$\widetilde{\Theta}^{(2)}(z) = I_2 + (z-\mu) \begin{bmatrix} C \\ E \end{bmatrix} (\mu I - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} (zI - T_2)^{-1} \begin{bmatrix} \widetilde{C}_2^* & -\widetilde{E}_2^* \end{bmatrix}. \tag{3.28}$$

Now we substitute (3.26) into the formula (2.2) defining Θ and take into account (3.20) and (3.27) to get

$$\begin{aligned} \Theta(z) &= \Theta^{(1)}(z) + (z-\mu) \begin{bmatrix} C \\ E \end{bmatrix} (zI_n - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \widetilde{P}_{22} \\ &\quad \times \begin{bmatrix} -P_{21}P_{11}^{-1} & 1 \end{bmatrix} (I_n - \mu T^*)^{-1} \begin{bmatrix} C^* & -E^* \end{bmatrix} \\ &= \Theta^{(1)}(z) + (z-\mu) \begin{bmatrix} C \\ E \end{bmatrix} (zI_n - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \\ &\quad \times (\mu I - T_2)^{-1} \begin{bmatrix} \widetilde{C}_2^* & -\widetilde{E}_2^* \end{bmatrix}. \end{aligned}$$

Thus, (3.22) is equivalent to

$$\begin{aligned} \widetilde{\Theta}^{(2)}(z) &= I_2 + (z-\mu)\Theta^{(1)}(z)^{-1} \begin{bmatrix} C \\ E \end{bmatrix} (zI_n - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \\ &\quad \times (\mu I - T_2)^{-1} \begin{bmatrix} \widetilde{C}_2^* & -\widetilde{E}_2^* \end{bmatrix}. \end{aligned}$$

Comparing the last relation with (3.28) we conclude that to complete the proof it suffices to show that

$$\begin{aligned} &\Theta^{(1)}(z)^{-1} \begin{bmatrix} C \\ E \end{bmatrix} (zI_n - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} (\mu I - T_2)^{-1} \\ &= \begin{bmatrix} C \\ E \end{bmatrix} (\mu I - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} (zI - T_2)^{-1}. \end{aligned} \tag{3.29}$$

The explicit formula for $\Theta^{(1)}(z)^{-1}$ can be obtained similarly to (3.15):

$$\Theta^{(1)}(z)^{-1} = I_2 - (z-\mu)\begin{bmatrix} C_1 \\ E_1 \end{bmatrix}(\mu I - T_1)^{-1}P_{11}^{-1}(I - zT_1^*)^{-1}\begin{bmatrix} C_1^* & -E_1^* \end{bmatrix}. \tag{3.30}$$

Next, comparing the top block entries in the Stein identity (3.1) we get, due to decompositions (3.18) and (3.19),

$$\begin{bmatrix} P_{11} & P_{12} \end{bmatrix} - T_1^*\begin{bmatrix} P_{11} & P_{12} \end{bmatrix} T = E_1^*E - C_1^*C$$

which, being multiplied by $(I - zT_1^*)^{-1}$ on the left and by $(zI - T)^{-1}$ on the right, leads us to

$$\begin{aligned} &(I - zT_1^*)^{-1}(E_1^*E - C_1^*C)(zI - T)^{-1} \\ &= (I - zT_1^*)^{-1}T_1^*\begin{bmatrix} P_{11} & P_{12} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \end{bmatrix}(zI - T)^{-1}. \end{aligned} \tag{3.31}$$

Upon making use of (3.29) and (3.31) we have

$$\begin{aligned} &\Theta^{(1)}(z)^{-1}\begin{bmatrix} C \\ E \end{bmatrix}(zI_n - T)^{-1}\begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} C \\ E \end{bmatrix}(zI - T)^{-1}\begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \\ &\quad + (z-\mu)\begin{bmatrix} C_1 \\ E_1 \end{bmatrix}(\mu I - T_1)^{-1}\begin{bmatrix} I & P_{11}^{-1}P_{12} \end{bmatrix}(zI - T)^{-1}\begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} \\ &= -\begin{bmatrix} C_1 \\ E_1 \end{bmatrix}(zI - T_1)^{-1}P_{11}^{-1}P_{12} + \begin{bmatrix} C_2 \\ E_2 \end{bmatrix}(zI - T_2)^{-1} \\ &\quad + (z-\mu)\begin{bmatrix} C_1 \\ E_1 \end{bmatrix}(\mu I - T_1)^{-1}\left(P_{11}^{-1}P_{12}(zI - T_2)^{-1} - (zI - T_1)^{-1}P_{11}^{-1}P_{12}\right) \\ &= -\begin{bmatrix} C_1 \\ E_1 \end{bmatrix}(\mu I - T_1)^{-1}P_{11}^{-1}P_{12}(\mu I - T_2) + \begin{bmatrix} C_2 \\ E_2 \end{bmatrix}(zI - T_2)^{-1} \\ &= \begin{bmatrix} C \\ E \end{bmatrix}(\mu I - T)^{-1}\begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix}(zI - T_2)^{-1} \end{aligned}$$

which proves (3.29) and therefore, completes the proof of the lemma. □

Remark 3.7. The case when $\ell = 1$ in Lemma 3.6 will be of special interest. In this case,

$$P_{22} = \gamma_n, \quad \widetilde{P}_{22} = \widetilde{p}_{nn}, \quad T_2 = t_n, \quad C_2 = w_n, \quad E_2 = 1, \quad \widetilde{C}_2 = \widetilde{c}_n, \quad \widetilde{E}_2 = \widetilde{e}_n.$$

Then the formula (3.21) for $\widetilde{\Theta}^{(2)}$ simplifies to

$$\widetilde{\Theta}^{(2)}(z) = I_2 + \frac{z-\mu}{(1-\mu\bar{t}_n)(z - t_n)}\begin{bmatrix} \widetilde{c}_n \\ \widetilde{e}_n \end{bmatrix}\widetilde{p}_{nn}^{-1}\begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix}. \tag{3.32}$$

4. Fundamental Matrix Inequality

In this section we characterize the solution set $\mathbb{S}_{16}$ of Problem 1.6 in terms of certain Hermitian kernel. We start with some simple observations.

Proposition 4.1. *Let $K(z,\zeta)$ be a Hermitian kernel defined on $\Omega \subseteq \mathbb{C}$ and with* $\mathrm{sq}\_K = \kappa$. *Then*

1. *For every choice of an integer p, of a Hermitian $p \times p$ matrix A and of a $p \times 1$ vector-valued function B,*
$$\mathrm{sq}_- \begin{bmatrix} A & B(z) \\ B(\zeta)^* & K(z,\zeta) \end{bmatrix} \leq \kappa + p.$$
2. *If $\lambda_1, \dots, \lambda_p$ are points in Ω and if*
$$A = [K(\lambda_j, \lambda_i)]_{i,j=1}^p \quad \text{and} \quad B(z) = \begin{bmatrix} K(z,\lambda_1) \\ \vdots \\ K(z,\lambda_p) \end{bmatrix}, \tag{4.1}$$
then
$$\mathrm{sq}_- \begin{bmatrix} A & B(z) \\ B(\zeta)^* & K(z,\zeta) \end{bmatrix} = \kappa. \tag{4.2}$$

Proof. For the proof of the first statement we have to show that for every integer m and every choice of points $z_1, \dots, z_m \in \Omega$, the block matrix
$$M = \left[\begin{bmatrix} A & B(z_j) \\ B(z_i)^* & K(z_j, z_i) \end{bmatrix}\right]_{i,j=1}^m \tag{4.3}$$
has at most $\kappa + p$ negative eigenvalues. It is easily seen that M contains m block identical rows of the form
$$\begin{bmatrix} A & B(z_1) & A & B(z_2) & \dots & A & B(z_n) \end{bmatrix}.$$
Deleting all these rows but one and deleting also the corresponding columns, we come up with the $(m+p) \times (m+p)$ matrix
$$\widetilde{M} = \begin{bmatrix} A & B(z_1) & \dots & B(z_m) \\ B(z_1)^* & K(z_1,z_1) & \dots & K(z_1,z_m) \\ \vdots & \vdots & & \vdots \\ B(z_m)^* & K(z_m,z_1) & \dots & K(z_m,z_m) \end{bmatrix}$$
having the same number of positive and negative eigenvalues as M. The bottom $m \times m$ principal submatrix of $\widetilde{M}$ has at most κ negative eigenvalues since $\mathrm{sq}\_K = \kappa$. Since $\widetilde{M}$ is Hermitian, we have by the Cauchy's interlacing theorem (see, e.g., [4, p. 59]), that $\mathrm{sq}\_\widetilde{M} \leq \kappa + p$. Thus, $\mathrm{sq}\_M \leq \kappa + p$ which completes the proof of Statement 1.

If A and B are of the form (4.1), then the matrix M in (4.3) is of the form $[K(\zeta_j, \zeta_i)]_{i,j=1}^{m+pm}$ where all the points ζ_i live in Ω. Since $\mathrm{sq}\_K = \kappa$, it follows that $\mathrm{sq}\_M \leq \kappa$ for every choice of $z_1, \dots, z_m$ in Ω which means that the kernel

$\begin{bmatrix} A & B(z) \\ B(\zeta)^* & K(z,\zeta) \end{bmatrix}$ has at most κ negative squares on Ω. But it has at least κ negative squares since it contains the kernel $K(z,\zeta)$ as a principal block. Thus, (4.2) follows. $\square$

Theorem 4.2. *Let P, T, E and C be defined as in* (1.14) *and* (2.3)*, let w be a function meromorphic on $\mathbb{D}$ and let the kernel K_w be defined as in* (1.1)*. Then w is a solution of Problem* 1.6 *if and only if the kernel*

$$\mathbf{K}_w(z,\zeta) := \begin{bmatrix} P & (I - zT^*)^{-1}(E^* - C^*w(z)) \\ (E - w(\zeta)^*C)(I - \bar{\zeta}T)^{-1} & K_w(z,\zeta) \end{bmatrix} \tag{4.4}$$

has κ negative squares on $\mathbb{D} \cap \rho(w)$:

$$\mathrm{sq}_- \mathbf{K}_w(z,\zeta) = \kappa. \tag{4.5}$$

Proof of the necessity part. Let w be a solution of Problem 1.6, i.e., let w belong to the class $\mathcal{S}_{\kappa'}$ for some $\kappa' \leq \kappa$ and satisfy conditions (1.17) at all but $\kappa - \kappa'$ interpolation nodes.

First we consider the case when $w \in \mathcal{S}_\kappa$. Then w satisfies all the conditions (1.17) (i.e., w is also a solution to Problem 1.4). Furthermore, $\mathrm{sq}_- K_w = \kappa$ and by the second statement in Proposition 4.1, the kernel

$$\mathbf{K}^{(1)}(z,\zeta) := \begin{bmatrix} K_w(z_1,z_1) & \dots & K_w(z_n,z_1) & K_w(z,z_1) \\ \vdots & & \vdots & \vdots \\ K_w(z_1,z_n) & \dots & K_w(z_n,z_n) & K_w(z,z_n) \\ K_w(z_1,\zeta) & \dots & K_w(z_n,\zeta) & K_w(z,\zeta) \end{bmatrix} \tag{4.6}$$

has κ negative squares on $\mathbb{D} \cap \rho(w)$ for every choice of points $z_1, \dots, z_n \in \mathbb{D} \cap \rho(w)$. Since the limits $d_w(t_i)$ and $w(t_i) = w_i$ exist for $i = 1, \dots, n$, it follows that

$$[K_w(z_j,z_i)]_{i,j=1}^n = \left[\frac{1 - w(z_i)^* w(z_j)}{1 - \bar{z}_i z_j}\right]_{i,j=1}^n \longrightarrow P^w(t_1,\dots,t_n) \tag{4.7}$$

(by definition (1.18) of the matrix $P^w(t_1,\dots,t_n)$) and also

$$K_w(z_i,\zeta) = \frac{1 - w(\zeta)^* w(z_i)}{1 - \bar{\zeta} z_i} \longrightarrow \frac{1 - w(\zeta)^* w_i}{1 - \bar{\zeta} t_i} \qquad (i = 1, \dots, n).$$

Note that by the structure (2.3) of the matrices T, E and C,

$$(E - w(\zeta)^*C)(I - \bar{\zeta}T)^{-1} = \begin{bmatrix} \frac{1 - w(\zeta)^* w_1}{1 - \bar{\zeta} t_1} & \dots & \frac{1 - w(\zeta)^* w_n}{1 - \bar{\zeta} t_n} \end{bmatrix}$$

which, being combined with the previous relation, gives

$$\begin{bmatrix} K_w(z_1,\zeta) & \dots & K_w(z_n,\zeta) \end{bmatrix} \longrightarrow (E - w(\zeta)^*C)(I - \bar{\zeta}T)^{-1}. \tag{4.8}$$

Now we take the limit in (4.6) as $z_i \to t_i$ for $i = 1, \dots, n$; on account of (4.7) and (4.8), the limit kernel has the form

$$\mathbf{K}^{(2)}(z,\zeta) := \begin{bmatrix} P^w(t_1,\dots,t_n) & (I - zT^*)^{-1}(E^* - C^*w(z)) \\ (E - w(\zeta)^*C)(I - \bar{\zeta}T)^{-1} & K_w(z,\zeta) \end{bmatrix}.$$

Since $\mathbf{K}^{(2)}$ is the limit of a family of kernels each of which has κ negative squares, $\mathrm{sq}_-\mathbf{K}^{(2)} \le \kappa$. It remains to note that the kernel $\mathbf{K}_w$ defined in (4.4) is expressed in terms of $\mathbf{K}^{(2)}$ as

$$\mathbf{K}_w(z,\zeta) = \mathbf{K}^{(2)}(z,\zeta) + \begin{bmatrix} P - P^w(t_1,\dots,t_n) & 0 \\ 0 & 0 \end{bmatrix}$$

and since the second term on the right-hand side is positive semidefinite (due to the first series of conditions in (1.17); see also (1.20)),

$$\mathrm{sq}_-\mathbf{K}_w \le \mathrm{sq}_-\mathbf{K}^{(2)} \le \kappa.$$

On the other hand, since $\mathbf{K}_w$ contains the kernel K_w as a principal submatrix, $\mathrm{sq}_-\mathbf{K}_w \ge \mathrm{sq}_- K_w = \kappa$ which eventually leads us to (4.5). Note that in this part of the proof we have not used the fact that $\mathrm{sq}_- P = \kappa$.

Now we turn to the general case: let $w \in \mathcal{S}_{\kappa'}$ for some $\kappa' \le \kappa$ and let conditions (1.17) be fulfilled at all but $\ell := \kappa - \kappa'$ interpolation nodes t_i's. We may assume without loss of generality that conditions (1.17) are satisfied at t_i for $i = 1,\dots,n-\ell$:

$$d_w(t_i) \le \gamma_i \quad \text{and} \quad w(t_i) = w_i \qquad (i = 1,\dots,n-\ell). \tag{4.9}$$

Let us consider conformal partitioning (3.18), (3.19) for matrices P, T, C and E and let us set for short

$$F_i(z) = (I - zT_i^*)^{-1}(E_i^* - C_i^* w(z)) \qquad (i = 1,2) \tag{4.10}$$

so that

$$\begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix} = (I - zT^*)^{-1}(E^* - C^* w(z)). \tag{4.11}$$

The matrix P_{11} is the Pick matrix of the truncated interpolation problem with the data t_i, w_i, γ_i $(i = 1,\dots,n-\ell)$ and with interpolation conditions (4.9). By the first part of the proof, the kernel

$$\widetilde{\mathbf{K}}_w(z,\zeta) := \begin{bmatrix} P_{11} & F_1(z) \\ F_1(\zeta)^* & K_w(z,\zeta) \end{bmatrix} \tag{4.12}$$

has κ' negative squares on $\mathbb{D} \cap \rho(w)$. Now we apply the first statement in Proposition 4.1 to

$$K(z,\zeta) = \widetilde{\mathbf{K}}_w(z,\zeta), \quad B(z) = \begin{bmatrix} P_{21} & F_2(z) \end{bmatrix} \quad \text{and} \quad A = P_{22} \tag{4.13}$$

to conclude that

$$\mathrm{sq}_- \begin{bmatrix} P_{22} & B(z) \\ B(\zeta)^* & \widetilde{\mathbf{K}}_w(z,\zeta) \end{bmatrix} \le \mathrm{sq}_-\widetilde{\mathbf{K}}_w + \ell = \kappa' + (\kappa - \kappa') = \kappa. \tag{4.14}$$

By (4.13) and (4.12), the latter kernel equals

$$\begin{bmatrix} P_{22} & B(z) \\ B(\zeta)^* & \widetilde{\mathbf{K}}_w(z,\zeta) \end{bmatrix} = \begin{bmatrix} P_{22} & P_{21} & F_2(z) \\ P_{12} & P_{11} & F_1(z) \\ F_2(\zeta)^* & F_1(\zeta)^* & K_w(z,\zeta) \end{bmatrix}.$$

Now it follows from (4.4) and (4.12) that

$$\mathbf{K}_w(z,\zeta) = U \begin{bmatrix} P_{22} & B(z) \\ B(\zeta)^* & \widetilde{\mathbf{K}}_w(z,\zeta) \end{bmatrix} U^*, \quad \text{where} \quad U = \begin{bmatrix} 0 & I_{n-\ell} & 0 \\ I_\ell & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, on account of (4.14), implies that $\mathrm{sq}_-\mathbf{K}_w \le \kappa$. Finally, since $\mathbf{K}_w$ contains P as a principal submatrix, $\mathrm{sq}_-\mathbf{K}_w \ge \mathrm{sq}_- P = \kappa$ which now implies (4.5) and completes the proof of the necessity part of the theorem. The proof of the sufficiency part will be given in Sections 6 and 7 (see Remarks 6.3 and 7.3 there). □

In the case when P is invertible, all the functions satisfying (4.5) can be described in terms of a linear fractional transformation.

Theorem 4.3. *Let the Pick matrix P be invertible and let $\Theta = [\Theta_{ij}]$ be the 2×2 matrix-valued function defined in* (2.2)*. A function w meromorphic on $\mathbb{D}$ is subject to FMI* (4.5) *if and only if it is of the form*

$$w(z) = \mathbf{T}_\Theta[\mathcal{E}] := \frac{\Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z)}{\Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z)} \tag{4.15}$$

for some Schur function $\mathcal{E} \in \mathcal{S}_0$.

Proof. The proof is about the same as in the definite case. Let $\mathbf{S}$ be the Schur complement of P in the kernel $\mathbf{K}_w$ defined in (4.4):

$$\mathbf{S}(z,\zeta) := K_w(z,\zeta) - (E - w(\zeta)^*C)(I - \bar{\zeta}T)^{-1}P^{-1}(I - zT^*)^{-1}(E^* - C^*w(z)).$$

Obvious equalities

$$K_w(z,\zeta) := \frac{1 - w(\zeta)^*w(z)}{1 - \bar{\zeta}z} = -\begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} J \begin{bmatrix} w(z) \\ 1 \end{bmatrix}$$

where J is the matrix introduced in (2.5), and

$$E - w(\zeta)^*C = -\begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} J \begin{bmatrix} C \\ -E \end{bmatrix}$$

allows us to represent $\mathbf{S}$ in the form

$$\begin{aligned} \mathbf{S}(z,\zeta) &= -\begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} \left\{ \frac{J}{1 - z\bar{\zeta}} + \begin{bmatrix} C \\ -E \end{bmatrix} (I - \bar{\zeta}T)^{-1}P^{-1} \right. \\ &\qquad \left. \times\, (I - zT^*)^{-1} \begin{bmatrix} C^* & -E^* \end{bmatrix} \right\} \begin{bmatrix} w(z) \\ 1 \end{bmatrix} \end{aligned}$$

or, on account of identity (3.16), as

$$\mathbf{S}(z,\zeta) = -\begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} \frac{\Theta(\zeta)^{-*}J\Theta(z)^{-1}}{1 - z\bar{\zeta}} \begin{bmatrix} w(z) \\ 1 \end{bmatrix}.$$

By the standard Schur complement argument,

$$\mathrm{sq}_-\mathbf{K}_w = \mathrm{sq}_- P + \mathrm{sq}_-\mathbf{S}$$

which implies, since sq_$P = \kappa$, that (4.5) holds if and only if the kernel $\mathbf{S}$ is positive definite on $\rho(w) \cap \mathbb{D}$:

$$-\begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} \frac{\Theta(\zeta)^{-*} J \Theta(z)^{-1}}{1 - z\bar{\zeta}} \begin{bmatrix} w(z) \\ 1 \end{bmatrix} \succeq 0. \tag{4.16}$$

It remains to show that (4.16) holds if and only if w is of the form (4.15). To show the "only if" part, let us consider meromorphic functions u and v defined by

$$\begin{bmatrix} u(z) \\ v(z) \end{bmatrix} := \Theta(z)^{-1} \begin{bmatrix} w(z) \\ 1 \end{bmatrix}. \tag{4.17}$$

Then inequality (4.16) can be written in terms of these functions as

$$-\begin{bmatrix} u(\zeta)^* & v(\zeta)^* \end{bmatrix} \frac{J}{1 - \bar{\zeta} z} \begin{bmatrix} u(z) \\ v(z) \end{bmatrix} = \frac{v(\zeta)^* v(z) - u(\zeta)^* u(z)}{1 - \bar{\zeta} z} \succeq 0. \tag{4.18}$$

As it follows from definition (4.17), u and v are analytic on $\rho(w) \cap \mathbb{D}$. Moreover,

$$v(z) \neq 0 \quad \text{for every } \ z \in \rho(w) \cap \mathbb{D}. \tag{4.19}$$

Indeed, assuming that $v(\xi) = 0$ at some point $\xi \in \mathbb{D}$, we conclude from (4.18) that $u(\xi) = 0$ and then (4.17) implies that $\det \Theta(\xi)^{-1} = 0$ which is a contradiction. Due to (4.19), we can introduce the meromorphic function

$$\mathcal{E}(z) = \frac{u(z)}{v(z)} \tag{4.20}$$

which is analytic on $\rho(w) \cap \mathbb{D}$. Writing (4.18) in terms of $\mathcal{E}$ as

$$v(\zeta)^* \cdot \frac{1 - \mathcal{E}(\zeta)^* \mathcal{E}(z)}{1 - \bar{\zeta} z} \cdot v(z) \succeq 0 \quad (z, \zeta \in \rho(w) \cap \mathbb{D}),$$

we then take advantage of (4.19) to conclude that

$$\frac{1 - \mathcal{E}(\zeta)^* \mathcal{E}(z)}{1 - \bar{\zeta} z} \succeq 0 \quad (z, \zeta \in \rho(w) \cap \mathbb{D}).$$

The latter means that $\mathcal{E}$ is (after an analytic continuation to the all of $\mathbb{D}$) a Schur function. Finally, it follows from (4.17) that

$$\begin{bmatrix} w \\ 1 \end{bmatrix} = \Theta \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \Theta_{11} u + \Theta_{12} v \\ \Theta_{21} u + \Theta_{22} v \end{bmatrix}$$

which in turn implies

$$w = \frac{\Theta_{11} u + \Theta_{12} v}{\Theta_{21} u + \Theta_{22} v} = \frac{\Theta_{11} \mathcal{E} + \Theta_{12}}{\Theta_{21} \mathcal{E} + \Theta_{22}} = \mathbf{T}_\Theta[\mathcal{E}].$$

Now let $\mathcal{E}$ be a Schur function. Then the function

$$V(z) = \Theta_{21}(z) \mathcal{E}(z) + \Theta_{22}(z)$$

does not vanish identically. Indeed, since Θ is rational and $\Theta(\mu) = I_2$, it follows that $\Theta_{22}(z) \approx 1$ and $\Theta_{21}(z) \approx 0$ if z is close enough to μ. Since $|\mathcal{E}(z)| \le 1$ everywhere in $\mathbb{D}$, the function V does not vanish on $\mathcal{U}_\delta = \{z \in \mathbb{D} : \ |z-\mu| < \delta\}$ if δ is small enough. Thus, formula (4.15) makes sense and can be written equivalently as

$$\begin{bmatrix} w(z) \\ 1 \end{bmatrix} = \Theta(z) \begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix} \cdot \frac{1}{V(z)}$$

Then it is readily seen that

$$\begin{aligned} \frac{1-\mathcal{E}(\zeta)^*\mathcal{E}(z)}{1-\bar{\zeta}z} &= -\begin{bmatrix} \mathcal{E}(\zeta)^* & 1 \end{bmatrix} \frac{J}{1-\bar{\zeta}z} \begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix} \\ &= -\frac{1}{V(\zeta)^*V(z)} \cdot \begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} \frac{\Theta(\zeta)^{-*}J\Theta(z)^{-1}}{1-z\bar{\zeta}} \begin{bmatrix} w(z) \\ 1 \end{bmatrix} \end{aligned}$$

for $z, \zeta \in \rho(w) \cap \mathbb{D}$. Since $\mathcal{E}$ is a Schur function, the latter kernel is positive on $\rho(w) \cap \mathbb{D}$ and since $V \not\equiv 0$, (4.16) follows. $\square$

Remark 4.4. *Combining Theorems* 4.2 *and* 4.3 *we get the necessity part in Theorem* 2.2*.*

Indeed, by the necessity part in Theorem 4.2, any solution w of Problem 1.6 satisfies (4.5); then by Theorem 4.3, $w = \mathbf{T}_\Theta[\mathcal{E}]$ for some $\mathcal{E} \in \mathcal{S}_0$.

In the case when $\kappa = 0$, Theorem 4.2 was established in [12].

Theorem 4.5. *Let the Pick matrix P be positive semidefinite. Then a function w defined on $\mathbb{D}$ is a solution to Problem* 1.1 (*i.e., belongs to the Schur class $\mathcal{S}_0$ and meets conditions* (1.6)) *if and only if*

$$\mathbf{K}_w(z,\zeta) \succeq 0 \quad (z, w \in \mathbb{D}) \tag{4.21}$$

where $\mathbf{K}_w(z,\zeta)$ is the kernel defined in (4.4)*.*

Under the a priori assumption that w is a Schur function, condition (4.21) can be replaced by a seemingly weaker matrix inequality

$$\mathbf{K}_w(z,z) \ge 0 \quad \text{for every } \ z \in \mathbb{D}$$

which is known in interpolation theory as a Fundamental Matrix Inequality (FMI) of V.P. Potapov. We will follow this terminology and will consider relation (4.5) as an indefinite analogue of V.P. Potapov's FMI. It is appropriate to note that a variation of the Potapov's method was first applied to the Nevanlinna-Pick problem (with finitely many interpolation nodes inside the unit disk) for generalized Schur functions in [10]. We conclude this section with another theorem concerning the classical case which will be useful for the subsequent analysis.

Theorem 4.6.

(1) *If the Pick matrix P is positive definite then all the solutions w to Problem* 1.1 *are parametrized by the formula* (2.10) *with the coefficient matrix Θ defined as in* (2.2) *with $\mathcal{E}$ being a free Schur class parameter.*

(2) *If P is positive semidefinite and singular, then Problem* 1.1 *has a unique solution w which is a Blaschke product of degree* $r = \operatorname{rank} P$. *Furthermore, this unique solution can be represented as*

$$w(z) = \frac{x^*(I - zT_2^*)^{-1}E^*}{x^*(I - zT_2^*)^{-1}C^*} \tag{4.22}$$

where T, C and E are defined as in (2.3) *and where x is any nonzero vector such that $Px = 0$.*

These results are well known and have been established using different methods in [1, 12, 3, 2, 11]. In regard to methods used in the present paper, note that the first statement follows immediately from Theorems 4.5 and 4.3. This demonstrates how the Potapov's method works in the definite case (and this is exactly how the result was established in [12]). The second statement also can be derived from Theorem 4.5: if w solves Problem 1.1, then the kernel $\mathbf{K}_w(z,\zeta)$ defined in (4.4) is positive definite. Multiplying it by the vector $\begin{bmatrix} x \\ 1 \end{bmatrix}$ on the right and by its adjoint on the left we come to the positive definite kernel

$$\begin{bmatrix} x^*Px & x^*(I - zT^*)^{-1}(E^* - C^*w(z)) \\ (E - w(\zeta)^*C)(I - \bar{\zeta}T)^{-1}x & K_w(z,\zeta) \end{bmatrix} \succeq 0.$$

Thus, for every $x \neq 0$ such that $Px = 0$, we also have

$$x^*(I - zT^*)^{-1}(E^* - C^*w(z)) \equiv 0.$$

Solving the latter identity for w we arrive at formula (4.22). The numerator and the denominator in (4.22) do not vanish identically due to conditions (3.13). Since x can be chosen so that $n - \operatorname{rank} P - 1$ its coordinates are zeros, the rational function w is of McMillan degree $r = \operatorname{rank} P$. Due to the Stein identity (3.1), w is inner and therefore, it is a finite Blachke product of degree r.

5. Parameters and interpolation conditions

In this section we prove Theorem 2.3. It will be done in several steps formulated as separate theorems. In what follows, $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ will stand for the functions

$$U_{\mathcal{E}}(z) = \Theta_{11}(z)\mathcal{E}(z) + \Theta_{12}(z), \qquad V_{\mathcal{E}}(z) = \Theta_{21}(z)\mathcal{E}(z) + \Theta_{22}(z) \tag{5.1}$$

for a fixed Schur function $\mathcal{E}$, so that

$$\begin{bmatrix} U_{\mathcal{E}}(z) \\ V_{\mathcal{E}}(z) \end{bmatrix} = \Theta(z) \begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix} \tag{5.2}$$

and (2.10) takes the form

$$w(z) := \mathbf{T}_\Theta[\mathcal{E}] = \frac{U_{\mathcal{E}}(z)}{V_{\mathcal{E}}(z)}. \tag{5.3}$$

Substituting (3.10) into (5.2) and setting

$$\Psi(z) = (zI - T)^{-1}\left(\widetilde{E}^* - \widetilde{C}^*\mathcal{E}(z)\right) \tag{5.4}$$

for short, we get

$$\begin{aligned} U_{\mathcal{E}}(z) &= \mathcal{E}(z) - (z-\mu)C(\mu I - T)^{-1}\Psi(z), &(5.5)\\ V_{\mathcal{E}}(z) &= 1 - (z-\mu)E(\mu I - T)^{-1}\Psi(z). &(5.6)\end{aligned}$$

Furthermore, for w of the form (5.3), we have

$$\frac{1 - w(\zeta)^* w(z)}{1-\bar{\zeta}z} = \frac{1}{V_{\mathcal{E}}(\zeta)^* V_{\mathcal{E}}(z)} \cdot \frac{V_{\mathcal{E}}(\zeta)^* V_{\mathcal{E}}(z) - U_{\mathcal{E}}(\zeta)^* U_{\mathcal{E}}(z)}{1-\bar{\zeta}z}. \tag{5.7}$$

Note that

$$\begin{aligned} V_{\mathcal{E}}(\zeta)^* V_{\mathcal{E}}(z) - U_{\mathcal{E}}(\zeta)^* U_{\mathcal{E}}(z) &= -\begin{bmatrix} U_{\mathcal{E}}(\zeta)^* & V_{\mathcal{E}}(\zeta)^* \end{bmatrix} J \begin{bmatrix} U_{\mathcal{E}}(z) \\ V_{\mathcal{E}}(z) \end{bmatrix}\\ &= \begin{bmatrix} \mathcal{E}(\zeta)^* & 1 \end{bmatrix} \Theta(\zeta)^* J \Theta(z) \begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix}\\ &= 1 - \mathcal{E}(\zeta)^*\mathcal{E}(z) + (1-\bar{\zeta}z)\Psi(\zeta)^* P \Psi(z), \end{aligned}$$

where the second equality follows from (5.2), and the third equality is a consequence of (3.17) and definition (5.4) of Ψ. Now (5.7) takes the form

$$\frac{1 - w(\zeta)^* w(z)}{1-\bar{\zeta}z} = \frac{1}{V_{\mathcal{E}}(\zeta)^* V_{\mathcal{E}}(z)} \left(\frac{1-\mathcal{E}(\zeta)^*\mathcal{E}(z)}{1-\bar{\zeta}z} + \Psi(\zeta)^* P \Psi(z)\right). \tag{5.8}$$

Remark 5.1. *Equality* (5.8) *implies that for every $\mathcal{E} \in \mathcal{S}_0$ and $\Theta \in \mathcal{W}_\kappa$, the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ belongs to the generalized Schur class $\mathcal{S}_{\kappa'}$ for some $\kappa' \le \kappa$.*

Indeed, it follows from (5.8) that $\mathrm{sq}\_K_w \le \mathrm{sq}\_K_{\mathcal{E}} + \mathrm{sq}\_P = 0 + \kappa$.

Upon evaluating (5.8) at $\zeta = z$ we get

$$\frac{1 - |w(z)|^2}{1-|z|^2} = \frac{1}{|V_{\mathcal{E}}(z)|^2} \left(\frac{1-|\mathcal{E}(z)|^2}{1-|z|^2} + \Psi(z)^* P \Psi(z)\right) \tag{5.9}$$

and realize that boundary values of $w(t_i)$ and $d_w(t_i)$ can be calculated from asymptotic formulas for Ψ, $U_{\mathcal{E}}$, $V_{\mathcal{E}}$ and $\mathcal{E}$ as z tends to one of the interpolation nodes t_i. These asymptotic relations are presented in the next lemma.

Lemma 5.2. *Let $\mathcal{E}$ be a Schur function, let Ψ, $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ be defined as in* (5.4), (5.5) *and* (5.6), *respectively , and let t_i be an interpolation node. Then the following asymptotic relations hold as z tends to t_i nontangentially:*

$$\begin{aligned} (z-t_i)\Psi(z) &= \mathbf{e}_i\,(\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)) + O(|z-t_i|), &(5.10)\\ (z-t_i)U_{\mathcal{E}}(z) &= w_i\,(\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)) + O(|z-t_i|), &(5.11)\\ (z-t_i)V_{\mathcal{E}}(z) &= (\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)) + O(|z-t_i|). &(5.12)\end{aligned}$$

Proof. Recall that $\mathbf{e}_i$ be the ith column in the identity matrix I_n. Since

$$(z-t_i)(zI-T)^{-1} = \mathbf{e}_i\mathbf{e}_i^* + O(|z-t_i|) \quad \text{as } z \to t_i,$$

and since $\mathcal{E}(z)$ is uniformly bounded on $\mathbb{D}$, we have by (5.4),

$$\begin{aligned}(z-t_i)\Psi(z) &= (z-t_i)(zI-T)^{-1}\left(\widetilde{E}^* - \widetilde{C}^*\mathcal{E}(z)\right)\\ &= \mathbf{e}_i\mathbf{e}_i^*\left(\widetilde{E}^* - \widetilde{C}^*\mathcal{E}(z)\right) + O(|z-t_i|)\end{aligned}$$

which proves (5.10), since $\mathbf{e}_i^*\widetilde{C}^* = \widetilde{c}_i^*$ and $\mathbf{e}_i^*\widetilde{E}^* = \widetilde{e}_i^*$ by (3.2).

Now we plug in the asymptotic relation (5.10) into the formulas (5.5) and (5.10) for $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ and make use of evident equalities

$$C(\mu I-T)^{-1}\mathbf{e}_i = \frac{w_i}{\mu - t_i} \quad \text{and} \quad E(\mu I-T)^{-1}\mathbf{e}_i = \frac{1}{\mu-t_i} \tag{5.13}$$

to get (5.11) and (5.12):

$$\begin{aligned}(z-t_i)U_{\mathcal{E}}(z) &= (z-t_i)\mathcal{E}(z) - (z-t_i)(z-\mu)C(\mu I-T)^{-1}\Psi(z)\\ &= (\mu-z)C(\mu I-T)^{-1}\mathbf{e}_i\left(\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)\right) + O(|z-t_i|)\\ &= \frac{\mu-z}{\mu-t_i}w_i\left(\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)\right) + O(|z-t_i|)\\ &= w_i\left(\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)\right) + O(|z-t_i|),\\ (z-t_i)V_{\mathcal{E}}(z) &= (z-t_i) - (z-t_i)(z-\mu)E(\mu I-T)^{-1}\Psi(z)\\ &= (\mu-z)E(\mu I-T)^{-1}\mathbf{e}_i\left(\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)\right) + O(|z-t_i|)\\ &= \left(\widetilde{e}_i^* - \widetilde{c}_i^*\mathcal{E}(z)\right) + O(|z-t_i|).\end{aligned}$$

Lemma 5.3. *Let $w \in \mathcal{S}_\kappa$, let $t_0 \in \mathbb{T}$, and let us assume that the limit*

$$d := \lim_{j\to\infty}\frac{1-|w(r_jt_0)|^2}{1-r_j^2} < \infty \tag{5.14}$$

exists and is finite for some sequence of numbers $r_j \in (0,\,1)$ such that $\lim_{j\to\infty} r_j = 1$. *Then the nontangential limits $d_w(t_0)$ and $w(t_0)$ (defined as in* (1.3) *and* (1.4)) *exist and moreover*

$$d_w(t_0) = d \quad \text{and} \quad |w(t_0)| = 1. \tag{5.15}$$

Proof. Since w is a generalized Schur function, it admits the Krein-Langer representation (1.9) and identity (1.11) holds at every point $z \in \mathbb{D}$. In particular,

$$\frac{1-|w(r_jt_0)|^2}{1-r_j^2} = \frac{1}{|B(r_jt_0)|^2}\left(\frac{1-|S(r_jt_0)|^2}{1-r_j^2} - \frac{1-|B(r_jt_0)|^2}{1-r_j^2}\right). \tag{5.16}$$

Since B is a finite Blaschke product, it is analytic at t_0 and the limit $d_B(t_0) := \lim\limits_{z\to t_0}\dfrac{1-|B(z)|^2}{1-|z|^2}$ exists and is finite. Assumption (5.14) implies therefore that the limit

$$\lim_{j\to\infty}\frac{1-|S(r_jt_0)|^2}{1-r_j^2} = d + d_B(t_0)$$

exists and is finite. Since $S \in \mathcal{S}_0$, we then conclude by the Carathéodory-Julia theorem (see, e.g., [17, 18, 20]) that the nontangential limits $d_S(t_0)$ and $S(t_0)$ exist and moreover,

$$d_S(t_0) = d + d_B(t_0) \quad \text{and} \quad |S(t_0)| = 1. \tag{5.17}$$

Now we pass to limits in (1.9) and (1.11) as z tends to t_0 nontangentially to get

$$w(t_0) := \lim_{z \to t_0} w(z) = \frac{S(t_0)}{B(t_0)} \quad \text{and} \quad d_w(t_0) := \lim_{z \to t_0} \frac{1 - |w(z)|^2}{1 - |z|^2} = d_S(t_0) - d_B(t_0)$$

and relations (5.17) imply now (5.15) and complete the proof. □

Theorem 5.4. *If $\mathcal{E} \in \mathcal{S}_0$ meets condition $\mathbf{C}_1$ at t_i (i.e., the nontangential boundary limit $\lim_{z\to t_i} \mathcal{E}(z)$ is not equal to $\eta_i = \dfrac{\widetilde{e}_i^*}{\widetilde{c}_i^*}$ or fails to exist), then the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ is subject to*

$$\lim_{z \to t_i} w(z) = w_i \quad \textit{and} \quad \lim_{z \to t_i} \frac{1 - |w(z)|^2}{1 - |z|^2} = \gamma_i. \tag{5.18}$$

Proof. By the assumption of the theorem, there exists $\varepsilon > 0$ and a sequence of points $\{r_\alpha t_i\}_{\alpha=1}^{\infty}$ tending to t_i radially ($0 < r_\alpha < 1$ and $r_\alpha \to 1$) such that

$$|\widetilde{e}_i^* - \widetilde{c}_i^* \mathcal{E}(r_\alpha t_i)| \geq \varepsilon \quad \text{for every } \ \alpha. \tag{5.19}$$

Since $\mathbf{e}_i^* P \mathbf{e}_i = \gamma_i$ by the definition (1.14) of P, it follows from (5.10) that

$$|z - t_i|^2 \Psi(z)^* P \Psi(z) = |\widetilde{e}_i^* - \widetilde{c}_i^* \mathcal{E}(z)|^2 \gamma_i + O(|z - t_i|).$$

Furthermore, relation

$$|z - t_i|^2 \cdot |V_{\mathcal{E}}(z)|^2 = |\widetilde{e}_i^* - \widetilde{c}_i^* \mathcal{E}(z)|^2 + O(|z - t_i|)$$

is a consequence of (5.12) and, since $\mathcal{E}$ is uniformly bounded on $\mathbb{D}$, it is clear that

$$\lim_{z \to t_i} |z - t_i|^2 \cdot \frac{1 - |\mathcal{E}(z)|^2}{1 - |z|^2} = 0.$$

Now we substitute the three last relations into (5.9) and let $z = r_\alpha t_i \to t_i$; due to (5.19) we have

$$\begin{aligned} \lim_{z = r_\alpha t_i \to t_i} \frac{1 - |w(z)|^2}{1 - |z|^2} &= \lim_{z = r_\alpha t_i \to t_i} \frac{|z - t_i|^2 \cdot \dfrac{1 - |\mathcal{E}(z)|^2}{1 - |z|^2} + |z - t_i|^2 \Psi(z)^* P \Psi(z)}{|z - t_i|^2 \cdot |V_{\mathcal{E}}(z)|^2} \\ &= \frac{0 + \gamma_i}{1} = \gamma_i. \end{aligned}$$

Since w is a generalized Schur function (by Remark 5.1), we can apply Lemma 5.3 to conclude that the nontangential limit $d_w(t_i)$ exists and equals γ_i. This proves the second relation in (5.18). Furthermore, by (5.11) and (5.12) and in view of (5.19),

$$\lim_{z = r_\alpha t_i \to t_i} w(z) = \lim_{z \to t_i} \frac{(z - t_i) U_{\mathcal{E}}(z)}{(z - t_i) V_{\mathcal{E}}(z)} = w_i. \tag{5.20}$$

Again by Lemma 5.3, the nontangential limit $w(t_i)$ exists; therefore, it is equal to the subsequential limit (5.20), that is, to w_i. This proves the first relation in (5.18) and completes the proof of the theorem. □

The next step will be to handle condition $\mathbf{C}_2$ (see (2.14)). We need an auxiliary result.

Lemma 5.5. *Let $t_0 \in \mathbb{T}$ and let $\mathcal{E}$ be a Schur function such that*

$$\lim_{z\to t_0} \mathcal{E}(z) = \mathcal{E}_0 \ \ (|\mathcal{E}_0| = 1) \quad \text{and} \quad \lim_{z\to t_0} \frac{1-|\mathcal{E}(z)|^2}{1-|z|^2} = \infty. \tag{5.21}$$

Then

$$\lim_{z\to t_0} \frac{1-|\mathcal{E}(z)|^2}{1-|z|^2} \cdot \left|\frac{z-t_0}{\mathcal{E}(z)-\mathcal{E}_0}\right|^2 = 0 \quad \text{and} \quad \lim_{z\to t_0} \frac{z-t_0}{\mathcal{E}(z)-\mathcal{E}_0} = 0. \tag{5.22}$$

Proof. Since $|\mathcal{E}_0| = 1$, we have

$$\begin{aligned} 2\mathrm{Re}\,(1-\mathcal{E}(z)\overline{\mathcal{E}}_0) &= (1-\mathcal{E}(z)\overline{\mathcal{E}}_0) + (1-\mathcal{E}_0\overline{\mathcal{E}(z)}) \\ &= |1-\mathcal{E}(z)\overline{\mathcal{E}}_0|^2 + 1 - |\mathcal{E}_0|^2 \cdot |\mathcal{E}(z)|^2 \\ &\geq 1-|\mathcal{E}(z)|^2 \end{aligned}$$

and thus,

$$|\mathcal{E}(z)-\mathcal{E}_0| = |1-\mathcal{E}(z)\overline{\mathcal{E}_0}| \geq \mathrm{Re}\,(1-\mathcal{E}(z)\overline{\mathcal{E}}_0) \geq \frac{1}{2}\left(1-|\mathcal{E}(z)|^2\right). \tag{5.23}$$

Furthermore, for every z in the Stoltz domain

$$\Gamma_a(t_0) = \{z \in \mathbb{D} : \ |z-t_0| < a(1-|z|)\}, \quad a > 1,$$

it holds that

$$\frac{1-|z|^2}{|z-t_0|} \geq \frac{1-|z|}{|z-t_0|} > \frac{1}{a},$$

which together with (5.23) leads us to

$$\left|\frac{\mathcal{E}(z)-\mathcal{E}_0}{z-t_0}\right| \geq \frac{1}{2}\cdot\frac{1-|\mathcal{E}(z)|^2}{|z-t_0|} = \frac{1}{2}\cdot\frac{1-|\mathcal{E}(z)|^2}{1-|z|^2}\cdot\frac{1-|z|^2}{|z-t_0|} > \frac{1}{2a}\cdot\frac{1-|\mathcal{E}(z)|^2}{1-|z|^2}$$

which is equivalent to

$$\frac{1-|\mathcal{E}(z)|^2}{1-|z|^2}\cdot\left|\frac{z-t_0}{\mathcal{E}(z)-\mathcal{E}_0}\right| \leq 2a. \tag{5.24}$$

Note that the denominator $\mathcal{E}(z)-\mathcal{E}_0$ in the latter inequality does not vanish: assuming that $\mathcal{E}(z_0) = \mathcal{E}_0$ at some point $z_0 \in \mathbb{D}$, we would have by the maximum modulus principle (since $|\mathcal{E}_0| = 1$) that $\mathcal{E}(z) \equiv \mathcal{E}_0$ which would contradict the second assumption in (5.21). Finally, by this latter assumption, $d_{\mathcal{E}}(t_0) = \infty$ and relations (5.22) follow immediately from (5.24). □

Theorem 5.6. *Let $\mathcal{E} \in \mathcal{S}_0$ meet condition $\mathbf{C}_2$ at t_i:*

$$\lim_{z\to t_i} \mathcal{E}(z) = \eta_i = \frac{\widetilde{e}_i^*}{\widetilde{c}_i^*} \quad \textit{and} \quad \lim_{z\to t_i} \frac{1-|\mathcal{E}(z)|^2}{1-|z|^2} = \infty. \tag{5.25}$$

Then the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ is subject to relations (5.18).

Proof. Let for short

$$\Delta_i(z) := \frac{\widetilde{e}_i^* - \widetilde{c}_i^* \mathcal{E}(z)}{t_i - z}$$

and note that

$$\Delta_i(z) \neq 0 \quad (z \in \mathbb{D}). \tag{5.26}$$

To see this we argue as in the proof of the previous lemma: assuming that $\mathcal{E}(z_0) = \eta_i$ at some point $z_0 \in \mathbb{D}$, we would have by the maximum modulus principle (since $|\eta_i| = 1$) that $\mathcal{E}(z) \equiv \eta_i$ which would contradict the second assumption in (5.25). Furthermore, since $|\eta_i| = 1$ and due to assumptions (5.25), we can apply Lemma 5.5 (with $\mathcal{E}_0 = \eta_i$ and $t_0 = t_i$) to conclude that

$$\lim_{z\to t_0} \frac{1-|\mathcal{E}(z)|^2}{1-|z|^2} \cdot \frac{1}{|\Delta_i(z)|^2} = 0 \tag{5.27}$$

and

$$\lim_{z\to t_0} \Delta_i(z)^{-1} = 0. \tag{5.28}$$

Now we divide both parts in asymptotic relations (5.10)–(5.12) by $(\widetilde{e}_i^* - \widetilde{c}_i^* \mathcal{E}(z))$ and write the obtained equalities in terms of Δ_i as

$$\begin{aligned} \Delta_i(z)^{-1}\Psi(z) &= \mathbf{e}_i + \Delta_i(z)^{-1} \cdot O(1), \\ \Delta_i(z)^{-1}U_{\mathcal{E}}(z) &= w_i + \Delta_i(z)^{-1} \cdot O(1), \\ \Delta_i(z)^{-1}V_{\mathcal{E}}(z) &= 1 + \Delta_i(z)^{-1} \cdot O(1). \end{aligned}$$

By (5.28), the following nontangential limits exist

$$\lim_{z\to t_i} \Delta_i(z)^{-1}\Psi(z) = \mathbf{e}_i, \quad \lim_{z\to t_i} \Delta_i(z)^{-1}U_{\mathcal{E}}(z) = w_i, \quad \lim_{z\to t_i} \Delta_i(z)^{-1}V_{\mathcal{E}}(z) = 1$$

and we use these limits along with (5.27) to pass to limits in (5.9):

$$\begin{aligned} \lim_{z\to t_i} \frac{1-|w(z)|^2}{1-|z|^2} &= \lim_{z\to t_i} \frac{|\Delta_i(z)|^{-2}\dfrac{1-|\mathcal{E}(z)|^2}{1-|z|^2} + |\Delta_i(z)|^{-2}\Psi(z)^* P \Psi(z)}{|\Delta_i(z)|^{-2}|V_{\mathcal{E}}(z)|^2} \\ &= \frac{0 + \mathbf{e}_i^* P \mathbf{e}_i}{1} = \gamma_i. \end{aligned}$$

Finally,

$$\lim_{z\to t_i} w(z) = \lim_{z\to t_i} \frac{\Delta_i(z)^{-1}U_{\mathcal{E}}(z)}{\Delta_i(z)^{-1}V_{\mathcal{E}}(z)} = \frac{w_i}{1} = w_i,$$

which completes the proof. □

Theorem 5.7. *Let $\widetilde{p}_{ii}$ be the ith diagonal entry of $P^{-1} = [\widetilde{p}_{ij}]_{i,j=1}^{n}$, let $\mathcal{E} \in \mathcal{S}_0$ be subject to*

$$\lim_{z \to t_i} \mathcal{E}(z) = \eta_i \quad \text{and} \quad \lim_{z \to t_i} \frac{1 - |\mathcal{E}(z)|^2}{1 - |z|^2} = d_{\mathcal{E}}(t_i) < \infty. \tag{5.29}$$

Let us assume that

$$d_{\mathcal{E}}(t_i) \neq \frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2}. \tag{5.30}$$

Then the function $w := \mathbf{T}_{\Theta}[\mathcal{E}]$ satisfies

$$\lim_{z \to t_i} w(z) = w_i \tag{5.31}$$

and the nontangential limit $d_w(t_i) := \lim_{z \to t_i} \frac{1 - |w(z)|^2}{1 - |z|^2}$ is finite. Moreover,

$$d_w(t_i) < \gamma_i \quad \text{if} \quad d_{\mathcal{E}}(t_i) > -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2} \tag{5.32}$$

and

$$d_w(t_i) > \gamma_i \quad \text{if} \quad d_{\mathcal{E}}(t_i) < -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2}. \tag{5.33}$$

In other words, $d_w(t_i) < \gamma_i$ if $\mathcal{E}$ meets condition $\mathbf{C}_3$ and $d_w(t_i) > \gamma_i$ if $\mathcal{E}$ meets condition $\mathbf{C}_4$ at t_i.

Proof. By the Carathéodory-Julia theorem (for Schur functions), conditions (5.29) imply that the following nontangential limits exist

$$\lim_{z \to t_i} \mathcal{E}'(z) = \lim_{z \to t_i} \frac{\mathcal{E}(z) - \eta_i}{z - t_i} = \bar{t}_i \eta_i d_{\mathcal{E}}(t_i)$$

and the following asymptotic equality holds

$$\mathcal{E}(z) = \eta_i + (z - t_i)\bar{t}_i \eta_i d_{\mathcal{E}}(t_i) + o(|z - t_i|) \quad \text{as} \ z \to t_i. \tag{5.34}$$

We shall show that the functions Ψ, $U_{\mathcal{E}}$ and $V_{\mathcal{E}}$ defined in (5.4), (5.5), (5.6) admit the nontangential boundary limits at every interpolation node t_i:

$$\Psi(t_i) = \frac{\bar{t}_i}{\widetilde{e}_i} \left(P^{-1}\mathbf{e}_i - \mathbf{e}_i(\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i)) \right), \tag{5.35}$$

$$U_{\mathcal{E}}(t_i) = -\frac{\bar{t}_i w_i}{\widetilde{e}_i}(\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i)) \quad \text{and} \quad V_{\mathcal{E}}(t_i) = -\frac{\bar{t}_i}{\widetilde{e}_i}(\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i)). \tag{5.36}$$

To prove (5.35) we first multiply both parts in the Stein identity (3.4), by $\mathbf{e}_i$ on the right and obtain

$$P^{-1}\mathbf{e}_i - TP^{-1}T^*\mathbf{e}_i = \widetilde{E}^*\widetilde{e}_i - \widetilde{C}^*\widetilde{c}_i$$

which can be written equivalently, since $T^*\mathbf{e}_i = \bar{t}_i\mathbf{e}_i$ and $\widetilde{c}_i = \widetilde{e}_i\eta_i$, as

$$\widetilde{E}^* - \widetilde{C}^*\eta_i = \frac{\bar{t}_i}{\widetilde{e}_i}(t_i I - T)P^{-1}\mathbf{e}_i. \tag{5.37}$$

Substituting (5.34) into (5.4) and making use of (5.37) we get

$$\begin{aligned}\Psi(z) &= (zI-T)^{-1}\left(\widetilde{E}^*-\widetilde{C}^*\eta_i\right)-(z-t_i)(zI-T)^{-1}\widetilde{C}^*\eta_i d_{\mathcal{E}}(t_i)\bar{t}_i+o(1)\\ &= \frac{\bar{t}_i}{\widetilde{e}_i}(zI-T)^{-1}(t_iI-T)P^{-1}\mathbf{e}_i\\ &\quad -(z-t_i)(zI-T)^{-1}\widetilde{C}^*\eta_i d_{\mathcal{E}}(t_i)\bar{t}_i+o(1). \qquad (5.38)\end{aligned}$$

Since the following limits exist

$$\lim_{z\to t_i}(zI-T)^{-1}(t_iI-T)=I-\mathbf{e}_i\mathbf{e}_i^*, \quad \lim_{z\to t_i}(z-t_i)(zI-T)^{-1}=\mathbf{e}_i\mathbf{e}_i^*,$$

we can pass to the limit in (5.38) as $z\to t_i$ nontangentially to get

$$\Psi(t_i)=\frac{\bar{t}_i}{\widetilde{e}_i}(I-\mathbf{e}_i\mathbf{e}_i^*)P^{-1}\mathbf{e}_i-\mathbf{e}_i\mathbf{e}_i^*\widetilde{C}^*\eta_i d_{\mathcal{E}}(t_i)\bar{t}_i. \qquad (5.39)$$

Since $\mathbf{e}_i^*P^{-1}\mathbf{e}_i=\widetilde{p}_{ii}$ and $\mathbf{e}_i^*\widetilde{C}^*\eta_i=\widetilde{c}_i^*\eta_i=\widetilde{e}_i^*$, the right-hand side expression in (5.39) coincides with that in (5.35).

Making use of (5.34) and (5.35) we pass to the limits in (5.5) and (5.6) as $z\to t_i$ nontangentially:

$$\begin{aligned}U_{\mathcal{E}}(t_i) &= \mathcal{E}(t_i)-(t_i-\mu)C(\mu I-T)^{-1}\Psi(t_i)\\ &= \eta_i-\frac{1-\mu\bar{t}_i}{\widetilde{e}_i}C(\mu I-T)^{-1}\left(P^{-1}\mathbf{e}_i-\mathbf{e}_i(\widetilde{p}_{ii}+|\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i))\right), \qquad (5.40)\\ V_{\mathcal{E}}(t_i) &= 1-(t_i-\mu)E(\mu I-T)^{-1}\Psi(t_i)\\ &= 1-\frac{1-\mu\bar{t}_i}{\widetilde{e}_i}E(\mu I-T)^{-1}\left(P^{-1}\mathbf{e}_i-\mathbf{e}_i(\widetilde{p}_{ii}+|\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i))\right). \qquad (5.41)\end{aligned}$$

Note that by (3.2),

$$\begin{aligned}\frac{1-\mu\bar{t}_i}{\widetilde{e}_i}C(\mu I-T)^{-1}P^{-1}\mathbf{e}_i &= \frac{1-\mu\bar{t}_i}{\widetilde{e}_i}\widetilde{C}(I-\mu T^*)^{-1}\mathbf{e}_i=\frac{\widetilde{c}_i}{\widetilde{e}_i}=\eta_i, \qquad (5.42)\\ \frac{1-\mu\bar{t}_i}{\widetilde{e}_i}E(\mu I-T)^{-1}P^{-1}\mathbf{e}_i &= \frac{1-\mu\bar{t}_i}{\widetilde{e}_i}\widetilde{E}(I-\mu T^*)^{-1}\mathbf{e}_i=\frac{\widetilde{e}_i}{\widetilde{e}_i}=1. \qquad (5.43)\end{aligned}$$

Making use of these two equalities we simplify (5.40) and (5.41) to

$$U_{\mathcal{E}}(t_i)=\frac{1-\mu\bar{t}_i}{\widetilde{e}_i}C(\mu I-T)^{-1}\mathbf{e}_i(\widetilde{p}_{ii}+|\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i))$$

and

$$V_{\mathcal{E}}(t_i)=\frac{1-\mu\bar{t}_i}{\widetilde{e}_i}E(\mu I-T)^{-1}\mathbf{e}_i(\widetilde{p}_{ii}+|\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i)),$$

respectively, and it is readily seen from (5.13) that the two latter equalities coincide with those in (5.36).

Now we conclude from (5.3) and (5.36) that the nontangential boundary limits $w(t_i)$ exist for $i=1,\dots,n$ and

$$w(t_i)=\lim_{z\to t_i}w(z)=\lim_{z\to t_i}\frac{U_{\mathcal{E}}(z)}{V_{\mathcal{E}}(z)}=\frac{U_{\mathcal{E}}(t_i)}{V_{\mathcal{E}}(t_i)}=w_i$$

which proves (5.31). Furthermore, since the nontangential boundary limits $d_{\mathcal{E}}(t_i)$ and

$$|V_{\mathcal{E}}(t_i)|^2 = \frac{(\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i))^2}{|\widetilde{e}_i|^2} \tag{5.44}$$

exist (by the second assumption in (5.29) and the second relation in (5.36)), we can pass to the limit in (5.9) as z tends to t_i nontangentially:

$$d_w(t_i) = \frac{d_{\mathcal{E}}(t_i) + \Psi(t_i)^* P \Psi(t_i)}{|V_{\mathcal{E}}(t_i)|^2}.$$

By (5.44) and (5.35) we write $d_w(t_i)$ as follows

$$\frac{|\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i) + \left(\mathbf{e}_i^* P^{-1} - (\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i))\mathbf{e}_i^*\right) P \left(P^{-1}\mathbf{e}_i - \mathbf{e}_i(\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i))\right)}{(\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i))^2}$$

and elementary algebraic transformations based on equalities $\mathbf{e}_i^* P^{-1} \mathbf{e}_i = \widetilde{p}_{ii}$, $\mathbf{e}_i^* P \mathbf{e}_i = \gamma_i$ and $\mathbf{e}_i^* \mathbf{e}_i = 1$ lead us to

$$d_w(t_i) = \gamma_i - \frac{1}{\widetilde{p}_{ii} + |\widetilde{e}_i|^2 d_{\mathcal{E}}(t_i)}. \tag{5.45}$$

Statements (5.32) and (5.33) follow immediately from (5.45). □

As we have already mentioned in Introduction, Theorem 2.1 is known for the case $\kappa = 0$ (see [19]) At this point we already can recover this result.

Theorem 5.8. *Let the Pick matrix P be positive definite and let T, E, C, $\Theta(z)$ and η_i be defined as in* (2.3), (2.2) *and* (2.12). *Then all solutions w of Problem* 1.2 *are parametrized by the formula* (2.10) *when the parameter $\mathcal{E}$ belongs to the Schur class $\mathcal{S}_0$ and satisfies condition $\mathbf{C}_1 \vee \mathbf{C}_2$ at each interpolation node: either $\mathcal{E}$ fails to admit the nontangential boundary limit η_i at t_i or*

$$\mathcal{E}(t_i) = \eta_i \quad \textit{and} \quad d_{\mathcal{E}}(t_i) = \infty.$$

Proof. Any solution w of Problem 1.2 is a solution of Problem 1.1 and then by Statement 1 in Theorem 4.6, it is of the form $w = \mathbf{T}_\Theta[\mathcal{E}]$ for some Schur class function $\mathcal{E}$. Since $P > 0$, the diagonal entries $\widetilde{p}_{ii}$ of P^{-1} are positive. Therefore, the cases specified in (2.16)–(2.18) (conditions $\mathbf{C}_4 - \mathbf{C}_6$ cannot occur in this situation, whereas condition $\mathbf{C}_3$ simplifies to

$$\mathbf{C}_3 : \quad \mathcal{E}(t_i) = \eta_i \quad \text{and} \quad d_{\mathcal{E}}(t_i) < \infty.$$

In other words, any function $\mathcal{E} \in \mathcal{S}_0$ satisfies exactly one of the conditions $\mathbf{C}_1$, $\mathbf{C}_2$ or $\mathbf{C}_3$ at each one of interpolation nodes. Therefore, once $\mathcal{E}$ does not meet condition $\mathbf{C}_1$ or condition $\mathbf{C}_2$ at at least one interpolation node t_i, it meets condition $\mathbf{C}_3$ at t_i. Therefore, it holds for the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ that $d_w(t_i) < \gamma_i$ (by Theorem 5.7) and therefore w is not a solution of Problem 1.2. On the other hand, if $\mathcal{E}$ meets condition $\mathbf{C}_1 \vee \mathbf{C}_2$ at every interpolation node, then $w = \mathbf{T}_\Theta[\mathcal{E}]$ satisfies interpolation conditions (5.18) (by Theorems 5.4 and 5.6) that means that w is a solution of Problem 1.2. □

Remark 5.9. It is useful to note that for the one-point interpolation problem (i.e., when $n=1$), definition (3.3) takes the form

$$\begin{bmatrix} \widetilde{c}_1 \\ \widetilde{e}_1 \end{bmatrix} = \begin{bmatrix} w_1 \\ 1 \end{bmatrix} (\mu - t_1)^{-1}\gamma_1^{-1}(I-\mu\bar{t}_1) = -\bar{t}_1 \begin{bmatrix} w_1 \\ 1 \end{bmatrix} \gamma_1^{-1}$$

and therefore the number $\eta_1 := \frac{\widetilde{c}_1}{\widetilde{e}_1}$ in this case is equal to w_1.

Now we turn back to the indefinite case. Theorems 5.10 and 5.11 below treat the case when condition (5.30) is dropped. For notational convenience we let $i=n$ and

$$T_1 = \begin{bmatrix} t_1 & & \\ & \ddots & \\ & & t_{n-1} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} w_1 & \dots & w_{n-1} \end{bmatrix}$$

so that decompositions

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & t_n \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & w_n \end{bmatrix} \tag{5.46}$$

are conformal with partitioning

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & \gamma_n \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{p}_{nn} \end{bmatrix}. \tag{5.47}$$

Theorem 5.10. *Let $\widetilde{p}_{nn} < 0$ and let $\mathcal{E}$ be a Schur function such that*

$$\lim_{z\to t_n} \mathcal{E}(z) = \eta_n \quad \text{and} \quad d_{\mathcal{E}}(t_n) = -\frac{\widetilde{p}_{nn}}{|\widetilde{e}_n|^2}. \tag{5.48}$$

Then the function

$$w := \mathbf{T}_{\Theta}[\mathcal{E}] \tag{5.49}$$

is subject to one of the following:

1. *The nontangential boundary limit $w(t_n)$ does not exist.*
2. *The latter limit exists and $w(t_n) \neq w_n$.*
3. *The latter limit exists, is equal to w_n and $d_w(t_n) = \infty$.*

Proof. Since $\mathcal{E}$ is a Schur function, conditions (5.48) form a well-posed one-point interpolation problem (similar to Problem 1.2). By Theorem 5.8, $\mathcal{E}$ admits a representation

$$\mathcal{E} = \mathbf{T}_{\widehat{\Theta}}[\widehat{\mathcal{E}}] \tag{5.50}$$

with the coefficient matrix $\widehat{\Theta}$ defined via formula (2.2), but with P, T, E and C replaced by $-\frac{\widetilde{p}_{nn}}{|\widetilde{e}_n|^2}$, t_n, 1 and η_n, respectively:

$$\widehat{\Theta}(z) = I_2 - \frac{z-\mu}{(z-t_n)(1-\mu\bar{t}_n)} \begin{bmatrix} \eta_n \\ 1 \end{bmatrix} \frac{|\widetilde{e}_n|^2}{\widetilde{p}_{nn}} \begin{bmatrix} \eta_n^* & -1 \end{bmatrix} \tag{5.51}$$

and a parameter $\widehat{\mathcal{E}} \in \mathcal{S}_0$ satisfying one of the following three conditions:

(a) The limit $\widehat{\mathcal{E}}(t_n)$ does not exist.
(b) The limit $\widehat{\mathcal{E}}(t_n)$ exists and is not equal to η_n.
(c) It holds that

$$\widehat{\mathcal{E}}(t_n) = \eta_n \quad \text{and} \quad d_{\widehat{\mathcal{E}}}(t_n) = \infty. \tag{5.52}$$

We shall show that conditions (a), (b) and (c) for the parameter $\widehat{\mathcal{E}}$ are equivalent to statements (1), (2) and (3), respectively, in the formulation of the theorem. This will complete the proof.

Note that η_n appearing in (a) and (b) is the same as in (5.48), due to Remark 5.9. Since $\eta_n = \dfrac{\widetilde{c}_n}{\widetilde{e}_n}$, we can write (5.51) as

$$\widehat{\Theta}(z) = I_2 - \frac{z-\mu}{(z-t_n)(1-\mu\bar{t}_n)} \begin{bmatrix} \widetilde{c}_n \\ \widetilde{e}_n \end{bmatrix} \frac{1}{\widetilde{p}_{nn}} \begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix}$$

The inverse of $\widehat{\Theta}$ equals

$$\widehat{\Theta}(z)^{-1} = I_2 + \frac{z-\mu}{(z-t_n)(1-\mu\bar{t}_n)} \begin{bmatrix} \widetilde{c}_n \\ \widetilde{e}_n \end{bmatrix} \frac{1}{\widetilde{p}_{nn}} \begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix} \tag{5.53}$$

and coincides with the function $\widehat{\Theta}^{(2)}$ in (3.32). Therefore, by Lemma 3.6 and by Remark 3.7,

$$\Theta(z) = \Theta^{(1)}(z)\widehat{\Theta}(z)^{-1} \tag{5.54}$$

where $\Theta^{(1)}$ is given in (3.20). Substituting (5.51) into (5.49) (that is, representing w as a result of composition of two linear fractional transformations) and taking into account (5.54) we get

$$w := \mathbf{T}_\Theta[\mathcal{E}] = \mathbf{T}_\Theta[\mathbf{T}_{\widehat{\Theta}}[\widehat{\mathcal{E}}]] = \mathbf{T}_{\Theta\widehat{\Theta}}[\widehat{\mathcal{E}}] = \mathbf{T}_{\Theta^{(1)}}[\widehat{\mathcal{E}}].$$

Thus, upon setting

$$U_{\widehat{\mathcal{E}}}(z) = \Theta_{11}^{(1)}(z)\widehat{\mathcal{E}}(z) + \Theta_{12}^{(1)}(z), \qquad V_{\widehat{\mathcal{E}}}(z) = \Theta_{21}^{(1)}(z)\widehat{\mathcal{E}}(z) + \Theta_{22}^{(1)}(z), \tag{5.55}$$

we have

$$w = \mathbf{T}_{\Theta^{(1)}}[\widehat{\mathcal{E}}] = \frac{\Theta_{11}^{(1)}\widehat{\mathcal{E}} + \Theta_{12}^{(1)}}{\Theta_{21}^{(1)}\widehat{\mathcal{E}} + \Theta_{22}^{(1)}} = \frac{U_{\widehat{\mathcal{E}}}}{V_{\widehat{\mathcal{E}}}}. \tag{5.56}$$

Note that $\Theta^{(1)}$ is a rational function analytic and invertible at t_n. It follows immediately from (5.56) that if the boundary limit $\widehat{\mathcal{E}}(t_n)$ does not exist, then the boundary $w(t_n)$ does not exist either. Thus, $(a) \Rightarrow (1)$. The rest is broken into two steps.

Step 1: *Let the nontangential boundary limit $\widehat{\mathcal{E}}(t_n)$ exists. Then so do the limits $U_{\widehat{\mathcal{E}}}(t_n)$, $V_{\widehat{\mathcal{E}}}(t_n)$ and $w(t_n)$, and moreover,*

$$V_{\widehat{\mathcal{E}}}(t_n) := \lim_{z\to t_n} V_{\widehat{\mathcal{E}}}(z) \neq 0 \tag{5.57}$$

and

$$w(t_n) = w_n \quad \text{if and only if} \quad \widehat{\mathcal{E}}(t_n) = \eta_n. \tag{5.58}$$

Proof of Step 1. Existence of the limits $U_{\widehat{\mathcal{E}}}(t_n)$ and $V_{\widehat{\mathcal{E}}}(t_n)$ is clear since $\Theta^{(1)}$ is analytic at t_n. Assume that $V_{\widehat{\mathcal{E}}}(t_n) = 0$. Then $U_{\widehat{\mathcal{E}}}(t_n) = 0$, since otherwise, the function w of the form (5.56) would not be bounded in a neighborhood of $t_n \in \mathbb{T}$ which cannot occur since w is a generalized Schur function. If $V_{\widehat{\mathcal{E}}}(t_n) = U_{\widehat{\mathcal{E}}}(t_n) = 0$, then it follows from (5.55) that

$$\Theta^{(1)}(t_n)\begin{bmatrix}\widehat{\mathcal{E}}(t_n)\\ 1\end{bmatrix} = \begin{bmatrix}U_{\widehat{\mathcal{E}}}(t_n)\\ V_{\widehat{\mathcal{E}}}(t_n)\end{bmatrix} = 0$$

and thus, the matrix $\Theta^{(1)}(t_n)$ is singular which is a contradiction. Now it follows from (5.56) and (5.57) that the limit $w(t_n)$ exists. This completes the proof of $(a) \Leftrightarrow (1)$. The proof of (5.58) rests on the equality

$$\begin{bmatrix} w_n^* & -1 \end{bmatrix}\Theta^{(1)}(t_n) = \frac{\bar{t}_n}{\widetilde{p}_{nn}}\begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix}. \tag{5.59}$$

Indeed, it follows from (5.56) and (5.59) that

$$\begin{aligned} w(t_n) - w_n &= \frac{U_{\widehat{\mathcal{E}}}(t_n) - w_n V_{\widehat{\mathcal{E}}}(t_n)}{V_{\widehat{\mathcal{E}}}(t_n)} \\ &= \frac{w_n}{V_{\widehat{\mathcal{E}}}(t_n)}\cdot\begin{bmatrix} w_n^* & -1 \end{bmatrix}\Theta^{(1)}(t_n)\begin{bmatrix}\widehat{\mathcal{E}}(t_n)\\ 1\end{bmatrix} \\ &= \frac{\bar{t}_n w_n}{\widetilde{p}_{nn} V_{\widehat{\mathcal{E}}}(t_n)}\begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix}\begin{bmatrix}\widehat{\mathcal{E}}(t_n)\\ 1\end{bmatrix} \\ &= \frac{\bar{t}_n w_n}{\widetilde{p}_{nn}\widetilde{c}_n^* V_{\widehat{\mathcal{E}}}(t_n)}\left(\widehat{\mathcal{E}}(t_n) - \eta_n\right) \end{aligned}$$

which clearly implies (5.58). It remains to prove (5.59). To this end, note that by (3.11),

$$\operatorname{Res}_{z=t_n}\Theta(z) = -\begin{bmatrix} w_n \\ 1 \end{bmatrix}\begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix}$$

and it is readily seen from (5.53) that

$$\operatorname{Res}_{z=t_n}\widehat{\Theta}(z)^{-1} = t_n\begin{bmatrix} \widetilde{c}_n \\ \widetilde{e}_n \end{bmatrix}\frac{1}{\widetilde{p}_{nn}}\begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix}.$$

Taking into account that $\Theta^{(1)}$ is analytic at t_n and that Θ and $\widehat{\Theta}^{-1}$ have simple poles at t_n, we compare the residues of both parts in (5.54) at t_n to arrive at

$$-\begin{bmatrix} w_n \\ 1 \end{bmatrix}\begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix} = \frac{t_n}{\widetilde{p}_{nn}}\Theta^{(1)}(t_n)\begin{bmatrix} \widetilde{c}_n \\ \widetilde{e}_n \end{bmatrix}\begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix},$$

which implies (since $\widetilde{e}_n \neq 0$)

$$\begin{bmatrix} w_n \\ 1 \end{bmatrix} = \Theta^{(1)}(t_n)\begin{bmatrix} \widetilde{c}_n \\ \widetilde{e}_n \end{bmatrix}\frac{t_n}{\widetilde{p}_{nn}}.$$

Equality of adjoints in the latter equality gives

$$\begin{bmatrix} w_n^* & -1 \end{bmatrix} = \begin{bmatrix} w_n^* & 1 \end{bmatrix} J = \frac{\bar{t}_n}{\widetilde{p}_{nn}} \begin{bmatrix} \widetilde{c}_n^* & -\widetilde{e}_n^* \end{bmatrix} J\Theta^{(1)}(t_n)^* J$$

which is equivalent to (5.59), since $\Theta^{(1)}(t_n)$ is J-unitary and thus, $J\Theta^{(1)}(t_n)^*J = \Theta^{(1)}(t_n)^{-1}$. This completes the proof of (5.58) which implies in particular, that $(b) \Leftrightarrow (2)$.

Step 2: $(c) \Leftrightarrow (3)$.

Proof of Step 2. Equality $w(t_n) = w_n$ is equivalent to the first condition in (5.52) by (5.58). To complete the proof, it suffices to show that if $\widehat{\mathcal{E}}(t_n) = \eta_n$, then

$$d_w(t_n) = \infty \quad \text{if and only if} \quad d_{\widehat{\mathcal{E}}}(t_n) = \infty. \tag{5.60}$$

To this end, we write a virtue of relation (5.9) in terms of the parameter $\widehat{\mathcal{E}}$:

$$\frac{1-|w(z)|^2}{1-|z|^2} = \frac{1}{|V_{\widehat{\mathcal{E}}}(z)|^2}\left(\frac{1-|\widehat{\mathcal{E}}(z)|^2}{1-|z|^2} + \widehat{\Psi}(z)^* P \widehat{\Psi}(z)\right) \tag{5.61}$$

where

$$\widehat{\Psi}(z) = (zI - T_1)^{-1}(\mu I - T_1)P_{11}^{-1}(I - \mu T_1^*)^{-1}\left(E_1^* - C_1^*\widehat{\mathcal{E}}(z)\right). \tag{5.62}$$

Note that to get (5.62) we represent the right-hand side expression in (5.4) in terms of C and E (rather than $\widetilde{C}$ and $\widetilde{E}$; this can be achieved with help of (3.3)) and then replace P, T, E, C and $\mathcal{E}$ in the obtained formula by P_{11}, T_1, E_1, C_1 and $\widehat{\mathcal{E}}$, respectively. Since the nontangential boundary limit

$$\widehat{\Psi}(t_n) = (t_n I - T_1)^{-1}(\mu I - T_1)P_{11}^{-1}(I - \mu T_1^*)^{-1}\left(E_1^* - C_1^*\eta_n\right)$$

exists and is finite, equivalence (5.60) follows from (5.61). □

Theorem 5.11. *Let $\widetilde{p}_{nn} = 0$ and let $\mathcal{E}$ be a Schur function such that*

$$\mathcal{E}(t_n) = \eta_n \quad \textit{and} \quad d_{\mathcal{E}}(t_n) = 0. \tag{5.63}$$

Then the function $w := \mathbf{T}_\Theta[\mathcal{E}]$ admits finite nontangential boundary limits $d_w(t_n)$ and $w(t_n) \neq w_n$.

Proof. Conditions (5.63) state a one-point boundary interpolation problem for Schur functions $\mathcal{E}$ with the Pick matrix equals $d_{\mathcal{E}}(t_n) = 0$. Then by Statement 2 in Theorem 4.6, the only function $\mathcal{E}$ satisfying conditions (5.63) is the constant function $\mathcal{E}(z) \equiv \eta_n$ (the Blaschke product of degree zero). Since $|\eta_n| = 1$, the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ is rational and unimodular on $\mathbb{T}$. Therefore, it is equal to the ratio of two finite Blaschke products and therefore, the limits $w(t)$ and $d_w(t)$ exist at every point $t \in \mathbb{T}$. We shall use decompositions (5.46) and (5.47) with understanding that $\widetilde{p}_{nn} = 0$, so that

$$\widetilde{P}_{21}P_{12} = 1 \quad \text{and} \quad P^{-1}\mathbf{e}_n = \begin{bmatrix} \widetilde{P}_{12} \\ 0 \end{bmatrix}. \tag{5.64}$$

We shall also make use the formula

$$P_{21}(I-\bar{t}_nT_1)^{-1}=(E_1-w_n^*C_1) \tag{5.65}$$

that follows from the Stein identity (3.1) upon substituting partitioning (5.46), (5.47) and comparison the $(1,2)$ block entries.

In the current context, the formula (5.4) for Ψ simplifies, on account of (5.37), to

$$\begin{aligned}\Psi(z) &= (zI-T)^{-1}\left(\widetilde{E}^*-\widetilde{C}^*\eta_n\right)\\ &= \frac{\bar{t}_n}{\widetilde{e}_n}(zI-T)^{-1}(t_nI-T)P^{-1}\mathbf{e}_n\end{aligned}$$

Now we substitute the latter equality into (5.5) and (5.6) and use formulas (5.42) and (5.43) (for $i=n$) to get

$$U_{\mathcal{E}}(z)=\frac{1-z\bar{t}_n}{\widetilde{e}_n}C(zI-T)^{-1}P^{-1}\mathbf{e}_n,\quad V_{\mathcal{E}}(z)=\frac{1-z\bar{t}_n}{\widetilde{e}_n}E(zI-T)^{-1}P^{-1}\mathbf{e}_n.$$

Taking into account the second equality in (5.64), rewrite the latter two formulas in terms of partitioning (5.46) and (5.47) as

$$U_{\mathcal{E}}(z)=\frac{1-z\bar{t}_n}{\widetilde{e}_n}C_1(zI-T_1)^{-1}\widetilde{P}_{12},\quad V_{\mathcal{E}}(z)=\frac{1-z\bar{t}_n}{\widetilde{e}_n}E_1(zI-T_1)^{-1}\widetilde{P}_{12}. \tag{5.66}$$

Thus,

$$w(z):=\frac{U_{\mathcal{E}}(z)}{V_{\mathcal{E}}(z)}=\frac{C_1(zI-T_1)^{-1}\widetilde{P}_{12}}{E_1(zI-T_1)^{-1}\widetilde{P}_{12}}.$$

We shall show that the denominator on the right-hand side in the latter formula does not vanish at $z=t_n$, so that

$$w(t_n):=\lim_{z\to t_n}\frac{C_1(zI-T_1)^{-1}\widetilde{P}_{12}}{E_1(zI-T_1)^{-1}\widetilde{P}_{12}}=\frac{C_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}{E_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}. \tag{5.67}$$

Then we will have, on account of (5.65) and the first equality in (5.64),

$$\begin{aligned}w_n-w(t_n) &= w_n-\frac{C_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}{E_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}\\ &= \frac{(w_nE_1-C_1)(t_nI-T_1)^{-1}\widetilde{P}_{12}}{E_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}\\ &= \frac{w_nt_n(E_1-w_n^*C_1)(I-\bar{t}_nT_1)^{-1}\widetilde{P}_{12}}{E_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}\\ &= \frac{w_nt_n\widetilde{P}_{21}P_{12}}{E_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}=\frac{w_nt_n}{E_1(t_nI-T_1)^{-1}\widetilde{P}_{12}}\neq 0\end{aligned} \tag{5.68}$$

and thus $w(t_n)\neq w_n$. Thus, it remains to show that the denominator in (5.67) is not zero. Assume that $E_1(t_nI-T_1)^{-1}\widetilde{P}_{12}=0$. Since the limit in (5.67) exists (recall that w is the ratio of two finite Blaschke products), the latter assumption forces $C_1(t_nI-T_1)^{-1}\widetilde{P}_{12}=0$ and therefore, equality

$$(w_nE_1-C_1)(t_nI-T_1)^{-1}\widetilde{P}_{12}=0.$$

But it was already shown in calculation (5.68) that

$$(w_nE_1-C_1)(t_nI-T_1)^{-1}\widetilde{P}_{12}=w_nt_n\neq 0$$

and the obtained contradiction completes the proof. □

Recall that the interpolation node t_n in Theorems 5.10 and 5.11 was chosen just for notational convenience and can be replaced by any interpolation node t_i. It means that Theorems 5.10 and 5.11 prove Statements (4) and (5) in Theorem 2.3. Furthermore, Theorem 5.7 proves the "if" parts in Statements (4) and (5) in Theorem 2.3, whereas Theorems 5.4 and 5.6 prove the "if" part in Statement (1) in Theorem 2.3. Finally since conditions $\mathbf{C}_1$-$\mathbf{C}_6$ are disjoint, the "only if" parts in Statements (1), (2) and (3) are obvious. This completes the proof of Theorem 2.3.

6. Negative squares of the function $w=\mathbf{T}_\Theta[\mathcal{E}]$

In this section we prove Theorems 2.9 and 2.5. We assume without loss of generality that (maybe after an appropriate rearrangement of the interpolation nodes) a fixed parameter $\mathcal{E}\in\mathcal{S}_0$ satisfies condition $\mathbf{C}_{1-3}$ at interpolation nodes $t_1,\dots,t_{n-\ell}$ and conditions $\mathbf{C}_{4-6}$ at the remaining ℓ points. Thus, we assume that

$$\lim_{z\to t_i}\mathcal{E}(z)=\eta_i \quad\text{and}\quad \lim_{z\to t_i}\frac{1-|\mathcal{E}(z)|^2}{1-|z|^2}\leq -\frac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2}\quad (i=n-\ell+1,\dots,n). \tag{6.1}$$

Let

$$P^{-1}=\begin{bmatrix}\widetilde{P}_{11} & \widetilde{P}_{12}\\ \widetilde{P}_{21} & \widetilde{P}_{22}\end{bmatrix}\quad\text{with}\quad \widetilde{P}_{22}\in\mathbb{C}^{\ell\times\ell}. \tag{6.2}$$

Note that under the above assumption, the matrix $\mathcal{P}$ in the formulation of Theorem 2.9 coincides with $\widetilde{P}_{22}$ in the decomposition (6.2). Thus, to prove Theorem 2.9, it suffices to show that there exists a Schur function $\mathcal{E}$ satisfying conditions (6.1) if and only if the matrix $\widetilde{P}_{22}$ is negative semidefinite.

Proof of Theorem 2.9. Since $|\eta_i|=1$, conditions (6.1) form a well-posed boundary Nevanlinna-Pick problem (similar to Problem 1.1) in the Schur class $\mathcal{S}_0$. This problem has a solution $\mathcal{E}$ if and only if the corresponding Pick matrix

$$\mathbb{P}=[\mathbb{P}_{ij}]_{i,j=n-\ell+1}^n\quad\text{with the entries}\quad \mathbb{P}_{ij}=\begin{cases}\dfrac{1-\eta_i^*\eta_j}{1-\bar{t}_it_j} & \text{for}\quad i\neq j,\\ -\dfrac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2} & \text{for}\quad i=j,\end{cases} \tag{6.3}$$

is positive semidefinite. Furthermore, there exist infinitely many functions $\mathcal{E}$ satisfying (6.1) if $\mathbb{P}$ is positive definite and there is a unique such function (which is a Blaschke product of degree equals rank $\mathbb{P}$) if $\mathbb{P}$ is singular. Thus, to complete the proof, it suffices to show that

$$\mathbb{P} > 0 \Longleftrightarrow \widetilde{P}_{22} < 0, \quad \mathbb{P} \geq 0 \Longleftrightarrow \widetilde{P}_{22} \leq 0 \quad \text{and} \quad \operatorname{rank} \mathbb{P} = \operatorname{rank} \widetilde{P}_{22}. \tag{6.4}$$

To this end, note that

$$\bar{t}_i \widetilde{e}_i^* \cdot \mathbb{P}_{ij} \cdot t_j \widetilde{e}_j = -\widetilde{p}_{ij} \quad (i, j = n - \ell + 1, \ldots, n) \tag{6.5}$$

where $\widetilde{p}_{ij}$ is the ijth entry in P^{-1}. Indeed, if $i \neq j$, then (6.5) follows from (6.3), (3.6) and definition (2.12) of η_i. If $i = j$, then (6.5) follows directly from (6.3). By (6.2), $[\widetilde{p}_{ij}]_{i,j=\ell+1}^n = \widetilde{P}_{22}$, which allows us to rewrite equalities (6.5) in the matrix form as

$$\mathbf{C}^* \, \mathbb{P} \, \mathbf{C} = -\widetilde{P}_{22} \quad \text{where} \quad \mathbf{C} = \operatorname{diag}\left(t_{\ell+1}\widetilde{e}_{\ell+1}, \; t_{\ell+2}\widetilde{e}_{\ell+2}, \; \ldots, \; t_n \widetilde{e}_n\right). \tag{6.6}$$

Since the matrix $\mathbf{C}$ is invertible, all the statements in (6.4) follow from (6.6). This completes the proof of Theorem 2.9. $\square$

To prove Theorem 2.5 we shall use the following result (see [5, Lemma 2.4] for the proof).

Lemma 6.1. *Let $P \in \mathbb{C}^{n \times n}$ be an invertible Hermitian matrix and let*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} \widetilde{P}_{11} & \widetilde{P}_{12} \\ \widetilde{P}_{21} & \widetilde{P}_{22} \end{bmatrix} \tag{6.7}$$

be two conformal decompositions of P and of P^{-1} with $P_{22}, \widetilde{P}_{22} \in \mathbb{C}^{\ell \times \ell}$. Furthermore, let $\widetilde{P}_{22}$ be negative semidefinite. Then

$$\operatorname{sq}_- P_{11} = \operatorname{sq}_- P - \ell.$$

Proof of Theorem 2.5. We start with several remarks. We again assume (without loss of generality) that a picked parameter $\mathcal{E} \in \mathcal{S}_0$ satisfies condition $\mathbf{C}_{1-3}$ at $t_1, \ldots, t_{n-\ell}$ and conditions (6.1) at the remaining ℓ interpolation nodes. Under these non-restrictive assumptions we will show that the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ belongs to the class $\mathcal{S}_{\kappa-\ell}$. Throughout the proof, we shall be using partitioning (3.18), (3.19). Note that by Theorem 2.9, the block $\widetilde{P}_{22}$ is necessarily negative semidefinite. Then by Lemma 6.1, $\operatorname{sq}_- P_{11} = \kappa - \ell$. Furthermore, since $\mathcal{E}$ meets condition $\mathbf{C}_{1-3}$ at $t_1, \ldots, t_{n-\ell}$, the function $w = \mathbf{T}_\Theta[\mathcal{E}]$ satisfies interpolation conditions (1.17) at each of these points. Then by Remark 1.5, w has at least $\operatorname{sq}_- P_{11} = \kappa - \ell$ negative squares.

It remains to show that it has at most $\kappa - \ell$ negative squares. This will be done separately for the cases when $\widetilde{P}_{22}$ is negative definite and when $\widetilde{P}_{22}$ is negative semidefinite and singular.

Conditions (6.1) mean that $\mathcal{E}$ is a solution of a boundary Nevanlinna–Pick interpolation problem with the data set consisting of ℓ interpolation nodes t_i, unimodular numbers η_i and nonnegative numbers $\mathbb{P}_{ii} = -\dfrac{\widetilde{p}_{ii}}{|\widetilde{e}_i|^2}$ for $i = n - \ell + 1, \ldots, n$. The Pick matrix $\mathbb{P}$ of the problem is defined in (6.3).

Case 1: $\widetilde{P}_{22} < 0$: In this case $\mathbb{P} > 0$ (by (6.6)) and by the first statement in Theorem 4.6, $\mathcal{E}$ admits a representation

$$\mathcal{E} = \mathbf{T}_{\widehat{\Theta}}[\widehat{\mathcal{E}}] \tag{6.8}$$

for some $\widehat{\mathcal{E}} \in \mathcal{S}_0$ where, according to (2.2), the coefficient matrix $\widehat{\Theta}$ in (6.8) is of the form

$$\widehat{\Theta}(z) = I_2 + (z-\mu)\begin{bmatrix} M \\ E_2 \end{bmatrix}(zI - T_2)^{-1}\mathbb{P}^{-1}(I-\mu T_2^*)^{-1}\begin{bmatrix} M^* & -E_2^* \end{bmatrix} \tag{6.9}$$

where the matrices

$$T_2 = \operatorname{diag}(t_{n-\ell+1}, \dots, t_n), \quad E_2 = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix} \tag{6.10}$$

are exactly the same as in (3.18), (3.19)) and

$$M = \begin{bmatrix} \eta_{n-\ell+1} & \eta_{n-\ell+2} & \dots & \eta_n \end{bmatrix} \tag{6.11}$$

Self-evident equalities

$$\begin{bmatrix} \eta_i \\ 1 \end{bmatrix} \cdot \frac{1}{z-t_i} \cdot t_i \widetilde{e}_i = -\begin{bmatrix} \widetilde{c}_i \\ \widetilde{e}_i \end{bmatrix} \cdot \frac{1}{1-z\bar{t}_i} \quad (i = n-\ell+1, \dots, n)$$

can be written in the matrix form as

$$\begin{bmatrix} M \\ E_2 \end{bmatrix}(zI - T_2)^{-1}\mathbf{C} = -\begin{bmatrix} \widetilde{C}_2 \\ \widetilde{E}_2 \end{bmatrix}(I - zT_2^*)^{-1} \tag{6.12}$$

where $\mathbf{C}$ is defined in (6.6), whereas

$$\widetilde{E}_2 = \begin{bmatrix} \widetilde{e}_{n-\ell+1} & \dots & \widetilde{e}_n \end{bmatrix} \quad \text{and} \quad \widetilde{C}_2 = \begin{bmatrix} \widetilde{c}_{n-\ell+1} & \dots & \widetilde{c}_n \end{bmatrix}$$

are the matrices from the two last partitionings in (3.19). On account of (6.12) and (6.6), we rewrite the formula (6.9) as

$$\widehat{\Theta}(z) = I_2 - (z-\mu)\begin{bmatrix} \widetilde{C}_2 \\ \widetilde{E}_2 \end{bmatrix}(I - zT_2^*)^{-1}\widetilde{P}_{22}^{-1}(\mu I - T_2)^{-1}\begin{bmatrix} \widetilde{C}_2^* & -\widetilde{E}_2^* \end{bmatrix}.$$

Then its inverse can be represented as

$$\widehat{\Theta}(z)^{-1} = I_2 + (z-\mu)\begin{bmatrix} \widetilde{C}_2 \\ \widetilde{E}_2 \end{bmatrix}(I - \mu T_2^*)^{-1}\widetilde{P}_{22}^{-1}(zI - T_2)^{-1}\begin{bmatrix} \widetilde{C}_2^* & -\widetilde{E}_2^* \end{bmatrix}$$

and coincides with the function $\widetilde{\Theta}^{(2)}$ from (3.21). Therefore, by Lemma 3.6,

$$\Theta(z) = \Theta^{(1)}(z)\widehat{\Theta}(z)^{-1} \tag{6.13}$$

where $\Theta^{(1)}$ is given in (3.20). Note that

$$\Theta^{(1)} \in \mathcal{W}_{\kappa_1} \quad \text{where} \;\; \kappa_1 = \operatorname{sq}_- P_{11} = \kappa - \ell. \tag{6.14}$$

Substituting (6.8) into (2.10) (that is, representing w as a result of composition of two linear fractional transformations) and taking into account (6.13) we get

$$w := \mathbf{T}_\Theta[\mathcal{E}] = \mathbf{T}_\Theta[\mathbf{T}_{\widehat{\Theta}}[\widehat{\mathcal{E}}]] = \mathbf{T}_{\Theta\widehat{\Theta}}[\widehat{\mathcal{E}}] = \mathbf{T}_{\Theta^{(1)}}[\widehat{\mathcal{E}}].$$

Since $\mathcal{E} \in \mathcal{S}_0$ and due to (6.13), the last equality guarantees (by Remark 5.1) that w has at most $\kappa_1 = \kappa - \ell$ negative squares which completes the proof of Case 1.

Case 2: $\widetilde{P}_{22} \leq 0$ is singular: In this case $\mathbb{P}$ is positive semidefinite and singular (again, by (6.6)) and by the second statement in Theorem 4.6, $\mathcal{E}$ admits a representation

$$\mathcal{E}(z) = \frac{x^*(I - zT_2^*)^{-1}E_2^*}{x^*(I - zT_2^*)^{-1}M^*} \tag{6.15}$$

where x is any nonzero vector such that $\mathbb{P}x = 0$. Letting $y := \mathbf{C}^{-1}x$ we have (due to (6.6))

$$\widetilde{P}_{22}y = 0 \tag{6.16}$$

and, on account of (6.12), we can rewrite (6.15) as

$$\mathcal{E}(z) = \frac{y^*\mathbf{C}^*(I - zT_2^*)^{-1}E_2^*}{y^*\mathbf{C}^*(I - zT_2^*)^{-1}M^*} = \frac{y^*(zI - T_2)^{-1}\widetilde{E}_2^*}{y^*(zI - T_2)^{-1}\widetilde{C}_2^*}. \tag{6.17}$$

Since $\mathcal{E}$ is a finite Blaschke product (again by the second statement in Theorem 4.6) it satisfies the symmetry relation $\mathcal{E}(z) = (\overline{\mathcal{E}(1/\bar{z})})^{-1}$ which together with (6.17) gives another representation for $\mathcal{E}$:

$$\mathcal{E}(z) = \frac{\widetilde{C}_2(I - zT_2^*)^{-1}y}{\widetilde{E}_2(I - zT_2^*)^{-1}y}. \tag{6.18}$$

We will use the latter formula and (5.8) to get an explicit expression for the kernel $K_w(z, w)$. Setting

$$u(z) = \widetilde{C}_2(I - zT_2^*)^{-1}y \quad \text{and} \quad v(z) = \widetilde{E}_2(I - zT_2^*)^{-1}y$$

for short and making use of the second Stein identity in (3.4) we have

$$\begin{aligned} v(\zeta)^*v(z) - u(\zeta)^*u(z) &= y^*(I - \bar{\zeta}T_2)^{-1}\left[\widetilde{E}_2^*\widetilde{E}_2 - \widetilde{C}_2^*\widetilde{C}_2\right](I - zT_2^*)^{-1}y \\ &= y^*(I - \bar{\zeta}T_2)^{-1}\left[\widetilde{P}_{22} - T_2\widetilde{P}_{22}T_2^*\right](I - zT_2^*)^{-1}y \end{aligned}$$

which reduces, due to (6.16), to

$$v(\zeta)^*v(z) - u(\zeta)^*u(z) = -(1 - z\bar{\zeta})y^*(I - \bar{\zeta}T_2)^{-1}T_2\widetilde{P}_{22}T_2^*(I - zT_2^*)^{-1}y.$$

Upon dividing both parts in the latter equality by $(1 - z\bar{\zeta})v(z)v(\zeta)^*$ we arrive at

$$\frac{1 - \mathcal{E}(\zeta)^*\mathcal{E}(z)}{1 - \bar{\zeta}z} = -\frac{y^*}{v(\zeta)^*}(I - \bar{\zeta}T_2)^{-1}T_2\widetilde{P}_{22}T_2^*(I - zT_2^*)^{-1}\frac{y}{v(z)}. \tag{6.19}$$

Next, we substitute the explicit formula (6.18) for $\mathcal{E}$ into (5.4) to get

$$\begin{aligned} \Psi(z) &= (zI - T)^{-1}\left(\widetilde{E}^* - \widetilde{C}^*\mathcal{E}(z)\right) \\ &= (zI - T)^{-1}(\widetilde{E}^*\widetilde{E}_2 - \widetilde{C}^*\widetilde{C}_2)(I - zT_2^*)^{-1} \cdot \frac{y}{v(z)}. \end{aligned} \tag{6.20}$$

Substituting partitionings (3.18), (3.19) into the Stein identity (3.4) and comparing the right block entries we get

$$\begin{bmatrix} \widetilde{P}_{12} \\ \widetilde{P}_{22} \end{bmatrix} - T \begin{bmatrix} \widetilde{P}_{12} \\ \widetilde{P}_{22} \end{bmatrix} T_2^* = \widetilde{E}\widetilde{E}_2^* - \widetilde{C}\widetilde{C}_2^*$$

which implies

$$\begin{aligned} &(zI-T)^{-1}\left\{\widetilde{E}\widetilde{E}_2^* - \widetilde{C}\widetilde{C}_2^*\right\}(I-zT_2^*)^{-1} \\ &= (zI-T)^{-1}\begin{bmatrix} \widetilde{P}_{12} \\ \widetilde{P}_{22} \end{bmatrix} + \begin{bmatrix} \widetilde{P}_{12} \\ \widetilde{P}_{22} \end{bmatrix} T_2^*(I-zT_2^*)^{-1}. \end{aligned}$$

Now we substitute the last equality into (5.4) and take into account (6.16) to get

$$\Psi(z) = (zI-T)^{-1}\begin{bmatrix} \widetilde{P}_{12} \\ 0 \end{bmatrix} \cdot \frac{y}{v(z)} + \begin{bmatrix} \widetilde{P}_{12} \\ \widetilde{P}_{22} \end{bmatrix} T_2^*(I-zT_2^*)^{-1} \cdot \frac{y}{v(z)}$$

On account of partitionings (3.18), the latter equality leads us to

$$\begin{aligned} \Psi(\zeta)^* P \Psi(z) \;=\; & \frac{y^*}{v(\zeta)^*}\Big(\widetilde{P}_{12}^*(\bar{\zeta}I - T_1^*)^{-1}P_{11}(zI-T_1)^{-1}\widetilde{P}_{12} \\ & +(I-\bar{\zeta}T_2)^{-1}T_2\widetilde{P}_{22}T_2^*(I-zT_2^*)^{-1}\Big)\frac{y}{v(z)}. \end{aligned} \tag{6.21}$$

Upon substituting (6.19) and (6.21) into (5.8) we get

$$\frac{1-w(\zeta)^*w(z)}{1-\bar{\zeta}z} = \frac{y^*}{V_{\mathcal{E}}(\zeta)^*v(\zeta)^*} \cdot \widetilde{P}_{12}^*(\bar{\zeta}I - T_1^*)^{-1}P_{11}(zI-T_1)^{-1}\widetilde{P}_{12} \cdot \frac{y}{V_{\mathcal{E}}(z)v(z)}.$$

Thus, the kernel $K_w(z,\zeta)$ admits a representation

$$K_w(z,\zeta) = R(\zeta)^*P_{11}R(z) \quad \text{where} \quad R(z) = \frac{y\widetilde{P}_{21}T_1^*(I-zT_1^*)^{-1}}{v(z)V_{\mathcal{E}}(z)}$$

and thus,

$$\mathrm{sq}_-K_w \le \mathrm{sq}_-P_{11} = \kappa - \ell$$

which completes the proof of the theorem. □

Remark 6.2. At this point Theorem 2.2 is completely proved: the necessity part follows from Theorem 4.3 and from the necessity part in Theorem 4.2; the sufficiency part follows (as was explained in Introduction) from Corollary 2.4 and Theorem 2.5 which have been already proved.

Remark 6.3. We also proved the sufficiency part in Theorem 4.2 when the Pick matrix P is invertible.

Indeed, in this case, every solution w to the FMI (4.5) is of the form (4.15), by Theorem 4.3. But every function of this form solves Problem 1.6, by Theorem 2.2.

7. The degenerate case

In this section we study Problem 1.6 in the case when the Pick matrix P of the problem (defined in (1.14)) is singular. In the course of the study we will prove Theorem 2.1 and will complete the proof of Theorem 4.2.

Theorem 7.1. *Let the Pick matrix P defined in* (1.14) *be singular with* $\operatorname{rank} P = \ell < n$. *Then there is a unique generalized Schur function w such that*

$$\operatorname{sq}_-\mathbf{K}_w(z,\zeta) = \kappa \tag{7.1}$$

where $\mathbf{K}_w(z,\zeta)$ is the kernel defined in (4.4)*. Furthermore,*

1. *This unique function w is the ratio of two finite Blaschke products*
$$w(z) = \frac{B_1(z)}{B_2(z)} \tag{7.2}$$
with no common zeroes and such that
$$\deg B_1 + \deg B_2 = \operatorname{rank} P. \tag{7.3}$$
2. *This unique function w belongs to the generalized Schur class $\mathcal{S}_{\kappa'}$ where $\kappa' = \deg B_2 \le \kappa$ and satisfies conditions*
$$d_w(t_i) \le \gamma_i \quad \textit{and} \quad w(t_i) = w_i \qquad (i = 1, \dots, n) \tag{7.4}$$
at all but $\kappa - \kappa'$ interpolation nodes (*that is, w is a solution to Problem* 1.6).
3. *The function w satisfies conditions*
$$d_w(t_i) = \gamma_i \quad \textit{and} \quad w(t_i) = w_i$$
at at least $n - \operatorname{rank} P$ interpolation nodes.

Proof. Without loss of generality we can assume that the top $\ell \times \ell$ principal submatrix P_{11} of P is invertible and has κ negative eigenvalues. We consider conformal partitionings

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{7.5}$$

and

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad \det\, P_{11} \ne 0, \quad \operatorname{sq}_- P_{11} = \kappa = \operatorname{sq} P. \tag{7.6}$$

Since $\operatorname{rank} P_{11} = \operatorname{rank} P$, it follows that $P_{22} - P_{21}P_{11}^{-1}P_{12}$ the Schur complement of P_{11} in P, is the zero matrix, i.e.,

$$P_{22} = P_{21}P_{11}^{-1}P_{12}. \tag{7.7}$$

Furthermore, it is readily seen that the ith row of the block P_{21} in (7.6) can be written in the form

$$\mathbf{e}_i^* P_{21} = \begin{bmatrix} \dfrac{1 - w_{\ell+i}^* w_1}{1 - \bar{t}_{\ell+i} t_1} & \dots & \dfrac{1 - w_{\ell+i}^* w_\ell}{1 - \bar{t}_{\ell+i} t_\ell} \end{bmatrix} = \left(E_1 - w_{\ell+i}^* C_1\right)\left(I - \bar{t}_{\ell+i} T_1\right)^{-1}$$

and similarly, the jth column in P_{12} is equal to

$$P_{12}\mathbf{e}_j = \left(I - t_{\ell+j}T_1^*\right)^{-1}\left(E_1^* - C_1^* w_{\ell+j}\right) \tag{7.8}$$

(recall that $\mathbf{e}_j$ stands for the jth column of the identity matrix of an appropriate size). Taking into account that the ijth entry in P_{22} is equal to $\dfrac{1 - w_{\ell+i}^* w_{\ell+j}}{1 - \bar{t}_{\ell+i}t_{\ell+j}}$ (if $i \neq j$) or to $\gamma_{\ell+i}$ (if $i = j$) we write the equality (7.7) entrywise and get the equalities

$$\frac{1 - w_i^* w_j}{1 - \bar{t}_i t_j} = \left(E_1 - w_i^* C_1\right)\left(I - \bar{t}_i T_1\right)^{-1} P_{11}^{-1}\left(I - t_j T_1^*\right)^{-1}\left(E_1^* - w_j C_1^*\right) \tag{7.9}$$

for $i \neq j \in \{\ell+1, \ldots, n\}$ and the equalities

$$\gamma_i = \left(E_1 - w_i^* C_1\right)\left(I - \bar{t}_i T_1\right)^{-1} P_{11}^{-1}\left(I - t_i T_1^*\right)^{-1}\left(E_1^* - w_i C_1^*\right) \tag{7.10}$$

for $i = \ell+1, \ldots, n$. The rest of the proof is broken into a number of steps.

Step 1: *If w is a meromorphic function such that* (7.1) *holds, then it is necessarily of the form*

$$w = \mathbf{T}_{\Theta^{(1)}}[\mathcal{E}] := \frac{\Theta_{11}^{(1)}\mathcal{E} + \Theta_{12}^{(1)}}{\Theta_{21}^{(1)}\mathcal{E} + \Theta_{22}^{(1)}} \tag{7.11}$$

for some Schur function $\mathcal{E} \in \mathcal{S}_0$, where $\Theta^{(1)}$ is given in (3.20).

Proof of Step 1. Write the kernel $\mathbf{K}_w(z,\zeta)$ in the block form as

$$\mathbf{K}_w(z,\zeta) = \begin{bmatrix} P_{11} & P_{12} & F_1(z) \\ P_{21} & P_{22} & F_2(z) \\ F_1(\zeta)^* & F_2(\zeta)^* & K_w(z,\zeta) \end{bmatrix} \tag{7.12}$$

where F_1 and F_2 are given in (4.10). The kernel

$$\begin{aligned} \mathbf{K}_w^1(z,\zeta) \;&:=\; \begin{bmatrix} P_{11} & F_1(z) \\ F_1(\zeta)^* & K_w(z,\zeta) \end{bmatrix} \\ &=\; \begin{bmatrix} P_{11} & (I - zT_1^*)^{-1}(E_1^* - C_1^* w(z)) \\ (E_1 - w(\zeta)^* C_1)(I - \bar{\zeta}T_1)^{-1} & K_w(z,\zeta) \end{bmatrix} \end{aligned}$$

is contained in $\mathbf{K}_w(z,\zeta)$ as a principal submatrix and therefore, $\text{sq}\_\mathbf{K}_w^1 \leq \kappa$. On the other hand, $\mathbf{K}_w^1$ contains P_{11} as a principal submatrix and therefore $\text{sq}\_\mathbf{K}_w^1 \geq \text{sq}\_P_{11} = \kappa$. Thus,

$$\text{sq}\_\mathbf{K}_w^1 = \kappa. \tag{7.13}$$

Recall that P_{11} is an invertible Hermitian matrix with κ negative eigenvalues and satisfies the first Stein identity in (3.4). Then we can apply Theorem 4.3 (which is already proved for the case when the Pick matrix is invertible) to the FMI (7.13). Upon this application we conclude that w is of the form (7.11) with some $\mathcal{E} \in \mathcal{S}_0$ and $\Theta^{(1)}$ of the form (3.20).

Step 2: *Every function of the form* (7.11) *solves the following truncated Problem* 1.6*: it belongs to the generalized Schur class $\mathcal{S}_{\kappa'}$ for some $\kappa' \le \kappa$ and satisfies conditions*

$$d_w(t_i) \le \gamma_i \quad \textit{and} \quad w(t_i) = w_i \qquad (i = 1, \dots, \ell)$$

at all but $\kappa - \kappa'$ interpolation nodes.

Proof of Step 2. The Pick matrix for the indicated truncated interpolation problem is P_{11} which is invertible and has κ negative eigenvalues. Thus, we can apply Theorem 2.2 (which is already proved for the nondegenerate case) to get the desired statement.

The rational function $\Theta^{(1)}$ is analytic and J-unitary at t_i for every $i = \ell + 1, \dots, n$. Then we can consider the numbers a_i and b_i defined by

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \Theta^{(1)}(t_i)^{-1} \begin{bmatrix} w_i \\ 1 \end{bmatrix} \quad \text{for } \ i = \ell + 1, \dots, n. \tag{7.14}$$

It is clear from (7.14) that $|a_i| + |b_i| > 0$. Furthermore,

Step 3: *It holds that*

$$|a_i| = |b_i| \ne 0 \quad \textit{and} \quad \frac{a_i}{b_i} = \frac{a_j}{b_j} \quad \textit{for} \quad i, j = \ell + 1, \dots, n. \tag{7.15}$$

Proof of Step 3. Let $i \in \{\ell + 1, \dots, n\}$. Since the matrix $\Theta^{(1)}(t_i)^{-1}$ is J-unitary and since $|w_i| = 1$, we conclude from (7.14) that

$$\begin{aligned} |a_i|^2 - |b_i|^2 = \begin{bmatrix} a_i^* & b_i^* \end{bmatrix} J \begin{bmatrix} a_i \\ b_i \end{bmatrix} &= \begin{bmatrix} w_i^* & 1 \end{bmatrix} \Theta^{(1)}(t_i)^{-*} J \Theta^{(1)}(t_i)^{-1} \begin{bmatrix} w_i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} w_i^* & 1 \end{bmatrix} J \begin{bmatrix} w_i \\ 1 \end{bmatrix} = |w_i|^2 - 1 = 0. \end{aligned} \tag{7.16}$$

Thus, $|a_i| = |b_i|$ and, since $|a_i| + |b_i| > 0$, the first statement in (7.15) follows. Similarly to (7.16), we have

$$a_i^* a_j - b_i^* b_j = \begin{bmatrix} w_i^* & 1 \end{bmatrix} \Theta^{(1)}(t_i)^{-*} J \Theta^{(1)}(t_j)^{-1} \begin{bmatrix} w_j \\ 1 \end{bmatrix} \tag{7.17}$$

for every choice of $i, j \in \{\ell + 1, \dots, n\}$. By a virtue of formula (3.16),

$$\frac{\Theta^{(1)}(\zeta)^{-*} J \Theta^{(1)}(z)^{-1} - J}{1 - z\bar{\zeta}} = \begin{bmatrix} C_1 \\ -E_1 \end{bmatrix} (I - \bar{\zeta} T_1)^{-1} P_{11}^{-1} (I - z T_1^*)^{-1} \begin{bmatrix} C_1^* & -E_1^* \end{bmatrix}. \tag{7.18}$$

Substituting the latter formula (evaluated at $\zeta = t_i$ and $z = t_j$) into the right-hand side expression in (7.17) and taking into account that $\begin{bmatrix} w_i^* & 1 \end{bmatrix} J \begin{bmatrix} w_j \\ 1 \end{bmatrix} = w_i^* w_j - 1$, we get

$$\begin{aligned} a_i^* a_j - b_i^* b_j &= w_i^* w_j - 1 + (1 - \bar{t}_i t_j) (E_1 - w_i^* C_1) (I - \bar{t}_i T_1)^{-1} P_{11}^{-1} \\ &\qquad \times (I - t_j T_1^*)^{-1} (E_1^* - w_j C_1^*). \end{aligned}$$

The latter expression is equal to zero, by (7.9). Therefore, $a_i^*a_j = b_i^*b_j$ and consequently,

$$\frac{a_j}{b_j} = \frac{b_i^*}{a_i^*} = \frac{a_i}{b_i}$$

where the second equality holds since $|a_i| = |b_i|$.

Step 4: *Let a_i and b_i be defined as in* (7.14). *Then the row vectors*

$$A = \begin{bmatrix} a_{\ell+1} & \dots & a_n \end{bmatrix}, \qquad B = \begin{bmatrix} b_{\ell+1} & \dots & b_n \end{bmatrix} \tag{7.19}$$

can be represented as follows:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C \\ E \end{bmatrix} (\mu I - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ I \end{bmatrix} (\mu I - T_2). \tag{7.20}$$

Proof of Step 4. First we substitute the formula (3.30) for the inverse of $\Theta^{(1)}$ into (7.14) to get

$$\begin{bmatrix} a_i \\ b_i \end{bmatrix} = \begin{bmatrix} w_i \\ 1 \end{bmatrix} + (t_i - \mu) \begin{bmatrix} C_1 \\ E_1 \end{bmatrix} (\mu I - T_1)^{-1} P_{11}^{-1} (I - t_i T_1^*)^{-1} (E_1^* - C_1^* w_i)$$

for $i = \ell + 1, \dots, n$ and then we make use of (7.8) and of the vector $\mathbf{e}_i$ to write the latter equalities in the form

$$\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{e}_i = \begin{bmatrix} w_{\ell+i} \\ 1 \end{bmatrix} - \begin{bmatrix} C_1 \\ E_1 \end{bmatrix} (\mu I - T_1)^{-1} P_{11}^{-1} P_{12} \mathbf{e}_i (\mu - t_{\ell+i})$$

for $i = 1, \dots, n - \ell$. Now we transform the right-hand side expression in the latter equality as follows

$$\begin{aligned}
\begin{bmatrix} A \\ B \end{bmatrix} \mathbf{e}_i &= \begin{bmatrix} C_2 \\ E_2 \end{bmatrix} \mathbf{e}_i - \begin{bmatrix} C_1 \\ E_1 \end{bmatrix} (\mu I - T_1)^{-1} P_{11}^{-1} P_{12} (\mu I - T_2) \mathbf{e}_i \\
&= \left(\begin{bmatrix} C_2 \\ E_2 \end{bmatrix} (\mu I - T_2)^{-1} - \begin{bmatrix} C_1 \\ E_1 \end{bmatrix} (\mu I - T_1)^{-1} P_{11}^{-1} P_{12} \right) (\mu I - T_2) \mathbf{e}_i \\
&= \begin{bmatrix} C \\ E \end{bmatrix} (\mu I - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ I \end{bmatrix} (\mu I - T_2) \mathbf{e}_i
\end{aligned}$$

and since the latter equality holds for every $i \in \{1, \dots, n - \ell\}$, (7.20) follows.

Remark 7.2. Comparing (7.20) and (3.29) we conclude that

$$\begin{bmatrix} A \\ B \end{bmatrix} = \Theta^{(1)}(z)^{-1} \begin{bmatrix} C \\ E \end{bmatrix} (zI - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix} (zI - T_2).$$

By the symmetry principle, $\Theta^{(1)}(z)^{-1} = J\Theta^{(1)}(1/\bar{z})^* J$ and thus, the latter identity can be written equivalently as

$$\begin{bmatrix} A \\ -B \end{bmatrix} (zI - T_2)^{-1} = \Theta^{(1)}(/\bar{z})^* \begin{bmatrix} C \\ -E \end{bmatrix} (zI - T)^{-1} \begin{bmatrix} -P_{11}^{-1}P_{12} \\ 1 \end{bmatrix}.$$

Taking adjoints and replacing z by $1/\bar{z}$ in the resulting identity we obtain eventually

$$(I - zT_2^*)^{-1}\begin{bmatrix} A^* & -B^* \end{bmatrix} = \begin{bmatrix} -P_{21}P_{11}^{-1} & 1 \end{bmatrix} (I - zT^*)^{-1}\begin{bmatrix} C^* & -E^* \end{bmatrix} \Theta^{(1)}(z). \tag{7.21}$$

Step 5: *A function w of the form* (7.11) *satisfies the FMI* (7.1) *only if the corresponding parameter $\mathcal{E}$ is the unimodular constant*

$$\mathcal{E}(z) \equiv \mathcal{E}_0 := \frac{a_{\ell+1}}{b_{\ell+1}} = \cdots = \frac{a_n}{b_n}. \tag{7.22}$$

Proof of Step 5. Let us consider the Schur complement $\mathbf{S}$ of the block P_{11} in (7.12):

$$\mathbf{S}(z,\zeta) = \begin{bmatrix} P_{22} & F_2(z) \\ F_2(\zeta)^* & K_w(z,\zeta) \end{bmatrix} - \begin{bmatrix} P_{21} \\ F_1(\zeta)^* \end{bmatrix} P_{11}^{-1}\begin{bmatrix} P_{12} & F_1(z) \end{bmatrix}$$

Since

$$\mathrm{sq}_-\mathbf{K}_w = \mathrm{sq}_-P_{11} + \mathrm{sq}_-\mathbf{S} = \kappa + \mathrm{sq}_-\mathbf{S},$$

it follows that the FMI (7.1) is equivalent to positivity of $\mathbf{S}$ on $\rho(w) \cap \mathbb{D}$:

$$\mathbf{S}(z,\zeta) \succeq 0. \tag{7.23}$$

Since the "11″ block in $\mathbf{S}(z,\zeta)$ equals $P_{22} - P_{21}P_{11}^{-1}P_{12}$ which is the zero matrix (by (7.7)), the positivity condition (7.23) guarantees the the nondiagonal entries in $\mathbf{S}$ vanish everywhere in $\mathbb{D}$:

$$F_2(z) - P_{21}P_{11}^{-1}F_1(z) \equiv 0. \tag{7.24}$$

By (4.11), the latter identity can be written as

$$\begin{bmatrix} -P_{21}P_{11}^{-1} & I \end{bmatrix} (I - zT^*)^{-1}(E^* - C^*w(z)) \equiv 0. \tag{7.25}$$

We already know from Step 1, that w is of the form (7.11) for some $\mathcal{E} \in \mathcal{S}_0$. Now we will show that (7.25) holds for w of the form (7.11) if and only if the corresponding parameter $\mathcal{E}$ is subject to

$$A^*\mathcal{E}(z) \equiv B^* \tag{7.26}$$

where A and B are given in (7.19). Indeed, it is easily seen that for w of the form (7.11), it holds that

$$E^* - C^*w = \left(\Theta_{21}^{(1)}\mathcal{E} + \Theta_{22}^{(1)}\right)^{-1}\begin{bmatrix} -C^* & E^* \end{bmatrix}\begin{bmatrix} \Theta_{11}^{(1)} & \Theta_{12}^{(1)} \\ \Theta_{21}^{(1)} & \Theta_{22}^{(1)} \end{bmatrix}\begin{bmatrix} \mathcal{E} \\ 1 \end{bmatrix}$$

and therefore, identity (7.25) can be written equivalently in terms of the parameter $\mathcal{E}$ as

$$\begin{bmatrix} -P_{21}P_{11}^{-1} & I \end{bmatrix} (I - zT^*)^{-1}\begin{bmatrix} C^* & -E^* \end{bmatrix}\Theta^{(1)}(z)\begin{bmatrix} \mathcal{E}(z) \\ 1 \end{bmatrix} \equiv 0$$

which is, due to (7.21), the same as

$$(I - zT_2^*)^{-1}\begin{bmatrix} A^* & B^* \end{bmatrix} J \begin{bmatrix} \mathcal{E}(z) \\ I \end{bmatrix} \equiv 0.$$

The latter identity is clearly equivalent to (7.26). Writing (7.26) entrywise we get the system of equalities

$$a_i^*\mathcal{E}(z) \equiv b_i^* \qquad (i = \ell+1, \dots, n).$$

This system is consistent, by (7.15), and it clearly admits a unique solution $\mathcal{E}_0$ defined as in (7.22). Combining Step 1 and Step 5, we can already conclude that the FMI (7.1) has at most one solution: the only candidate is the function

$$w = \mathbf{T}_{\Theta^{(1)}}[\mathcal{E}_0] \tag{7.27}$$

where $\mathcal{E}_0$ is the unimodular constant defined in (7.22). The next step will show that this function indeed is a solution to the FMI (7.1).

Step 6: *The function* (7.27) *satisfies the FMI* (7.1) *and interpolation conditions*

$$d_w(t_i) = \gamma_i \quad \textit{and} \quad w(t_i) = w_i \quad \textit{for} \;\; i = \ell+1, \dots, n. \tag{7.28}$$

Proof of Step 6. First we note that since $\Theta^{(1)}$ is a rational J-inner function of McMillan degree ℓ and since $\mathcal{E}_0$ is a unimodular constant, the function w of the form (7.27) is a rational function of degree ℓ which is unimodular on $\mathbb{T}$. Therefore, w is the ratio of two finite Blachke products satisfying (7.3). Since w belongs to $\mathcal{S}_{\kappa'}$ (by Step 2), it has κ' poles inside $\mathbb{D}$ and thus, the denominator B_2 in (7.2) is a finite Blachke product of order κ'.

It was shown in the proof of Step 5 that equation (7.26) is equivalent to (7.24)) and thus, for the function w of the form (7.27), it holds that

$$F_2(z) \equiv P_{21}P_{11}^{-1}F_1(z) \tag{7.29}$$

which is the same, due to definitions (4.10), as

$$(I - zT_2^*)^{-1}(E_2^* - C_2^*w(z)) \equiv P_{21}P_{11}^{-1}(I - zT_1^*)^{-1}(E_1^* - C_1^*w(z)). \tag{7.30}$$

Next we show that for w of the form (7.27) it holds that

$$K_w(z,\zeta) \equiv F_1(\zeta)^*P_{11}^{-1}F_1(z) \tag{7.31}$$

or, which is the same,

$$\frac{1 - w(\zeta)^*w(z)}{1 - \bar{\zeta}z} \equiv (E_1 - w(\zeta)^*C_1)(I - \bar{\zeta}T_1)^{-1}P_{11}^{-1}(I - zT_1^*)^{-1}(E_1^* - C_1^*w(z)). \tag{7.32}$$

Indeed, on account of (7.18),

$$\begin{aligned}
&(E_1 - w(\zeta)^*C_1)(I - \bar{\zeta}T_1)^{-1}P_{11}^{-1}(I - zT_1^*)^{-1}(E_1^* - C_1^*w(z)) \\
&= \begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} \frac{\Theta^{(1)}(\zeta)^{-*}J\Theta^{(1)}(z)^{-1} - J}{1 - z\bar{\zeta}} \begin{bmatrix} w(z) \\ 1 \end{bmatrix} \\
&= \frac{1 - w(z)w(\zeta)^*}{1 - z\bar{\zeta}} + \begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} \frac{\Theta^{(1)}(\zeta)^{-*}J\Theta^{(1)}(z)^{-1}}{1 - z\bar{\zeta}} \begin{bmatrix} w(z) \\ 1 \end{bmatrix}.
\end{aligned} \tag{7.33}$$

Representation (7.27) is equivalent to

$$\begin{bmatrix} w(z) \\ 1 \end{bmatrix} = \Theta^{(1)}(z) \begin{bmatrix} \mathcal{E}_0 \\ 1 \end{bmatrix} \frac{1}{v(z)}, \quad \text{where } \; v(z) = \Theta_{21}^{(1)}(z)\mathcal{E}_0 + \Theta_{22}^{(1)}(z),$$

and therefore,

$$\begin{bmatrix} w(\zeta)^* & 1 \end{bmatrix} \Theta^{(1)}(\zeta)^{-*} J \Theta^{(1)}(z)^{-1} \begin{bmatrix} w(z) \\ 1 \end{bmatrix} = \frac{|\mathcal{E}_0|^2 - 1}{v(z)v(\zeta)^*} \equiv 0,$$

since $|\mathcal{E}_0| = 1$. On account of this latter equality, (7.33) implies (7.31). By (7.7), (7.29) and (7.31), the kernel $\mathbf{K}_w(z,\zeta)$ defined in (4.4) and partitioned as in (7.12), can be represented also in the form

$$\mathbf{K}_w(z,\zeta) = \begin{bmatrix} P_{11} \\ P_{21} \\ F_1(\zeta)^* \end{bmatrix} P_{11}^{-1} \begin{bmatrix} P_{11} & P_{12} & F_1(z) \end{bmatrix}$$

and the latter representation implies that $\mathrm{sq}_-\mathbf{K}_w = \mathrm{sq}_- P_{11} = \kappa$, i.e., that w of the form (7.27) satisfies the FMI (7.1). It remains to check that w satisfies interpolation conditions (7.28). Since w is a ratio of two finite Blaschke products, it is analytic on $\mathbb{T}$. Let t_i ($\ell < i \le n$) be an interpolation node. Comparing the residues at $z = t_i$ of both parts in the identity (7.30) we get

$$-t_i \mathbf{e}_i \mathbf{e}_i^* \left(E_2^* - C_2^* w(t_i)\right) = 0$$

which is equivalent to

$$1 - w_i^* w(t_i) = 0$$

or, since $|w_i| = 1$, to the second condition in (7.28). On the other hand, letting $z, \zeta \to t_i$ in (7.32) and taking into account that $w(t_i) = w_i$, we get

$$\begin{aligned} d_w(t_i) &= (E_1 - w(t_i)^* C_1)(I - \bar{t}_i T_1)^{-1} P_{11}^{-1} (I - t_i T_1^*)^{-1} (E_1^* - C_1^* w(t_i)) \\ &= (E_1 - w_i^* C_1)(I - \bar{t}_i T_1)^{-1} P_{11}^{-1} (I - t_i T_1^*)^{-1} (E_1^* - C_1^* w_i) \end{aligned}$$

which together with (7.10) implies the first condition in (7.28).

The first statement of the Theorem is proved. Statement 2 follows by Step 2 and (7.28): the function w meets interpolation conditions (7.4) at all but $\kappa - \kappa'$ interpolation nodes (and all the exceptional nodes are in $\{t_1, \dots, t_\ell\}$). Statement 3 follows from (7.28). □

Remark 7.3. Statement 2 in Theorem 7.1 completes the proof of sufficiency part in Theorem 4.2: if P is singular, then a (unique) solution of the FMI (4.5) solves Problem 1.6.

8. An example

In this section we present a numerical example illustrating the preceding analysis. The data set of the problem is as follows:

$$t_1 = 1, \quad t_2 = -1, \quad w_1 = 1, \quad w_2 = -1, \quad \gamma_1 = 1, \quad \gamma_2 = 0. \tag{8.1}$$

Then the matrices (2.3) take the form

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C \\ E \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and since $\dfrac{1-w_1^*w_2}{1-\bar{t}_1t_2}=1$ we have also

$$P=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix} \quad \text{and} \quad P^{-1}=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}.$$

It is readily seen that P is invertible and has one negative eigenvalue. Thus, Problems 1.3, 1.4 and 1.6 take the following form.

Problem 1.4: *Find all functions* $w\in\mathcal{S}_1$ *such that*

$$w(1)=1,\quad d_w(1)\le 1,\quad w(-1)=-1,\quad d_w(-1)\le 0. \tag{8.2}$$

Problem 1.3: *Find all functions* $w\in\mathcal{S}_1$ *that satisfy conditions* (8.2) *with equalities in the second and in the fourth conditions.*

Problem 1.6: *Find all functions* w *such that either*

1. $w\in\mathcal{S}_1$ *and satisfies all the conditions in* (8.2) *or*
2. $w\in\mathcal{S}_0$ *and satisfies the two first conditions in* (8.2) *or*
3. $w\in\mathcal{S}_0$ *and satisfies the two last conditions in* (8.2).

Letting $\mu=i$, we get by (2.2) the following formula for $\Theta(z)$

$$\begin{aligned}
&I_2+(z-i)\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{z-1} & 0\\ 0 & \frac{1}{z+1}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\frac{1}{1-i} & 0\\ 0 & \frac{1}{1+i}\end{bmatrix}\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix}\\
=\ &\frac{1}{2(z^2-1)}\begin{bmatrix}(i-1)z^2+2(1+2i)z-1-i & (3i-1)z^2+2z+i-1\\ (i+1)z^2-2z+1+3i & (1-i)z^2+2(2i-1)z+1+i\end{bmatrix}
\end{aligned}$$

and thus, by Theorem 2.2, all the solutions w to Problem 1.6 are parametrized by the linear fractional formula

$$w(z)=\frac{\big[(i-1)z^2+2(1+2i)z-1-i\big]\,\mathcal{E}(z)+(3i-1)z^2+2z+i-1}{\big[(i+1)z^2-2z+1+3i\big]\,\mathcal{E}(z)+(1-i)z^2+2(2i-1)z+1+i} \tag{8.3}$$

when the parameter $\mathcal{E}$ runs through the Schur class $\mathcal{S}_0$. Furthermore, formula (3.3) in the present setting gives

$$\begin{aligned}
\begin{bmatrix}\widetilde{c}_1 & \widetilde{c}_2\\ \widetilde{e}_1 & \widetilde{e}_2\end{bmatrix} &= \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\frac{1}{i-1} & 0\\ 0 & \frac{1}{i+1}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1-i & 0\\ 0 & 1+i\end{bmatrix}\\
&= \begin{bmatrix}1 & 1-i\\ -1 & -1-i\end{bmatrix}
\end{aligned}\tag{8.4}$$

and since the diagonal entries of P^{-1} are $\widetilde{p}_{11}=0$ and $\widetilde{p}_{11}=-1$, we also have

$$\eta_1:=\frac{\widetilde{c}_1}{\widetilde{e}_1}=-1,\quad \eta_2:=\frac{\widetilde{c}_2}{\widetilde{e}_2}=i,\quad \frac{\widetilde{p}_{11}}{|\widetilde{e}_1|^2}=0,\quad \frac{\widetilde{p}_{22}}{|\widetilde{e}_2|^2}=-\frac{1}{2}.$$

By Theorem 2.7, every function w of the form (8.3) also solves Problem 1.4, unless the parameter $\mathcal{E}$ is subject to

$$\mathcal{E}(1)=-1\quad\text{and}\quad d_{\mathcal{E}}(1)=0 \tag{8.5}$$

or to

$$\mathcal{E}(-1)=i\quad\text{and}\quad d_{\mathcal{E}}(-1)\le\frac{1}{2}. \tag{8.6}$$

On the other hand, Theorem 2.6 tells us that every function w of the form (8.3) solves Problem 1.3, unless the parameter $\mathcal{E}$ is subject to

$$\mathcal{E}(1) = -1 \quad \text{and} \quad d_{\mathcal{E}}(1) < \infty$$

or to

$$\mathcal{E}(-1) = i \quad \text{and} \quad d_{\mathcal{E}}(-1) < \infty.$$

Thus, every parameter $\mathcal{E} \in \mathcal{S}_0$ satisfying conditions (8.5) or (8.6) leads to a solution w of Problem 1.6 which is not a solution to Problem 1.4. For these special solutions, it looks curious to track which conditions in (8.2) are satisfied and which are not. This will also illustrate propositions 4 and 5 in Theorem 2.3.

First we note that there is only one Schur function $\mathcal{E} \equiv -1$ satisfying conditions (8.5) (this is the case indicated in the fifth part in Theorem 2.3). The corresponding function w obtained via (8.3), equals

$$w(z) = \frac{2iz^2 - 4iz + 2i}{-2iz^2 + 4iz - 2i} \equiv -1.$$

It satisfies all the conditions in (8.2) but the first one.

All other "special" solutions of Problem 1.6 are exactly all Schur functions satisfying the two first conditions in (8.2). Every such function does not satisfy at least one of the two last conditions in (8.2). We present several examples omitting straightforward computations:

Example 1: The function

$$\mathcal{E}(z) = \frac{2iz + 2}{(1-i)z - 1 - 3i}$$

belongs to $\mathcal{S}_0$ and satisfies $\mathcal{E}(-1) = i$ and $d_{\mathcal{E}}(-1) = \frac{1}{2}$ (i.e., it meets condition (2.17) at t_2). Substituting this parameter into (8.3) we get the function

$$w(z) = \frac{z - i}{iz + 1 - 2i}$$

which belongs to $\mathcal{S}_0$ and satisfies (compare with (8.2))

$$w(1) = 1, \quad d_w(1) = 1, \quad w(-1) = \frac{1+i}{3i-1}, \quad d_w(-1) = \infty.$$

Example 2: The function

$$\mathcal{E}(z) = \frac{(3-i)z - (1+i)}{-(1+i)z + 3i - 1}$$

belongs to $\mathcal{S}_0$ and satisfies (as in Example 1) $\mathcal{E}(-1) = i$ and $d_{\mathcal{E}}(-1) = \frac{1}{2}$. Substituting this parameter into (8.3) we get the function $w(z) \equiv 1$ which belongs to $\mathcal{S}_0$ and satisfies (compare with (8.2))

$$w(1) = 1, \quad d_w(1) = 0, \quad w(-1) = 1, \quad d_w(-1) = 0.$$

Example 3: The function

$$\mathcal{E}(z) = \frac{[(3+i)z + 1 - i]\, e^{\frac{z-1}{z+1}} - 2iz - 2}{-2(1+iz)e^{\frac{z-1}{z+1}} + (i-1)z + 3i + 1}$$

belongs to $\mathcal{S}_0$ and satisfies $\mathcal{E}(-1) = i$ and $d_{\mathcal{E}}(-1) = \frac{1}{2}$. Substituting this parameter into (8.3) we get the function

$$w(z) = \frac{[(2-i)z - 1]\, e^{\frac{z-1}{z+1}} - z + i}{(z-i)e^{\frac{z-1}{z+1}} - iz + 2i - 1}$$

which belongs to $\mathcal{S}_0$ and fails to have a boundary nontangential limit at $t_2 = -1$.

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A Truncated Matricial Moment Problem on a Finite Interval

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Abstract. The main goal of this paper is to study the truncated matricial moment problem on a finite closed interval by using the FMI method of V.P. Potapov. The solvability of the problem is characterized by the fact that two block Hankel matrices built from the data of the problem are nonnegative Hermitian. An essential step to solve the problem under consideration is to derive an effective coupling identity between both block Hankel matrices (see Proposition 2.2). In the case that these block Hankel matrices are both positive Hermitian we parametrize the set of solutions via a linear fractional transformation the generating matrix-valued function of which is a matrix polynomial whereas the set of parameters consists of distinguished pairs of meromorphic matrix-valued functions.

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0. Introduction and preliminaries

In the 1970's V.P. Potapov developed a particular approach to handle matrix versions of classical interpolation and moment problems. His method is based on transforming the original problems into equivalent matrix inequalities. Using this strategy several matricial interpolation problems could be successfully treated by V.P. Potapov's associates (see, e.g., Dubovoj [Du]; Dyukarev/Katsnelson [DK]; Dyukarev [Dy1]; Golinskii [G1], [G2]; Katsnelson [Ka1] - [Ka3]; Kovalishina [Ko1] - [Ko2]). V.P. Potapov's approach was enriched by L.A. Sakhnovich who introduced a method of operator identities which serves to unify the particular instances of V.P. Potapov's procedure under one framework (see [IS], [S2], [BS]). These operator identities have the form

$$AS - SA^* = i\Pi J\Pi^*$$

and are also called Sylvester identities or Ljapunov identities. In this connection it should be mentioned that the problem of reducing a nonselfadjoint operator to a diagonal form has already lead L.A. Sakhnovich to a relation of the above-described Sylvester-Ljapunov type (see formula (3) in [S1]).

In this paper, we apply V.P. Potapov's approach in combination with L.A. Sakhnovich's method of operator identities to the truncated matrix moment problem on a finite closed interval. Hereby, we assume that an even number of moment matrices is prescribed. (The odd case will be treated somewhere else.) The scalar version of this problem was studied by M.G. Krein [Kr2] (see also [KN, Ch. 4]) using different methods. An important feature of scalar moment problems connected with certain subintervals of the real axis is that their solvability is characterized by the fact that several matrices built from the set of prescribed moments have to be simultaneously nonnegative Hermitian (see, e.g., Chapters 4,5, and 8 in [KN]).

What concerns the matrix case there can be observed an intensive treatment of the matricial version of the classical Stieltjes moment problem and somehow related interpolation problems in various classes of holomorphic matrix functions (see, e.g., [DK], [Dy1]-[Dy6], [BS], [B]). A closer look at this work shows that the solvability of the problem under consideration is guaranteed if and only if two distinguished block matrices built from the data are simultaneously nonnegative Hermitian. In the case that both block matrices are even positive Hermitian the set of solutions can be described via an appropriate linear fractional transformation which is constructed via a clever coupling of the two above mentioned positive Hermitian block matrices.

According to the matrix moment problem studied in this paper we will again meet the situation that there are solutions if and only if two block Hankel matrix built from the data are nonnegative Hermitian. Each of these block Hankel matrices satisfies a certain Ljapunov type identity (see Proposition 2.1). An essential point in the paper is to find an effective algebraic coupling between both block Hankel matrices. The desired coupling will be realized in Proposition 2.2.

A first main result (see Theorem 1.2) indicates that (after Stieltjes transform) the original matrix moment problem is equivalent to a system of two fundamental matrix inequalities (FMI) of Potapov type. Our proof of this uses the theory of the matricial Nevanlinna class of holomorphic functions in the upper half-plane which have a nonnegative Hermitian imaginary part. (Essential statements on this class of matrix-valued functions are summarized in an appendix.) In particular, the generalized inversion formula of Stieltjes-Perron type stated in Theorem 8.6 occupies a key position in our strategy.

Assuming positive Hermitian block Pick matrices we will parametrize the set of all solutions of the system of FMI's of Potapov type. (It should be mentioned that these block Pick matrices are called information blocks by V.P. Potapov and his associates.) In the first step, we will treat the two inequalities of the system by the factorization method of V.P. Potapov. The main difficulty is hidden in the second step. One has to find a suitable coupling between the solutions of the two single inequalities (see Proposition 6.10). Hereby, we will essentially use the algebraic coupling identity obtained in Proposition 2.2. In Section 7, we will characterize the case that the considered truncated matrix moment problem on a finite closed interval has a solution (see Theorem 1.3).

1. The moment problem

Throughout this paper, let p, q, and r be positive integers. We will use $\mathbb{C}, \mathbb{R}, \mathbb{N}_0$, and $\mathbb{N}$ to denote the set of all complex numbers, the set of all real numbers, the set of all nonnegative integers, and the set of all positive integers, respectively. For every nonnegative integers m and n, let $\mathbb{N}_{m,n}$ designate the set of all integers k which satisfy $m \leq k \leq n$. The notation $\mathbb{C}^{p\times q}$ stands for the set of all complex $p \times q$ matrices. If $A \in \mathbb{C}^{q\times q}$, then let $\operatorname{Re} A$ and $\operatorname{Im} A$ be the real part of A and the imaginary part of A, respectively: $\operatorname{Re} A := \frac{1}{2}(A + A^*)$ and $\operatorname{Im} A := \frac{1}{2i}(A - A^*)$. For all $A \in \mathbb{C}^{p\times q}$, we will use A^+ to denote the Moore-Penrose inverse of A. Further, for each $A \in \mathbb{C}^{p\times q}$, let $\|A\|_E$ (respectively, $\|A\|$) be the Euclidean norm (respectively, operator norm) of A. The notation $\mathbb{C}_H^{q\times q}$ stands for the set of all Hermitian complex $q \times q$ matrices. If A and B are complex $q \times q$ matrices and if we write $A \geq B$ or $B \leq A$, then we mean that A and B are Hermitian complex matrices for which the matrix $A - B$ is nonnegative Hermitian. Further, let $\Pi_+ := \{w \in \mathbb{C} : \operatorname{Im} w \in (0, +\infty)\}$, let $\Pi_- := \{w \in \mathbb{C} : \operatorname{Im} w \in (-\infty, 0)\}$, and we will write $\mathfrak{B}$ for the Borel σ-algebra on $\mathbb{R}$ (respectively, $\tilde{\mathfrak{B}}$ for the Borel σ-algebra on $\mathbb{C}$). The Borel σ-algebra on $\mathbb{C}^{p\times q}$ will be denoted by $\tilde{\mathfrak{B}}_{p\times q}$. If $\mathcal{X}$ and $\mathcal{Y}$ are nonemtpy sets, if $\mathcal{Z}$ is a nonempty subset of $\mathcal{X}$, and if $f : \mathcal{X} \to \mathcal{Y}$ is a mapping, then $\operatorname{Rstr.}_{\mathcal{Z}} f$ stands for the restriction of f onto $\mathcal{Z}$. Further, if $\mathcal{Z}$ is a nonempty subset of $\mathbb{C}$ and if f is a matrix-valued function defined on $\mathcal{Z}$, then for each $z \in \mathcal{Z}$ the notation $f^*(z)$ is short for $(f(z))^*$.

The matricial generalization of M.G. Krein's classical moment problem considered in this paper is formulated using the notion of nonnegative Hermitian $q \times q$ measure. Let Λ be a nonempty set and let $\mathfrak{A}$ be a σ-algebra on Λ. A matrix-valued function μ whose domain is the σ-algebra $\mathfrak{A}$ and whose values belong to the set $\mathbb{C}_{\geq}^{q\times q}$ of all nonnegative Hermitian complex matrices is called nonnegative Hermitian $q \times q$ measure on $(\Lambda, \mathfrak{A})$ if it is countably additive, i.e., if μ satisfies

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$$

for each infinite sequence $(A_j)_{j=1}^{\infty}$ of pairwise disjoint sets that belong to $\mathfrak{A}$. We will use $\mathcal{M}_{\geq}^q(\Lambda, \mathfrak{A})$ to denote the set of all nonnegative Hermitian $q \times q$ measures on $(\Lambda, \mathfrak{A})$. Let $\mu = (\mu_{jk})_{j,k=1}^q$ belong to $\mathcal{M}_{\geq}^q(\Lambda, \mathfrak{A})$. Then every entry function μ_{jk} of μ is a complex-valued measure on $(\Lambda, \mathfrak{A})$. For each complex-valued function f defined on Λ which is, for all $j \in \mathbb{N}_{1,q}$ and all $k \in \mathbb{N}_{1,q}$, integrable with respect to the variation $|\mu_{jk}|$ of μ_{jk}, the integral

$$\int_\Lambda f d\mu := \left(\int_\Lambda f d\mu_{jk}\right)_{j,k=1}^q \tag{1.1}$$

is defined. We will also write $\int_\Lambda f(\lambda)\mu(d\lambda)$ for this integral.

Now let us formulate the matricial version of M.G. Krein's moment problem.

Let a and b be real numbers with $a < b$, let l be a nonnegative integer, and let $(s_j)_{j=0}^l$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{M}_\geq^q\left[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^l\right]$ of all nonnegative Hermitian $q\times q$ measures σ which are defined on the Borel σ-algebra $\mathfrak{B}\cap[a,b]$ on the interval $[a,b]$ and which satisfy

$$\int_{[a,b]} t^j\sigma(dt) = s_j$$

for each integer j with $0 \leq j \leq l$.

In this paper, we turn our attention to the case of an even number of given moments, i.e., to the situation that $l = 2n+1$ holds with some nonnegative integer n. (The case of an odd number of given moments will be discussed somewhere else.) According to the idea which was used by M.G. Krein and A.A. Nudelman in the scalar case $q = 1$ (see [KN, IV, §7]), by Stieltjes transformation we will translate the moment problem into the language of the class $\mathcal{R}_q[a,b]$ of matrix-valued functions $S : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{q\times q}$ which satisfy the following four conditions:

(i) S is holomorphic in $\mathbb{C}\setminus[a,b]$.
(ii) For each $w \in \Pi_+$, the matrix $\operatorname{Im} S(w)$ is nonnegative Hermitian.
(iii) For each $t \in (-\infty,a)$, the matrix $S(t)$ is nonnegative Hermitian.
(iv) For each $t \in (b,+\infty)$, the matrix $-S(t)$ is nonnegative Hermitian.

Let us observe that, according to the investigations of M.G. Krein and A.A. Nudelman, one can show that the class $\tilde{\mathcal{R}}_q[a,b]$ of all matrix-valued functions $S : \Pi_+ \cup (\mathbb{R}\setminus[a,b]) \to \mathbb{C}^{q\times q}$ which satisfy (ii), (iii), (iv), and

(i$'$) : S is holomorphic in Π_+ and continuous in $H := \Pi_+ \cup (\mathbb{R}\setminus[a,b])$.

admits the representation $\tilde{\mathcal{R}}_q[a,b] = \{\mathrm{Rstr.}_H S : S \in \mathcal{R}_q[a,b]\}$.

The following theorem describes the interrelation between the set

$$\mathcal{M}_\geq^q([a,b],\mathfrak{B}\cap[a,b])$$

of all nonnegative Hermitian $q\times q$ measures defined on $\mathfrak{B}\cap[a,b]$ and the set $\mathcal{R}_q[a,b]$.

Theorem 1.1.

(a) *For each $\sigma \in \mathcal{M}_\geq^q([a,b],\mathfrak{B}\cap[a,b])$, the matrix-valued function $S^{[\sigma]} : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{q\times q}$ defined by*

$$S^{[\sigma]}(z) := \int_{[a,b]} \frac{1}{t-z}\sigma(dt) \tag{1.2}$$

belongs to $\mathcal{R}_q[a,b]$.

(b) *For each $S \in \mathcal{R}_q[a,b]$, there exists a unique nonnegative Hermitian measure $\sigma \in \mathcal{M}_{\geq}^{q}([a,b], \mathfrak{B} \cap [a,b])$ such that*

$$S(z) = \int_{[a,b]} \frac{1}{t-z} \sigma(dt) \tag{1.3}$$

is satisfied for all $z \in \mathbb{C} \setminus [a,b]$.

Theorem 1.1 can be proved by modifying the proof in the case $q = 1$. This scalar case is considered in [KN, Appendix, Ch. 3]. According to Theorem 1.1, the mapping $f : \mathcal{M}_{\geq}^{q}([a,b], \mathfrak{B} \cap [a,b]) \to \mathcal{R}_q[a,b]$ given by $f(\sigma) := S^{[\sigma]}$ is bijective. For every nonnegative Hermitian measure $\sigma \in \mathcal{M}_{\geq}^{q}([a,b], \mathfrak{B} \cap [a,b])$, the matrix-valued function $S^{[\sigma]} : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q \times q}$ defined by (1.2) is called the *Stieltjes transform of* σ. Conversely, if a matrix-valued function $S \in \mathcal{R}_q[a,b]$ is given, then the unique $\sigma \in \mathcal{M}_{\geq}^{q}([a,b], \mathfrak{B} \cap [a,b])$ which satisfies (1.3) for all $z \in \mathbb{C} \setminus [a,b]$ is said to be the *Stieltjes measure of* S.

With these notations the matricial version of M.G. Krein's moment problem can be reformulated:

Let a and b be real numbers with $a < b$, let l be a nonnegative integer, and let $(s_j)_{j=0}^{l}$ be a sequence of complex $q \times q$ matrices. Describe then the set $\mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{l}\right]$ of the Stieltjes transforms of all nonnegative Hermitian measures which belong to $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_j)_{j=0}^{l}\right]$.

The consideration of this reformulated version of the moment problem has the advantage that one can apply function-theoretic methods. Because of

$$\mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{l}\right] \subseteq \mathcal{R}_q[a,b]$$

it is an interpolation problem in the class $\mathcal{R}_q[a,b]$. Note that

$$\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_j)_{j=0}^{l}\right] \neq \emptyset$$

if and only if $\mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{l}\right] \neq \emptyset$. As already mentioned in this paper we will consider the case that an even number of moments is given. We will show that, for every nonnegative integer n, the set $\mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$ can be characterized as the set of solutions of an appropriately constructed system of two fundamental matrix inequalities of Potapov-type. To state this result we give some further notation. We will use I_q to designate the identity matrix which belongs to $\mathbb{C}^{q \times q}$. The notation $0_{p \times q}$ stands for the null matrix which belongs to $\mathbb{C}^{p \times q}$. If the size of an identity matrix or a null matrix is obvious, we will omit the indexes. For all $j \in \mathbb{N}_0$ and all $k \in \mathbb{N}_0$, let δ_{jk} be the Kronecker symbol, i.e., let $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$. For each $n \in \mathbb{N}_0$, let

$$T_n := (\delta_{j,k+1} I_q)_{j,k=0}^{n} \tag{1.4}$$

and let $R_{T_n} : \mathbb{C} \to \mathbb{C}^{(n+1)q \times (n+1)q}$ be defined by

$$R_{T_n}(z) := (I - zT_n)^{-1}. \tag{1.5}$$

Observe that, for each $n \in \mathbb{N}_0$, the matrix-valued function R_{T_n} can be represented via

$$R_{T_n}(z) = \begin{pmatrix} I_q & 0 & 0 & \dots & 0 & 0 \\ zI_q & I_q & 0 & \dots & 0 & 0 \\ z^2 I_q & zI & I_q & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z^n I_q & z^{n-1} I_q & z^{n-2} I_q & \dots & zI_q & I_q \end{pmatrix} \tag{1.6}$$

for each $z \in \mathbb{C}$. Let $v_0 := I_q$ and, for each $n \in \mathbb{N}$, let

$$v_n := \begin{pmatrix} I_q \\ 0_{nq \times q} \end{pmatrix}. \tag{1.7}$$

For each $n \in \mathbb{N}_0$ and each sequence $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices, we will call

$$\tilde{H}_{1,n} := \begin{pmatrix} s_0 & s_1 & s_2 & \dots & s_n \\ s_1 & s_2 & s_3 & \dots & s_{n+1} \\ s_2 & s_3 & s_4 & \dots & s_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n} \end{pmatrix},$$

$$\left(\text{respectively,} \quad \tilde{H}_{2,n} := \begin{pmatrix} s_1 & s_2 & s_3 & \dots & s_{n+1} \\ s_2 & s_3 & s_4 & \dots & s_{n+2} \\ s_3 & s_4 & s_5 & \dots & s_{n+3} \\ \vdots & \vdots & \vdots & & \vdots \\ s_{n+1} & s_{n+2} & s_{n+3} & \dots & s_{2n+1} \end{pmatrix} \right)$$

the *first* (respectively, *second*) *block Hankel matrix associated with* $(s_j)_{j=0}^{2n+1}$. Moreover, for all real numbers a and b which satisfy $a < b$, for each nonnegative integer n, and for each sequence $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices, we will call

$$H_{1,n} := -a\tilde{H}_{1,n} + \tilde{H}_{2,n} \quad (\text{respectively, } H_{2,n} := b\tilde{H}_{1,n} - \tilde{H}_{2,n}) \tag{1.8}$$

the *first* (respectively, *second*) *block Hankel matrix associated with the interval* $[a, b]$ *and the sequence* $(s_j)_{j=0}^{2n+1}$. For each $n \in \mathbb{N}_0$ and each sequence $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices, one can easily see that the matrices

$$\tilde{u}_n := -\begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{pmatrix}, \quad u_{1,n} := \tilde{u}_n - aT_n\tilde{u}_n, \text{ and } \quad u_{2,n} := -\tilde{u}_n + bT_n\tilde{u}_n \tag{1.9}$$

satisfy the identities $\tilde{u}_n = -\tilde{H}_{1,n}v_n$, $u_{1,n} = [R_{T_n}(a)]^{-1}\tilde{u}_n$, $u_{2,n} = -[R_{T_n}(b)]^{-1}\tilde{u}_n$, and $R_{T_n}(a)u_{1,n} = -R_{T_n}(b)u_{2,n}$.

Now we formulate the first main result of this paper.

Theorem 1.2. *Let a and b be real numbers which satisfy $a < b$, let n be a nonnegative integer, and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices. Let $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ be a $q \times q$ matrix-valued function, and let $\tilde{S}_1 : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ and $\tilde{S}_2 : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ be defined by*

$$\tilde{S}_1(z) := (z-a)S(z) \quad \text{and} \quad \tilde{S}_2(z) := (b-z)S(z). \tag{1.10}$$

Then S belongs to $\mathcal{R}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ if and only if the following conditions are satisfied:

(i) *S is holomorphic in $\mathbb{C} \setminus [a,b]$.*
(ii) *For each $z \in \mathbb{C} \setminus \mathbb{R}$, the matrices*

$$K_{1,n}^{[S]}(z) := \begin{pmatrix} H_{1,n} & R_{T_n}(z)(v_n\tilde{S}_1(z) - u_{1,n}) \\ \left(R_{T_n}(z)(v_n\tilde{S}_1(z) - u_{1,n})\right)^* & \frac{\tilde{S}_1(z)-\tilde{S}_1^*(z)}{z-\overline{z}} \end{pmatrix} \tag{1.11}$$

and

$$K_{2,n}^{[S]}(z) := \begin{pmatrix} H_{2,n} & R_{T_n}(z)(v_n\tilde{S}_2(z) - u_{2,n}) \\ \left(R_{T_n}(z)(v_n\tilde{S}_2(z) - u_{2,n})\right)^* & \frac{\tilde{S}_2(z)-\tilde{S}_2^*(z)}{z-\overline{z}} \end{pmatrix} \tag{1.12}$$

are both nonnegative Hermitian.

We will use Theorem 1.2 in order to describe the case that $\mathcal{M}_{\geq}^q\left[[a,b];(s_j)_{j=0}^{2n+1}\right] \neq \emptyset$. More precisely, in Section 7, we will prove the following result which in the scalar case $q = 1$ is due to M.G. Krein (see [Kr2, Theorem A_{2r+1}, p. 30], [KN, Theorem 2.1, p. 91]).

Theorem 1.3. *Let a and b be real numbers with $a < b$, let n be a nonnegative integer, and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices. Then $\mathcal{M}_{\geq}^q\left[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n+1}\right]$ is nonempty if and only if the block Hankel matrices $H_{1,n}$ and $H_{2,n}$ are both nonnegative Hermitian.*

Let the assumptions of Theorem 1.2 be satisfied. Then we will say that the matrix-valued function $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ is a *solution of the system of the fundamental matrix inequalities of Potapov-type associated with* $[a,b]$ *and* $(s_j)_{j=0}^{2n+1}$ if S is holomorphic in $\mathbb{C} \setminus [a,b]$ and if the matrix inequalities $K_{1,n}^{[S]}(z) \geq 0$ and $K_{2,n}^{[S]}(z) \geq 0$ are satisfied for every choice of z in $\mathbb{C} \setminus \mathbb{R}$. Further, if a complex $q \times q$ matrix-valued function S defined on $\mathbb{C} \setminus [a,b]$ is given, we will then continue to use the notations $\tilde{S}_1$ and $\tilde{S}_2$ to denote the matrix-valued functions which are also defined on $\mathbb{C} \setminus [a,b]$ and which are given by (1.10). We call $\tilde{S}_1$ (respectively, $\tilde{S}_2$) the *first* (respectively, *second*) *matrix-valued function associated canonically with* S. Note that M.G. Krein and A.A. Nudelman [KN, Appendix, Ch. 3] stated that, in the case $q = 1$, the functions $\tilde{S}_1$ and $\tilde{S}_2$ can be used to characterize the class $\mathcal{R}_1[a,b]$. This result will be proved below for the class $\mathcal{R}_q[a,b]$ (see Lemma 3.6).

At the end of this section, let us note that in an appendix (Section 8) we will summarize some results on the class $\mathcal{R}_q$ of all matrix-valued functions $F : \Pi_+ \to \mathbb{C}^{q\times q}$ which are holomorphic in Π_+ and which satisfy $\operatorname{Im} F(w) \geq 0$ for all $w \in \Pi_+$. Every function F which belongs to $\mathcal{R}_q$ admits a unique integral representation which in the scalar case is due to R. Nevanlinna (see Theorem 8.1). In view of this integral representation, the subclasses $\mathcal{R}'_q$ and $\mathcal{R}_{0,q}$ of $\mathcal{R}_q$ are of particular interest (see Section 8).

2. Main algebraic identities

In this section we will single out essential identities connecting the block matrices introduced in Section 1 (see formulas (1.4)–(1.9)).

Observe that if $n \in \mathbb{N}_0$ and if $(s_j)_{j=0}^{2n+1}$ is a sequence of complex $q \times q$ matrices such that $H_{1,n} \geq 0$ and $H_{2,n} \geq 0$, then the equation

$$\tilde{H}_{1,n} = \frac{1}{b-a}(H_{1,n} + H_{2,n}) \tag{2.1}$$

shows that $\tilde{H}_{1,n}$ is nonnegative Hermitian as well. Moreover, from $H_{1,n}^* = H_{1,n}$, $H_{2,n}^* = H_{2,n}$, and $\tilde{H}_{1,n}^* = \tilde{H}_{1,n}$ it follows $\tilde{H}_{2,n}^* = \tilde{H}_{2,n}$ and $s_j^* = s_j$ for each $j \in \mathbb{N}_{0,2n+1}$.

Proposition 2.1. (Ljapunov type identities) *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence of Hermitian complex $q \times q$ matrices. For each $k \in \{1,2\}$, then*

$$H_{k,n}T_n^* - T_n H_{k,n} = u_{k,n}v_n^* - v_n u_{k,n}^*. \tag{2.2}$$

Proof. Since $H_{1,n}$ and $H_{2,n}$ are Hermitian block Hankel matrices which satisfy

$$T_n H_{1,n} v_n = -u_{1,n} - v_n s_0 \quad \text{and} \quad T_n H_{2,n} v_n = -u_{2,n} + v_n s_0$$

equation (2.2) follows by a straightforward calculation. □

Now we state an essential coupling formula between the block Hankel matrices $H_{1,n}$ and $H_{2,n}$.

Proposition 2.2. (Coupling Identity) *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of Hermitian complex $q \times q$ matrices. Then*

$$H_{2,n} + [R_{T_n}(b)]^{-1} R_{T_n}(a) H_{1,n} = (a-b) R_{T_n}(a) v_n u_{1,n}^* [R_{T_n}(a)]^*. \tag{2.3}$$

Proof. For every choice of w and ζ in $\mathbb{C}$, the identities $[R_{T_n}(w)]^{-1} = I - wT_n$ and

$$(I - \zeta T_n) R_{T_n}(w) = R_{T_n}(w) \cdot (I - \zeta T_n) \tag{2.4}$$

hold obviously. Therefore we can conclude

$$H_{2,n} + [R_{T_n}(b)]^{-1} R_{T_n}(a) H_{1,n} = R_{T_n}(a) \left[(I - aT_n) H_{2,n} + (I - bT_n) H_{1,n} \right]. \tag{2.5}$$

From (1.8) we obtain

$$\begin{aligned}
&(I - aT_n)H_{2,n} + (I - bT_n)H_{1,n} = (b-a)(\tilde{H}_{1,n} - T_n\tilde{H}_{2,n}) \\
&= (b-a)(\tilde{H}_{1,n} - T_n\tilde{H}_{2,n})(I - aT_n^*)[R_{T_n}(a)]^* \\
&= (b-a)\left[\tilde{H}_{1,n} - a\tilde{H}_{1,n}T_n^* - T_n\tilde{H}_{2,n} + aT_n\tilde{H}_{2,n}T_n^*\right][R_{T_n}(a)]^*. \quad (2.6)
\end{aligned}$$

Since $s_j = s_j^*$ holds for each integer j with $0 \le j \le 2n+1$, a straightforward calculation yields

$$\tilde{H}_{1,n} - a\tilde{H}_{1,n}T_n^* - T_n\tilde{H}_{2,n} + aT_n\tilde{H}_{2,n}T_n^* = -v_n u_{1,n}^*. \quad (2.7)$$

From (2.5), (2.6), and (2.7) we get finally (2.3). □

In the paper [DC] the first two authors studied the problem of Nevanlinna-Pick interpolation in the class $\mathcal{R}_q[a,b]$ by using V.P. Potapov's method. A closer look at the paper [DC] shows that there are direct analogues of Propositions 2.1 and 2.2, respectively. More precisely, Proposition 2.1 corresponds to an unnumbered formula at the top of p. 1271 in [DC]. Moreover, formula (11) at p. 1271 in [DC] is the direct analogue of Proposition 2.2. It should be observed that in the context of interpolation problems in the Stieltjes class a similar situation already occurred. What concerns analogues of Proposition 2.1 we refer to formulas (4) and (5) in [Dy2] and formula (2.8) in [Dy3], whereas coupling identities of fundamental importance are stated in equation formula (1) of [Dy2] and formula (2.1) of [Dy3].

3. From the moment problem to the system of fundamental matrix inequalities of Potapov-type

In this section, we will show that every matrix-valued function which belongs to $\mathcal{R}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ is a solution of the system of the fundamental matrix inequalities associated with $[a,b]$ and $(s_j)_{j=0}^{2n+1}$. First we recall some results of the integration theory of nonnegative Hermitian measures (for details, see [Kt] and [R]).

Let $(\Lambda,\mathfrak{A})$ be a measurable space. For each subset A of Λ, we will write 1_A for the indicator function of the set A (defined on Λ). If ν is a nonnegative real-valued measure on $(\Lambda,\mathfrak{A})$, then let $p\times q - \mathcal{L}^1(\Lambda,\mathfrak{A},\nu;\mathbb{C})$ denote the class of all $\mathfrak{A}-\mathfrak{B}_{p\times q}$-measurable complex $p\times q$ matrix-valued functions $\Phi = (\varphi_{jk})_{\substack{j=1,\dots,p\\k=1,\dots,q}}$ defined on Λ for which every entry function φ_{jk} is integrable with respect to ν.

Now let $\mu \in \mathcal{M}_{\ge}^q(\Lambda,\mathfrak{A})$. Then every entry function μ_{jk} of $\mu = (\mu_{jk})_{j,k=1}^n$ is a complex-valued measure on $(\Lambda,\mathfrak{A})$. In particular, $\mu_{11},\mu_{22},\dots,\mu_{qq}$ are finite nonnegative real-valued measures on $(\Lambda,\mathfrak{A})$. Moreover, μ is absolutely continuous with respect to the so-called trace measure $\tau := \sum_{j=1}^q \mu_{jj}$ of μ, i.e., for each

$A \in \mathfrak{A}$ which satisfies $\tau(A) = 0$ it follows $\mu(A) = 0_{q\times q}$. The corresponding Radon-Nikodym derivatives $\frac{d\mu_{jk}}{d\tau}$ are thus well defined up to sets of zero τ-measure. Setting $\mu'_\tau := \left(\frac{d\mu_{jk}}{d\tau}\right)_{j,k=1}^q$ we have then

$$\mu(A) = \left(\int_A \frac{d\mu_{jk}}{d\tau} d\tau\right)_{j,k=1}^q = \int_A \mu'_\tau d\tau$$

for each $A \in \mathfrak{A}$. An ordered pair $[\Phi, \Psi]$ consisting of an $\mathfrak{A} - \tilde{\mathfrak{B}}_{p\times q}$-measurable complex $p \times q$ matrix-valued function $\Phi = (\varphi_{jk})_{\substack{j=1,\dots,p\\k=1,\dots,q}}$ defined on Λ and an $\mathfrak{A} - \tilde{\mathfrak{B}}_{p\times q}$-measurable complex $r \times q$ matrix-valued function $\Psi = (\psi_{lk})_{\substack{l=1,\dots,r\\k=1,\dots,q}}$ defined on Λ is said to be left-integrable with respect to μ if $\Phi\mu'_\tau\Psi^*$ belongs to $p\times q - \mathcal{L}^1(\Lambda, \mathfrak{A}, \mu; \mathbb{C})$. In this case, for each $A \in \mathfrak{A}$, the ordered pair $[1_A\Phi, 1_A\Psi]$ is also left-integrable with respect to μ and the integral of $[\Phi, \Psi]$ over A is defined by

$$\int_A \Phi d\mu\Psi^* := \int_\Lambda (1_A\Phi)\mu'_\tau(1_A\Psi)^* d\tau.$$

We will also write $\int_A \Phi(\lambda)\mu(d\lambda)\Psi^*(\lambda)$ for this integral. Let us consider an arbitrary σ-finite nonnegative real-valued measure ν on $(\Lambda, \mathfrak{A})$ such that μ is absolutely continuous with respect to ν and let $\mu'_\nu := \left(\frac{d\mu_{jk}}{d\nu}\right)_{j,k=1}^q$ be a version of the matrix-valued function of the corresponding Radon-Nikodym derivatives. For each ordered pair $[\Phi, \Psi]$ of an $\mathfrak{A} - \tilde{\mathfrak{B}}_{p\times q}$-measurable matrix-valued function $\Phi : \Lambda \to \mathbb{C}^{p\times q}$ and an $\mathfrak{A} - \tilde{\mathfrak{B}}_{r\times q}$-measurable matrix-valued function $\Psi : \Lambda \to \mathbb{C}^{r\times q}$ which is left-integrable with respect to μ and each $A \in \mathfrak{A}$, then

$$\int_A \Phi d\mu\Psi^* = \int_A (1_A\Phi)\mu'_\nu(1_A\Psi)^* d\nu$$

holds. We will use $p\times q - \mathcal{L}^2(\Lambda, \mathfrak{A}, \mu)$ to denote the set of all $\mathfrak{A} - \tilde{\mathfrak{B}}_{p\times q}$-measurable mappings $\Phi : \Lambda \to \mathbb{C}^{p\times q}$ for which the pair $[\Phi, \Phi]$ is left-integrable with respect to μ. Note that if $\Phi \in p \times q - \mathcal{L}^2(\Lambda, \mathfrak{A}, \mu)$ and if $\Psi \in r \times q - \mathcal{L}^2(\Lambda, \mathfrak{A}, \mu)$, then the pair $[\Phi, \Psi]$ is left-integrable with respect to μ. If $\Phi : \Lambda \to \mathbb{C}^{p\times q}$ is an $\mathfrak{A} - \tilde{\mathfrak{B}}_{p\times q}$-measurable mapping for which a set $N \in \mathfrak{A}$ with $\mu(N) = 0$ and a nonnegative real number C exist such that $\|\Phi(\lambda)\| \le C$ holds for each $\lambda \in \Lambda \setminus N$, then Φ belongs to $p\times q - \mathcal{L}^2(\Lambda, \mathfrak{A}, \mu)$. For all complex-valued functions f and g which are defined on Λ and for which the function $h := f\overline{g}$ is integrable with respect to $|\mu_{jk}|$ for every choice of j and k in $\mathbb{N}_{1,q}$, the pair $[fI_q, gI_q]$ is left-integrable with respect to μ and, in view of (1.1),

$$\int_A (fI_q)d\mu(gI_q)^* = \int_A (f\overline{g})d\mu$$

holds for all $A \in \mathfrak{A}$.

Remark 3.1. *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Lambda, \mathfrak{A})$ and let $\Phi \in p \times q - \mathcal{L}^2(\Lambda, \mathfrak{A}, \mu)$. Then $\mu_{[\Phi]} : \mathfrak{A} \to \mathbb{C}^{p\times p}$ given by*

$$\mu_{[\Phi]}(A) := \int_A \Phi d\mu \Phi^*$$

belongs to $\mathcal{M}_{\geq}^{p}(\Lambda, \mathfrak{A})$. If $\Psi : \Lambda \to \mathbb{C}^{r\times p}$ is $\mathfrak{A} - \tilde{\mathfrak{B}}_{r\times p}$-measurable and if $\Theta : \Lambda \to \mathbb{C}^{t\times p}$ is $\mathfrak{A} - \tilde{\mathfrak{B}}_{t\times p}$-measurable, then $[\Psi, \Theta]$ is left-integrable with respect to $\mu_{[\Phi]}$ if and only if $[\Psi\Phi, \Theta\Phi]$ is left-integrable with respect to μ. Moreover, in this case,

$$\int_\Lambda \Psi d\mu_{[\Phi]} \Theta^* = \int_\Lambda \Psi\Phi d\mu (\Theta\Phi)^*.$$

Remark 3.2. *Let $\mu \in \mathcal{M}_{\geq}^{q}(\Lambda, \mathfrak{A})$ and let $\mathfrak{C} := \{\lambda \in \Lambda : \{\lambda\} \in \mathfrak{A}\}$. Then one can easily see that $\mathfrak{C}_\mu := \{\lambda \in \mathfrak{C} : \mu(\{\lambda\}) \neq 0\}$ is a countable subset of Λ.*

Throughout this paper, we assume now that a and b are real numbers which satisfy $a < b$. Let us turn our attention to nonnegative Hermitian $q \times q$ measures on the Borel σ-algebra $\mathfrak{B} \cap [a, b]$ on the closed finite interval $[a, b]$. For each $\sigma \in \mathcal{M}_{\geq}^{q}([a, b], \ \mathfrak{B} \cap [a, b])$ and each $j \in \mathbb{N}_0$, let

$$s_j^{[\sigma]} := \int_{[a,b]} t^j \sigma(dt). \tag{3.1}$$

Further, for all $\sigma \in \mathcal{M}_{\geq}^{q}([a, b], \mathfrak{B} \cap [a, b])$ and all $m \in \mathbb{N}_0$, let $\tilde{H}_{1,m}^{[\sigma]}$ (respectively, $\tilde{H}_{2,m}^{[\sigma]}$) denote the first (respectively, second) block Hankel matrix associated with $(s_j^{[\sigma]})_{j=0}^{2m+1}$ and let $H_{1,m}^{[\sigma]}$ (respectively, $H_{2,m}^{[\sigma]}$) be the first (respectively, second) block Hankel matrix associated with the interval $[a, b]$ and the sequence $(s_j^{[\sigma]})_{j=0}^{2m+1}$, i.e., the matrices $\tilde{H}_{1,m}^{[\sigma]}, \tilde{H}_{2,m}^{[\sigma]}, H_{1,m}^{[\sigma]}$, and $H_{2,m}^{[\sigma]}$ are given by

$$\tilde{H}_{1,m}^{[\sigma]} := (s_{j+k}^{[\sigma]})_{j,k=0}^{m}, \quad \tilde{H}_{2,m}^{[\sigma]} := (s_{j+k+1}^{[\sigma]})_{j,k=0}^{m}, \tag{3.2}$$

$$H_{1,m}^{[\sigma]} := -a\tilde{H}_{1,m}^{[\sigma]} + \tilde{H}_{2,m}^{[\sigma]}, \quad \text{and} \quad H_{2,m}^{[\sigma]} := b\tilde{H}_{1,m}^{[\sigma]} - \tilde{H}_{2,m}^{[\sigma]}. \tag{3.3}$$

For each $m \in \mathbb{N}_0$, let the $(m + 1)q \times q$ matrix polynomial E_m be defined by

$$E_m(z) := \begin{pmatrix} I_q \\ zI_q \\ z^2 I_q \\ \dots \\ z^m I_q \end{pmatrix}. \tag{3.4}$$

Obviously, $E_m(0) = v_m$ for each $m \in \mathbb{N}_0$. Further, for each $m \in \mathbb{N}_0$ and each $z \in \mathbb{C}$, from (1.6) and (1.7) it follows immediately

$$E_m(z) = R_{T_m}(z)v_m. \tag{3.5}$$

Now we state important integral representations for the block Hankel matrices introduced in (3.2) and (3.3).

Lemma 3.3. *Let $\sigma \in \mathcal{M}_{\geq}^{q}([a,b], \mathfrak{B} \cap [a,b])$. For each $m \in \mathbb{N}_0$, then*

$$\int_{[a,b]} E_m(t)\sigma(dt)E_m^*(t) = \tilde{H}_{1,m}^{[\sigma]}, \qquad \int_{[a,b]} tE_m(t)\sigma(dt)E_m^*(t) = \tilde{H}_{2,m}^{[\sigma]},$$

$$\int_{[a,b]} \sqrt{t-a}E_m(t)\sigma(dt)\left(\sqrt{t-a}E_m(t)\right)^* = H_{1,m}^{[\sigma]},$$

and

$$\int_{[a,b]} \sqrt{b-t}E_m(t)\sigma(dt)\left(\sqrt{b-t}E_m(t)\right)^* = H_{2,m}^{[\sigma]}.$$

In particular, for each $m \in \mathbb{N}_0$, the matrices $\tilde{H}_{1,m}^{[\sigma]}$, $H_{1,m}^{[\sigma]}$, and $H_{2,m}^{[\sigma]}$ are nonnegative Hermitian, and the matrix $\tilde{H}_{2,m}^{[\sigma]}$ is Hermitian.

Lemma 3.3 can be proved by straightforward calculation. We omit the details.

From Lemma 3.3 we get immediately a necessary condition for the existence of a solution of the matricial version of M.G. Krein's moment problem.

Remark 3.4. *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_j)_{j=0}^{2n+1}\right] \neq \emptyset$. From Lemma 3.3 one can easily see then that all the matrices $\tilde{H}_{1,n}$, $H_{1,n}$, and $H_{2,n}$ are nonnegative Hermitian and that the matrix $\tilde{H}_{2,n}$ is Hermitian. In particular, $s_j^* = s_j$ for all $j \in \mathbb{N}_{0,2n+1}$.*

Lemma 3.5. *Let $S \in \mathcal{R}_q[a,b]$, and let σ be the Stieltjes measure of S.*

(a) *For $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\frac{\tilde{S}_1(z) - \tilde{S}_1^*(z)}{z - \overline{z}} = \int_{[a,b]} \left(\frac{\sqrt{t-a}}{\overline{t-z}} I_q\right) \sigma(dt) \left(\frac{\sqrt{t-a}}{\overline{t-z}} I_q\right)^* \tag{3.6}$$

and

$$\frac{\tilde{S}_2(z) - \tilde{S}_2^*(z)}{z - \overline{z}} = \int_{[a,b]} \left(\frac{\sqrt{b-t}}{\overline{t-z}} I_q\right) \sigma(dt) \left(\frac{\sqrt{b-t}}{\overline{t-z}} I_q\right)^*. \tag{3.7}$$

(b) *The matrix-valued functions $\tilde{S}_1$ and $\tilde{S}_2$ are both holomorphic in $\mathbb{C} \setminus [a,b]$ and, for each $w \in \Pi_+$, the matrices $\operatorname{Im} \tilde{S}_1(w)$ and $\operatorname{Im} \tilde{S}_2(w)$ are both nonnegative Hermitian.*

Proof. (a) Let $z \in \mathbb{C} \setminus \mathbb{R}$. In view of (1.3) and (1.10) we obtain

$$\frac{\tilde{S}_1(z) - \tilde{S}_1^*(z)}{z - \overline{z}} = \frac{1}{z - \overline{z}} \int_{[a,b]} \left(\frac{z-a}{t-z} - \frac{\overline{z}-a}{t-\overline{z}}\right) \sigma(dt).$$

Since

$$\frac{z-a}{t-z} - \frac{\overline{z}-a}{t-\overline{z}} = \frac{(z-\overline{z})(t-a)}{(t-z)(\overline{t-z})}$$

is valid for each $t \in [a,b]$, it follows (3.6). Analogously, (3.7) can be verified.

(b) Obviously, the right-hand sides of (3.6) and (3.7) are both nonnegative Hermitian for each $z \in \mathbb{C} \setminus \mathbb{R}$. For each $k \in \{1, 2\}$ and each $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\operatorname{Im} \tilde{S}_k(z) = \frac{\tilde{S}_k(z) - \tilde{S}_k^*(z)}{z - \overline{z}} \cdot \operatorname{Im} z. \tag{3.8}$$

Thus the assertion stated in part (b) follows immediately. □

If $S : \mathbb{C} \setminus [a, b] \to \mathbb{C}^{q \times q}$ is given, then, as already mentioned in [KN, Appendix, Ch. 3] for the case $q = 1$, the first matrix-valued function $\tilde{S}_1$ and the second matrix-valued function $\tilde{S}_2$ associated canonically with S can be used to characterize the case that S belongs to the class $\mathcal{R}_q[a, b]$.

Lemma 3.6. *Let S be a complex $q \times q$ matrix-valued function defined on $\mathbb{C} \setminus [a, b]$. Then the following statements are equivalent:*

(i) *S belongs to $\mathcal{R}_q[a, b]$.*

(ii) *The matrix-valued functions $\tilde{S}_1$ and $\tilde{S}_2$ are both holomorphic in $\mathbb{C} \setminus [a, b]$ and the inequalities $\operatorname{Im} \tilde{S}_1(w) \geq 0$ and $\operatorname{Im} \tilde{S}_2(w) \geq 0$ hold for all $w \in \Pi_+$.*

Proof. Lemma 3.5 shows that (ii) is necessary for (i). Now suppose (ii). Since

$$S(z) = \frac{1}{b - a} \left(\tilde{S}_1(z) + \tilde{S}_2(z) \right)$$

is satisfied for all $z \in \mathbb{C} \setminus [a, b]$, the function S is holomorphic in $\mathbb{C} \setminus [a, b]$ and satisfies $\operatorname{Im} S(w) \geq 0$ for all $w \in \Pi_+$. Now let $t \in (-\infty, a)$. Then we get

$$\operatorname{Im} S(t) = \lim_{\varepsilon \to 0+0} \operatorname{Im} S(t + i\varepsilon) \geq 0 \tag{3.9}$$

and

$$(t - a) \operatorname{Im} S(t) = \operatorname{Im} \tilde{S}_1(t) = \lim_{\varepsilon \to 0+0} \operatorname{Im} \tilde{S}_1(t + i\varepsilon) \geq 0. \tag{3.10}$$

Since $t - a < 0$ holds, from (3.9) and (3.10) we obtain $\operatorname{Im} S(t) = 0$. Further, for each $\varepsilon \in (0, +\infty)$ we have then

$$0 \leq \operatorname{Im} \tilde{S}_1(t + i\varepsilon) = (t - a) \operatorname{Im} S(t + i\varepsilon) + \varepsilon \operatorname{Re} S(t + i\varepsilon) \leq \varepsilon \operatorname{Re} S(t + i\varepsilon)$$

and consequently

$$S(t) = \operatorname{Re} S(t) = \lim_{\varepsilon \to 0+0} \operatorname{Re} S(t + i\varepsilon) \geq 0.$$

Similarly, it follows $-S(x) \geq 0$ for all $x \in (b, +\infty)$. Hence, (i) is verified. □

Let $S : \mathbb{C} \setminus [a, b] \to \mathbb{C}^{q \times q}$ be holomorphic in $\mathbb{C} \setminus [a, b]$. Then Lemma 3.6 and (3.8) show that if the right lower $q \times q$ blocks of the matrices $K_{1,n}^{[S]}$ and $K_{2,n}^{[S]}$, given by (1.11) and (1.12), are nonnegative Hermitian for each $z \in \Pi_+$, then the function S necessarily belongs to $\mathcal{R}_q[a, b]$. Thus the inequalities $K_{1,n}^{[S]}(z) \geq 0$ and $K_{2,n}^{[S]}(z) \geq 0$ holding for each $z \in \Pi_+$ ensure that S belongs to $\mathcal{R}_q[a, b]$.

Now we are going to discuss the right upper $(n+1)q \times q$ blocks of the matrices $K_{1,n}^{[S]}(z)$ and $K_{2,n}^{[S]}(z)$. Before doing this let us observe that, in view of (1.4) and

(1.6), for each $n \in \mathbb{N}_0$ the matrix-valued function $R_{T_n} : \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by (1.5) can be represented via

$$R_{T_n}(z) = \sum_{j=0}^{n} z^j T_n^j \tag{3.11}$$

for all $z \in \mathbb{C}$ and that the identities

$$R_{T_n}(w)R_{T_n}(z) = R_{T_n}(z)R_{T_n}(w), \quad (I - wT_n)R_{T_n}(z) = R_{T_n}(z)(I - wT_n),$$

and

$$\begin{aligned} R_{T_n}(w) - R_{T_n}(z) &= R_{T_n}(z)\big((I - zT_n) - (I - wT_n)\big)R_{T_n}(w) \\ &= R_{T_n}(z)(wT_n - zT_n)R_{T_n}(w) = (w - z)R_{T_n}(z)T_nR_{T_n}(w) \end{aligned} \tag{3.12}$$

are satisfied for every choice of w and z in $\mathbb{C}$.

Lemma 3.7. *Let $n \in \mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B}\cap[a,b]; (s_j)_{j=0}^{2n+1}\right]$ is nonempty. Let $S \in \mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$ and let σ be the Stieltjes measure of S. For each $z \in \mathbb{C}\setminus\mathbb{R}$, then*

$$K_{1,n}^{[S]}(z) = \int_{[a,b]} \sqrt{t-a}\begin{pmatrix} E_n(t) \\ \frac{1}{t-z}I_q \end{pmatrix} \sigma(dt) \left[\sqrt{t-a}\begin{pmatrix} E_n(t) \\ \frac{1}{t-z}I_q \end{pmatrix}\right]^*$$

and

$$K_{2,n}^{[S]}(z) = \int_{[a,b]} \sqrt{b-t}\begin{pmatrix} E_n(t) \\ \frac{1}{t-z}I_q \end{pmatrix} \sigma(dt) \left[\sqrt{b-t}\begin{pmatrix} E_n(t) \\ \frac{1}{t-z}I_q \end{pmatrix}\right]^*.$$

Proof. From $S \in \mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$ we get $\sigma \in \mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B}\cap[a,b]; (s_j)_{j=0}^{2n+1}\right]$. In view of Lemma 3.3 and Lemma 3.5 it is sufficient to verify that

$$R_{T_n}(z)[v_n\tilde{S}_1(z) - u_{1,n}] = \int_{[a,b]} \left(\sqrt{t-a}\,E_n(t)\right)\sigma(dt)\left(\frac{\sqrt{t-a}}{\overline{t-z}}I_q\right)^* \tag{3.13}$$

and

$$R_{T_n}(z)[v_n\tilde{S}_2(z) - u_{2,n}] = \int_{[a,b]} \left(\sqrt{b-t}\,E_n(t)\right)\sigma(dt)\left(\frac{\sqrt{b-t}}{\overline{t-z}}I_q\right)^* \tag{3.14}$$

are satisfied for all $z \in \mathbb{C}\setminus\mathbb{R}$. Let $z \in \mathbb{C}\setminus\mathbb{R}$. Using (1.10) and (1.9) we can conclude

$$R_{T_n}(z)[v_n\tilde{S}_1(z) - u_{1,n}] = (z-a)R_{T_n}(z)v_nS(z) - R_{T_n}(z)(I - aT_n)u_n. \tag{3.15}$$

In view of (1.3) we have

$$S(z) = \int_{[a,b]} \left(\frac{1}{t-z}I_q\right)\sigma(dt)I_q^*. \tag{3.16}$$

From Lemma 3.3, (1.5), and (1.7) we see immediately that

$$u_n = -\tilde{H}_{1,n}v_n = -\int_{[a,b]} E_n(t)\sigma(dt)I_q^* = -\int_{[a,b]} R_{T_n}(t)v_n\sigma(dt)I_q^* \tag{3.17}$$

holds. Because of (3.15), (3.16), and (3.17) we infer then

$$
\begin{aligned}
&R_{T_n}(z)[v_n\tilde{S}_1(z)-u_{1,n}]\\
&=\int_{[a,b]}\frac{z-a}{t-z}R_{T_n}(z)v_n\sigma(dt)I_q^*+\int_{[a,b]}R_{T_n}(z)(I-aT_n)R_{T_n}(t)v_n\sigma(dt)I_q^*. \quad (3.18)
\end{aligned}
$$

For each real number t, from (3.12) we obtain

$$
R_{T_n}(z)(I-aT_n)R_{T_n}(t)=R_{T_n}(z)R_{T_n}(t)-\frac{a}{t-z}\big(R_{T_n}(t)-R_{T_n}(z)\big). \quad (3.19)
$$

Consequently, equations (3.18) and (3.19) provide us

$$
\begin{aligned}
&R_{T_n}(z)\big(v_n\tilde{S}_1(z)-u_{1,n}\big)\\
&=\int_{[a,b]}\left(\frac{1}{t-z}\left[zR_{T_n}(z)+(t-z)R_{T_n}(z)R_{T_n}(t)-aR_{T_n}(t)\right]v_n\right)\sigma(dt)I_q^*. \quad (3.20)
\end{aligned}
$$

Using (1.5), for every choice of t in $\mathbb{R}$, we obtain

$$
\begin{aligned}
&zR_{T_n}(z)+(t-z)R_{T_n}(z)R_{T_n}(t)=R_{T_n}(z)\big(z(I-tT_n)+(t-z)I\big)R_{T_n}(t)\\
&=R_{T_n}(z)t(I-zT_n)R_{T_n}(t)=tR_{T_n}(t). \quad (3.21)
\end{aligned}
$$

From (3.20) and (3.21) it follows

$$
\begin{aligned}
R_{T_n}(z)\big(v_n\tilde{S}_1(z)-u_{1,n}\big)&=\int_{[a,b]}\frac{t-a}{t-z}R_{T_n}(t)v_n\sigma(dt)I_q^*\\
&=\int_{[a,b]}\sqrt{t-a}R_{T_n}(t)v_n\sigma(dt)\left(\frac{\sqrt{t-a}}{\overline{t-z}}I_q\right)^*. \quad (3.22)
\end{aligned}
$$

The equations (3.22) and $R_{T_n}v_n=E_n$ imply (3.13). Analogously, (3.14) can be proved. □

If $n\in\mathbb{N}_0$ and if $(s_j)_{j=0}^{2n+1}$ is a sequence of complex $q\times q$ matrices, then we will use the notation $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ to denote the set of all solutions of the system of the fundamental matrix inequalities of Potapov-type associated with $[a,b]$ and $(s_j)_{j=0}^{2n+1}$, i.e., the set of all matrix-valued functions $S:\mathbb{C}\setminus[a,b]\to\mathbb{C}^{q\times q}$ which are holomorphic in $\mathbb{C}\setminus[a,b]$ and for which the matrices $K_{1,n}^{[S]}(z)$ and $K_{2,n}^{[S]}(z)$ are both nonnegative Hermitian for every choice of z in $\mathbb{C}\setminus\mathbb{R}$.

Proposition 3.8. *Let $n\in\mathbb{N}_0$ and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q\times q$ matrices. Then $\mathcal{R}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ is a subset of the set $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ of all solutions of the system of the fundamental matrix inequalities of Potapov-type associated with $[a,b]$ and $(s_j)_{j=0}^{2n+1}$.*

Proof. Apply Lemma 3.7. □

4. From the system of fundamental matrix inequalities to the moment problem

Throughout this section, we again assume that a and b are real numbers which satisfy $a < b$. Further, let n be a nonnegative integer and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices. We will continue to work with the notations given above. In particular, if a matrix-valued function $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ is given, then let $\tilde{S}_1$ (respectively, $\tilde{S}_2$) be the first (respectively, second) matrix-valued function which is associated canonically with S (see (1.10)). We will again use the notation $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ to denote the set of all solutions of the system of the fundamental matrix inequalities of Potapov-type associated with $[a,b]$ and $(s_j)_{j=0}^{2n+1}$.

Remark 4.1. *If $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ is nonempty, then $H_{1,n} \geq 0$, $H_{2,n} \geq 0$, $\tilde{H}_{1,n} \geq 0$, $\tilde{H}_{2,n}^* = \tilde{H}_{2,n}$, and in particular $s_j^* = s_j$ for each $j \in \mathbb{N}_{0,2n+1}$.*

Remark 4.2. *Suppose $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right] \neq \emptyset$. Let $S \in \mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$. Then $\tilde{S}_1$ and $\tilde{S}_2$ are both holomorphic in $\mathbb{C} \setminus [a,b]$. Moreover, for each $k \in \{1,2\}$ and each $w \in \Pi_+$, from $K_{k,n}^{[S]}(w) \geq 0$ and (3.8) it follows immediately $\operatorname{Im} \tilde{S}_k(w) \geq 0$.*

Lemma 4.3. $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right] \subseteq \mathcal{R}_q[a,b]$.

Proof. Use Remark 4.2 and Lemma 3.6. □

Lemma 4.4. *Suppose that $s_j^* = s_j$ holds for each $j \in \mathbb{N}_{0,2n+1}$. Let $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ be a matrix-valued function. For each $k \in \{1,2\}$, let $F_{k,n} : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{(n+1)q\times(n+1)q}$ be defined by*

$$F_{k,n}(w) := H_{k,n} T_n^* R_{T_n}^*(\overline{w}) + R_{T_n}(w)\left(v_n \tilde{S}_k(w) - u_{k,n}\right) v_n^* R_{T_n}^*(\overline{w}). \tag{4.1}$$

For each $k \in \{1,2\}$ and for every choice of z in $\mathbb{C} \setminus \mathbb{R}$, then

$$\triangle_n(z) K_{k,n}^{[S]}(z) \triangle_n^*(z) = Q_{k,n}^{[S]}(z) \tag{4.2}$$

and

$$\Gamma_n(z) Q_{k,n}^{[S]}(z) \Gamma_n^*(z) = K_{k,n}^{[S]}(z) \tag{4.3}$$

where $K_{k,n}^{[S]}(z), Q_{k,n}^{[S]}(z), \triangle_n(z)$, and $\Gamma_n(z)$ are given by (1.11), (1.12),

$$Q_{k,n}^{[S]}(z) := \begin{pmatrix} H_{k,n} & F_{k,n}(z) \\ F_{k,n}^*(z) & \frac{F_{k,n}(z) - F_{k,n}^*(z)}{z - \overline{z}} \end{pmatrix}, \tag{4.4}$$

$$\triangle_n(z) := \begin{pmatrix} I_{(n+1)q} & 0 \\ R_{T_n}(\overline{z}) T_n & R_{T_n}(\overline{z}) v_n \end{pmatrix}, \tag{4.5}$$

and

$$\Gamma_n(z) := \begin{pmatrix} I_{(n+1)q} & 0 \\ -v_n^* R_{T_n}(\overline{z}) T_n & v_n^* \end{pmatrix}.$$

Proof. Let $k \in \{1,2\}$ and let $z \in \mathbb{C} \setminus \mathbb{R}$. First we verify (4.2). Let $\hat{Q}^{[S]}_{k,n}(z) := \triangle_n(z) K^{[S]}_{k,n}(z) \triangle^*_n(z)$ and let $\hat{Q}^{[S]}_{k,n}$ be partitioned into $(n+1)q \times (n+1)q$ blocks via

$$\hat{Q}^{[S]}_{k,n}(z) = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}.$$

Obviously, $A(z) = H_{k,n}$, $B(z) = F_{k,n}(z)$, and $C(z) = F^*_{k,n}(z)$. Moreover, from (1.11), (1.12), and (4.5) we see easily that

$$\begin{aligned}
&D(z) = (R_{T_n}(\overline{z}) \cdot T_n, R_{T_n}(\overline{z}) v_n) \\
&\cdot \begin{pmatrix} H_{k,n} & R_{T_n}(z)\left(v_n \tilde{S}_k(z) - u_{k,n}\right) \\ \left(R_{T_n}(z)\left(v_n \tilde{S}_k(z) - u_{k,n}\right)\right)^* & \frac{\tilde{S}_k(z) - \tilde{S}^*_k(z)}{z - \overline{z}} \end{pmatrix} \begin{pmatrix} T^*_n R^*_{T_n}(\overline{z}) \\ v^*_n R^*_{T_n}(\overline{z}) \end{pmatrix} \\
&= R_{T_n}(\overline{z}) T_n H_{k,n} T^*_n R^*_{T_n}(\overline{z}) + R_{T_n}(\overline{z}) v_n \frac{\tilde{S}_k(z) - \tilde{S}^*_k(z)}{z - \overline{z}} v^*_n R^*_{T_n}(\overline{z}) \\
&+ R_{T_n}(\overline{z}) T_n R_{T_n}(z)\left(v_n \tilde{S}_k(z) - u_{k,n}\right) v^*_n R^*_{T_n}(\overline{z}) \\
&+ \left(R_{T_n}(\overline{z}) T_n R_{T_n}(z)\left(v_n \tilde{S}_k(z) - u_{k,n}\right) v^*_n R^*_{T_n}(\overline{z})\right)^*.
\end{aligned}$$

Using (3.12) we can conclude

$$\begin{aligned}
&R_{T_n}(\overline{z}) T_n R_{T_n}(z)\left(v_n \tilde{S}_k(z) - u_{k,n}\right) v^*_n R^*_{T_n}(\overline{z}) \\
&= \frac{R_{T_n}(z) - R_{T_n}(\overline{z})}{z - \overline{z}} \left(v_n \tilde{S}_k(z) - u_{k,n}\right) v^*_n R^*_{T_n}(\overline{z}) \\
&= \frac{1}{z - \overline{z}} \Big(R_{T_n}(z)\left(v_n \tilde{S}_k(z) - u_{k,n}\right) v^*_n R^*_{T_n}(\overline{z}) \\
&- R_{T_n}(\overline{z}) v_n \tilde{S}_k(z) v^*_n R^*_{T_n}(\overline{z}) + R_{T_n}(\overline{z}) u_{k,n} v^*_n R^*_{T_n}(\overline{z}) \Big)
\end{aligned}$$

and therefore

$$\begin{aligned}
D(z) &= \frac{1}{z - \overline{z}} \Big((z - \overline{z}) R_{T_n}(\overline{z}) T_n H_{k,n} T^*_n R^*_{T_n}(\overline{z}) \\
&+ R_{T_n}(z)\left(v_n \tilde{S}_k(z) - u_{k,n}\right) v^*_n R^*_{T_n}(\overline{z}) \\
&+ R_{T_n}(\overline{z}) u_{k,n} v^*_n R^*_{T_n}(\overline{z}) - R_{T_n}(\overline{z}) v_n \left(v_n \tilde{S}_k(z) - u_{k,n}\right)^* R^*_{T_n}(z) \\
&- R_{T_n}(\overline{z}) v_n u^*_{k,n} R^*_{T_n}(\overline{z}) \Big).
\end{aligned} \tag{4.6}$$

Proposition 2.1 provides us

$$\begin{aligned}
&R_{T_n}(\overline{z}) u_{k,n} v^*_n R^*_{T_n}(\overline{z}) - R_{T_n}(\overline{z}) v_n u^*_{k,n} R^*_{T_n}(\overline{z}) \\
&= R_{T_n}(\overline{z})(u_{k,n} v^*_n - v_n u^*_{k,n}) R^*_{T_n}(\overline{z}) \\
&= R_{T_n}(\overline{z})(H_{k,n} T^*_n - T_n H_{k,n}) R^*_{T_n}(\overline{z}).
\end{aligned} \tag{4.7}$$

Hence from (4.6) and (4.7) we infer

$$D(z) = \frac{1}{z-\overline{z}}\Big(R_{T_n}(\overline{z})(I-\overline{z}T_n)H_{k,n}T_n^* R_{T_n}^*(\overline{z})$$
$$-R_{T_n}(\overline{z})T_n H_{k,n}(I-zT_n^*)R_{T_n}^*(\overline{z}) + R_{T_n}(z)\left(v_n\tilde{S}_k(z)-u_{k,n}\right)v_n^* R_{T_n}^*(\overline{z})$$
$$-R_{T_n}(\overline{z})v_n\left(v_n\tilde{S}_k(z)-u_{k,n}\right)^* R_{T_n}^*(z)\Big).$$

In view of $R_{T_n}(z)(I-zT_n)=I$ and $(I-\overline{z}T_n^*)R_{T_n}^*(\overline{z})=I$, we get

$$D(z) = \frac{1}{z-\overline{z}}\left(H_{k,n}T_n^* R_{T_n}^*(\overline{z}) - R_{T_n}(\overline{z})T_n H_{k,n}\right.$$
$$\left.+R_{T_n}(z)\left(v_n\tilde{S}_k(z)-u_{k,n}\right)v_n^* R_{T_n}^*(\overline{z}) - R_{T_n}(\overline{z})v_n\left(v_n\tilde{S}_k(z)-u_{k,n}\right)^* R_{T_n}^*(z)\right)$$
$$= \frac{F_{k,n}(z)-F_{k,n}^*(z)}{z-\overline{z}}.$$

Consequently, (4.2) is verified. In view of (1.6) and (1.7), we have $v_n^* R_{T_n}(z)v_n = I_q$ and therefore $\Gamma_n(z)\bigtriangleup_n(z) = I$. Thus from (4.2) it follows finally (4.3). □

Proposition 4.5. *Let $S : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{q\times q}$ be a matrix-valued function which is holomorphic in $\mathbb{C}\setminus[a,b]$. For $k\in\{1,2\}$, let $F_{k,n} : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{(n+1)q\times(n+1)q}$ for each $w\in\mathbb{C}\setminus[a,b]$ be defined by (4.1). Then S is a solution of the system of the fundamental matrix-inequalities of Potapov-type associated with $[a,b]$ and $(s_j)_{j=0}^{2n+1}$ if and only if for each $z\in\mathbb{C}\setminus\mathbb{R}$ the matrix $Q_{k,n}^{[S]}(z)$ given by (4.4) is nonnegative Hermitian.*

Proof. Apply Remark 4.1 and Lemma 4.4. □

Lemma 4.6. *Suppose $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]\neq\emptyset$. Let $S\in\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ and let $F_{1,n}$ and $F_{2,n}$ be the matrix-valued functions which are defined on $\mathbb{C}\setminus[a,b]$ and which are given by (4.1). For each $k\in\{1,2\}$, then $F_{k,n}^{\square} := \mathrm{Rstr.}_{\Pi_+}F_{k,n}$ belongs to $\mathcal{R}_{0,(n+1)q}$ and the spectral measure $\mu_{k,n}$ of $F_{k,n}^{\square}$ satisfies the inequality $\mu_{k,n}(\mathbb{R})\leq H_{k,n}$.*

Proof. Apply Proposition 4.5 and Lemma 8.9. □

In the following we will use results stated in the appendix (Section 8). In particular, we will consider matrix-valued functions which belong to the classes $\mathcal{R}_q, \mathcal{R}'_q$, and $\mathcal{R}_{0,q}$ which are described there.

Lemma 4.7. *Suppose $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]\neq\emptyset$. Let S belong to $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ and let σ denote the Stieltjes measure of S. Then $H_{1,n}^{[\sigma]}\leq H_{1,n}$, $H_{2,n}^{[\sigma]}\leq H_{2,n}$, and $\tilde{H}_{1,n}^{[\sigma]}\leq\tilde{H}_{1,n}$.*

Proof. First we observe that from Lemma 4.3 we know that S belongs to $\mathcal{R}_q[a,b]$. Let $k \in \{1,2\}$. In view of (3.5), the function $F_{k,n} : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{(n+1)q\times(n+1)q}$ given by (4.1) admits the representation

$$F_{k,n}(z) = \Psi_{k,n}(z) + E_n(z)\tilde{S}_k(z)E_n^*(\overline{z}) \tag{4.8}$$

where $E_n : \mathbb{C} \to \mathbb{C}^{(n+1)q\times q}$ and $\Psi_{k,n} : \mathbb{C} \to \mathbb{C}^{(n+1)q\times(n+1)q}$ are given by (3.4) and

$$\Psi_{k,n}(w) := H_{1,n}T_n^* R_{T_n}^*(\overline{w}) - R_{T_n}(w)u_{k,n}v_n^* R_{T_n}^*(\overline{w}).$$

From (1.6), (3.4), and (3.5) we see easily that $\Psi_{k,n}$ and E_n are matrix polynomials. In particular, $\Psi_{k,n}$ and E_n are both holomorphic in $\mathbb{C}$. For every real number x, Remark 4.1 and Proposition 2.1 yield

$$\begin{aligned}
&\Psi_{k,n}(x) - \Psi_{k,n}^*(x) \\
&= R_{T_n}(x)\left((I - xT_n)H_{k,n}T_n^* - u_{k,n}v_n^* - T_nH_{k,n}(I - xT_n^*) + v_nu_{k,n}^*\right)R_{T_n}^*(x) \\
&= R_{T_n}(x)\left(H_{k,n}T_n^* - T_nH_{k,n} - (u_{k,n}v_n^* - v_nu_{k,n}^*)\right)R_{T_n}^*(x) = 0.
\end{aligned}$$

According to Lemma 8.13, the function $S_k^{\square} := \mathrm{Rstr.}_{\Pi_k}\tilde{S}_k$ belongs to $\mathcal{R}_q'$. Let ρ_k denote the spectral measure of $S_k^{\square}$. For all real numbers α and β which satisfy $\alpha < \beta$, $\rho_k(\{\alpha\}) = 0$, and $\rho_k(\{\beta\}) = 0$, (4.8) and Theorem 8.6 provide us then

$$\int_{\mathbb{R}} 1_{(\alpha,\beta)}E_n d\rho_k E_n^* = \frac{1}{\pi}\lim_{\varepsilon\to 0+0}\int_{[\alpha,\beta]} \mathrm{Im}\, F_{k,n}(x+i\varepsilon)\lambda(dx). \tag{4.9}$$

According to Lemma 4.6, the matrix-valued function $F_{k,n}^{\square} := \mathrm{Rstr.}_{\Pi_+}F_{k,n}$ belongs to $\mathcal{R}_{0,(n+1)q}$ and the spectral measure $\mu_{k,n}$ of $F_{k,n}^{\square}$ fulfills

$$\mu_{k,n}(\mathbb{R}) \leq H_{k,n}. \tag{4.10}$$

For all real numbers α and β which satisfy $\alpha < \beta$, $\mu_{k,n}(\{\alpha\}) = 0$, and $\mu_{k,n}(\{\beta\}) = 0$, from Theorem 8.2 and (4.9) we infer

$$\mu_{k,n}((\alpha,\beta)) = \frac{1}{\pi}\lim_{\varepsilon\to 0+0}\int_{[\alpha,\beta]} \mathrm{Im}\, F_{k,n}(x+i\varepsilon)\lambda(dx) = \int_{\mathbb{R}} 1_{(\alpha,\beta)}E_n d\rho_k E_n^*. \tag{4.11}$$

In view of Remark 3.2, there are sequences $(\alpha_m)_{n=1}^{\infty}$ and $(\beta_m)_{m=1}^{\infty}$ of real numbers which satisfy the following conditions:

(i) For all $m \in \mathbb{N}$, the inequalities $\alpha_{m+1} < \alpha_m < 0 < \beta_m < \beta_{m+1}$ hold.
(ii) $\lim_{m\to\infty}\alpha_m = -\infty$ and $\lim_{m\to\infty}\beta_m = +\infty$.
(iii) For all $m \in \mathbb{N}$,

$$\mu_{k,n}(\{\alpha_m\}) = 0,\ \mu_{k,n}(\{\beta_m\}) = 0,\ \rho_k(\{\alpha_m\}) = 0, \quad \text{and} \quad \rho_k(\{\beta_m\}) = 0.$$

Because of (ii) there is an $m_0 \in \mathbb{N}$ such that

$$\alpha_m < a < b < \beta_m \tag{4.12}$$

for all integers m with $m \geq m_0$. In view of (4.11) it follows

$$\mu_{k,n}(\mathbb{R}) = \lim_{m\to\infty}\mu_{k,n}((\alpha_m,\beta_m)) = \lim_{m\to\infty}\int_{\mathbb{R}} 1_{(\alpha_m,\beta_m)}E_n d\rho_k E_n^*. \tag{4.13}$$

Let $\sigma_1 : \mathfrak{B}\cap[a,b] \to \mathbb{C}^{q\times q}$ and $\sigma_2 : \mathfrak{B}\cap[a,b] \to \mathbb{C}^{q\times q}$ be given by (8.24) and (8.25). Further, let $\theta_1 : \mathfrak{B} \to \mathbb{C}^{q\times q}$ and $\theta_2 : \mathfrak{B} \to \mathbb{C}^{q\times q}$ be defined by (8.27). By virtue of Lemma 8.13 we have then $\rho_1 = \theta_1$ and $\rho_2 = \theta_2$. In view of (4.12) and Remarks 3.3 and 8.12, for each integer m with $m \geq m_0$, we have then

$$\begin{aligned}
&\int_{\mathbb{R}} 1_{(\alpha_m,\beta_m)}(t)E_n(t)\rho_1(dt)E_n^*(t) = \int_{\mathbb{R}} 1_{[a,b]}(t)E_n(t)\theta_1(dt)E_n^*(t) \\
&= \int_{[a,b]} E_n(t)\sigma_1(dt)E_n^*(t) = \int_{[a,b]} \big(R_{T_n}(t)v_n\big)\sigma_1(dt)\big(R_{T_n}(t)v_n\big)^* \\
&= \int_{[a,b]} E_n(t)\sigma_1(dt)E_n^*(t) = \tilde{H}_{1,n}^{[\sigma_1]} = H_{1,n}^{[\sigma]}
\end{aligned} \tag{4.14}$$

and similarly

$$\int_{\mathbb{R}} 1_{(\alpha_m,\beta_m)}(t)E_n(t)\rho_2(dt)E_n^*(t) = \tilde{H}_{1,n}^{[\sigma_2]} = H_{2,n}^{[\sigma]}.$$

Hence, for each $k \in \{1,2\}$, from (4.14), (4.13), and (4.10) we see

$$H_{k,n}^{[\sigma]} = \lim_{m\to\infty} \int_{\mathbb{R}} 1_{(\alpha_m,\beta_m)}(t)E_n(t)\rho_k(dt)E_n^*(t) = \mu_{k,n}(\mathbb{R}) \leq H_{k,n}.$$

Finally, taking into account (2.1), we obtain then

$$\tilde{H}_{1,n}^{[\sigma]} = \frac{1}{b-a}\left(H_{1,n}^{[\sigma]} + H_{2,n}^{[\sigma]}\right) \leq \frac{1}{b-a}(H_{1,n} + H_{2,n}) = \tilde{H}_{1,n}.$$

□

Remark 4.8. *Let $m \in \mathbb{N}$ and let $u \in \mathbb{C}^{(n+1)q\times m}$. Then $P_u : \mathbb{R} \to \mathbb{C}^{q\times q}$ defined by*

$$P_u(y) := \sum_{j=0}^{n} i^j u^* T_n^j u y^j$$

is the restriction of a matrix polynomial onto $\mathbb{R}$. In view of (3.11) *it admits the representation $P_u(y) = u^* R_{T_n}(iy)u$ for each $y \in \mathbb{R}$. Hence if*

$$\lim_{y\to+\infty} u^* R_{T_n}(iy)u = 0 \quad \textit{or} \quad \lim_{y\to-\infty} u^* R_{T_n}(iy)u = 0,$$

then $P_u(y) = 0$ for all $y \in \mathbb{R}$ and consequently $u^*u = P_u(0) = 0$, i.e., $u = 0$.

Lemma 4.9. *Suppose $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right] \neq \emptyset$. Let S belong to $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$. For each $k \in \{1,2\}$, then*

$$\lim_{y\to+\infty} R_{T_n}(iy)\cdot\left(v_n\tilde{S}_k(iy) - u_{k,n}\right) = 0. \tag{4.15}$$

Proof. Let $k \in \{1,2\}$. For each $y \in (0,+\infty)$, we have then $K_{k,n}^{[S]}(iy) \geq 0$ and therefore, in view of Remark 8.8,

$$0 \leq \left\| R_{T_n}(iy)\left(v_n\tilde{S}_k(iy) - u_{k,n}\right)\right\|^2 \leq \|H_{k,n}\| \cdot \left\| \frac{\tilde{S}_k(iy) - \tilde{S}_k^*(iy)}{2iy} \right\|$$
$$\leq \|H_{k,n}\| \frac{\|\tilde{S}_k(iy)\|}{y}. \tag{4.16}$$

From Lemma 4.3 we know that S belongs to $\mathcal{R}_q[a,b]$. Thus from Remark 8.11 we see that letting $y \to +\infty$ the right-hand side of (4.16) converges to 0. The proof is complete. □

In the following, for each $\sigma \in \mathcal{M}_{\geq}^q([a,b], \mathfrak{B} \cap [a,b])$ and each $m \in \mathbb{N}_0$, let

$$u_m^{[\sigma]} := -\begin{pmatrix} s_0^{[\sigma]} \\ s_1^{[\sigma]} \\ \vdots \\ s_m^{[\sigma]} \end{pmatrix}, \quad u_{1,m}^{[\sigma]} := u_m^{[\sigma]} - aT_m u_m^{[\sigma]}, \text{ and } u_{2,m}^{[\sigma]} := -u_m^{[\sigma]} + bT_m u_m^{[\sigma]}$$

where $s_j^{[\sigma]}, j \in \mathbb{N}_0$, are given (3.1).

Lemma 4.10. *Suppose $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right] \neq \emptyset$. Let S belong to $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ and let σ denote the Stieltjes measure of S. Then $s_0^{[\sigma]} = s_0$, $u_{1,n}^{[\sigma]} = u_{1,n}$, and $u_{2,n}^{[\sigma]} = u_{2,n}$.*

Proof. Let $k \in \{1,2\}$. Using Lemma 4.9 we get (4.15). Obviously, σ belongs to $\mathcal{M}_{\geq}^q\left[[a,b], \mathfrak{B} \cap [a,b]; (s_j^{[\sigma]})_{j=0}^{2n+1}\right]$. Hence S belongs to $\mathcal{R}_q\left[[a,b]; (s_j^{[\sigma]})_{j=0}^{2n+1}\right]$. Applying Proposition 3.8 we obtain then that S belongs to $\mathcal{P}_q\left[[a,b]; (s_j^{[\sigma]})_{j=0}^{2n+1}\right]$. Thus Lemma 4.9 also yields

$$\lim_{y\to+\infty} R_{T_n}(iy) \cdot \left(v_n\tilde{S}_k(iy) - u_{k,n}^{[\sigma]}\right) = 0. \tag{4.17}$$

From (4.15) and (4.17) it follows then

$$\lim_{y\to+\infty} R_{T_n}(iy) \cdot \left(u_{k,n}^{[\sigma]} - u_{k,n}\right) = 0$$

and therefore

$$\lim_{y\to\infty} \left((u_{k,n}^{[\sigma]} - u_{k,n})^* \cdot R_{T_n}(iy) \cdot (u_{k,n}^{[\sigma]} - u_{k,n})\right) = 0.$$

In view of Remark 4.8 this implies $u_{k,n}^{[\sigma]} = u_{k,n}$ and in particular $s_0^{[\sigma]} = s_0$. □

Remark 4.11. *Let $m \in \mathbb{N}$ and let $(c_j)_{j=0}^{2m}$ be a sequence of complex $q \times q$ matrices which satisfies the following two conditions:*

(i) $c_0 = 0$.

(ii) *The block Hankel matrix $C_m := (c_{j+k})_{j,k=0}^{2m}$ is nonnegative Hermitian.*

Then Remark 8.8 *shows that the matrix* c_{2m} *is nonnegative Hermitian and that* $c_j = 0$ *holds for each* $j \in \mathbb{N}_{0,2m-1}$.

Proposition 4.12. $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right] \subseteq \mathcal{R}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$.

Proof. Assume that S belongs to $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$. By virtue of Lemma 4.3, the matrix-valued function S belongs then to $\mathcal{R}_q[a,b]$. Let σ denote the Stieltjes measure of S. From Lemma 4.10 we see that

$$s_0^{[\sigma]} = s_0 \tag{4.18}$$

holds. Lemma 4.7 shows that

$$H_{1,n}^{[\sigma]} \leq H_{1,n}, \quad H_{2,n}^{[\sigma]} \leq H_{2,n}, \tag{4.19}$$

and

$$\tilde{H}_{1,n} - \tilde{H}_{1,n}^{[\sigma]} \geq 0 \tag{4.20}$$

hold. The inequalities stated in (4.19) imply in particular

$$-as_{2n}^{[\sigma]} + s_{2n+1}^{[\sigma]} \leq -as_{2n} + s_{2n+1} \tag{4.21}$$

and

$$bs_{2n}^{[\sigma]} - s_{2n+1}^{[\sigma]} \leq bs_{2n} - s_{2n+1}. \tag{4.22}$$

In the case $n = 0$, from (4.18), (4.21), and (4.22) we obtain then $s_1^{[\sigma]} = s_1$ and consequently $S \in \mathcal{R}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$. Now suppose $n \geq 1$. In view of (4.18) and (4.20), application of Remark 4.11 to the block Hankel matrix $C_n^{[\sigma]} := \tilde{H}_{1,n} - \tilde{H}_{1,n}^{[\sigma]}$ provides us

$$s_j^{[\sigma]} = s_j \quad \text{for all } j \in \mathbb{N}_{0,2n-1}. \tag{4.23}$$

In particular, it follows

$$-as_0^{[\sigma]} + s_1^{[\sigma]} = -as_0 + s_1. \tag{4.24}$$

Hence using the first inequality in (4.19), (4.24), and Remark 4.11, for every integer j with $0 \leq j \leq 2n-1$, we obtain

$$-as_j^{[\sigma]} + s_{j+1}^{[\sigma]} = -as_j + s_{j+1}. \tag{4.25}$$

Combining (4.23) and (4.25) for $j = 2n-1$ we infer

$$s_{2n}^{[\sigma]} = s_{2n}. \tag{4.26}$$

From (4.21), (4.22), and (4.26) we can conclude then

$$s_{2n+1}^{[\sigma]} = s_{2n+1}. \tag{4.27}$$

Since S belongs to $\mathcal{R}_q[a,b]$ the equalities (4.23), (4.26), and (4.27) imply finally that S belongs to $\mathcal{R}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$. □

Now we obtain a proof of our first main result of this paper (see Theorem 1.2), which shows that the sets $\mathcal{R}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ and $\mathcal{P}_q\left[[a,b];(s_j)_{j=0}^{2n+1}\right]$ coincide.

Proof of Theorem 1.2. Apply Propositions 3.8 and 4.12. □

5. Nonnegative column pairs

Let J be a $p \times p$ signature matrix, i.e., J is a complex $p \times p$ matrix which satisfies $J^* = J$ and $J^2 = I$. A complex $p \times p$ matrix A is said to be J-contractive (respectively, J-expansive) if $J - A^*JA \geq 0$ (respectively, $A^*JA - J \geq 0$). If A is a complex $p \times p$ matrix, then A is J-contractive (respectively, J-expansive) if and only if A^* is J-contractive (respectively, J-expansive) (see, e.g., [DFK, Theorem 1.3.3]). Moreover, if A is a nonsingular complex $p \times p$ matrix, then A is J-contractive if and only if A^{-1} is J-expansive (see, e.g., [DFK, Lemma 1.3.15]). A complex $p \times p$ matrix is said to be J-unitary if $J - A^*JA = 0$. If A is a J-unitary complex $p \times p$ matrix, then A is nonsingular and the matrices A^* and A^{-1} are J-unitary as well.

A matrix-valued entire function $W : \mathbb{C} \to \mathbb{C}^{p \times p}$ is said to belong to the Potapov class $\mathfrak{P}_J(\Pi_+)$ if

$$J - W^*(z)JW(z) \geq 0 \tag{5.1}$$

is satisfied for all $z \in \Pi_+$. A matrix-valued function W that belongs to $\mathfrak{P}_J(\Pi_+)$ is called a J-inner function of $\mathfrak{P}_J(\Pi_+)$ if

$$J - W^*(x)JW(x) = 0$$

holds for all $x \in \mathbb{R}$.

Lemma 5.1. *Let J be a $p \times p$ signature matrix and let W be a J-inner function of $\mathfrak{P}_J(\Pi_+)$.*

(a) *For each $z \in \mathbb{C}$, the matrix $W(z)$ is nonsingular and*

$$[W(z)]^{-1} = JW^*(\overline{z})J \tag{5.2}$$

and

$$J - [W(z)]^{-*}J[W(z)]^{-1} = J(J - W(\overline{z})JW^*(\overline{z}))J. \tag{5.3}$$

(b) *For each $z \in \Pi_- := \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta \in (-\infty, 0)\}$,*

$$W^*(z)JW(z) - J \geq 0. \tag{5.4}$$

(c) *For each $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$\frac{W^*(z)JW(z) - J}{i(z - \overline{z})} \geq 0. \tag{5.5}$$

Proof. Let $W^\sharp : \mathbb{C} \to \mathbb{C}^{p \times p}$ be given by $W^\sharp(z) := W^*(\overline{z})$. Obviously, $W^\sharp$ and $V := J - W^\sharp JW$ are entire matrix-valued functions. For each $x \in \mathbb{R}$, we have $V(x) = 0$. The Identity Theorem for holomorphic functions yields $V(z) = 0$ and hence

$$JW^*(\overline{z})JW(z) = J^2 = I$$

for all $z \in \mathbb{C}$. Thus (5.2) and (5.3) follow. Let $z \in \Pi_-$. Then $\overline{z} \in \Pi_+$ and we get that $W(\overline{z})$ is J-contractive. Consequently, (5.3) shows that $[W(z)]^{-1}$ is J-contractive. This implies (5.4) and (5.5). For each $z \in \Pi_+$, inequality (5.5) follows from (5.1). $\square$

For our further considerations, the $2q \times 2q$ signature matrix

$$\tilde{J}_q := \begin{pmatrix} 0 & -iI_q \\ iI_q & 0 \end{pmatrix} \tag{5.6}$$

is of particular interest. Indeed, on the one hand, we work with the class $\mathcal{R}_q[a,b]$ and, on the other hand, for all complex $q \times q$ matrices C we have

$$\begin{pmatrix} C \\ I_q \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} C \\ I_q \end{pmatrix} = 2 \operatorname{Im} C. \tag{5.7}$$

For each Hermitian complex $(p+q) \times (p+q)$ matrix J in [FKK, Definition 51] the notion of a J-nonnegative pair is introduced. We are going to modify this definition for our purpose in this paper. In the following, we continue to suppose that a and b are real numbers which satisfy $a < b$.

Definition 5.2. *Let P and Q be $q \times q$ complex matrix-valued functions which are meromorphic in $\mathbb{C} \setminus [a,b]$. Then $\binom{P}{Q}$ is called a column pair which is nonnegative with respect to $-\tilde{J}_q$ and $[a,b]$ if there exists a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [a,b]$ such that the following four conditions are satisfied:*

(i) *The matrix-valued functions P and Q are holomorphic in $\mathbb{C} \setminus ([a,b] \cup \mathcal{D})$.*

(ii) *For all $z \in \mathbb{C} \setminus ([a,b] \cup \mathcal{D})$, rank $\binom{P(z)}{Q(z)} = q$.*

(iii) *For all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$,*

$$\frac{1}{2 \operatorname{Im} z} \begin{pmatrix} (z-a)P(z) \\ Q(z) \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} (z-a)P(z) \\ Q(z) \end{pmatrix} \geq 0.$$

(iv) *For all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$,*

$$\frac{1}{2 \operatorname{Im} z} \begin{pmatrix} (b-z)P(z) \\ Q(z) \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} (b-z)P(z) \\ Q(z) \end{pmatrix} \geq 0.$$

In the following, let $\mathcal{P}(-\tilde{J}_q, [a,b])$ denote the set of all column pairs which are nonnegative with respect to $-\tilde{J}_q$ and $[a,b]$.

Remark 5.3. *Let $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q \times q}$ be a matrix-valued function, and let $\tilde{S}_1$ (respectively, $\tilde{S}_2$) be the first (respectively, second) matrix-valued function associated canonically with S, i.e., $\tilde{S}_1 : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q \times q}$ and $\tilde{S}_2 : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q \times q}$ are given by (1.10). For each $k \in \{1,2\}$ and each $z \in \mathbb{C} \setminus \mathbb{R}$ from (5.7) one gets immediately*

$$\frac{1}{2 \operatorname{Im} z} \begin{pmatrix} \tilde{S}_k(z) \\ I \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} \tilde{S}_k(z) \\ I \end{pmatrix} = \frac{\tilde{S}_k(z) - [\tilde{S}_k(z)]^*}{z - \overline{z}}. \tag{5.8}$$

If $\binom{P}{Q}$ belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$ and if F is a $q \times q$ complex matrix-valued function which is meromorphic in $\mathbb{C} \setminus [a,b]$ and for which the complex-valued function $\det F$ does not vanish identically, then it is readily checked that $\binom{PF}{QF}$

also belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$. Pairs $\binom{P_1}{Q_1}$ and $\binom{P_2}{Q_2}$ which belong to $\mathcal{P}(-\tilde{J}_q, [a,b])$ are said to be equivalent if there exists a $q \times q$ complex matrix-valued function F which is meromorphic in $\mathbb{C} \setminus [a,b]$ such that the following conditions are satisfied:

(i) The function $\det F$ does not vanish identically.

(ii) The identities $P_2 = P_1 F$ and $Q_2 = Q_1 F$ hold.

One can easily see that this relation is really an equivalence relation on $\mathcal{P}(-\tilde{J}_q, [a,b])$. If $\binom{P}{Q} \in \mathcal{P}(-\tilde{J}_q, [a,b])$, then we will write $\left\langle \binom{P}{Q} \right\rangle$ for the equivalence class of all column pairs $\binom{R}{S} \in \mathcal{P}(-\tilde{J}_q, [a,b])$ which are equivalent to $\binom{P}{Q}$.

Remark 5.4. *From Remark* 5.3 *and Lemma* 3.5 *it is obvious that, for each $S \in \mathcal{R}_q[a,b]$, the matrix-valued function $\binom{S}{I_q}$ belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$.*

If f is a meromorphic matrix-valued function, then let $\mathbb{H}_f$ be the set of all points at which f is holomorphic.

The following two lemmas can be proved similarly as the implication "$(ii) \Rightarrow (i)$" in the proof of Lemma 3.6. That's why we omit the details of the proofs.

Lemma 5.5. *Let φ be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus [a,+\infty)$ and which fulfills $\operatorname{Im} \varphi(z) \geq 0$ for all $z \in \Pi_+ \cap \mathbb{H}_\varphi$. Suppose that the function $\varphi_1 : \mathbb{H}_\varphi \to \mathbb{C}^{q\times q}$ defined by $\varphi_1(w) := (w-a)\varphi(w)$ satisfies $\operatorname{Im} \varphi_1(z) \geq 0$ for all $z \in \Pi_+ \cap \mathbb{H}_\varphi$. Then, for each $x \in (-\infty, a) \cap \mathbb{H}_\varphi$, the matrix $\varphi(x)$ is nonnegative Hermitian.*

Lemma 5.6. *Let φ be a $q \times q$ matrix-valued function which is meromorphic in $\mathbb{C} \setminus (-\infty,b]$ and which fulfills $\operatorname{Im} \varphi(z) \geq 0$ for all $z \in \Pi_+ \cap \mathbb{H}_\varphi$. Suppose that the function $\varphi_2 : \mathbb{H}_\varphi \to \mathbb{C}^{q\times q}$ defined by $\varphi_2(w) := (b-w)\varphi(w)$ satisfies $\operatorname{Im} \varphi_2(z) \geq 0$ for all $z \in \Pi_+ \cap \mathbb{H}_\varphi$. Then, for each $x \in (b, +\infty) \cap \mathbb{H}_\varphi$, the matrix $-\varphi(x)$ is nonnegative Hermitian.*

Proposition 5.7. *Let P and Q be $q \times q$ matrix-valued functions which are meromorphic in $\mathbb{C} \setminus [a,b]$. Suppose that $\binom{P}{Q}$ is a column pair which is nonnegative with respect to $-\tilde{J}_q$ and $[a,b]$ and that the function $\det Q$ does not vanish identically in $\mathbb{C} \setminus [a,b]$. Then $S := PQ^{-1}$ belongs to $\mathcal{R}_q[a,b]$.*

Proof. Obviously, $\begin{pmatrix} S \\ I_q \end{pmatrix}$ belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$. Hence there is a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [a,b]$ such that S is holomorphic in $\mathbb{C} \setminus ([a,b] \cup \mathcal{D})$ and that

$$\frac{1}{2 \operatorname{Im} z} \begin{pmatrix} (z-a)S(z) \\ I \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} (z-a)S(z) \\ I \end{pmatrix} \geq 0 \tag{5.9}$$

and

$$\frac{1}{2 \operatorname{Im} z} \begin{pmatrix} (b-z)S(z) \\ I \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} (b-z)S(z) \\ I \end{pmatrix} \geq 0 \tag{5.10}$$

are satisfied for all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. Let $\varphi_1 : \mathbb{H}_S \to \mathbb{C}^{q\times q}$ and $\varphi_2 : \mathbb{H}_S \to \mathbb{C}^{q\times q}$ be given by $\varphi_1(z) := (z-a)S(z)$ and $\varphi_2(z) := (b-z)S(z)$ for all $z \in \mathbb{H}_S$. For each $z \in \Pi_+ \setminus \mathcal{D}$, from (5.9) and (5.10) we obtain

$$\frac{1}{\operatorname{Im} z}\operatorname{Im}\,\varphi_1(z) = \frac{1}{2\operatorname{Im} z}\begin{pmatrix}(z-a)S(z)\\ I\end{pmatrix}^*(-\tilde{J}_q)\begin{pmatrix}(z-a)S(z)\\ I\end{pmatrix} \geq 0$$

and

$$\frac{1}{\operatorname{Im} z}\operatorname{Im}\,\varphi_2(z) = \frac{1}{2\operatorname{Im} z}\begin{pmatrix}(b-z)S(z)\\ I\end{pmatrix}^*(-\tilde{J}_q)\begin{pmatrix}(b-z)S(z)\\ I\end{pmatrix} \geq 0.$$

Thus we have

$$\operatorname{Im}\,\varphi_1(z) \geq 0 \quad \text{and} \quad \operatorname{Im}\,\varphi_2(z) \geq 0 \tag{5.11}$$

for all $z \in \Pi_+ \setminus \mathcal{D}$. Hence, in view of $S = \frac{1}{b-a}(\varphi_1 + \varphi_2)$, it follows

$$\operatorname{Im} S(z) \geq 0 \tag{5.12}$$

for all $z \in \Pi_+ \setminus \mathcal{D}$. Applying Lemma 5.5 and Lemma 5.6, from (5.11) we can conclude

$$S(x) \in \mathbb{C}_{\geq}^{q\times q} \quad \text{for all} \quad x \in (-\infty, a) \setminus \mathcal{D} \tag{5.13}$$

and

$$-S(x) \in \mathbb{C}_{\geq}^{q\times q} \quad \text{for all} \quad x \in (b, +\infty) \setminus \mathcal{D}. \tag{5.14}$$

Because of (5.12) one can easily see that S is holomorphic in Π_+ (compare, e.g., [DFK, Lemma 2.1.9]). In other words,

$$\operatorname{Rstr.}_{\Pi_+} S \in \mathcal{R}_q. \tag{5.15}$$

Let $x_0 \in (-\infty, a) \setminus \mathcal{D}$. Then there is a positive real number η such that $(x_0 - \eta, x_0 + \eta) \subseteq (-\infty, a) \setminus \mathcal{D}$. In view of (5.13) and the symmetry principle, we see then that S is holomorphic in Π_- and satisfies $S(z) = S^*(\overline{z})$ for all $z \in \Pi_-$. It remains to show that S is holomorphic at each point which belongs to $\mathbb{R} \setminus [a,b]$. Let $x_0 \in \mathbb{R} \setminus [a,b]$. Then there exists a positive real number η such that $(x_0 - \eta, x_0 + \eta) \subseteq \mathbb{R} \setminus [a,b], (x_0 - \eta, x_0) \cap \mathcal{D} = \emptyset$ and $(x_0, x_0 + \eta) \cap \mathcal{D} = \emptyset$. In view of (5.15), let (α, β, ν) be the Nevanlinna parametrization of $\operatorname{Rstr.}_{\Pi_+} S$ (see Section 8). Using Proposition 8.3 we obtain

$$S(z) = \alpha + \beta z + \int_{\mathbb{R}\setminus E} \frac{1+tz}{t-z}\nu(dt)$$

for all $z \in \Pi_+ \cup E \cup \Pi_-$ where $E := (x_0 - \eta, x_0) \cup (x_0, x_0 + \eta)$. The matrix-valued function $\psi : \Pi_+ \cup (x_0 - \eta, x_0 + \eta) \cup \Pi_- \to \mathbb{C}^{q\times q}$ defined by

$$\psi(z) := \alpha + \beta z + \int_{\mathbb{R}\setminus(x_0-\eta,x_0+\eta)} \frac{1+tz}{t-z}\nu(dt) \tag{5.16}$$

is holomorphic in $\Pi_+ \cup (x_0 - \eta, x_0 + \eta) \cup \Pi_-$. Then

$$S(z) = \psi(z) + \frac{1+x_0 z}{x_0 - z}\nu(\{x_0\}) \tag{5.17}$$

for all $z \in \Pi_+ \cup E \cup \Pi_-$. Let $u \in \mathbb{C}^p$. From equation (5.17) we see that

$$u^* S(x) u = u^* \psi(x) u + \frac{1 + x_0 x}{x_0 - x} u^* \nu(\{x_0\}) u$$

holds for each $x \in E$. If $u^* \nu(\{x_0\}) u > 0$, then this would imply

$$\lim_{x \to x_0 + 0} u^* S(x) u = -\infty$$

and

$$\lim_{x \to x_0 - 0} u^* \left(-S(x)\right) u = -\infty$$

in contradiction to (5.13) and (5.14), respectively. Hence we get $u^* \nu(\{x_0\}) u = 0$ and consequently $\nu(\{x_0\}) = 0$. Thus $S(z) = \psi(z)$ is satisfied for every choice of z in $\Pi_+ \cup E \cup \Pi_-$. Since x_0 was arbitrarily chosen from $\mathbb{R} \setminus [a, b]$ we see that S has no poles in $\mathbb{R} \setminus [a, b]$ as well. Hence S belongs to $\mathcal{R}_q[a, b]$. □

6. Description of the solution set in the positive definite case

In this section, we suppose again that a and b are real numbers which satisfy $a < b$. Further, let n be a nonnegative integer. Let $\mathcal{M}_{\geq}^q\left([a, b], \mathfrak{B} \cap [a, b]\right)$ denote the set of all nonnegative Hermitian $q \times q$ measures defined on $\mathfrak{B} \cap [a, b]$. For all $\sigma \in \mathcal{M}_{\geq}^q\left([a, b], \mathfrak{B} \cap [a, b]\right)$ and all nonnegative integers j, let $s_j^{[\sigma]}$ be given by (3.1). From Lemma 3.3 we know that, for each $\sigma \in \mathcal{M}_{\geq}^q\left([a, b], \mathfrak{B} \cap [a, b]\right)$ and for every nonnegative integer m, the matrices $H_{1,m}^{[\sigma]}$ and $H_{2,m}^{[\sigma]}$ given by (3.2) and (3.3) are both nonnegative Hermitian. Hence, in view of the considerations in Section 1, if $(s_j)_{j=0}^{2n+1}$ is a sequence of complex $q \times q$ matrices such that the solution set $\mathcal{R}_q\left[[a, b]; (s_j)_{j=0}^{2n+1}\right]$ of the (reformulated) matricial version of M.G. Krein's moment problem is nonempty, then the first block Hankel matrix $H_{1,n}$ and the second block Hankel matrix $H_{2,n}$ associated with the interval $[a, b]$ and the sequence $(s_j)_{j=0}^{2n+1}$ are both nonnegative Hermitian. In this section, we will give a parametrization of the set $\mathcal{R}_q\left[[a, b]; (s_j)_{j=0}^{2n+1}\right]$ under the assumption that the block Hankel matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian.

For our following considerations we will apply the description of the set $\mathcal{R}_q\left[[a, b]; (s_j)_{j=0}^{2n+1}\right]$ given in Theorem 1.2 where the matrix-valued functions $K_{1,n}^{[S]}$ and $K_{2,n}^{[S]}$ given by (1.11) and (1.12) are used. However, first we are going now to present a class of measures $\sigma \in \mathcal{M}_{\geq}^q\left([a, b], \mathfrak{B} \cap [a, b]\right)$ for which the block Hankel matrices $H_{1,m}^{[\sigma]}$ and $H_{2,m}^{[\sigma]}$ are positive Hermitian for every nonnegative integer m. Let λ denote the Lebesgue measure defined on $\mathfrak{B} \cap [a, b]$ and let $\mathcal{L}^1\left([a, b], \mathfrak{B} \cap [a, b], \lambda; \mathbb{C}\right)$ designate the set of all $(\mathfrak{B} \cap [a, b]) - \tilde{\mathfrak{B}}$-measurable complex-valued functions which are defined on $[a, b]$ and which are integrable with respect to λ.

Lemma 6.1. *Let $X = (X_{jk})_{j,k=1}^q : [a, b] \to \mathbb{C}^{q \times q}$ be a $q \times q$ matrix-valued function every entry function X_{jk} of which belongs to $\mathcal{L}^1\left([a, b], \mathfrak{B} \cap [a, b], \lambda; \mathbb{C}\right)$ and which* *satisfies* $\lambda\left(\{t \in [a,b] : X(t) \in \mathbb{C}^{q\times q} \setminus \mathbb{C}^{q\times q}_{>}\}\right) = 0$. *Then* $\mu : \mathfrak{B} \cap [a,b] \to \mathbb{C}^{q\times q}$ *defined by*

$$\mu(B) := \int_B X d\lambda$$

belongs to $\mathcal{M}^q_{\geq}([a,b], \mathfrak{B} \cap [a,b])$ *and, for every nonnegative integer* m*, the block Hankel matrices* $H^{[\mu]}_{1,m}$ *and* $H^{[\mu]}_{2,m}$ *are both positive Hermitian.*

Proof. Let m be a nonnegative integer. From Lemma 3.3 we see that the representations

$$H^{[\mu]}_{1,m} = \int_{[a,b]} \sqrt{t-a} E_m(t)\mu(dt) \left[\sqrt{t-a} E_m(t)\right]^* = \int_{[a,b]} (t-a)E_m(t)X(t)E^*_m(t)\lambda(dt)$$

and

$$H^{[\mu]}_{2,m} = \int_{[a,b]} (t-b)E_m(t)X(t)E^*_m(t)\lambda(dt)$$

hold where E_m is the matrix polynomial which is for each $z \in \mathbb{C}$ given by (3.4). Let $x \in \mathbb{C}^{(m+1)q\times 1} \setminus \{0\}$. Then one can easily see that the set $M_x := \{t \in [a,b] : E^*_m(t)x = 0\}$ is finite. In particular, $\lambda(M_x \cup \{a,b\}) = 0$. Hence we obtain

$$\lambda\left(\{t \in [a,b] : (t-a)x^* E_m(t)X(t)E^*_m(t)x \in (-\infty, 0]\}\right) = 0$$

and consequently

$$x^* H^{[\mu]}_{1,m} x = \int_{[a,b]} (t-a)\left(E^*_m(t)x\right)^* X(t)E^*_m(t)x\, \lambda(dt) \in (0, +\infty).$$

Analogously, one can see that $x^* H^{[\mu]}_{2,m} x \in (0, +\infty)$ holds. □

Observe that the constant matrix-valued function $X : [a,b] \to \mathbb{C}^{q\times q}$ with value $\frac{1}{b-a} I_q$ is a simple example for a matrix-valued function which satisfies the assumptions of Lemma 6.1. In particular, there exists a sequence $(r_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices such that the block Hankel matrices $(-ar_{j+k} + r_{j+k+1})_{j,k=0}^n$ and $(br_{j+k} - r_{j+k+1})_{j,k=0}^n$ are both positive Hermitian.

Recall that Theorem 1.2 shows that a given matrix-valued function $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ belongs to $\mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$ if and only if S is a solution of the system of the fundamental matrix inequalities of Potapov-type associated with the interval $[a,b]$ and the sequence $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices, i.e., if and only if S is a holomorphic function for which the matrices $K^{[S]}_{1,n}(z)$ and $K^{[S]}_{2,n}(z)$ given by (1.11) and (1.12) are both nonnegative Hermitian for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Remark 6.2. *Suppose that* $(s_j)_{j=0}^{2n+1}$ *is a sequence of complex* $q\times q$ *matrices such that the matrices* $H_{1,n}$ *and* $H_{2,n}$ *are both positive Hermitian. Let* $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ *be a matrix-valued function. In view of Remark* 8.8, *one can easily see that the* *matrices* $K_{1,n}^{[S]}(z)$ *and* $K_{2,n}^{[S]}(z)$ *are both nonnegative Hermitian for all* $z \in \mathbb{C} \setminus \mathbb{R}$ *if and only if for each* $k \in \{1,2\}$ *and each* $z \in \mathbb{C} \setminus \mathbb{R}$ *the matrix*

$$\begin{aligned}\tilde{C}_{k,n}^{[S]}(z) &:= \frac{\tilde{S}_k(z) - [\tilde{S}_k(z)]^*}{z - \overline{z}} \\ &- \left(v_n \tilde{S}_k(z) - u_{k,n}\right)^* [R_{T_n}(z)]^* H_{k,n}^{-1} R_{T_n}(z) \left(v_n \tilde{S}_k(z) - u_{k,n}\right)\end{aligned} \tag{6.1}$$

is nonnegative Hermitian.

In the following, we again use the notation $\tilde{J}_q$ for the signature matrix given by (5.6).

Lemma 6.3. *Let* $(s_j)_{j=0}^{2n+1}$ *be a sequence of complex* $q \times q$ *matrices and let* $k \in \{1,2\}$. *Suppose that the block Hankel matrix* $H_{k,n}$ *is positive Hermitian. Then* $\tilde{U}_{k,n} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ *defined by*

$$\tilde{U}_{k,n}(z) := I_{2q} + i(z-a)(u_{k,n}, v_n)^* [R_{T_n}(\overline{z})]^* H_{k,n}^{-1} R_{T_n}(a) \cdot (u_{k,n}, v_n) \tilde{J}_q \tag{6.2}$$

is a $2q \times 2q$ *matrix polynomial of degree not greater than* $n+1$. *Furthermore, the following statements hold:*

(a) *For all* $z \in \mathbb{C}$,

$$\begin{aligned}&\tilde{J}_q - \tilde{U}_{k,n}(z) \cdot \tilde{J}_q \cdot [\tilde{U}_{k,n}(z)]^* \\ &= -i(z - \overline{z})(u_{k,n}, v_n)^* [R_{T_n}(\overline{z})]^* H_{k,n}^{-1} R_{T_n}(\overline{z}) \cdot (u_{k,n}, v_n).\end{aligned} \tag{6.3}$$

In particular, for each $w \in \Pi_+$,

$$\tilde{J}_q - \tilde{U}_{k,n}(w) \cdot \tilde{J}_q \cdot [\tilde{U}_{k,n}(w)]^* \geq 0. \tag{6.4}$$

Moreover, for each real number x,

$$\tilde{J}_q - \tilde{U}_{k,n}(x) \cdot \tilde{J}_q \cdot [\tilde{U}_{k,n}(x)]^* = 0. \tag{6.5}$$

(b) *For all* $z \in \mathbb{C}$, *the matrix* $\tilde{U}_{k,n}(z)$ *is nonsingular and the identities*

$$\begin{aligned}&[\tilde{U}_{k,n}(z)]^{-1} \\ &= I_{2q} - i(z-a)(u_{k,n}, v_n)^* [R_{T_n}(a)]^* H_{k,n}^{-1} R_{T_n}(z) \cdot (u_{k,n}, v_n) \tilde{J}_q\end{aligned}$$

and

$$\begin{aligned}&\tilde{J}_q - [\tilde{U}_{k,n}(z)]^{-*} \tilde{J}_q [\tilde{U}_{k,n}(z)]^{-1} \\ &= i(z - \overline{z}) \cdot \tilde{J}_q (u_{k,n}, v_n)^* [R_{T_n}(z)]^* H_{k,n}^{-1} R_{T_n}(z) \cdot (u_{k,n}, v_n) \tilde{J}_q\end{aligned} \tag{6.6}$$

hold.

Proof. For all $z \in \mathbb{C}$ we have $R_{T_n}(z) = \sum_{j=0}^{n} z^j T_n^j$. Hence one can easily see that $\tilde{U}_{k,n}$ is a matrix polynomial of degree not greater than $n+1$. Obviously, for each $w \in \mathbb{C}$, the identities

$$R_{T_n}(w) \cdot (I - wT_n) = I \quad \text{and} \quad (I - wT_n) R_{T_n}(w) = I \tag{6.7}$$

are satisfied. From Proposition 2.1 we obtain

$$H_{k,n}T_n^* - T_n H_{k,n} = i(u_{k,n}, v_n)\tilde{J}_q(u_{k,n}, v_n)^*. \tag{6.8}$$

Let $z \in \mathbb{C}$. Using (6.7) and (6.8) a straightforward calculation provides us

$$\begin{aligned}\tilde{J}_q - \tilde{U}_{k,n}(z)\tilde{J}_q[\tilde{U}_{k,n}(z)]^* &= i(u_{k,n}, v_n)^*[R_{T_n}(\overline{z})]^* H_{k,n}^{-1} R_{T_n}(a)\\ \cdot\Omega_{k,n}(z,a)\cdot[R_{T_n}(a)]^* H_{k,n}^{-1} R_{T_n}(\overline{z})\cdot(u_{k,n}, v_n)\end{aligned} \tag{6.9}$$

where

$$\begin{aligned}\Omega_{k,n}(z,a) &:= -(z-a)(I-\overline{z}T_n)H_{k,n}(I-aT_n^*)\\ &+(\overline{z}-a)(I-aT_n)H_{k,n}(I-zT_n^*) + |z-a|^2(H_{k,n}T_n^* - T_n H_{k,n}).\end{aligned}$$

A further straightforward calculation shows that $\Omega_{k,n}(z,a)$ can be represented via

$$\Omega_{k,n}(z,a) = (\overline{z}-z)(I-aT_n)H_{k,n}(I-aT_n^*). \tag{6.10}$$

In view of (6.7), (6.9), and (6.10) it follows (6.3) and hence (6.4) and (6.5). Part (a) is proved. Application of Lemma 5.1 and part (a) yield the proof of part (b). □

Lemma 6.4. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q\times q$ matrices such that matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. For each $k \in \{1,2\}$, then $\tilde{U}_{k,n} : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ defined by (6.2) is a $\tilde{J}_q$-inner function of the Potapov class $\mathfrak{P}_{\tilde{J}_q}(\Pi_+)$.*

Proof. If A is a complex $2q \times 2q$ matrix, then A^* is $\tilde{J}_q$-contractive (respectively, $\tilde{J}_q$-unitary) if and only if A is $\tilde{J}_q$-contractive (respectively, $\tilde{J}_q$-unitary). Hence from Lemma 6.3 the assertion follows immediately. □

Lemma 6.5. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. Let $k \in \{1,2\}$ and let $\tilde{U}_{k,n}$: $\mathbb{C} \to \mathbb{C}^{2q\times 2q}$ be defined by (6.2). Let $S : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{q\times q}$ be a matrix-valued function. Further, let $\tilde{S}_k : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{q\times q}$ and $\tilde{C}_{k,n}^{[S]} : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{q\times q}$ be given by (1.10) and (6.1). For all $z \in \mathbb{C}\setminus\mathbb{R}$, then the matrix $\tilde{C}_{k,n}^{[S]}(z)$ admits the representation*

$$\tilde{C}_{k,n}^{[S]}(z) = \frac{1}{i(z-\overline{z})}\begin{pmatrix}\tilde{S}_k(z)\\ I\end{pmatrix}^*[\tilde{U}_{k,n}(z)]^{-*}\tilde{J}_q[\tilde{U}_{k,n}(z)]^{-1}\begin{pmatrix}\tilde{S}_k(z)\\ I\end{pmatrix}. \tag{6.11}$$

Proof. Let $z \in \mathbb{C}\setminus\mathbb{R}$. From Remark 5.3 we see that (5.8) is true. Further, we have

$$v_n\tilde{S}_k(z) - u_{k,n} = \frac{1}{i}(u_{k,n}, v_n)\tilde{J}_q\begin{pmatrix}\tilde{S}_k(z)\\ I\end{pmatrix}. \tag{6.12}$$

Because of Lemma 6.3, equation (6.6) is valid. Using (6.1), (5.8), (6.12), and (6.6), we obtain finally (6.11). □

Remark 6.6. *Let M be a complex $q \times q$ matrix, let*

$$A_1 := \begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \quad \text{and let} \quad A_2 := \begin{pmatrix} I & M \\ 0 & I \end{pmatrix}.$$

Then

$$A_1^* \tilde{J}_q A_1 = \tilde{J}_q + \operatorname{diag}\big(i(M^* - M), 0\big)$$

and

$$A_2^* \tilde{J}_q A_2 = \tilde{J}_q + \operatorname{diag}\big(0, i(M - M^*)\big).$$

In particular, A_1 is $\tilde{J}_q$-unitary if and only if $M^* = M$. Moreover, A_2 is $\tilde{J}_q$-unitary if and only if $M^* = M$.

Lemma 6.7. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. Let*

$$M_{1,n} := (a-b) v_n^* [R_{T_n}(a)]^* H_{2,n}^{-1} R_{T_n}(a) v_n, \tag{6.13}$$

$$M_{2,n} := (a-b) u_{1,n}^* [R_{T_n}(a)]^* H_{1,n}^{-1} R_{T_n}(a) u_{1,n}, \tag{6.14}$$

$$A_{1,n} := \begin{pmatrix} I & 0 \\ M_{1,n} & I \end{pmatrix}, \quad \text{and} \quad A_{2,n} := \begin{pmatrix} I & -M_{2,n} \\ 0 & I \end{pmatrix}. \tag{6.15}$$

Let $k \in \{1,2\}$ and let $\tilde{U}_{k,n} : \mathbb{C} \to \mathbb{C}^{2q \times 2q}$ be given by (6.2). Then

$$U_{k,n} := \tilde{U}_{k,n} A_{k,n} \tag{6.16}$$

is a $2q \times 2q$ matrix polynomial of degree not greater than $n+1$. Moreover, $U_{k,n}$ is a $\tilde{J}_q$-inner function of the class $\mathfrak{P}_{\tilde{J}_q}(\Pi_+)$. For each $z \in \mathbb{C}$, the matrix $U_{k,n}(z)$ is nonsingular. Moreover, for each $z \in \mathbb{C}$, the identities

$$U_{k,n}(z) \tilde{J}_q [U_{k,n}(z)]^* = \tilde{U}_{k,n}(z) \tilde{J}_q [\tilde{U}_{k,n}(z)]^* \tag{6.17}$$

and

$$[U_{k,n}(z)]^{-*} \tilde{J}_q [U_{k,n}(z)]^{-1} = [\tilde{U}_{k,n}(z)]^{-*} \tilde{J}_q [\tilde{U}_{k,n}(z)]^{-1} \tag{6.18}$$

are satisfied.

Proof. Obviously, the matrices $M_{1,n}$ and $-M_{2,n}$ are both Hermitian. Remark 6.6 shows then that $A_{1,n}$ and $A_{2,n}$ are $\tilde{J}_q$-unitary. Consequently, all the matrices $A_{1,n}^*$, $A_{2,n}^*$, $A_{1,n}^{-1}$, and $A_{2,n}^{-1}$ are also $\tilde{J}_q$-unitary. Thus (6.17) and (6.18) follow for each $z \in \mathbb{C}$. In view of Lemma 6.3 the proof is finished. □

Proposition 6.8. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. Let $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q \times q}$ be a matrix-valued function. Then S belongs to $\mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$ if and only if S is holomorphic in $\mathbb{C} \setminus [a,b]$ and the matrix inequality*

$$\frac{1}{i(z-\overline{z})} \begin{pmatrix} \tilde{S}_k(z) \\ I \end{pmatrix}^* [U_{k,n}(z)]^{-*} \tilde{J}_q [U_{k,n}(z)]^{-1} \begin{pmatrix} \tilde{S}_k(z) \\ I \end{pmatrix} \geq 0 \tag{6.19}$$

is satisfied for each $k \in \{1,2\}$ and each $z \in \mathbb{C} \setminus \mathbb{R}$.

Proof. Use Theorem 1.2, Remark 6.2, Lemma 6.5, and Lemma 6.7. □

Remark 6.9. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. Straightforward calculations show that the matrix-valued functions $U_{1,n}$ and $U_{2,n}$ admit for each $z \in \mathbb{C}$ the block representations*

$$U_{1,n}(z) = \begin{pmatrix} U_{11;n}^{(1)}(z) & U_{12;n}^{(1)}(z) \\ \\ U_{21;n}^{(1)}(z) & U_{22;n}^{(1)}(z) \end{pmatrix} \quad \text{and} \quad U_{2,n}(z) = \begin{pmatrix} U_{11;n}^{(2)}(z) & U_{12;n}^{(2)}(z) \\ \\ U_{21;n}^{(2)}(z) & U_{22;n}^{(2)}(z) \end{pmatrix}$$

where

$$\begin{aligned}
U_{11;n}^{(1)}(z) &:= I + (z-a)u_{1,n}^*[R_{T_n}(\overline{z})]^* H_{1,n}^{-1} R_{T_n}(a)(u_{1,n}M_{1,n} - v_n),\\
U_{12;n}^{(1)}(z) &:= (z-a)u_{1,n}^*[R_{T_n}(\overline{z})]^* H_{1,n}^{-1} R_{T_n}(a)u_{1,n},\\
U_{21;n}^{(1)}(z) &:= M_{1,n} + (z-a)v_n^*[R_{T_n}(\overline{z})]^* H_{1,n}^{-1} R_{T_n}(a)(u_{1,n}M_{1,n} - v_n),\\
U_{22;n}^{(1)}(z) &:= I + (z-a)v_n^*[R_{T_n}(\overline{z})]^* H_{1,n}^{-1} R_{T_n}(a)u_{1,n},\\
U_{11;n}^{(2)}(z) &:= I - (z-a)u_{2,n}^*[R_{T_n}(\overline{z})]^* H_{2,n}^{-1} R_{T_n}(a)v_n,\\
U_{12;n}^{(2)}(z) &:= -M_{2,n} + (z-a)u_{2,n}^*[R_{T_n}(\overline{z})]^* H_{2,n}^{-1} R_{T_n}(a)(v_nM_{2,n} + u_{2,n}),\\
U_{21;n}^{(2)}(z) &:= -(z-a)v_n^*[R_{T_n}(\overline{z})]^* H_{2,n}^{-1} R_{T_n}(a)v_n,
\end{aligned}$$

and

$$U_{22;n}^{(2)}(z) := I + (z-a)v_n^*[R_{T_n}(\overline{z})]^* H_{2,n}^{-1} R_{T_n}(a) \cdot (v_nM_{2,n} + u_{2,n}).$$

Proposition 6.10. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. Then $V_n : \mathbb{C} \to \mathbb{C}^{2q\times 2q}$ defined for all $z \in \mathbb{C}$ by*

$$V_n(z) := \begin{pmatrix} V_{11;n}(z) & V_{12;n}(z) \\ V_{21;n}(z) & V_{22;n}(z) \end{pmatrix} \tag{6.20}$$

and

$$V_{11;n}(z) := I_q - (z-a)u_{2,n}^*[R_{T_n}(\overline{z})]^* H_{2,n}^{-1} R_{T_n}(a)v_n, \tag{6.21}$$

$$V_{12;n}(z) := u_{1,n}^*[R_{T_n}(\overline{z})]^* H_{1,n}^{-1} R_{T_n}(a)u_{1,n}, \tag{6.22}$$

$$V_{21;n}(z) := -(b-z)(z-a)v_n^*[R_{T_n}(\overline{z})]^* H_{2,n}^{-1} R_{T_n}(a)v_n, \tag{6.23}$$

$$V_{22;n}(z) := I_q + (z-a)v_n^*[R_{T_n}(\overline{z})]^* H_{1,n}^{-1} R_{T_n}(a)u_{1,n} \tag{6.24}$$

is a $2q \times 2q$ matrix polynomial of degree not greater than $n+2$. Moreover, the following statements hold:

(a) *For each $z \in \mathbb{C} \setminus \{a\}$, the identity*

$$V_n(z) = \begin{pmatrix} \frac{1}{z-a} I_q & 0 \\ 0 & I_q \end{pmatrix} \cdot U_{1,n}(z) \cdot \begin{pmatrix} (z-a)I_q & 0 \\ 0 & I_q \end{pmatrix} \tag{6.25}$$

is satisfied where $U_{1,n}$ is given by (6.2), (6.15), and (6.16).

(b) *For each $z \in \mathbb{C} \setminus \{b\}$, the identity*

$$V_n(z) = \begin{pmatrix} \frac{1}{b-z} I_q & 0 \\ 0 & I_q \end{pmatrix} \cdot U_{2,n}(z) \cdot \begin{pmatrix} (b-z)I_q & 0 \\ 0 & I_q \end{pmatrix} \tag{6.26}$$

is satisfied where $U_{2,n}$ is given by (6.2), (6.15), *and* (6.16).

(c) *For all $z \in \mathbb{C}$, the matrix $V_n(z)$ is nonsingular.*

Proof. We use the notations given above. For each $z \in \mathbb{C} \setminus \{a\}$, we see then that

$$V_{12;n}(z) = \frac{1}{z-a} U^{(1)}_{12;n}(z) \quad \text{and} \quad V_{22;n}(z) = U^{(1)}_{22;n}(z)$$

are satisfied. Hence to prove part (a) it is sufficient to verify that

$$V_{11;n}(z) = U^{(1)}_{11;n}(z) \quad \text{and} \quad V_{21;n}(z) = (z-a)U^{(1)}_{21;n}(z) \tag{6.27}$$

hold for each $z \in \mathbb{C} \setminus \{a\}$. For every choice of w and ζ in $\mathbb{C}$, we have

$$R_{T_n}(w)R_{T_n}(\zeta) = R_{T_n}(\zeta)R_{T_n}(w). \tag{6.28}$$

From (6.28), (2.4), and (1.9), we obtain then

$$\begin{aligned}
&[R_{T_n}(b)]^{-1}R_{T_n}(a)R_{T_n}(\overline{z})u_{1,n} = R_{T_n}(\overline{z})[R_{T_n}(b)]^{-1}R_{T_n}(a)u_{1,n} \\
&= R_{T_n}(\overline{z})[R_{T_n}(b)]^{-1}\tilde{u}_n = -R_{T_n}(\overline{z})u_{2,n}
\end{aligned} \tag{6.29}$$

for each $z \in \mathbb{C}$. This implies

$$\begin{aligned}
&U^{(1)}_{11,n}(z) - V_{11,n}(z) = (z-a)\big(u^*_{1,n}[R_{T_n}(\overline{z})]^* H^{-1}_{1,n} R_{T_n}(a)(u_{1,n}M_{1,n} - v_n) \\
&+ u^*_{2,n}[R_{T_n}(\overline{z})]^* H^{-1}_{2,n} R_{T_n}(a) v_n\big) \\
&= (z-a)u^*_{1,n}[R_{T_n}(\overline{z})]^* \left(H^{-1}_{1,n} R_{T_n}(a)(u_{1,n}M_{1,n} - v_n)\right. \\
&\left. - [R_{T_n}(a)]^*[R_{T_n}(b)]^{-*} H^{-1}_{2,n} R_{T_n}(a) v_n\right)
\end{aligned} \tag{6.30}$$

for each $z \in \mathbb{C}$. Since $H_{1,n}$ and $H_{2,n}$ are Hermitian matrices from (6.13) and Lemma 2.2 we can conclude

$$\begin{aligned}
&R_{T_n}(a)(u_{1,n}M_{1,n} - v_n) \\
&= ((a-b)R_{T_n}(a)u_{1,n}v^*_n[R_{T_n}(a)]^* - H_{2,n})\, H^{-1}_{2,n} R_{T_n}(a) v_n \\
&= \left[(H_{2,n} + [R_{T_n}(b)]^{-1}R_{T_n}(a)H_{1,n})^* - H_{2,n}\right] H^{-1}_{2,n} R_{T_n}(a) v_n \\
&= H_{1,n}[R_{T_n}(a)]^*[R_{T_n}(b)]^{-*} H^{-1}_{2,n} R_{T_n}(a) v_n.
\end{aligned} \tag{6.31}$$

Thus from (6.30) and (6.31) we see that the first equation in (6.27) holds for all $z \in \mathbb{C}$. Using (2.4) we get

$$\begin{aligned}
&(b-z)I_q + (z-a)[R_{T_n}(a)]^*[R_{T_n}(b)]^{-*} \\
&= [(b-z)(I - aT^*_n) + (z-a)(I - bT^*_n)]\,[R_{T_n}(a)]^* \\
&= (b-a)(I - zT^*_n)[R_{T_n}(a)]^* = (b-a)[R_{T_n}(\overline{z})]^{-*}[R_{T_n}(a)]^*
\end{aligned}$$

and consequently

$$(z-a)[R_{T_n}(\overline{z})]^*[R_{T_n}(a)]^*[R_{T_n}(b)]^{-*} = (b-a)[R_{T_n}(a)]^* - (b-z)[R_{T_n}(\overline{z})]^* \tag{6.32}$$

for each $z \in \mathbb{C}$. Hence, for every complex number z, from (6.13), (6.31), and (6.32) it follows

$$\begin{aligned}
&(z-a)U^{(1)}_{21,n}(z) - V_{21,n}(z)\\
&= (z-a)\Big[M_{1,n} + (z-a)v_n^*[R_{T_n}(\overline{z})]^* H_{1,n}^{-1} R_{T_n}(a)(u_{1,n}M_{1,n} - v_n)\\
&+(b-z)v_n^*[R_{T_n}(\overline{z})]^* H_{2,n}^{-1} R_{T_n}(a)v_n\Big]\\
&= (z-a)v_n^*\Big((a-b)[R_{T_n}(a)]^* + (z-a)[R_{T_n}(\overline{z})]^*[R_{T_n}(a)]^*[R_{T_n}(b)]^{-*}\\
&+(b-z)[R_{T_n}(\overline{z})]^*\Big)H_{2,n}^{-1}R_{T_n}(a)v_n = 0
\end{aligned}$$

and therefore the second equation in (6.27). Thus part (a) is proved. Obviously,

$$V_{11;n}(z) = U^{(2)}_{11;n}(z) \quad \text{and} \quad V_{21;n}(z) = (b-z)U^{(2)}_{21;n}(z)$$

are valid for all $z \in \mathbb{C}$. Hence, to check part (b) it remains to show that

$$V_{12;n}(z) = \frac{1}{b-z}U^{(2)}_{12;n}(z) \quad \text{and} \quad V_{22;n}(z) = U^{(2)}_{22;n}(z) \tag{6.33}$$

hold for all $z \in \mathbb{C} \setminus \{b\}$. For each $z \in \mathbb{C}$, from (6.14) and (6.22) we see that

$$\begin{aligned}
&U^{(2)}_{12;n}(z) - (b-z)V_{12,n}(z) = (b-a)u_{1,n}^*[R_{T_n}(a)]^*H_{1,n}^{-1}R_{T_n}(a)u_{1,n}\\
&+(z-a)u_{2,n}^*[R_{T_n}(\overline{z})]^*H_{2,n}^{-1}R_{T_n}(a)(v_nM_{2,n}+u_{2,n})\\
&-(b-z)u_{1,n}^*[R_{T_n}(\overline{z})]^*H_{1,n}^{-1}R_{T_n}(a)u_{1,n}
\end{aligned}\tag{6.34}$$

is valid. Because of (6.29) we have

$$[R_{T_n}(b)]^{-1}R_{T_n}(a)R_{T_n}(a)u_{1,n} = -R_{T_n}(a)u_{2,n}. \tag{6.35}$$

Using (6.14), Lemma 2.2, and (6.35) we infer

$$\begin{aligned}
&R_{T_n}(a)(v_nM_{2,n}+u_{2,n})\\
&= (a-b)R_{T_n}(a)v_nu_{1,n}^*[R_{T_n}(a)]^*H_{1,n}^{-1}R_{T_n}(a)u_{1,n} + R_{T_n}(a)u_{2,n}\\
&= \Big(H_{2,n} + [R_{T_n}(b)]^{-1}R_{T_n}(a)H_{1,n}\Big)H_{1,n}^{-1}R_{T_n}(a)u_{1,n} + R_{T_n}(a)u_{2,n}\\
&= H_{2,n}H_{1,n}^{-1}R_{T_n}(a)u_{1,n}.
\end{aligned}\tag{6.36}$$

In view of (6.29) it follows

$$\begin{aligned}
&u_{2,n}^*[R_{T_n}(\overline{z})]^*H_{2,n}^{-1}R_{T_n}(a)(v_nM_{n,2}+u_{2,n})\\
&= -u_{1,n}^*[R_{T_n}(\overline{z})]^*[R_{T_n}(a)]^*[R_{T_n}(b)]^{-*}H_{1,n}^{-1}R_{T_n}(a)u_{1,n}
\end{aligned}\tag{6.37}$$

for each $z \in \mathbb{C}$. From (6.34), (6.37), and (6.32) we see then that

$$\begin{aligned}
&U^{(2)}_{12;n} - (b-z)V_{12;n}(z) \\
&= u^*_{1,n}\Big((b-a)[R_{T_n}(a)]^* - (z-a)[R_{T_n}(\overline{z})]^*[R_{T_n}(a)]^*[R_{T_n}(b)]^{-*} \\
&-(b-z)[R_{T_n}(\overline{z})]^*\Big)H^{-1}_{1,n}R_{T_n}(a)u_{1,n} = 0
\end{aligned}$$

holds for all $z \in \mathbb{C}$. Hence the first identity in (6.33) is verified for each $z \in \mathbb{C} \setminus \{b\}$. The second one follows immediately as well. Indeed, for each $z \in \mathbb{C}$, identity (6.36) implies

$$\begin{aligned}
&U^{(2)}_{22;n} - V_{22;n}(z) \\
&= (z-a)v^*_n[R_{T_n}(\overline{z})]^*\Big[H^{-1}_{2,n}R_{T_n}(a)(v_nM_{2,n} + u_{2,n}) - H^{-1}_{1,n}R_{T_n}(a)u_{1,n}\Big] = 0.
\end{aligned}$$

Thus part (b) is proved. Lemma 6.7 shows that $U_{1,n}(z)$ and $U_{2,n}(z)$ are nonsingular for each $z \in \mathbb{C}$. In view of (6.25) and (6.26), part (c) is also verified. □

In the following, let $V_n, V_{11;n}, V_{12;n}, V_{21;n}$, and $V_{22;n}$ be the matrix polynomials given in (6.20) - (6.24), let $W_{jk;n} := \mathrm{Rstr.}_{\mathbb{C}\setminus[a,b]}V_{jk;n}$ for $j,k \in \{1,2\}$ and let $W_n := \mathrm{Rstr.}_{\mathbb{C}\setminus[a,b]}V_n$.

Lemma 6.11. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian. Let $\binom{P}{Q} \in \mathcal{P}(-\tilde{J}_q,[a,b])$, let $P_1 := W_{11;n}P + W_{12;n}Q$ and let $Q_1 := W_{21;n}P + W_{22;n}Q$. Then* $\det P_1$ *and* $\det Q_1$ *are complex-valued functions which are meromorphic in $\mathbb{C} \setminus [a,b]$ and which do not vanish identically. Moreover, the column pair $\binom{P_1}{Q_1}$ belongs to $\mathcal{P}(-\tilde{J}_q,[a,b])$.*

Proof. According to Definition 5.2, P and Q are $q \times q$ complex matrix-valued functions for which there exists a discrete subset $\mathcal{D}$ of $\mathbb{C} \setminus [a,b]$ such that the conditions (i), (ii), (iii), and (iv) in Definition 5.2 are satisfied. First we are going to show that $\binom{P_1}{Q_1}$ also belongs to $\mathcal{P}(-\tilde{J}_q,[a,b])$. In view of Proposition 6.10 and (i), P_1 and Q_1 are meromorphic in $\mathbb{C} \setminus [a,b]$ and holomorphic in $\mathbb{C} \setminus ([a,b] \cup \mathcal{D})$. By virtue of part (c) of Proposition 6.10 and (ii), we get

$$\operatorname{rank}\begin{pmatrix} P_1(z) \\ Q_1(z) \end{pmatrix} = \operatorname{rank}\left[W_n(z)\begin{pmatrix} P(z) \\ Q(z) \end{pmatrix}\right] = \operatorname{rank}\begin{pmatrix} P(z) \\ Q(z) \end{pmatrix} = q \tag{6.38}$$

for each $z \in \mathbb{C} \setminus ([a,b] \cup \mathcal{D})$. According to Lemma 6.7, for each $k \in \{1,2\}$, the matrix-valued function $U_{k,n}$ given by (6.2) and (6.16) is a $\tilde{J}_q$-inner function of the class $\mathfrak{P}_{\tilde{J}_q}(\Pi_+)$, and hence from Lemma 5.1 we obtain

$$\frac{[U_{k,n}(z)]^*\tilde{J}_qU_{k,n}(z)}{i(z-\overline{z})} \geq \frac{\tilde{J}_q}{i(z-\overline{z})} \tag{6.39}$$

for each $z \in \mathbb{C} \setminus \mathbb{R}$. Using Proposition 6.10 we can see that

$$\begin{aligned}\begin{pmatrix}(z-a)P_1(z)\\ Q_1(z)\end{pmatrix} &= \begin{pmatrix}(z-a)I_q & 0\\ 0 & I_q\end{pmatrix} V_n(z)\begin{pmatrix}P(z)\\ Q(z)\end{pmatrix}\\ &= U_{1,n}(z)\begin{pmatrix}(z-a)I_q & 0\\ 0 & I_q\end{pmatrix}\begin{pmatrix}P(z)\\ Q(z)\end{pmatrix} = U_{1,n}(z)\begin{pmatrix}(z-a)P(z)\\ Q(z)\end{pmatrix} \qquad (6.40)\end{aligned}$$

and

$$\begin{pmatrix}(b-z)P_1(z)\\ Q_1(z)\end{pmatrix} = U_{2,n}(z)\begin{pmatrix}(b-z)P(z)\\ Q(z)\end{pmatrix} \qquad (6.41)$$

are satisfied for all $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. From (6.40), (6.39), and (iii) it follows

$$\frac{1}{2 \operatorname{Im} z}\begin{pmatrix}(z-a)P_1(z)\\ Q_1(z)\end{pmatrix}^*(-\tilde{J}_q)\begin{pmatrix}(z-a)P_1(z)\\ Q_1(z)\end{pmatrix} \geq 0$$

for each $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathcal{D})$. Similarly, using (6.41), (6.39), and (iv) we get

$$\frac{1}{2 \operatorname{Im} z}\begin{pmatrix}(b-z)P_1(z)\\ Q_1(z)\end{pmatrix}^*(-\tilde{J}_q)\begin{pmatrix}(b-z)P_1(z)\\ Q_1(z)\end{pmatrix} \geq 0$$

for all $z \in \mathbb{C}\setminus(\mathbb{R}\cup\mathcal{D})$. Hence $\binom{P_1}{Q_1}$ belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$. Now let $z \in \mathbb{C}\setminus(\mathbb{R}\cup\mathcal{D})$. From Lemma 6.7 we know that $\det U_{1,n}$ does not vanish in $\mathbb{C}$. Therefore, in view of (6.40) we have

$$\begin{pmatrix}(z-a)P(z)\\ Q(z)\end{pmatrix} = [U_{1,n}(z)]^{-1}\begin{pmatrix}(z-a)P_1(z)\\ Q_1(z)\end{pmatrix}. \qquad (6.42)$$

From (iii) and (6.42) we can conclude

$$\frac{1}{i(z-\overline{z})}\begin{pmatrix}(z-a)P_1(z)\\ Q_1(z)\end{pmatrix}^*[U_{1,n}(z)]^{-*}\tilde{J}_q[U_{1,n}(z)]^{-1}\begin{pmatrix}(z-a)P_1(z)\\ Q_1(z)\end{pmatrix} \geq 0.$$

For each $g \in \mathcal{N}[P_1(z)] := \{h \in \mathbb{C}^q : P_1(z)h = 0\}$, this implies

$$\frac{1}{i(z-\overline{z})}\begin{pmatrix}0\\ Q_1(z)g\end{pmatrix}^*[U_{1,n}(z)]^{-*}\tilde{J}_q[U_{1,n}(z)]^{-1}\begin{pmatrix}0\\ Q_1(z)g\end{pmatrix} \geq 0.$$

Since

$$\frac{1}{i(z-\overline{z})}\begin{pmatrix}0\\ Q_1(z)g\end{pmatrix}^*\tilde{J}_q\begin{pmatrix}0\\ Q_1(z)g\end{pmatrix} = 0$$

holds for all $g \in \mathbb{C}^q$, we see then that

$$\begin{pmatrix}0\\ Q_1(z)g\end{pmatrix}^*\frac{\tilde{J}_q - [U_{1,n}(z)]^{-*}\tilde{J}_q[U_{1,n}(z)]^{-1}}{i(z-\overline{z})}\begin{pmatrix}0\\ Q_1(z)g\end{pmatrix} \leq 0 \qquad (6.43)$$

is true for each $g \in \mathcal{N}[P_1(z)]$. On the other hand, Lemma 6.7 and part (b) of Lemma 6.3 provide us

$$\begin{aligned}&\frac{\tilde{J}_q - [U_{1,n}(z)]^{-*}\tilde{J}_q[U_{1,n}(z)]^{-1}}{i(z-\overline{z})}\\ &= \tilde{J}_q(u_{1,n}, v_n)^*[R_{T_n}(z)]^*H_{1,n}^{-1}R_{T_n}(z)(u_{1,n}, v_n)\tilde{J}_q. \qquad (6.44)\end{aligned}$$

Since the matrix $H_{1,n}$ is positive Hermitian, the right-hand side of (6.44) is nonnegative Hermitian. In view of (6.43), for each $g \in \mathcal{N}[P_1(z)]$, thus we get

$$\begin{pmatrix} 0 \\ Q_1(z)g \end{pmatrix}^* \tilde{J}_q(u_{1,n}, v_n)^*[R_{T_n}(z)]^* H_{1,n}^{-1} R_{T_n}(z)(u_{1,n}, v_n)\tilde{J}_q \begin{pmatrix} 0 \\ Q_1(z)g \end{pmatrix} = 0$$

and, in view of $\det R_{T_n}(z) \neq 0$, then

$$0 = (u_{1,n}, v_n)\tilde{J}_q \begin{pmatrix} 0 \\ Q_1(z)g \end{pmatrix} = -iu_{1,n}Q_1(z)g.$$

According to (1.9) this implies $s_0 Q_1(z)g = 0$ for all $g \in \mathcal{N}[P_1(z)]$. Since $H_{1,n}$ is positive Hermitian, the matrix s_0 is nonsingular. Hence

$$\begin{pmatrix} P_1(z) \\ Q_1(z) \end{pmatrix} g = 0$$

for all $g \in \mathcal{N}[P_1(z)]$. Thus (6.38) shows $\mathcal{N}[P_1(z)] = \{0\}$. Hence the matrix $P_1(z)$ is nonsingular. Analogously, one can check that the matrix $Q_1(z)$ is nonsingular. The proof is complete. $\square$

Now we are able to prove the main result of this section.

Theorem 6.12. *Let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian.*

(a) *For each $\binom{P}{Q} \in \mathcal{P}(-\tilde{J}_q, [a,b])$, the matrix-valued function*

$$S := (W_{11;n}P + W_{12;n}Q)(W_{21;n}P + W_{22;n}Q)^{-1}$$

belongs to $\mathcal{R}\left[[a,b]; (s_j)_{j=1}^{2n+1}\right]$.

(b) *For each $S \in \mathcal{R}_q\left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$, there is a column pair $\binom{P}{Q} \in \mathcal{P}(-\tilde{J}_q, [a,b])$ of matrix-valued functions P and Q which are holomorphic in $\mathbb{C} \setminus [a,b]$ such that S admits the representation*

$$S = (W_{11;n}P + W_{12;n}Q)(W_{21;n}P + W_{22;n}Q)^{-1}.$$

(c) *If $\binom{P_1}{Q_1}$ and $\binom{P_2}{Q_2}$ belong to $\mathcal{P}(-\tilde{J}_q, [a,b])$, then*

$$\begin{aligned}&(W_{11;n}P_1 + W_{12;n}Q_1)(W_{21;n}P_1 + W_{22;n}Q_1)^{-1}\\ &= (W_{11;n}P_2 + W_{12;n}Q_2)(W_{21;n}P_2 + W_{22;n}Q_2)^{-1}\end{aligned} \tag{6.45}$$

if and only if

$$\left\langle \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} \right\rangle. \tag{6.46}$$

Proof. (a) Let $\binom{P}{Q} \in \mathcal{P}(-\tilde{J}_q, [a,b])$. By virtue of Lemma 6.11, then $\binom{P_1}{Q_1}$ defined by $P_1 := W_{11;n}P + W_{12;n}Q$ and $Q_1 := W_{21;n}P + W_{22;n}Q$ also belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$ and, moreover, the function $\det Q_1$ does not vanish identically in $\mathbb{C} \setminus [a,b]$. From Lemma 5.7 it follows that $S := P_1 Q_1^{-1}$ belongs to $\mathcal{R}_q[a,b]$. One can easily see that the column pair $\binom{\tilde{P}}{\tilde{Q}}$ given by $\tilde{P} := PQ_1^{-1}$ and $\tilde{Q} := QQ_1^{-1}$ also belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$. Obviously,

$$\begin{pmatrix} S \\ I_q \end{pmatrix} = \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} Q_1^{-1} = W_n \begin{pmatrix} P \\ Q \end{pmatrix} Q_1^{-1} = W_n \begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix}$$

holds. In view of part (c) of Proposition 6.10, it follows

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = W_n^{-1} \begin{pmatrix} S \\ I_q \end{pmatrix}. \tag{6.47}$$

Proposition 6.10 yields that $\binom{\tilde{P}}{\tilde{Q}}$ is holomorphic in $\mathbb{C} \setminus [a,b]$. Since $\binom{\tilde{P}}{\tilde{Q}}$ belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$, we have then

$$\frac{1}{2 \operatorname{Im} z} \begin{pmatrix} (z-a)\tilde{P}(z) \\ \tilde{Q}(z) \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} (z-a)\tilde{P}(z) \\ \tilde{Q}(z) \end{pmatrix} \geq 0 \tag{6.48}$$

and

$$\frac{1}{2 \operatorname{Im} z} \begin{pmatrix} (b-z)\tilde{P}(z) \\ \tilde{Q}(z) \end{pmatrix}^* (-\tilde{J}_q) \begin{pmatrix} (b-z)\tilde{P}(z) \\ \tilde{Q}(z) \end{pmatrix} \geq 0 \tag{6.49}$$

for each $z \in \mathbb{C} \setminus \mathbb{R}$. From (6.47), (1.10), Lemma 6.7, and Proposition 6.10 we get

$$\begin{pmatrix} (z-a)\tilde{P}(z) \\ \tilde{Q}(z) \end{pmatrix} = [U_{1,n}(z)]^{-1} \begin{pmatrix} \tilde{S}_1(z) \\ I \end{pmatrix} \tag{6.50}$$

and

$$\begin{pmatrix} (b-z)\tilde{P}(z) \\ \tilde{Q}(z) \end{pmatrix} = [U_{2,n}(z)]^{-1} \begin{pmatrix} \tilde{S}_2(z) \\ I \end{pmatrix} \tag{6.51}$$

for each $z \in \mathbb{C} \setminus [a,b]$. Thus from (6.48), (6.49), (6.50), and (6.51) we see that inequality (6.19) is satisfied for each $k \in \{1,2\}$ and each $z \in \mathbb{C} \setminus \mathbb{R}$. Applying Proposition 6.8 it follows that S belongs to $\mathcal{R}_q \left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$.

(b) Now we consider an arbitrary matrix-valued function S which belongs to $\mathcal{R}_q \left[[a,b]; (s_j)_{j=0}^{2n+1}\right]$. Let

$$\tilde{P} := (I_q, 0) W_n^{-1} \begin{pmatrix} S \\ I \end{pmatrix} \quad \text{and} \quad \tilde{Q} := (0, I_q) W_n^{-1} \begin{pmatrix} S \\ I \end{pmatrix}. \tag{6.52}$$

From Proposition 6.10 we see that the matrix-valued function W_n^{-1} is holomorphic in $\mathbb{C} \setminus [a,b]$. Hence $\tilde{P}$ and $\tilde{Q}$ are also holomorphic in $\mathbb{C} \setminus [a,b]$ and we obtain

$$\operatorname{rank} \begin{pmatrix} \tilde{P}(z) \\ \tilde{Q}(z) \end{pmatrix} = \operatorname{rank} \begin{pmatrix} S(z) \\ I \end{pmatrix} = q \tag{6.53}$$

for each $z \in \mathbb{C} \setminus [a,b]$. Using (6.52), Lemma 6.7, and Proposition 6.10 it is readily checked that the identities (6.50) and (6.51) are fulfilled for all $z \in \mathbb{C} \setminus [a,b]$. Since from Proposition 6.8 we know that inequality (6.19) holds for each $k \in \{1,2\}$ and each $z \in \mathbb{C} \setminus \mathbb{R}$ it follows then that the inequalities (6.48) and (6.49) are satisfied for all $z \in \mathbb{C} \setminus \mathbb{R}$. In view of (6.53) thus we see that $\binom{\tilde{P}}{\tilde{Q}}$ belongs to $\mathcal{P}(-\tilde{J}_q, [a,b])$. From (6.52) we obtain

$$\begin{pmatrix} S \\ I \end{pmatrix} = W_n \begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = \begin{pmatrix} W_{11;n}\tilde{P} + W_{12;n}\tilde{Q} \\ W_{21;n}\tilde{P} + W_{22;n}\tilde{Q} \end{pmatrix}$$

and therefore

$$S = S \cdot I_q^{-1} = (W_{11;n}\tilde{P} + W_{12;n}\tilde{Q})(W_{21;n}\tilde{P} + W_{22;n}\tilde{Q})^{-1}.$$

(c) Let $\binom{P_1}{Q_1}$ and $\binom{P_2}{Q_2}$ belong to $\mathcal{P}(-\tilde{J}_q, [a,b])$. Obviously,

$$\begin{pmatrix} W_{11;n}P_k + W_{12;n}Q_k \\ W_{21;n}P_k + W_{22;n}Q_k \end{pmatrix} = W_n \begin{pmatrix} P_k \\ Q_k \end{pmatrix}$$

for each $k \in \{1,2\}$. In view of part (c) of Proposition 6.10 and Lemma 6.11 this implies

$$\begin{aligned} \binom{P_k}{Q_k} &= W_n^{-1} \begin{pmatrix} W_{11;n}P_k + W_{12;n}Q_k \\ W_{21;n}P_k + W_{22;n}Q_k \end{pmatrix} \\ &= W_n^{-1} \begin{pmatrix} (W_{11;n}P_k + W_{12;n}Q_k)(W_{21;n}P_k + W_{22;n}Q_k)^{-1} \\ I \end{pmatrix} (W_{21;n}P_k + W_{22;n}Q_k) \end{aligned} \tag{6.54}$$

for each $k \in \{1,2\}$. Now suppose that (6.45) holds. From (6.54) we get then

$$\begin{aligned} &\begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} \\ &= W_n^{-1} \begin{pmatrix} (W_{11;n}P_1 + W_{12,n}Q_1)(W_{21;n}P_1 + W_{22;n}Q_1)^{-1} \\ I \end{pmatrix} (W_{21;n}P_2 + W_{22;n}Q_2) \\ &= \begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} (W_{21;n}P_1 + W_{22;n}Q_1)^{-1}(W_{21;n}P_2 + W_{22;n}Q_2) = \begin{pmatrix} P_1F \\ Q_1F \end{pmatrix} \end{aligned}$$

where $F := (W_{21;n}P_1 + W_{22;n}Q_1)^{-1}(W_{21,n}P_2 + W_{22,n}Q_2)$ is a matrix-valued function which is meromorphic in $\mathbb{C} \setminus [a,b]$. Moreover, from Lemma 6.11 we know that $\det F$ does not vanish identically. Hence (6.46) holds.

Conversely, now assume that (6.46) is satisfied. Then there is a matrix-valued function F which is meromorphic in $\mathbb{C} \setminus [a,b]$ such that $\det F$ does not vanish identically and that $P_2 = P_1F$ and $Q_2 = Q_1F$ hold. Then (6.45) immediately follows. □

Corollary 6.13. *If $(s_j)_{j=0}^{2n+1}$ is a sequence of complex $q \times q$ matrices such that the matrices $H_{1,n}$ and $H_{2,n}$ are both positive Hermitian, then*

$$\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_j)_{j=0}^{2n+1}\right] \neq \emptyset.$$

Proof. In view of Remark 5.4, apply Theorem 6.12. □

7. A necessary and sufficient condition for the existence of a solution of the moment problem

In this section, we turn our attention to a characterization of the case that the matricial moment problem on a finite interval considered in this paper has a solution.

Remark 7.1. *Let $(s_j^{(1)})_{j=0}^{2n+1}$ and $(s_j^{(2)})_{j=0}^{2n+1}$ be sequences of complex $q \times q$ matrices, let α be a positive real number, and let $r_j := s_j^{(1)} + \alpha s_j^{(2)}$ for each integer j with $0 \leq j \leq 2n+1$. For $m \in \{1,2\}$, let*

$$\tilde{H}_{1,n}^{(m)} := (s_{j+k}^{(m)})_{j,k=0}^{n}, \quad \tilde{H}_{2,n}^{(m)} := (s_{j+k+1}^{(m)})_{j,k=0}^{n},$$

$$H_{1,n}^{(m)} := -a\tilde{H}_{1,n}^{(m)} + \tilde{H}_{2,n}^{(m)}, \quad \text{and} \quad H_{2,n}^{(m)} := b\tilde{H}_{1,n}^{(m)} - \tilde{H}_{2,n}^{(m)}.$$

Suppose that the block Hankel matrices $H_{1,n}^{(1)}$ and $H_{2,n}^{(1)}$ are both nonnegative Hermitian and that the block Hankel matrices $H_{1,n}^{(2)}$ and $H_{2,n}^{(2)}$ are both positive Hermitian. Then the block Hankel matrices $(-ar_{j+k} + r_{j+k+1})_{j,k=0}^{n}$ and $(br_{j+k} - r_{j+k+1})_{j,k=0}^{n}$ are positive Hermitian as well.

Now we verify a further main result which was already formulated in Theorem 1.3 (see Section 1).

Proof of Theorem 1.3 If $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b];\ (s_j)_{j=0}^{2n+1}\right]$ is nonempty, then Remark 3.4 shows that the block Hankel matrices $H_{1,n}$ and $H_{2,n}$ are both necessarily nonnegative Hermitian. Conversely, we suppose now that the matrices $H_{1,n}$ and $H_{2,n}$ are nonnegative Hermitian. In view of Lemma 6.1, let $(r_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$ matrices such that the block Hankel matrices $(-ar_{j+k} + r_{j+k+1})_{j,k=0}^{n}$ and $(br_{j+k} - r_{j+k+1})_{j,k=0}^{n}$ are both positive Hermitian. For each real number ε which satisfies $0 < \varepsilon \leq 1$ and each integer j which satisfies $0 \leq j \leq 2n+1$, let $s_{j,\varepsilon} := s_j + \varepsilon r_j$. According to Remark 7.1, for each $\varepsilon \in (0,1]$, the block Hankel matrices $(as_{j+k,\varepsilon} + s_{j+k+1,\varepsilon})_{j,k=0}^{n}$ and $(bs_{j+k,\varepsilon} - s_{j+k+1,\varepsilon})_{j,k=0}^{n}$ are both positive Hermitian. From Corollary 6.13 we see then that for each $\varepsilon \in (0,1]$ the set $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_{j,\varepsilon})_{j=0}^{2n+1}\right]$ is nonempty. Now let $(\varepsilon_m)_{m=1}^{\infty}$ be a sequence of real numbers belonging to the interval $(0,1]$ which satisfies

$$\lim_{m\to\infty} \varepsilon_m = 0.$$

For each positive integer m, we can choose then a nonnegative Hermitian $q \times q$ measure σ_m which belongs to $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_{j,\varepsilon_m})_{j=0}^{2n+1}\right]$. Using the notation given in (3.1), we have $s_j^{[\sigma_m]} = s_{j,\varepsilon_m}$ for all positive integers m and all integers j which satisfy $0 \leq j \leq 2n+1$. Obviously, it follows

$$\sigma_m([a,b]) = s_0^{[\sigma_m]} = s_{0,\varepsilon_m} = s_0 + \varepsilon_m r_0 \leq s_0 + r_0 \tag{7.1}$$

for all positive integers m. In view of (7.1), application of the matricial version of the Helly-Prohorov theorem (see [FK, Satz 9]) provides us that there are a subsequence $(\sigma_{m_k})_{k=1}^{\infty}$ of the sequence $(\sigma_m)_{m=1}^{\infty}$ and a nonnegative Hermitian $q \times q$

measure $\sigma \in \mathcal{M}_{\geq}^{q}[[a,b], \mathfrak{B} \cap [a,b]]$ such that $(\sigma_{m_k})_{k=1}^{\infty}$ converges weakly to σ, i.e., such that

$$\lim_{k\to\infty} \int_{[a,b]} f d\sigma_{m_k} = \int_{[a,b]} f d\sigma$$

is satisfied for all continuous complex-valued functions defined on $[a,b]$. Therefore we can conclude then

$$s_j^{[\sigma]} = \lim_{k\to\infty} s_j^{[\sigma_{m_k}]} = \lim_{k\to\infty} (s_j + \varepsilon_{m_k} r_j) = s_j$$

for every integer j which satisfies $0 \leq j \leq 2n+1$. Hence σ belongs to $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_j)_{j=0}^{2n+1}\right]$. In particular, $\mathcal{M}_{\geq}^{q}\left[[a,b], \mathfrak{B} \cap [a,b];\ (s_j)_{j=0}^{2n+1}\right] \neq \emptyset$. $\square$

Finally, let us give a remark concerning the scalar case $q = 1$. M.G. Krein [Kr2, Theorem 4.2, p. 48] (see also [KN, Theorem 4.1, p. 110–111]) showed that the set $\mathcal{M}_{\geq}^{1}\left[[a,b], \mathfrak{B} \cap [a,b]; (s_j)_{j=0}^{2n+1}\right]$ contains exactly one measure if and only if $H_{1,n}$ and $\tilde{H}_{2,n}$ are both nonnegative Hermitian matrices and at least one of them is nonsingular.

8. Appendix: Certain subclasses of holomorphic matrix-valued functions and a generalization of Stieltjes' inversion formula

Our investigations in this paper heavily lean on various classes of holomorphic matrix-valued functions. Therefore, we summarize now some material on this topic. For a comprehensive treatment of this subject we refer the reader to the paper [GT] and the references cited therein. Let $\mathcal{R}_q$ be the set of all matrix-valued functions $F : \Pi_+ \to \mathbb{C}^{q\times q}$ which are holomorphic in Π_+ and which satisfy $\operatorname{Im} F(w) \geq 0$ for each $w \in \Pi_+$. Obviously, if a and b are real numbers with $a < b$, then for each $S \in \mathcal{R}_q[a,b]$ the matrix-valued function $S^{\square} := \mathrm{Rstr.}_{\Pi_+} S$ belongs to $\mathcal{R}_q$. Every function F which belongs to $\mathcal{R}_q$ admits a unique integral representation which in the scalar case is due to R. Nevanlinna.

Theorem 8.1.

(a) *For every matrix-valued function F which belongs to the class $\mathcal{R}_q$, there are a unique Hermitian complex $q \times q$ matrix α, a unique nonnegative Hermitian complex matrix β, and a unique nonnegative Hermitian $q \times q$ measure $\nu \in \mathcal{M}_{\geq}^{q}(\mathbb{R}, \mathfrak{B} \cap \mathbb{R})$ such that*

$$F(z) = \alpha + \beta z + \int_{\mathbb{R}} \frac{1+tz}{t-z} \nu(dt) \tag{8.1}$$

is satisfied for each $z \in \Pi_+$.

(b) *Every matrix-valued function $F : \Pi_+ \to \mathbb{C}^{q\times q}$ for which there exist a Hermitian complex $q \times q$ matrix α, a nonnegative Hermitian complex $q \times q$ matrix β, and a nonnegative Hermitian $q \times q$ measure $\nu \in \mathcal{M}_{\geq}^{q}(\mathbb{R}, \mathfrak{B} \cap \mathbb{R})$ such that* (8.1) *is satisfied for all $z \in \Pi_+$ belongs to the class $\mathcal{R}_q$.*

This matricial version of Nevanlinna's famous theorem can be proved using the classical version of the theorem in the case $q = 1$ and the fact that, for each $F \in \mathcal{R}_q$ and each $u \in \mathbb{C}^q$, the function $f_u := u^* F u$ belongs to $\mathcal{R}_1$. We omit the details. For each $F \in \mathcal{R}_q$, we will call (α, β, ν) given in (8.1) the *Nevanlinna parametrization of* F and in particular the unique nonnegative Hermitian $q \times q$ measure ν on $\mathfrak{B} \cap \mathbb{R}$ described in part (a) of Theorem 8.1 the *Nevanlinna measure of* F.

Let λ denote the Lebesgue measure which is defined on $\mathfrak{B} \cap \mathbb{R}$. Further, let $\mathcal{B}_0$ designate the system of all bounded sets which belong to $\mathfrak{B} \cap \mathbb{R}$. Observe that for each $B \in \mathcal{B}_0$ and each $\nu \in \mathcal{M}^q_{\geq}(\mathbb{R}, \mathfrak{B} \cap \mathbb{R})$, it is readily checked that the function $f_B : \mathbb{R} \to \mathbb{C}^{q \times q}$ defined for each $t \in \mathbb{R}$ by

$$f_B(t) := 1_B(t)\sqrt{1 + t^2} I_q$$

belongs to $q \times q - \mathcal{L}^2(\mathbb{R}, \mathfrak{B} \cap \mathbb{R}, \nu)$. Now we formulate a matricial version of the Stieltjes-Perron inversion formula.

Theorem 8.2. *Let F belong to $\mathcal{R}_q$. Let ν be the Nevanlinna measure of F and let $\mu : \mathcal{B}_0 \to \mathbb{C}^{q \times q}$ be for all $B \in \mathcal{B}_0$ be defined by*

$$\mu(B) := \int_B \left(\sqrt{1 + t^2} I_q\right) \nu(dt) \left(\sqrt{1 + t^2} I_q\right)^* . \tag{8.2}$$

Further, let a and b be real numbers such that $a < b$. Then

$$\frac{1}{\pi} \lim_{\varepsilon \to 0+0} \int_{[a,b]} \operatorname{Im} F(x + i\varepsilon) \lambda(dx) = \mu((a, b)) + \frac{1}{2} \left(\mu(\{a\}) + \mu(\{b\})\right).$$

Since for each $u \in \mathbb{C}^p$ and each $F \in \mathcal{R}_q$ the function $f_u := u^* F u$ belongs to $\mathcal{R}_1$, Theorem 8.2 can be easily verified using the scalar version of Theorem 8.2, which is proved, e.g., in [KN, Appendix, Chapter 1]. We again omit the details.

Proposition 8.3. *Let M be a finite union of open intervals of $\mathbb{R}$ and let $\varphi : \Pi_+ \cup M \cup \Pi_- \to \mathbb{C}^{q \times q}$ be a matrix-valued function which satisfies the following conditions:*

(i) *φ is holomorphic in $\Pi_+ \cup M \cup \Pi_-$.*
(ii) $\mathrm{Rstr.}_{\Pi_+} \varphi \in \mathcal{R}_q$.
(iii) *For all $x \in M$, the matrix $\varphi(x)$ is Hermitian.*

Denote (α, β, ν) the Nevanlinna parametrization of $\mathrm{Rstr.}_{\Pi_+} \varphi$. *Then*

$$\varphi(z) = \alpha + \beta z + \int_{\mathbb{R} \setminus M} \frac{1 + tz}{t - z} \nu(dt)$$

for all $z \in \Pi_+ \cup M \cup \Pi_-$.

Proof. Let c and d be real numbers such that $c < d$ and $(c, d) \subseteq M$ hold. We show that $\nu((c, d)) = 0$. We consider an arbitrary vector $u \in \mathbb{C}^q$. Let $\varphi_u := u^* \varphi u$. Then $\tilde{\varphi}_u := \mathrm{Rstr.}_{\Pi_+} \varphi_u$ belongs to $\mathcal{R}_1$ and $\nu_u := u^* \nu u$ is the Nevanlinna measure of $\tilde{\varphi}_u$. Let $\rho := \frac{d-c}{4}$. Denoting $c_m := c + \frac{\rho}{m}$ and $d_m := d - \frac{\rho}{m}$ for all $m \in \mathbb{N}$ we obtain $[c_m, d_m] \subseteq (c, d)$ and $\bigcup_{m=1}^{\infty} [c_m, d_m] = (c, d)$. Then φ_u is bounded on the set $\mathcal{D}_m := \{z \in \mathbb{C} : \text{ Re } z \in [c_m, d_m], \text{ Im } z \in [0,1]\}$. Let $(\varepsilon_k)_{k=1}^{\infty}$ be a decreasing sequence of real numbers belonging to the interval $(0,1]$ and satisfying $\lim_{k\to\infty} \varepsilon_k = 0$. Using Theorem 8.2, Lebesgue's dominated convergence theorem and (iii) we can conclude then

$$\begin{aligned}
&\int_{(c_m,d_m)} (1+t^2)\nu_u(dt) + \frac{1}{2}\left((1+c_m^2)\nu_u(\{c_m\}) + (1+d_m^2)\nu_u(\{d_m\})\right) \\
&= \frac{1}{\pi} \lim_{k\to\infty} \int_{[c_m,d_m]} \text{Im}\, \varphi_u(x+i\varepsilon_k)\lambda(dx) = \frac{1}{\pi} \int_{[c_m,d_m]} \lim_{k\to\infty} \text{Im}\, \varphi_u(x+i\varepsilon_k)\lambda(dx) \\
&= 0
\end{aligned}$$

and consequently $\nu_u([c_m, d_m]) = 0$. Hence $\nu_u((c,d)) = 0$ follows. Since u was chosen arbitrarily in $\mathbb{C}^q$, we have then $\nu((c,d)) = 0$. Applying Theorem 8.1 the proof is finished. □

Remark 8.4. *Let $S \in \mathcal{R}_q[a,b]$. Using the Stieltjes-Perron inversion formula one can show similarly to the proof of Proposition* 8.3 *that the Nevanlinna measure ν of S satisfies $\nu(\mathbb{R} \setminus [a,b]) = 0$.*

Let $\mathcal{R}'_q$ be the set of all $F \in \mathcal{R}_q$ for which, if ν denotes the Nevanlinna measure associated with F, the matrix-valued function $f : \mathbb{R} \to \mathbb{C}^{q\times q}$ given by $f(t) := \sqrt{1+t^2}$ belongs to $q \times q - \mathcal{L}^2(\mathbb{R}, \mathfrak{B} \cap \mathbb{R}, \nu)$. In view of Remark 3.1, for each $F \in \mathcal{R}'_q$, then $\mu : \mathfrak{B} \cap \mathbb{R} \to \mathbb{C}^{q\times q}$ defined for all $B \in \mathfrak{B} \cap \mathbb{R}$ by (8.2) belongs to $\mathcal{M}_{\geq}^q(\mathbb{R}, \mathfrak{B} \cap \mathbb{R})$ and this nonnegative Hermitian measure μ is called the *spectral measure of* F. In order to prove a generalized version of Stieltjes' inversion formula we will use a result on integrals (with respect to nonnegative Hermitian measures), which depend on a parameter.

Proposition 8.5. *Let $\mathcal{K}$ be a metric space, let $(\Omega, \mathfrak{A})$ be a measurable space, let $\mu \in \mathcal{M}_{\geq}^q(\Omega, \mathfrak{A})$ and let $\zeta_0 \in \mathcal{K}$. Further, let $\Gamma : \mathcal{K} \times \Omega \to \mathbb{C}^{p\times q}$ and $\triangle : \mathcal{K} \times \Omega \to \mathbb{C}^{r\times q}$ be mappings which satisfy the following three conditions:*

(i) *For every choice of ζ in $\mathcal{K}$, the pair $[\Gamma_{\zeta\bullet}, \triangle_{\zeta\bullet}]$ consisting of the matrix-valued functions $\Gamma_{\zeta\bullet} : \Omega \to \mathbb{C}^{p\times q}$ and $\triangle_{\zeta\bullet} : \Omega \to \mathbb{C}^{r\times q}$ defined by $\Gamma_{\zeta\bullet}(\omega) := \Gamma(\zeta,\omega)$ and $\triangle_{\zeta\bullet}(\omega) := \triangle(\zeta,\omega)$ are both left-integrable with respect to μ.*

(ii) *For every choice of ω in Ω, the matrix-valued functions $\Gamma_{\bullet\omega} : \mathcal{K} \to \mathbb{C}^{p\times q}$ and $\triangle_{\bullet\omega} : \mathcal{K} \to \mathbb{C}^{r\times q}$ given by $\Gamma_{\bullet\omega}(\zeta) := \Gamma(\zeta,\omega)$ and $\triangle_{\bullet\omega}(\zeta) := \triangle(\zeta,\omega)$ are continuous in the point ζ_0.*

(iii) *There are real numbers C and D such that*

$$\|\Gamma(\zeta,\omega)\|_E \leq C \quad \text{and} \quad \| \triangle (\zeta,\omega)\|_E \leq D$$

hold for all $\zeta \in \mathcal{K}$ and all $\omega \in \Omega$.

Then the matrix-valued function $H : \mathcal{K} \to \mathbb{C}^{p\times r}$ defined by

$$H(\zeta) := \int_{\Lambda} \Gamma(\zeta,\omega)\mu(d\omega)\left(\triangle(\zeta,\omega)\right)^*$$

is continuous in ζ_0.

Proposition 8.5 follows from the corresponding result in the scalar case $p = q = r = 1$ (see, e.g., [E, 5.6]) by using standard techniques.

Now we turn our attention to the announced generalized inversion formula.

Theorem 8.6. *Let $F \in \mathcal{R}_q$ and let ν be the Nevanlinna measure of F. Let $\Phi : \mathbb{C} \to \mathbb{C}^{p\times q}$ be a matrix-valued function which is holomorphic in $\mathbb{C}$ and let $\Psi : \mathbb{C} \to \mathbb{C}^{p\times p}$ be a matrix-valued function which is continuous in $\mathbb{C}$ and which satisfies $\Psi^*(t) = \Psi(t)$ for all $t \in \mathbb{R}$. Let $G : \Pi_+ \to \mathbb{C}^{p\times p}$ be for each $w \in \Pi_+$ be defined by*

$$G(w) := \Psi(w) + \Phi(w)F(w)\Phi^*(\overline{w}), \tag{8.3}$$

and let a and b be real numbers which satisfy $a < b$. Then

$$\begin{aligned}\frac{1}{\pi}\lim_{\varepsilon\to 0+0}\int_{[a,b]} \operatorname{Im} G(x+i\varepsilon)\lambda(dx) &= \int_{(a,b)} \sqrt{1+t^2}\Phi(t)\nu(dt)\left(\sqrt{1+t^2}\Phi(t)\right)^* \\ &+\frac{1}{2}\left[(1+a^2)\Phi(a)\nu(\{a\})\Phi^*(a) + (1+b^2)\Phi(b)\nu(\{b\})\Phi^*(b)\right].\end{aligned} \tag{8.4}$$

If F moreover belongs to the subclass $\mathcal{R}'_q$ of $\mathcal{R}_q$ the right-hand side of (8.4) is equal to

$$\int_{(a,b)} \Phi d\mu \Phi^* + \frac{1}{2}\big(\Phi(a)\mu(\{a\})\Phi^*(a) + \Phi(b)\mu(\{b\})\Phi^*(b)\big)$$

where μ denotes the spectral measure of F.

Proof. Let $c := a-1$ and $d := b+1$. Since Φ is continuous on $\mathbb{C}$ the matrix-valued $\Phi_1 : \mathbb{R} \to \mathbb{C}^{p\times q}$ given by $\Phi_1(t) := 1_{[c,d]}(t)\Phi(t)$ is Borel measurable and bounded. Hence Φ_1 belongs to $p \times q - \mathcal{L}^2(\mathbb{R}, \mathfrak{B}\cap\mathbb{R}, \nu)$. Thus $\rho : \mathfrak{B}\cap\mathbb{R} \to \mathbb{C}^{p\times p}$ defined by

$$\rho(B) := \int_B 1_{[c,d]}\Phi d\nu \left(1_{[c,d]}\Phi\right)^* \tag{8.5}$$

belongs to $\mathcal{M}^p_{\geq}(\mathbb{R}, \mathfrak{B}\cap\mathbb{R})$. For each $z \in \mathbb{C}\setminus\mathbb{R}$ and each $t \in \mathbb{R}$ we have

$$\left|\frac{1+tz}{t-z}\right| \leq \frac{|1+z^2|}{|t-z|} + |z| \tag{8.6}$$

and $|t-z| \geq |\mathrm{Im}\, z| > 0$. Consequently, for each $\zeta \in \Pi_+$, the integral

$$g(\zeta) := \int_{\mathbb{R}} \frac{1+t\zeta}{t-\zeta}\rho(dt) \tag{8.7}$$

exists. In other words, $g : \Pi_+ \to \mathbb{C}^{p\times p}$ given by (8.7) is a well-defined matrix-valued function. Let

$$\mathcal{K} := \{z \in \mathbb{C} : a \leq \operatorname{Re} z \leq b \text{ and } -1 \leq \operatorname{Im} z \leq 1\}$$

and let $w \in \mathcal{K}$. Since Φ is holomorphic in $\mathbb{C}$ there is a matrix-valued function $\hat{\Phi}_w : \mathbb{C} \to \mathbb{C}^{p\times q}$ which is continuous on $\mathbb{C}$ such that

$$\Phi(z) = \Phi(w) + (z-w)\hat{\Phi}_w(z) \tag{8.8}$$

holds for all $z \in \mathbb{C}$. Obviously, the function $\varphi_{1,w} : \mathbb{R} \to \mathbb{C}^{p\times q}$ defined by

$$\varphi_{1,w}(t) := (1+tw)1_{[c,d]}(t)\hat{\Phi}_w(t)$$

is Borel measurable and bounded. Hence $\varphi_{1,w}$ belongs to $p \times q - \mathcal{L}^2(\mathbb{R}, \mathfrak{B} \cap \mathbb{R}, \nu)$. Similarly, we can see that $\varphi_{2,w} : \mathbb{R} \to \mathbb{C}^{q\times q}$ given by

$$\varphi_{2,w}(t) := 1_{[c,d]}(t)\Phi(\overline{w})$$

belongs to $p \times q - \mathcal{L}^2(\mathbb{R}, \mathfrak{B} \cap \mathbb{R}, \nu)$. For all $t \in \mathbb{R} \setminus [a,b]$ we have $|t-w| \geq 1$ and hence, in view of (8.6), the Borel measurable mapping $\chi_{1,w} : \mathbb{R} \to \mathbb{C}^{q\times q}$ given by

$$\chi_{1,w}(t) := \begin{cases} \frac{1+tw}{t-w} I_q & , \quad t \in \mathbb{R} \setminus [a,b] \\ 0_{q\times q} & , \quad t \in [a,b] \end{cases}$$

satisfies $\|\chi_{1,w}(t)\|_E \leq \sqrt{q(|1+w^2|+|w|)}$ for all $t \in \mathbb{R}$. Therefore $\chi_{1,w}$ belongs to $q \times q - \mathcal{L}^2(\mathbb{R}, \mathfrak{B} \cap \mathbb{R}, \nu)$. Clearly, we also have $1_{\mathbb{R}\setminus[a,b]} I_q \in q \times q - \mathcal{L}^2(\mathbb{R}, \mathfrak{B} \cap \mathbb{R}, \nu)$. Thus we can conclude that all the pairs $[\varphi_{1,w}, \varphi_{2,w}]$, $[\Phi, \varphi_{1,\overline{w}}]$, and $[\chi_{1,w}, 1_{\mathbb{R}\setminus[a,b]} I_q]$ are left-integrable with respect to ν. Therefore all the mappings $\varphi : \mathcal{K} \to \mathbb{C}^{p\times p}$, $\Theta : \mathcal{K} \to \mathbb{C}^{p\times p}$, and $\chi : \mathcal{K} \to \mathbb{C}^{q\times q}$ given by

$$\varphi(\zeta) := \int_{[c,d]} (1+t\zeta)\hat{\Phi}_\zeta(t)\nu(dt)\Phi^*(\overline{\zeta}), \tag{8.9}$$

$$\Theta(\zeta) := \int_{[c,d]} \Phi(t)\nu(dt)[(1+t\overline{\zeta})\hat{\Phi}_{\overline{\zeta}}(t)]^*, \tag{8.10}$$

and

$$\chi(\zeta) := \int_{\mathbb{R}\setminus[a,b]} \frac{1+t\zeta}{t-\zeta} I_q \nu(dt) I_q^* \tag{8.11}$$

are well defined. The functions $\Gamma : \mathcal{K} \times [c,d] \to \mathbb{C}^{p\times p}$ and $\triangle : \mathcal{K} \times [c,d] \to \mathbb{C}^{p\times p}$ given by

$$\Gamma(\zeta,t) := (1+t\zeta)\hat{\Phi}_\zeta(t) \quad \text{and} \quad \triangle(\zeta,t) := \Phi^*(\overline{\zeta})$$

are continuous. Since $\mathcal{K} \times [c,d]$ is a compact subset of $\mathbb{C} \times \mathbb{R}$, the matrix-valued functions Γ and $\triangle$ are both bounded. Moreover, we get that, for all $t \in [a,b]$, the functions $\Gamma_{\bullet t} : \mathcal{K} \to \mathbb{C}^{p\times p}$ and $\triangle_{\bullet t} : \mathcal{K} \to \mathbb{C}^{p\times p}$ given by $\Gamma_{\bullet t}(\zeta) := \Gamma(\zeta,t)$ and $\triangle_{\bullet t}(\zeta) := \triangle(\zeta,t)$ are continuous. Applying Proposition 8.5 we obtain then that φ given by (8.9) is continuous on $\mathcal{K}$. Similarly, we see that the matrix-valued functions Θ and χ given by (8.10) and (8.11) are also continuous on $\mathcal{K}$. In view of the assumption that ν is the Nevanlinna measure of F, let α be the unique Hermitian complex $q\times q$ matrix and β the unique nonnegative Hermitian complex $q \times q$ matrix such that (8.1) is satisfied for all $z \in \Pi_+$. Then let $h : \mathcal{K} \to \mathbb{C}^{p\times p}$ be defined by

$$h(\zeta) := \psi(\zeta) + \Phi(\zeta)\big(\alpha + \beta\zeta + \chi(\zeta)\big)\Phi^*\left(\overline{\zeta}\right) - \varphi(\zeta) - \Theta(\zeta). \tag{8.12}$$

Since all the matrix-valued functions $\psi, \Phi, \varphi, \Theta$, and χ are continuous on $\mathcal{K}$, the matrix-valued function h is also continuous on $\mathcal{K}$. Now we verify that $G(z) = h(z) + g(z)$ is satisfied for all $z \in \mathcal{K} \cap \Pi_+$. From (8.7), (8.5), and Remark 3.1 we get

$$g(\zeta) = \int_{\mathbb{R}} \left(\frac{1+t\zeta}{t-\zeta} I_q\right) \rho(dt) I_q = \int_{\mathbb{R}} \frac{1+t\zeta}{t-\zeta} \Phi(t)\nu(dt)\Phi^*(t) \tag{8.13}$$

for all $\zeta \in \Pi_+$. From (8.9) and (8.8) we can conclude

$$\varphi(\zeta) = \int_{[c,d]} \frac{1+t\zeta}{t-\zeta} \big(\Phi(t) + \Phi(\zeta)\big)\nu(dt)\Phi^*\left(\overline{\zeta}\right) \tag{8.14}$$

for all $\zeta \in \Pi_+ \cap \mathcal{K}$. Using (8.10) and (8.8) we infer

$$\Theta(\zeta) = \int_{[c,d]} \frac{1+t\zeta}{t-\zeta} \Phi(t)\nu(dt)\big(\Phi(t) - \Phi\left(\overline{\zeta}\right)\big)^* \tag{8.15}$$

for all $\zeta \in \Pi_+ \cap \mathcal{K}$. From (8.13), (8.14), (8.15), and (8.11) we obtain then

$$g(\zeta) - \varphi(\zeta) - \Theta(\zeta) = \Phi(\zeta) \left(\int_{[c,d]} \left(\frac{1+t\zeta}{t-\zeta} I_q \right) \nu(dt) I_q^* \right) \Phi^*\left(\overline{\zeta}\right) \tag{8.16}$$

for all $\zeta \in \Pi_+ \cap \mathcal{K}$. In view of (8.12), (8.16), (8.1), and (8.3), it follows

$$\begin{aligned} h(\zeta) + g(\zeta) &= \Psi(\zeta) + \Phi(\zeta) \cdot \left(\alpha + \beta\zeta + \int_{\mathbb{R}} \left(\frac{1+t\zeta}{t-\zeta} I_q \right) \nu(dt) I_q^* \right) \Phi^*\left(\overline{\zeta}\right) \\ &= \Psi(\zeta) + \Phi(\zeta) F(\zeta) \Phi^*\left(\overline{\zeta}\right) = G(\zeta) \end{aligned} \tag{8.17}$$

for all $\zeta \in \Pi_+ \cap \mathcal{K}$. Because of (8.7) and part (b) of Theorem 8.1 the function g belongs to $\mathcal{R}_p$ and ρ is the Nevanlinna measure of g. Applying Theorem 8.2 provides us

$$\begin{aligned} &\frac{1}{\pi} \lim_{\varepsilon\to 0+0} \int_{[a,b]} \operatorname{Im} g(x+i\varepsilon)\lambda(dx) \\ &= \int_{(a,b)} \left(\sqrt{1+t^2} I_p\right) \rho(dt) \left(\sqrt{1+t^2} I_p\right)^* + \frac{1}{2} \left[(1+a^2)\rho(\{a\}) + (1+b^2)\rho(\{b\})\right]. \end{aligned}$$

From (8.5) and Remark 3.1 it follows then

$$\begin{aligned} \frac{1}{2} \lim_{\varepsilon\to 0+0} \int_{[a,b]} \operatorname{Im} g(x+i\varepsilon)\lambda(dx) &= \int_{(a,b)} \left(\sqrt{1+t^2}\Phi(t)\right) \nu(dt) \left(\sqrt{1+t^2}\Phi(t)\right)^* \\ &+ \frac{1}{2} \left[(1+a^2)\Phi(a)\nu(\{a\})\Phi^*(a) + (1+b^2)\Phi(b)\nu(\{b\})\Phi^*(b)\right]. \end{aligned} \tag{8.18}$$

A straightforward calculation yields $\chi^*(t) = \chi(t)$ and $(\varphi(t)+\Theta(t))^* = \varphi(t)+\Theta(t)$ for every choice of t in $[a,b]$. By assumption we also have $\Psi^*(t) = \Psi(t)$ for each $t \in [a,b]$. Thus from (8.12) we can conclude that $\operatorname{Im} h(t) = 0$ holds for all $t \in [a,b]$. Since the matrix-valued function h is continuous on the compact subset $\mathcal{K}$ of $\mathbb{C}$, applying Lebesgue's dominated convergence theorem provides us then

$$\begin{aligned} \lim_{\varepsilon\to 0+0} \int_{[a,b]} \operatorname{Im} h(x+i\varepsilon)\lambda(dx) &= \int_{[a,b]} \lim_{\varepsilon\to 0+0} \operatorname{Im} h(x+i\varepsilon)\lambda(dx) \\ &= \int_{[a,b]} \operatorname{Im} h(x)\lambda(dx) = 0. \end{aligned} \tag{8.19}$$

From (8.17), (8.18), and (8.19) it finally (8.4). The rest follows easily. □

We thank the referee for providing us the following historical information. In the scalar case, a slightly different version of Theorem 8.6 was obtained by M.S. Livsic in his candidate dissertation [L]. The result of M.S. Livsic also appears as Lemma 2.1 in [Kr1]. An operator-valued version of the inversion formula in M.S. Livsic's form was obtained by Yu. L. Shmulyan in his second doctorate thesis (Kiev, Institute of Mathematics of the Ukrainian Academy of Sciences, 1970).

Now let us consider the class $\mathcal{R}_{0,q}$ of all matrix-valued functions F which belong to $\mathcal{R}_q$ and which satisfy

$$\sup_{y\in[1,+\infty)} y\|F(iy)\| < +\infty.$$

Using standard arguments one can check the inclusion $\mathcal{R}_{0,q} \subseteq \mathcal{R}'_q$. Every matrix-valued function F which belongs to $\mathcal{R}_{0,q}$ fulfills obviously

$$\lim_{y\to+\infty} F(iy) = 0 \tag{8.20}$$

and admits a particular integral representation.

Theorem 8.7.

(a) *For each $F \in \mathcal{R}_{0,q}$, there is a unique nonnegative Hermitian measure $\mu \in \mathcal{M}^q_{\geq}(\mathbb{R}, \mathfrak{B} \cap \mathbb{R})$ such that F admits the representation*

$$F(w) = \int_{\mathbb{R}} \frac{1}{t-w}\mu(dt) \tag{8.21}$$

for all $w \in \Pi_+$, namely the spectral measure of F, and

$$\mu(\mathbb{R}) = \lim_{y\to+\infty} y \operatorname{Im} F(iy) = -i \lim_{y\to+\infty} yF(iy) = i \lim_{y\to+\infty} yF^*(iy).$$

(b) *If $F : \Pi_+ \to \mathbb{C}^{q\times q}$ is a matrix-valued function for which there exists a $\mu \in \mathcal{M}^q_{\geq}(\mathbb{R}, \mathfrak{B}\cap\mathbb{R})$ such that (8.21) holds for all $w \in \Pi_+$, then F belongs to $\mathcal{R}_{0,q}$.*

Using the classical scalar version of Theorem 8.7 and the fact that, for each $F \in \mathcal{R}_{0,q}$ and each $u \in \mathbb{C}^q$, the function $f_u := u^*Fu$ belongs to $\mathcal{R}_{0,1}$ one gets easily a proof of Theorem 8.7. We omit the details.

In our considerations we encounter several situations in which we have to check that certain block matrices are nonnegative Hermitian. Hereby, the following well-known criterion is useful.

Remark 8.8. *Let $A \in \mathbb{C}^{p\times p}$, let $B \in \mathbb{C}^{p\times q}$, let $D \in \mathbb{C}^{q\times q}$, and let*

$$E := \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Albert [A] proved that the block matrix E is nonnegative Hermitian if and only if the following four conditions are satisfied:

(i) $A \geq 0$.
(ii) $AA^+B = B$.
(iii) $C = B^*$.
(iv) $D - CA^+B \geq 0$.

(For a slightly different but related version of a characterization of nonnegative Hermitian block matrices, we refer to a paper of Efimov and Potapov [EP]*). Moreover, it is readily checked that if E is nonnegative Hermitian, then the inequality $\|B\|^2 \le \|A\| \cdot \|D\|$ holds.*

Lemma 8.9. *Let M be a complex $q \times q$ matrix and $F : \Pi_+ \to \mathbb{C}^{q\times q}$ be a matrix-valued function which is holomorphic in Π_+ and which satisfies the inequality*

$$\begin{pmatrix} M & F(w) \\ F^*(w) & \frac{F(w)-F^*(w)}{w-\overline{w}} \end{pmatrix} \ge 0 \tag{8.22}$$

for each $w \in \Pi_+$. Then F belongs to the class $\mathcal{R}_{0,q}$ and fulfills

$$\sup_{y\in(0,+\infty)} y\|F(iy)\| \le \|M\|. \tag{8.23}$$

Moreover, the spectral measure μ of F satisfies $\mu(\mathbb{R}) \le M$.

Proof. Inequality (8.22) and Remark 8.8 provide us

$$\|F(iy)\|^2 \le \|M\| \left\| \frac{F(iy) - F^*(iy)}{2iy} \right\| \le \frac{1}{y} \|M\| \cdot \|F(iy)\|$$

for all $y \in (0, +\infty)$. Thus (8.23) follows. Because of (8.22) we also have

$$\mathrm{Im}\, F(w) = \frac{F(w) - F^*(w)}{w - w^*} \cdot \mathrm{Im}\, w \ge 0$$

for each $w \in \Pi_+$. Hence F belongs to $\mathcal{R}_{0,q}$. From (8.22) we obtain

$$\begin{pmatrix} M & -iyF(iy) \\ iyF^*(iy) & \frac{1}{2} iy\big(F(iy) - F^*(iy)\big) \end{pmatrix} \ge 0$$

for all $y \in (0, +\infty)$. Using part (a) of Theorem 8.7 we can conclude then

$$\begin{pmatrix} M & \mu(\mathbb{R}) \\ \mu(\mathbb{R}) & \mu(\mathbb{R}) \end{pmatrix} \ge 0.$$

From Remark 8.8 it follows finally

$$0 \le M - \mu(\mathbb{R})(\mu(\mathbb{R}))^+\mu(\mathbb{R}) = M - \mu(\mathbb{R}).$$

□

Remark 8.10. *Let $S \in \mathcal{R}_q[a,b]$. Using part (a) of Theorem* 1.1 *one can verify that $F := \mathrm{Rstr.}_{\Pi_+} S$ belongs to $\mathcal{R}_{0,q}$. In particular, from* (8.20) *it follows immediately*

$$\lim_{y\to\infty} S(iy) = 0.$$

Moreover, in view of Remark 8.4*, one can check that if σ denotes the Stieltjes measure of S, then the spectral measure μ of F satisfies $\mu(B) = \sigma(B \cap [a,b])$ for all $B \in \mathfrak{B} \cap \mathbb{R}$.*

Remark 8.11. *Let $S \in \mathcal{R}_q[a,b]$. From Remark* 8.10 *one can easily see that*

$$\lim_{y\to+\infty} \frac{\tilde{S}_1(iy)}{y} = 0 \quad \text{and} \lim_{y\to+\infty} \frac{\tilde{S}_1(iy)}{y} = 0.$$

We again work with the notations stated in (1.2) and (1.10).

If $S : \mathbb{C} \setminus [a,b] \to \mathbb{C}^{q\times q}$ is a matrix-valued function we have associated to it the matrix-valued functions $\tilde{S}_1$ and $\tilde{S}_2$ given by (1.10). Now we will introduce the corresponding construction for matrix measures. If $\sigma \in \mathcal{M}^q_{\geq}([a,b],\ \mathfrak{B} \cap [a,b])$ we will associate with it two particular measures σ_1 and σ_2 which belong to $\mathcal{M}^{\geq}_q([a,b], \mathfrak{B} \cap [a,b])$ and which are absolutely continuous with respect to σ.

Remark 8.12. *Let $\sigma \in \mathcal{M}^q_{\geq}([a,b], \mathfrak{B} \cap [a,b])$. From Remark* 3.1 *one can immediately see that $\sigma_1 : \mathfrak{B} \cap [a,b] \to \mathbb{C}^{q\times q}$ and $\sigma_2 : \mathfrak{B} \cap [a,b] \to \mathbb{C}^{q\times q}$ given by*

$$\sigma_1(B) := \int_{[a,b]} \Big(1_B(t)\sqrt{t-a}I_q\Big)\sigma(dt)\Big(1_B(t)\sqrt{t-a}I_q\Big)^* \tag{8.24}$$

and

$$\sigma_2(B) := \int_{[a,b]} \Big(1_B(t)\sqrt{b-t}I_q\Big)\,\sigma(dt)\,\Big(1_B(t)\sqrt{b-t}I_q\Big)^* \tag{8.25}$$

belong to $\mathcal{M}^q_{\geq}([a,b], \mathfrak{B} \cap [a,b])$ as well and satisfy $\sigma_1 + \sigma_2 = (b-a)\sigma$. Moreover, in view of Lemma 3.3, *it is readily checked that $\tilde{H}^{[\sigma_1]}_{1,m} = H^{[\sigma]}_{1,m}$ and $\tilde{H}^{[\sigma_2]}_{2,m} = H^{[\sigma]}_{2,m}$ hold for all $m \in \mathbb{N}_0$.*

Lemma 8.13. *Let $S \in \mathcal{R}_q[a,b]$, let σ be the Stieltjes measure associated with S, and let $\sigma_1 : \mathfrak{B} \cap [a,b] \to \mathbb{C}^{q\times q}$ and $\sigma_2 : \mathfrak{B} \cap [a,b] \to \mathbb{C}^{q\times q}$ be given by* (8.24) *and* (8.25)*. Then*

$$\tilde{S}_1 = S^{[\sigma_1]} - \sigma([a,b]) \quad \text{and} \quad \tilde{S}_2 = S^{[\sigma_2]} + \sigma([a,b]). \tag{8.26}$$

Moreover, for each $k \in \{1,2\}$, the matrix-valued function $S^{\square}_k := \mathrm{Rstr.}_{\Pi_+}\tilde{S}_k$ belongs to $\mathcal{R}'_q$ and $\theta_k : \mathfrak{B} \to \mathbb{C}^{q\times q}$ given by

$$\theta_k(B) := \sigma_k(B \cap [a,b]) \tag{8.27}$$

is the spectral measure of $S^{\square}_k$.

Proof. For each $z \in \mathbb{C} \setminus [a,b]$, we obtain $\frac{z-a}{t-z} = \frac{t-a}{t-z} - 1$ and hence, in view of Remark 3.1,

$$\begin{aligned}\tilde{S}_1(z) &= (z-a)\int_{[a,b]} \frac{1}{t-z}\sigma(dt) = \int_{[a,b]} \frac{z-a}{t-z} I_q\sigma(dt)I_q^* \\ &= \int_{[a,b]} \frac{\sqrt{t-a}}{t-z} I_q\sigma(dt)(\sqrt{t-a}I_q)^* - \int_{[a,b]} I_q\sigma(dt)I_q^* \\ &= \int_{[a,b]} \frac{1}{t-z}\sigma_1(dt) - \sigma([a,b]) = S^{[\sigma_1]}(z) - \sigma([a,b]).\end{aligned}$$

Therefore the first identity in (8.26) is verified. The second one follows analogously. Let $k \in \{1,2\}$. Theorem 1.1 yields that $S^{[\sigma_k]}$ belongs to $\mathcal{R}_q[a,b]$. From Remark 8.10 we get then that $F_k := \mathrm{Rstr.}_{\Pi_+}S^{[\sigma_k]}$ belongs to $\mathcal{R}_{0,q}$ and that θ_k is the spectral measure of F_k. Every constant $q \times q$ matrix-valued function defined on Π_+ and having a nonnegative Hermitian value belongs to $\mathcal{R}'_q$ and the spectral measure of which is exactly the zero measure belonging to $\mathcal{M}^q_{\geq}(\mathbb{R}, \mathfrak{B} \cap \mathbb{R})$. In view of (8.26) we see then that the matrix-valued function $S_k^{\square}$ belongs to $\mathcal{R}'_q$ and that the spectral measure of which is θ_k. $\square$

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Shift Operators Contained in Contractions, Schur Parameters and Pseudocontinuable Schur Functions

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Abstract. The main goal of the paper is to study the properties of the Schur parameters of the noninner functions of the Schur class $\mathcal{S}$ which admit a pseudocontinuation. To realize this aim we construct a model of completely nonunitary contraction in terms of Schur parameters of its characteristic function (see Chapters 2 and 3). By means of the constructed model a quantitative criterion of pseudocontinuability is established (see Chapter 4 and Sections 5.1 and 5.2). The properties of the Schur parameter sequences of pseudocontinuable noninner Schur functions are studied (see Sections 5.3 and 5.4).

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0. Introduction

Let T be a completely nonunitary contraction in a separable Hilbert space $\mathfrak{H}$ over $\mathbb{C}$. Then it is known (see, e.g., Arov [4]) that T can be considered as a fundamental operator of an appropriately chosen scattering system. Such system can be constructed with the aid of the Sz.-Nagy dilation of T. The procedure of constructing this dilation (see Sz.-Nagy/Foias [33], Foias/Frazho [25]) can be roughly described as follows. There are orthogonally supplemented two other Hilbert spaces to the originally given Hilbert space $\mathfrak{H}$ and in this spaces the shift and coshift associated with T act. These shift and coshift operators generate of outer channels. The scattering along these channels is described by the characteristic operator function (c.o.f.) of the contraction T. It is known (see Brodskii [12], Sz.-Nagy/Foias [33], Foias/Frazho [25]) that every holomorphic operator function in the unit disk the values of which are contractive operators can be represented as the c.o.f. of some completely nonunitary contraction.

In the study of contractions and their characteristic operator functions an important role is played not only by the outer channels of the scattering system but also by the inner ones. These channels are generated by the maximal shift and maximal coshift which are contained in T or more precisely these channels are subspaces in which these shift and coshift act. According to the role of these channels in the theory of scattering systems with loss we refer the reader to Arov [4], and [8], [9]. We mention that other problems are linked with these shifts and coshifts, too. For example in the cycle of papers [19] their connection with the asymptotical behavior of the semiradii of the Weyl matrix balls in the "infinite" Schur interpolation problem associated with the c.o.f. of the contraction T is shown. At the same time, in the papers [20], [10] their relations to the problem of extension of holomorphic contractive operator functions are discussed. There are close connections between these extensions and constructing Darlington representations of Schur functions (Arov [3]–[5]). In [10] it is shown that the pseudocontinuability of the c.o.f. of the contraction T is completely characterized by the mutual position of the maximal shift and maximal coshift contained in T. These maximal shift and maximal coshift have close contact with the theory of orthogonal polynomials on the unit circle (see Chapter 2, Comments).

We note that the holomorphy of the c.o.f. of a contraction is a consequence of the orthogonality of outer channels of a scattering system. In contrast to the outer ones the inner channels of a scattering system are not orthogonal in the general case. Therefore the scattering function which describes scattering through the inner channels is not holomorphic in the general case (see [8]).

The main goal of this paper is to describe interrelations between the maximal shift and the maximal coshift which are contained in a completely nonunitary contraction and to study the properties of the Schur parameters of the noninner functions of the Schur class $\mathcal{S}$ which admit a pseudocontinuation. The choice of Schur parameters is caused by profound interrelations between the Schur interpolation problem and the maximal shift and the maximal coshift contained in T.

In Chapter 1 we present a short survey of the basic facts from the theory of unitary colligations which are necessary for the later considerations. Hereby, in contrast to other approaches, particular attention will be drawn to the shifts and coshifts contained in T. We note that a detailed presentation of the theory of unitary colligations was given, for example, by Brodskii [12].

In Chapter 2 a model of a unitary colligation will be constructed in the language of Schur parameters of its c.o.f. The construction of this model is connected with the orthogonalization of a special vector system. This leads us to the construction of a (canonical) orthonormal basis in $\mathfrak{H}$ which is closely related to the contraction T. One of the basic technical difficulties on the way of constructing of the model will be mastered in Lemma 2.7 which is obviously of own interest. In this paper the model will be constructed for complex-valued characteristic operator functions. Exactly in this case one can succeed in the best possible way to trace the interrelations between the procedure of constructing the model, the procedure of orthogonalization and the Schur algorithm which was worked out by

I. Schur in his classical paper [31]. The operator case requires different methods. It will be treated in a separate paper. Section 2 ends with the description of connections between unitary colligations and Naimark dilations of operator-valued Borel measures on the unit circle. Actually, this connection permits us to associate our results with the approaches proposed by Geronimus [26], Gragg [27], Teplyaev [34], Constantinescu [14].

In Chapter 3, we present a model representation for the maximal shift V_T which is contained in a completely nonunitary contraction T (see Theorem 3.6). In Chapter 4, we indicate the connections between the mutual position of the subspaces in which the maximal shifts V_T and V_{T^*} are acting and the pseudocontinuability of the corresponding characteristic operator function (c.o.f.) of the contraction T.

We list the reasons why, in our opinion, the constructed model turns out to be a sufficiently effective tool to study the mutual interpendence between shifts and coshifts in a completely nonunitary contraction:

1) The model space is the space l^2 with the usual scalar product.
2) The model has a layered character which expresses the layered character of the stepwise Schur algorithm.
3) The coshift contained in T can be easily picked out from the model.

On the other side, the proposed model seems less useful at the investigation of questions which are not related to the construction of a canonical basis in $\mathfrak{H}$. What concerns other models for contractive operators in Hilbert space we refer the reader to Sz.-Nagy/Foias [33], Brodskii [12], Foias/Frazho [25] and Nikolski [29, v. 2].

The main part of this paper is Chapter 5. Using the constructed model, in Section 5.1, a quantitative description of the interrelation between the mutual position of V_T and V_{T^*} and the pseudocontinuability of the c.o.f. of the contraction T will be obtained. This quantitative characteristics are expressed in terms of properties of a particular sequence $(\sigma_n(\gamma))_{n=0}^{\infty}$ of Gram determinants (see Theorem 5.5). In this way, rational Schur functions are characterized in terms of their Schur parameters (see Theorem 5.9), a quantitative criterion of pseudocontinuability is established (see Theorem 5.10) and, moreover, a connection between pseudocontinuability and the nonnegative definiteness of a special matrix is indicated (see Theorem 5.13). In Section 5.3, the properties of the Schur parameter sequences of pseudocontinuable Schur functions are studied. In particular, if $\gamma = (\gamma_j)_{j=0}^{\infty}$ is the sequence of Schur parameters of a pseudocontinuable noninner Schur function then there exists a nonnegative integer $m_0(\gamma)$ such that for all $m \geq m_0(\gamma) + 1$ the Schur parameter γ_m is uniquely determined by the subsequent Schur parameters $\gamma_{m+1}, \gamma_{m+2}, \ldots$ (see Theorem 5.19). The character of this dependence is investigated. Examples are adduced.

The main results of the paper were announced without proofs in [21]–[23].

1. Shifts contained in contractions, unitary colligations and characteristic operator functions

1.1. Shifts contained in contractions and unitary colligations

Let T be a contraction acting in some Hilbert space $\mathfrak{H}$, i.e., $\|T\| \leq 1$ (in this paper all Hilbert spaces are assumed to be complex and separable, all operators are assumed to be linear). The operators

$$D_T := \sqrt{I_{\mathfrak{H}} - T^*T} \quad \textit{and} \quad D_{T^*} := \sqrt{I_{\mathfrak{H}} - TT^*}$$

are called *the defect operators* of T. The closures of their ranges

$$\mathcal{D}_T := \overline{D_T(\mathfrak{H})} \quad \textit{and} \quad \mathcal{D}_{T^*} := \overline{D_{T^*}(\mathfrak{H})}$$

are called the *defect spaces* of T. The dimensions of these spaces

$$\delta_T := \dim \mathcal{D}_T \quad \textit{and} \quad \delta_{T^*} := \dim \mathcal{D}_{T^*}$$

are called *the defect numbers* of the contraction T. In this way, the condition $\delta_T = 0$ (resp. $\delta_{T^*} = 0$) characterizes isometric (resp. coisometric) operators, whereas the conditions $\delta_T = \delta_{T^*} = 0$ characterize unitary operators. Note that an operator is called coisometric if its adjoint is isometric. Clearly, $TD_T^2 = D_{T^*}^2 T$. From here (see, e.g., Sz.-Nagy/Foias [33, Chapter I]) it follows that $TD_T = D_{T^*}T$. Passing the adjoint operators we obtain

$$T^*D_{T^*} = D_T T^* \; . \tag{1.1}$$

Starting from the contraction T we can always find Hilbert spaces $\mathfrak{F}$ and $\mathfrak{G}$ and operators $F : \mathfrak{F} \to \mathfrak{H}$, $G : \mathfrak{H} \to \mathfrak{G}$ and $S : \mathfrak{F} \to \mathfrak{G}$ such that the operator matrix

$$U = \begin{pmatrix} T & F \\ G & S \end{pmatrix} : \mathfrak{H} \oplus \mathfrak{F} \to \mathfrak{H} \oplus \mathfrak{G} \tag{1.2}$$

is unitary, i.e., the conditions $U^*U = I_{\mathfrak{H}\oplus\mathfrak{F}}$, $UU^* = I_{\mathfrak{H}\oplus\mathfrak{G}}$ are satisfied. Obviously, these identities can be rewritten in the form

$$\begin{array}{l} T^*T + G^*G = I_{\mathfrak{H}}, \; F^*F + S^*S = I_{\mathfrak{F}}, \; T^*F + G^*S = 0 \; , \\ TT^* + FF^* = I_{\mathfrak{H}}, \; GG^* + SS^* = I_{\mathfrak{G}}, \; TG^* + FS^* = 0 \; . \end{array} \tag{1.3}$$

As an example of such a construction one can consider the spaces $\mathfrak{F} := \mathcal{D}_{T^*}$, $\mathfrak{G} := \mathcal{D}_T$ and the operators

$$F := \mathrm{Rstr.}_{\mathcal{D}_{T^*}} D_{T^*} : \mathfrak{F} \to \mathfrak{H}, \; G := D_T : \mathfrak{H} \to \mathfrak{G}, \; S := \mathrm{Rstr.}_{\mathcal{D}_{T^*}} (-T^*) : \mathfrak{F} \to \mathfrak{G} \; .$$

Using (1.1) it is easily checked that the conditions (1.3) are fulfilled in this case. Note that in the general situation from (1.3) it follows $G^*G = D_T^2$, $FF^* = D_{T^*}^2$. Hence,

$$\overline{G^*(\mathfrak{G})} = \mathcal{D}_T, \quad \overline{F(\mathfrak{F})} = \mathcal{D}_{T^*} \tag{1.4}$$

Definition 1.1. *The ordered tuple*

$$\Delta = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; T, F, G, S) \tag{1.5}$$

consisting of three Hilbert spaces $\mathfrak{H}, \mathfrak{F}, \mathfrak{G}$ *and four operators* T, F, G, S *where*

$$T : \mathfrak{H} \to \mathfrak{H},\ \ F : \mathfrak{F} \to \mathfrak{H},\ \ G : \mathfrak{H} \to \mathfrak{G},\ \ S : \mathfrak{F} \to \mathfrak{G}$$

is called a unitary colligation (*or more short colligation*) *if the operator matrix* U *given via* (1.2) *is unitary.*

The operator T is called *the fundamental operator* of the colligation Δ. Clearly, the fundamental operator of a colligation is a contraction. The operation of representing a contraction T as fundamental operator of a unitary colligation is called embedding T in a colligation. The space $\mathfrak{H}$ of the colligation Δ is called *inner* and the spaces $\mathfrak{F}$ and $\mathfrak{G}$ are called *outer*. This embedding permits to use the spectral theory of unitary operators for the study of contractions (see, e.g., Sz.-Nagy/Foias [33])

The spaces $\mathfrak{H}_{\mathfrak{F}} := \bigvee\limits_{n=0}^{\infty} T^n F(\mathfrak{F})$, $\mathfrak{H}_{\mathfrak{G}} := \bigvee\limits_{n=0}^{\infty} T^{*n} G^*(\mathfrak{G})$ and their orthogonal complements $\mathfrak{H}_{\mathfrak{F}}^{\perp} := \mathfrak{H} \ominus \mathfrak{H}_{\mathfrak{F}}$, $\mathfrak{H}_{\mathfrak{G}}^{\perp} := \mathfrak{H} \ominus \mathfrak{H}_{\mathfrak{G}}$ play an important role in the theory of colligations. Clearly,

$$\mathfrak{H} = \mathfrak{H}_{\mathfrak{F}} \oplus \mathfrak{H}_{\mathfrak{F}}^{\perp}, \mathfrak{H} = \mathfrak{H}_{\mathfrak{G}}^{\perp} \oplus \mathfrak{H}_{\mathfrak{G}}. \tag{1.6}$$

The spaces $\mathfrak{H}_{\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{G}}$ are called the spaces of *controllability* and *observability*, respectively. From (1.4) it follows that these spaces can also be defined in an alternate way, namely

$$\mathfrak{H}_{\mathfrak{F}} := \bigvee_{n=0}^{\infty} T^n \mathcal{D}_{T^*}, \quad \mathfrak{H}_{\mathfrak{G}} := \bigvee_{n=0}^{\infty} T^{*n} \mathcal{D}_T \ . \tag{1.7}$$

Consequently, the spaces $\mathfrak{H}_{\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{G}}$ do not depend on the concrete way of embedding T in a colligation. Note that $\mathfrak{H}_{\mathfrak{F}}$ is invariant with respect to T whereas $\mathfrak{H}_{\mathfrak{G}}$ is invariant with respect to T^*. This means that $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ and $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ are invariant with respect to T^* and T, respectively. Switching over to the kernel of the adjoint operators in the identities (1.7) we obtain

$$\mathfrak{H}_{\mathfrak{F}}^{\perp} = \bigcap_{n=0}^{\infty} \ker(D_{T^*} T^{*n}), \quad \mathfrak{H}_{\mathfrak{G}}^{\perp} = \bigcap_{n=0}^{\infty} \ker(D_T T^n) \ . \tag{1.8}$$

Theorem 1.2. *The identities* $\mathfrak{H}_{\mathfrak{G}}^{\perp}=\{h \in \mathfrak{H} :\ \|T^n h\| = \|h\|,\ n = 1, 2, 3, \dots\}$ *and* $\mathfrak{H}_{\mathfrak{F}}^{\perp}=\{h \in \mathfrak{H} :\ \|T^{*n} h\| = \|h\|,\ n = 1, 2, 3, \dots\}$ *hold true.*

Proof. For $n = 1, 2, 3, \dots$, clearly $\|T^n h\|^2=(T^{*n}T^n h, h) = (T^{*n-1}T^{n-1}h, h) - (T^{*n-1}D_T^2 T^{n-1}h, h)$. Now the first assertion follows from (1.8) and the identity $\|T^{n-1}h\|^2$-$\|T^n h\|^2$=$\|D_T T^{n-1}h\|^2$, $n = 1, 2, 3, \dots$. Analogously, the second assertion can be proved. $\square$

Corollary 1.3. *The space* $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is characterized by the following properties:*

(a) $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is invariant with respect to* T *(resp.* T^**).*
(b) $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{G}}^{\perp}} T$ *(resp.* $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}} T^*$*) is an isometric operator.*
(c) $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is the maximal subspace of* $\mathfrak{H}$ *having the properties* (a), (b).

From the foregoing consideration we immediately obtain the following result.

Theorem 1.4. *The identity* $\mathfrak{H}_{\mathfrak{G}}^{\perp} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp} = \{h \in \mathfrak{H} : \|T^{*n}h\| = \|h\| = \|T^n h\|,\ n = 1, 2, 3, \dots\}$ *holds true.*

Corollary 1.5. *The subspace* $\mathfrak{H}_{\mathfrak{G}}^{\perp} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp}$ *is maximal among all subspaces* $\mathfrak{H}'$ *of* $\mathfrak{H}$ *having the following properties:* $\mathfrak{H}'$ *reduces* T *and* $\mathrm{Rstr.}_{\mathfrak{H}'} T$ *is a unitary operator.*

A contraction T on $\mathfrak{H}$ is called *completely nonunitary* if there is no nontrivial reducing subspace $\mathfrak{L}$ of $\mathfrak{H}$ for which the operator $\mathrm{Rstr.}_{\mathfrak{L}} T$ is unitary. Consequently, a contraction is completely nonunitary if and only if $\mathfrak{H}_{\mathfrak{G}}^{\perp} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp} = \{0\}$. The colligation Δ given in (1.5) is called *simple* if $\mathfrak{H} = \mathfrak{H}_{\mathfrak{F}} \vee \mathfrak{H}_{\mathfrak{G}}$ Hence, the colligation Δ is simple if and only if its fundamental operator T is a completely nonunitary contraction.

Taking into account the Wold decomposition for isometric operators (see, e.g., Sz.-Nagy/Foias [33, Chapter I]) from Corollary 1.3 we infer the following result:

Theorem 1.6. *Let* T *be a completely nonunitary contraction in* $\mathfrak{H}$*. Then the subspace* $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is characterized by the following properties:*

(a) *The subspace* $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is invariant with respect to* T *(resp.* T^**).*
(b) *The operator* $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{G}}^{\perp}} T$ *(resp.* $\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}} T^*$*) is a unilateral shift.*
(c) $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ *(resp.* $\mathfrak{H}_{\mathfrak{F}}^{\perp}$*) is the maximal subspace of* $\mathfrak{H}$ *having the properties* (a), (b).

We say that a unilateral shift $V : \mathfrak{L} \to \mathfrak{L}$ *is contained* in the contraction T if $\mathfrak{L}$ is a subspace of $\mathfrak{H}$ which is invariant with respect to T and $\mathrm{Rstr.}_{\mathfrak{L}} T = V$ is satisfied.

Definition 1.7. *Let* T *be a completely nonunitary contraction in* $\mathfrak{H}$*. Then the shift* $V_T := \mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{G}}^{\perp}} T$ *is called the maximal shift contained in* T*.*

By a *coshift* we mean an operator the adjoint of which is a unilateral shift. We say that a coshift $\widetilde{V} : \widetilde{\mathfrak{L}} \to \widetilde{\mathfrak{L}}$ *is contained* in T if the unilateral shift $\widetilde{V}^*$ is contained in T^*. Then from Theorem 1.6 it follows that the operator $V_{T^*} = \mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}} T^*$ is the maximal shift contained in T^*. If $\mathfrak{H}_{\mathfrak{G}}^{\perp} = \{0\}$ (resp. $\mathfrak{H}_{\mathfrak{F}}^{\perp} = \{0\}$) we will say that the shift V_T (resp. V_{T^*}) has multiplicity zero.

Definition 1.8. *Let* T *be a completely nonunitary contraction in* $\mathfrak{H}$*. Then the coshift* $\widetilde{V}_T := (V_{T^*})^*$ *is called the maximal coshift contained in* T*.*

Theorem 1.9. *Let* T *be a completely nonunitary contraction in* $\mathfrak{H}$*. Then the multiplicities of the maximal shifts* V_T *and* V_{T^*} *are not greater than* δ_{T^*} *and* δ_T*, respectively.*

Proof. It is sufficient to show that the multiplicity of the shift V_T is not greater than δ_{T^*}. Then the second assertion follows immediately via replacing the contraction T by the contraction T^*. Let $\mathfrak{L}_0$ be the generating wandering subspace for the shift V_T and let $P_{\mathfrak{L}_0}$ be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{L}_0$. Then $\mathfrak{H}_{\mathfrak{G}}^{\perp} = \bigoplus_{n=0}^{\infty} V_T^n \mathfrak{L}_0 = \bigoplus_{n=0}^{\infty} T^n \mathfrak{L}_0$. According to the decomposition $\mathfrak{H} = \mathfrak{H}_{\mathfrak{G}}^{\perp} \oplus \mathfrak{H}_{\mathfrak{G}}$ the operator T is given by the block matrix

$$T = \begin{pmatrix} V_T & R \\ 0 & T_{\mathfrak{G}} \end{pmatrix} . \tag{1.9}$$

To prove the inequality $\dim \mathfrak{L}_0 \leq \delta_{T^*}$ it is sufficient to verify the identity

$$\mathfrak{L}_0 = \overline{P_{\mathfrak{L}_0} \mathcal{D}_{T^*}} \ , \tag{1.10}$$

which is equivalent to $\mathfrak{L}_0 \cap \ker D_{T^*} = \{0\}$. We set $\mathfrak{N}_0 := \mathfrak{L}_0 \cap \ker D_{T^*}$ and $\mathfrak{M}_0 := T^*\mathfrak{N}_0$. From (1.9) follows $T^*\mathfrak{L}_0 \perp \mathfrak{H}_{\mathfrak{G}}^{\perp}$. This implies

$$\mathfrak{M}_0 \perp \mathfrak{H}_{\mathfrak{G}}^{\perp} \ . \tag{1.11}$$

We will show that for each $h \in \mathfrak{M}_0$ the identities

$$\|T^n h\| = \|h\| \ , \ n = 1, 2, 3, \dots \tag{1.12}$$

hold true. Indeed, $h = T^*f$ for some $f \in \mathfrak{N}_0$. Because of $\mathfrak{N}_0 \subseteq \ker D_{T^*}$, we have $Th = TT^*f = f$. Thus, $\|f\| = \|TT^*f\| = \|Th\| \leq \|h\| = \|T^*f\| \leq \|f\|$. Hence, (1.12) is proved for $n = 1$. Hereby $\|f\| = \|h\|$. If $n \in \{2, 3, \dots\}$ from the inclusions $\mathfrak{N}_0 \subseteq \mathfrak{L}_0 \subseteq \mathfrak{H}_{\mathfrak{G}}^{\perp}$ and Theorem 1.2 we obtain $\|T^n h\| = \|T^{n-1}Th\| = \|T^{n-1}f\| = \|f\| = \|h\|$. Now from (1.12) and Theorem 1.2 it follows $\mathfrak{M}_0 \subseteq \mathfrak{H}_{\mathfrak{G}}^{\perp}$. Combining this with (1.11) we obtain $\mathfrak{M}_0 = \{0\}$ and thus, $\mathfrak{N}_0 = \{0\}$. □

Corollary 1.10. *Let Δ be a simple unitary colligation of type* (1.5). *Denote $\mathfrak{L}_0$ and $\widetilde{\mathfrak{L}}_0$ the generating wandering subspaces for the maximal shifts V_T and V_{T^*}, respectively. Then $\overline{P_{\mathfrak{L}_0} F(\mathfrak{F})} = \mathfrak{L}_0$, $\overline{P_{\widetilde{\mathfrak{L}}_0} G^*(\mathfrak{G})} = \widetilde{\mathfrak{L}}_0$, where $P_{\mathfrak{L}_0}$ and $P_{\widetilde{\mathfrak{L}}_0}$ are the orthogonal projections from $\mathfrak{H}$ onto $\mathfrak{L}$ and $\widetilde{\mathfrak{L}}$, respectively.*

Proof. The validity of the first of the identities follows from (1.4) and (1.10). The second one is verified by changing T for T^*. □

Remark 1.11. *In* [19, part III] *it was shown that the multiplicity of the shift V_T coincides with δ_{T^*} if and only if the multiplicity of the shift V_{T^*} coincides with δ_T. Moreover, all remaining cases connected with the inequalities*

$$0 \leq \dim \mathfrak{L}_0 < \delta_{T^*}, \ 0 \leq \dim \widetilde{\mathfrak{L}}_0 < \delta_T$$

are possible.

1.2. Characteristic operator functions

Let $\mathfrak{F}$ and $\mathfrak{G}$ be Hilbert spaces.

Definition 1.12. *The symbol $\mathcal{S}(\mathbb{D};\mathfrak{F},\mathfrak{G})$ denotes the set of all operator-valued functions which are defined and holomorphic in $\mathbb{D}$ and the values of which are contractive operators acting between $\mathfrak{F}$ and $\mathfrak{G}$.*

Definition 1.13. *Let Δ be the unitary colligation given in* (1.5). *The operator function*

$$\theta_\Delta(\zeta) := S + \zeta G(I_{\mathfrak{H}} - \zeta T)^{-1}F \ , \ \zeta \in \mathbb{D} \ , \tag{1.13}$$

is called the characteristic operator function (*c.o.f.*) *of the colligation Δ.*

The next result is very important (see, e.g., Brodskii [12]):

Theorem 1.14. *The characteristic operator function θ_Δ of the unitary colligation Δ belongs to the class $\mathcal{S}(\mathbb{D};\mathfrak{F},\mathfrak{G})$. Conversely, suppose that θ is an operator function belonging to the class $\mathcal{S}(\mathbb{D};\mathfrak{F},\mathfrak{G})$. Then there exists a simple unitary colligation Δ of the form* (1.5) *such that θ is the characteristic operator function of Δ.*

Definition 1.15. *Let $\Delta_k = (\mathfrak{H}_k,\mathfrak{F},\mathfrak{G};T_k,F_k,G_k,S_k)$, $k = 1,2$, be unitary colligations. Then Δ_1 and Δ_2 are called unitarily equivalent if $S_1 = S_2$ and if there exists a unitary operator $Z : \mathfrak{H}_1 \to \mathfrak{H}_2$ which satisfies $ZT_1 = T_2Z$, $ZF_1 = F_2$, $G_2Z = G_1$.*

It can be easily seen that the characteristic operator functions of unitarily equivalent colligations coincide. In this connection it turns out to be important that the converse statement is also true (see, e.g., Brodskii [12]):

Theorem 1.16. *If the characteristic operator functions of two simple colligations coincide then the colligations are unitarily equivalent.*

1.3. Naimark dilations

Let us consider interrelations between unitary colligations and Naimark dilations of Borel measures on the unit circle $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$.

Let $\mathfrak{E}$ be a separable complex Hilbert space and denote $[\mathfrak{E}]$ the set of bounded linear operators in $\mathfrak{E}$. We consider a $[\mathfrak{E}]$-valued Borel measure μ on $\mathbb{T}$. More precisely, μ is defined on the σ-algebra $\mathcal{B}(\mathbb{T})$ of Borelian subsets of $\mathbb{T}$ and has the following properties:

(a) For $\Delta \in \mathcal{B}(\mathbb{T})$, $\mu(\Delta) \in [\mathfrak{E}]$.
(b) $\mu(\emptyset) = 0$.
(c) For $\Delta \in \mathcal{B}(\mathbb{T})$, $\mu(\Delta) \geq 0$.
(d) μ is σ-additive with respect to the strong operator convergence.

The set of all $[\mathfrak{E}]$-valued Borel measures on $\mathbb{T}$ is denoted by $\mathfrak{M}(\mathbb{T},\mathfrak{E})$.

Definition 1.17. *Let $\mu \in \mathfrak{M}(\mathbb{T},\mathfrak{E})$ and assume $\mu(\mathbb{T}) = I$. Denote by $(s_n)_{n\in\mathbb{Z}}$ the sequence of Fourier coefficients of μ, i.e.,*

$$s_n := \int_{\mathbb{T}} t^{-n}\mu(dt) \ , \ n \in \mathbb{Z} \ . \tag{1.14}$$

By a Naimark dilation of the measure μ we mean an ordered triple $(\mathfrak{K}, \mathcal{U}, \tau)$ having the following properties.

1. *$\mathfrak{K}$ is a separable Hilbert space over $\mathbb{C}$.*
2. *$\mathcal{U}$ is a unitary operator in $\mathfrak{K}$.*
3. *τ is an isometric operator from $\mathfrak{E}$ into $\mathfrak{K}$, the so-called embedding operator, i.e., $\tau : \mathfrak{E} \to \mathfrak{K}$ and $\tau^* \tau = I_{\mathfrak{E}}$.*
4. *For $n \in \mathbb{Z}$*

$$s_n = \tau^* \mathcal{U}^n \tau \ . \tag{1.15}$$

A Naimark dilation is called *minimal* if

$$\mathfrak{K} = \bigvee_{n \in \mathbb{Z}} \mathcal{U}^n \tau(\mathfrak{E}) \ . \tag{1.16}$$

Definition 1.18. *Two Naimark dilations $(\mathfrak{K}_j, \mathcal{U}_j, \tau_j)$, $j = 1, 2$, of a measure $\mu \in \mathfrak{M}(\mathbb{T}, \mathfrak{E})$, $\mu(\mathbb{T}) = I$, are called unitarily equivalent if there exists a unitary operator $Z : \mathfrak{K}_1 \to \mathfrak{K}_2$ which satisfies $\mathcal{U}_2 Z = Z \mathcal{U}_1$ and $Z\tau_1 = \tau_2$.*

Analogously with Theorem 1.16 one can prove

Lemma 1.19. *Any two minimal Naimark dilations $(\mathfrak{K}_j, \mathcal{U}_j, \tau_j)$, $j = 1, 2$, of a measure $\mu \in \mathfrak{M}(\mathbb{T}, \mathfrak{E})$, $\mu(\mathbb{T}) = I$, are unitarily equivalent (i.e., a minimal Naimark dilation is essentially unique).*

According to the construction of a Naimark dilation of the measure μ we consider two functions. The first of them has the form

$$\Phi(\zeta) = \int_{\mathbb{T}} \frac{t + \zeta}{t - \zeta} \mu(dt) \ , \ \zeta \in \mathbb{D} \ . \tag{1.17}$$

Obviously, $\Phi(\zeta)$ is holomorphic in $\mathbb{D}$. Moreover, $\Re[\Phi(\zeta)] = \frac{1}{2}[\Phi(\zeta) + \Phi^*(\zeta)] \geq 0$, $\zeta \in \mathbb{D}$, and, as it follows from (1.14), $\Phi(\zeta)$ has the Taylor series representation

$$\Phi(\zeta) = I + 2s_1\zeta + 2s_2\zeta^2 + \dots \ , \zeta \in \mathbb{D} \ . \tag{1.18}$$

Thus, $\Phi(\zeta)$ belongs to the Carathéodory class $\mathcal{C}(\mathbb{D}, \mathfrak{E})$ of all $[\mathfrak{E}]$-valued functions which are holomorphic in $\mathbb{D}$ and have nonnegative real part in $\mathbb{D}$.

The second of these functions $\theta(\zeta)$ is related to $\Phi(\zeta)$ via the Cayley transform:

$$\zeta\theta(\zeta) = (\Phi(\zeta) - I)(\Phi(\zeta) + I)^{-1} \ . \tag{1.19}$$

From the properties of θ and the well-known lemma of H.A. Schwarz it follows that $\theta(\zeta) \in \mathcal{S}(\mathbb{D}; \mathfrak{E})$ where $\mathcal{S}(\mathbb{D}; \mathfrak{E}) := \mathcal{S}(\mathbb{D}; \mathfrak{E}, \mathfrak{E})$. The functions $\Phi(\zeta)$ and $\theta(\zeta)$ are called the functions of classes $\mathcal{C}(\mathbb{D}; \mathfrak{E})$ and $\mathcal{S}(\mathbb{D}; \mathfrak{E})$, respectively, which are *associated* with the measure μ.

If

$$\theta(\zeta) = c_0 + c_1\zeta + c_2\zeta^2 + \dots \tag{1.20}$$

then from (1.18) and (1.19) we obtain

$$(c_0 + c_1\zeta + c_2\zeta^2 + \dots)(I + s_1\zeta + s_2\zeta^2 + \dots) = s_1 + s_2\zeta + s_3\zeta^2 + \dots .$$

Thus

$$s_1 = c_0\ ,\ \ s_n = c_0 s_{n-1} + c_1 s_{n-2} + \dots + c_{n-2}s_1 + c_{n-1}\ ,\ \ n \in \mathbb{N}\setminus\{1\}\ . \tag{1.21}$$

Theorem 1.20. *Let $\mu \in \mathfrak{M}(\mathbb{T}, \mathfrak{E})$ and assume $\mu(\mathbb{T}) = I$. Denote by $\theta(\zeta)$ the function from the class $\mathcal{S}(\mathbb{D}; \mathfrak{E})$ associated with μ.*

(a) *Let $\Delta = (\mathfrak{H}, \mathfrak{E}, \mathfrak{E}; T, F, G, S)$ be a simple unitary colligation which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Then the triple*

$$(\mathfrak{K}, \mathcal{U}, \tau) \tag{1.22}$$

where $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{E}$, $\mathcal{U} = \begin{pmatrix} T & F \\ G & S \end{pmatrix} : \mathfrak{H} \oplus \mathfrak{E} \to \mathfrak{H} \oplus \mathfrak{E}$ and τ is the operator of embedding $\mathfrak{E}$ into $\mathfrak{H} \oplus \mathfrak{E}$, i.e., $\tau e = (0, e) \in \mathfrak{H} \oplus \mathfrak{E}$ for each $e \in \mathfrak{E}$, is a minimal Naimark dilation of the measure μ.

(b) *Let $(\mathfrak{K}, \mathcal{U}, \tau)$ be a minimal Naimark dilation of the measure μ and $\tau(\mathfrak{E}) = \widetilde{\mathfrak{E}}$. Let $\mathfrak{H} = \mathfrak{K} \ominus \widetilde{\mathfrak{E}}$ and suppose that according to the decomposition $\mathfrak{K} = \mathfrak{H} \oplus \widetilde{\mathfrak{E}}$ the unitary operator $\mathcal{U}$ has the matrix representation*

$$\mathcal{U} = \begin{pmatrix} T & F \\ G & S \end{pmatrix} : \mathfrak{K} \oplus \widetilde{\mathfrak{E}} \to \mathfrak{K} \oplus \widetilde{\mathfrak{E}}.$$

Then the tuple $(\mathfrak{H}, \mathfrak{E}, \mathfrak{E}; T, F\tau, \tau^* G, \tau^* S\tau)$ is a minimal unitary colligation which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$.

Remark 1.21. *The spaces $\mathfrak{F}$ and $\mathfrak{G}$ in the unitary colligation* (1.5) *are different, whereas they are identified in the consideration of Naimark dilation. For this reason one has to distinguish between the unitary operator U from* (1.2) *and the unitary operator $\mathcal{U}$ from* (1.22). *The first of them acts between different spaces, whereas the second one acts in the space $\mathfrak{K}$.*

Proof. (a) Suppose that θ has the Taylor series representation (1.20). In view of

$$\theta(\zeta) = S + \zeta G(I - \zeta T)^{-1}F = S + \sum_{n=1}^{\infty} \zeta^n G T^{n-1} F\ ,\ \ \zeta \in \mathbb{D},$$

we obtain

$$c_0 = S\ ,\ \ c_n = GT^{n-1}F\ , n \in \mathbb{N}\ . \tag{1.23}$$

Observe that concerning the proof of the identities (1.15) it is enough to prove them for $n \in \mathbb{N}$. For this, it is sufficient to prove that

$$\mathcal{U}^n = \begin{pmatrix} * & x_n \\ * & y_n \end{pmatrix}\ ,\ \ n \in \mathbb{N} \tag{1.24}$$

where

$$x_n = T^{n-1}Fs_0 + T^{n-2}Fs_1 + \dots + TFs_{n-2} + Fs_{n-1} \tag{1.25}$$

and

$$y_n = s_n \ . \tag{1.26}$$

Indeed, if the identity (1.24) is verified, for $n \in \mathbb{N}$ we get

$$\tau^* \mathcal{U}^n \tau = (0, I) \begin{pmatrix} * & x_n \\ * & y_n \end{pmatrix} \begin{pmatrix} 0 \\ I \end{pmatrix} = y_n = s_n.$$

From (1.21) and (1.23) we infer that (1.24) is satisfied for $n = 1$. Applying the method of mathematical induction we assume that (1.24) is satisfied for n. Then

$$\begin{aligned} \begin{pmatrix} * & x_{n+1} \\ * & y_{n+1} \end{pmatrix} &= \mathcal{U}^{n+1} = \mathcal{U}\mathcal{U}^n \\ &= \begin{pmatrix} T & F \\ G & S \end{pmatrix} \cdot \begin{pmatrix} * & x_n \\ * & y_n \end{pmatrix} = \begin{pmatrix} * & Tx_n + Fy_n \\ * & Gx_n + Sy_n \end{pmatrix} . \end{aligned}$$

This, together with (1.21), (1.23), (1.25) and (1.26), yields the validity of (1.24) for $n+1$, too. Hence the triple (1.22) is a Naimark dilation of the measure μ. According to the proof of minimality, we note that from (1.24) it follows that $\bigvee_{n=0}^{m} \mathcal{U}^n \mathfrak{E} = (\bigvee_{n=0}^{m-1} T^n F(\mathfrak{E})) \oplus \mathfrak{E}$, $m \in \mathbb{N}$. Thus, $\bigvee_{n=0}^{\infty} \mathcal{U}^n \mathfrak{E} = (\bigvee_{n=0}^{\infty} T^n F(\mathfrak{E})) \oplus \mathfrak{E}$. Analogously, we get $\bigvee_{n=-\infty}^{0} \mathcal{U}^n \mathfrak{E} = (\bigvee_{n=0}^{\infty} T^{*n} G^*(\mathfrak{E})) \oplus \mathfrak{E}$. Now the minimality condition (1.16) follows from the simplicity of the colligation Δ. The assertion of (b) follows from the fact that the above considerations can be done in the reverse order. □

Remark 1.22. *Thus, the model of unitary colligations is also a model for the Naimark dilation of Borel measures on the unit circle.*

2. Construction of a model of a unitary colligation via the Schur parameters of its c.o.f. in the scalar case

In this chapter, a construction of a model of a simple unitary colligation Δ of type (1.5) will be given for the case $\dim \mathfrak{F} = \dim \mathfrak{G} = 1$. In this case, $\mathfrak{F}$ and $\mathfrak{G}$ can be identified with the field $\mathbb{C}$ of complex numbers. Then in view of Theorems 1.14 the corresponding c.o.f. $\theta_\Delta(\zeta)$ is characterized by the following conditions: $\theta_\Delta(\zeta)$ is defined and holomorphic in $\mathbb{D}$ and $\theta_\Delta(\mathbb{D}) \subseteq \overline{\mathbb{D}}$. The set of all functions having these properties will be denoted by $\mathcal{S}$. Thus, $\mathcal{S} = \mathcal{S}(\mathbb{D}; \mathfrak{F}, \mathfrak{G})$, if $\dim \mathfrak{F} = \dim \mathfrak{G} = 1$.

2.1. Schur algorithm, Schur parameters

Let $\theta(\zeta) \in \mathcal{S}$. Following I. Schur [31] we set $\theta_0(\zeta) := \theta(\zeta)$ and $\gamma_0 := \theta_0(0)$. Obviously, $|\gamma_0| \leq 1$. If $|\gamma_0| < 1$, we consider the function

$$\theta_1(\zeta) := \frac{1}{\zeta} \frac{\theta_0(\zeta) - \gamma_0}{1 - \overline{\gamma}_0 \theta_0(\zeta)} \ .$$

In view of the Lemma of H.A. Schwarz $\theta_1(\zeta) \in \mathcal{S}$. As above we set $\gamma_1 := \theta_1(0)$ and if $|\gamma_1| < 1$ we consider the function $\theta_2(\zeta) := \frac{1}{\zeta}\frac{\theta_1(\zeta)-\gamma_1}{1-\overline{\gamma}_1\theta_1(\zeta)}$. Further, we continue this procedure inductively. Namely, if in the jth step a function $\theta_j(\zeta)$ occurs for which $|\gamma_j| < 1$ where $\gamma_j := \theta_j(0)$ we set

$$\theta_{j+1}(\zeta) := \frac{1}{\zeta}\frac{\theta_j(\zeta) - \gamma_j}{1 - \overline{\gamma}_j\theta_j(\zeta)}$$

and continue this procedure. Then two cases are possible:

(1) The procedure can be carried out without end, i.e., $|\gamma_j| < 1$, $j = 0, 1, 2, \ldots$.
(2) There exists an $n \in \{0, 1, 2, \ldots\}$ such that $|\gamma_n| = 1$ and, if $n > 0$, then $|\gamma_j| < 1$, $j \in \{0, \ldots, n-1\}$.

Thus, a sequence $(\gamma_j)_{j=0}^{\omega}$ is associated with each function $\theta(\zeta) \in \mathcal{S}$. Hereby we have $\omega = \infty$ in the first case and $\omega = n$ in the second. From I. Schur's paper [31] it is known that the second case appears if and only if $\theta(\zeta)$ is a finite Blaschke product of degree n.

Definition 2.1. *The sequence $(\gamma_j)_{j=0}^{\omega}$ obtained by the above procedure is called the sequence of Schur parameters associated with the function $\theta(\zeta)$.*

The following two properties established by I. Schur in [31] determine the particular role which the Schur parameters play in the study of functions of class $\mathcal{S}$.

(a) There is a one-to-one correspondence between the set of functions $\theta(\zeta) \in \mathcal{S}$ and the set of corresponding sequences $(\gamma_j)_{j=0}^{\omega}$.
(b) For each sequence $(\gamma_j)_{j=0}^{\omega}$ which satisfies

$$\begin{cases} |\gamma_j| < 1\ ,\ j \in \{0, 1, 2, \ldots\}\ ,\ \ if\ \ \omega = \infty, \\ |\gamma_j| < 1\ ,\ j \in \{0, \ldots, \omega - 1\}\ ,\ |\gamma_\omega| = 1\ ,\ \ if\ \ 0 < \omega < \infty, \\ |\gamma_0| = 1\ ,\ \ if\ \ \omega = 0 \end{cases}$$

there exists a function $\theta(\zeta) \in \mathcal{S}$ such that the sequence $(\gamma_j)_{j=0}^{\omega}$ is the Schur parameter sequence of $\theta(\zeta)$.

Thus, the Schur parameters are independent parameters which determine the functions of class $\mathcal{S}$.

2.2. General form of the model

Let $\theta(\zeta) \in \mathcal{S}$. Assume that

$$\Delta = (\mathfrak{H}, \mathfrak{G}, \mathfrak{F}; T, F, G, S) \tag{2.1}$$

is a simple unitary colligation satisfying $\theta(\zeta) = \theta_\Delta(\zeta)$. In the considered case is $\mathfrak{F} = \mathfrak{G} = \mathbb{C}$. We take 1 as basis vector of the one-dimensional vector space $\mathbb{C}$. Set

$$\phi_1' := F(1)\ ,\ \widetilde{\phi}_1' := G^*(1)\ . \tag{2.2}$$

Then (see (1.7)) $\mathfrak{H}_{\mathfrak{F}} = \bigvee_{n=0}^{\infty} T^n\phi_1'$, $\mathfrak{H}_{\mathfrak{G}} = \bigvee_{n=0}^{\infty} T^{*n}\widetilde{\phi}_1'$. If $(f_\alpha)_{\alpha\in\mathcal{A}}$ is some family of vectors from $\mathcal{H}$ the symbol $\bigvee_{\alpha\in\mathcal{A}} f_\alpha$ denotes the smallest (closed) subspace of $\mathcal{H}$

which contains all vectors of this family. The orthogonalization of the sequence $(T^n\phi_1')_{n=0}^{\infty}$ emphasizes an important place in the construction of the model. First we assume that $(T^n\phi_1')_{n=0}^{\infty}$ is a sequence of linearly independent vectors. Then the Gram-Schmidt orthogonalization procedure uniquely determines an orthonormal basis $(\phi_k)_{k=1}^{\infty}$ of the subspace $\mathfrak{H}_{\mathfrak{F}}$ such that, for $n \in \mathbb{N}$ the conditions

$$\bigvee_{k=1}^{n} \phi_k = \bigvee_{k=0}^{n-1} T^k\phi_1' \ , \quad (T^{n-1}\phi_1', \phi_n) > 0 \tag{2.3}$$

are satisfied. Observe that for $n \in \{2, 3, \dots\}$ the second condition is equivalent to $(T\phi_{n-1}, \phi_n) > 0$. It is well known that the sequence $(\phi_k)_{k=1}^{\infty}$ is constructed in the following way. We set

$$\phi_k' := T^{k-1}\phi_1' \ , \ k \in \{1, 2, 3, \dots\} \tag{2.4}$$

and define inductively the sequence of vectors $(\widehat{\phi}_k)_{k=1}^{\infty}$ via

$$\widehat{\phi}_1 := \phi_1' \ , \ \widehat{\phi}_k := \phi_k' - \sum_{s=1}^{k-1} \lambda_{ks}\phi_s' \ , \ k \in \{2, 3, \dots\} \tag{2.5}$$

where the coefficients λ_{ks} are determined by the conditions $\widehat{\phi}_k \perp \phi_j'$, $j \in \{1, \dots, k-1\}$. This means that the sequence $(\lambda_{ks})_{s=1}^{k-1}$ yields a solution of the system of linear equations $\sum\limits_{s=1}^{k-1} \lambda_{ks}(\phi_s', \phi_j') = (\phi_k', \phi_j')$, $j \in \{1, \dots, k-1\}$. Now we set $\phi_k := \frac{1}{\|\widehat{\phi}_k\|}\widehat{\phi}_k$, $k \in \{1, 2, \dots\}$. Thus, $\phi_1 = \frac{1}{\|\phi_1'\|}\phi_1'$, $\phi_k = \frac{1}{\|\widehat{\phi}_k\|}T^{k-1}\phi_1' + u_{k-1}$, $k \in \{2, 3, \dots\}$ where $u_k \in \bigvee\limits_{j=1}^{k} \phi_j$. These relations are equivalent to $T^{k-1}\phi_1' = \|\widehat{\phi}_k\|(\phi_k - u_{k-1})$, $u_0 = 0$, $k \in \{1, 2, \dots\}$. From these identities we obtain for $k \in \mathbb{N}$

$$\begin{aligned} T\phi_k &= \frac{1}{\|\widehat{\phi}_k\|}T^k\phi_1' + Tu_{k-1} = \frac{1}{\|\widehat{\phi}_k\|}\|\widehat{\phi}_{k+1}\|(\phi_{k+1} - u_k) + Tu_{k-1} \\ &= \frac{\|\widehat{\phi}_{k+1}\|}{\|\widehat{\phi}_k\|}\phi_{k+1} + v_k \end{aligned}$$

where $v_k \in \bigvee\limits_{j=1}^{k} \phi_j$. Thus,

$$\phi_1' = \|\phi_1'\|\phi_1 \tag{2.6}$$

and $T\phi_k = \sum\limits_{j=1}^{k+1} t_{jk}\phi_j$ where

$$t_{k+1,k} = \frac{\|\widehat{\phi}_{k+1}\|}{\|\widehat{\phi}_k\|} \ , \ k \in \{1, 2, \dots\} \ . \tag{2.7}$$

Denote $T_{\mathfrak{F}}$ the restriction of T onto the invariant subspace $\mathfrak{H}_{\mathfrak{F}}$. From the above consideration it follows that the matrix of the operator $T_{\mathfrak{F}}$ with respect to the basis $(\phi_k)_{k=1}^{\infty}$ of $\mathfrak{H}_{\mathfrak{F}}$ has the form

$$\begin{pmatrix} t_{11} & t_{12} & \dots & t_{1n} & \dots \\ t_{21} & t_{22} & \dots & t_{2n} & \dots \\ 0 & t_{32} & \dots & t_{3n} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & t_{n+1,n} & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} . \tag{2.8}$$

We assume that $\mathfrak{H}_{\mathfrak{F}}^{\perp} = \mathfrak{H} \ominus \mathfrak{H}_{\mathfrak{F}} \neq \{0\}$. Remember that the maximal shift $V_{T^*} = \mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}} T^*$ acts in $\mathfrak{H}_{\mathfrak{F}}^{\perp}$. Denote $\widetilde{\mathfrak{L}}_0$ the generating wandering subspace for the shift V_{T^*}. Then

$$\mathfrak{H}_{\mathfrak{F}}^{\perp} = \bigoplus_{n=0}^{\infty} T^{*n} \widetilde{\mathfrak{L}}_0 \tag{2.9}$$

where in view of Theorem 1.9 we have $\dim \widetilde{\mathfrak{L}}_0 = 1$. In view of Corollary 1.10 there exists a unique unit vector $\psi_1 \in \widetilde{\mathfrak{L}}_0$ such that

$$(\widetilde{\phi}_1', \psi_1) > 0 \tag{2.10}$$

where $\widetilde{\phi}_1'$ is defined in (2.2). In view of (2.9) and (2.10) the sequence $(\psi_k)_{k\in\mathbb{N}}$ where $\psi_k := T^{*k-1}\psi_1$, $k \in \{1, 2, \dots\}$ is the unique orthonormal basis in $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ satisfying the conditions

$$(\widetilde{\phi}_1', \psi_1) > 0 \ , \ \psi_{k+1} = T^*\psi_k \ , \ k \in \{1, 2, \dots\} \ . \tag{2.11}$$

Definition 2.2. *The constructed orthonormal basis*

$$\phi_1, \phi_2, \dots ; \psi_1, \psi_2, \dots \tag{2.12}$$

of $\mathfrak{H}$ *which satisfies the conditions* (2.3) *and* (2.11) *is called canonical.*

From the form of the construction it is clear that the canonical basis is uniquely defined by the conditions (2.3) and (2.11). This allows us to identify in the following considerations operators and their matrix representations with respect to this basis. We note that we suppose in the sequel that the vectors of the canonical basis are ordered as in (2.12). From the above considerations it follows that the matrix of the operator T with respect to the canonical basis of $\mathfrak{H}$ has the block form

$$T = \begin{pmatrix} T_{\mathfrak{F}} & \widetilde{R} \\ 0 & \widetilde{V}_T \end{pmatrix} \ , \ \widetilde{V}_T = (V_{T^*})^* \ , \tag{2.13}$$

where the matrix of $T_{\mathfrak{F}}$ is given by (2.8),

$$\widetilde{R} = \begin{pmatrix} r_1' & 0 & \dots \\ r_2' & 0 & \dots \\ \vdots & \vdots & \\ r_n' & 0 & \dots \\ \vdots & \vdots & \end{pmatrix} , \ \widetilde{V}_T = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots \\ & & & \ddots \end{pmatrix} . \tag{2.14}$$

Hereby missing matrix elements are assumed to be zero. Because of (2.2) and (2.6) the matrix of the operator F with respect to the canonical basis has the form

$$F = \operatorname{col}\left(\|\phi_1'\|, 0, 0, \dots; 0, 0, 0, \dots\right) . \tag{2.15}$$

For the remaining elements of the unitary colligation Δ we obtain the following matrix representations

$$G = (g_1, g_2, g_3, \dots; g_\infty, 0, 0, \dots) \ , \ S = \theta(0) = \gamma_0 \ , \tag{2.16}$$

where, in accordance with the above notations, we get

$$\begin{aligned} g_k &= G\phi_k = (G\phi_k, 1) = (\phi_k, G^*(1)) = (\phi_k, \widetilde{\phi}_1') \ , \ k \in \{1, 2, \dots\}, \\ g_\infty &= G\psi_1 = (G\psi_1, 1) = (\psi_1, G^*(1)) = (\psi_1, \widetilde{\phi}_1') \ . \end{aligned}$$

The remaining entries in formula (2.16) are zero since from the colligation condition $TG^* + FS^* = 0$ we have for $k \in \{2, 3, \dots\}$

$$\begin{aligned} G\psi_k &= (G\psi_k, 1) = (T^*\psi_{k-1}, G^*(1)) = (\psi_{k-1}, TG^*(1)) = -(\psi_{k-1}, FS^*(1)) \\ &= -\gamma_0(\psi_{k-1}, F(1)) = -\gamma_0\|\phi_1'\|(\psi_{k-1}, \phi_1) = 0 \ . \end{aligned}$$

Expressing the matrix elements in (2.13)-(2.16) in terms of Schur parameters we obtain the final form of the model of a unitary colligation. Hereby we will see under which conditions the elements of the sequence $(T^n\phi_1')_{n=0}^\infty$ are linearly independent (see Corollary 2.6) and also when $\mathfrak{H} = \mathfrak{H}_{\mathfrak{F}}$ is satisfied (see Corollary 2.10).

From the colligation condition $F^*F + S^*S = I$ we get $\|F\|^2 = 1 - |\gamma_0|^2$. Therefore, from (2.15) we infer

$$\|\phi_1'\| = \|F\| = \sqrt{1 - |\gamma_0|^2} \ . \tag{2.17}$$

We determine the remaining elements in following order. First we determine $t_{n+1,n}$ in (2.8). After that we will find the sequence $(g_k)_{k=1}^\infty$ in (2.16). The knowledge of this elements will permit us to find all others.

2.3. Schur determinants and contractive operators. Computation of $t_{n+1,n}$

Clearly, to determine the sequence $(t_{k+1,k})_{k=1}^\infty$ it suffices, in view of (2.7), to find the sequence $(\|\widehat{\phi}_k\|)_{k=1}^\infty$. As it is known (see, e.g., Akhiezer/Glasman [2, Chapter I]) from (2.5) it follows

$$\|\widehat{\phi}_k\|^2 = \frac{\Gamma(\phi_1', \dots, \phi_k')}{\Gamma(\phi_1', \dots, \phi_{k-1}')} \ , \ k \in \{2, 3, \dots\} \ , \tag{2.18}$$

where $\Gamma(\phi'_1,\dots,\phi'_k)$ is the Gram determinant

$$\Gamma(\phi'_1,\dots,\phi'_k)=\begin{vmatrix}(\phi'_1,\phi'_1) & \dots & (\phi'_k,\phi'_1)\\ \vdots & & \vdots\\ (\phi'_1,\phi'_k) & \dots & (\phi'_k,\phi'_k)\end{vmatrix}.$$

Lemma 2.3. *Let $\theta(\zeta)\in\mathcal{S}$. Assume that $\theta(\zeta)$ has the Taylor series representation*

$$\theta(\zeta)=c_0+c_1\zeta+\dots+c_n\zeta^n+\dots\ ,\ \zeta\in\mathbb{D}\ . \tag{2.19}$$

Suppose that Δ is a simple unitary colligation of type (2.1) *which satisfies $\theta_\Delta=\theta$. Then the sequence* (2.4) *satisfies the conditions*

$$\Gamma(\phi'_1,\dots,\phi'_k)=\det(I-C^*_{k-1}C_{k-1})\ ,k\in\mathbb{N}\ , \tag{2.20}$$

where

$$C_k=\begin{pmatrix}c_0 & 0 & \dots & 0\\ c_1 & c_0 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ c_k & c_{k-1} & \dots & c_0\end{pmatrix}\ ,\ k\in\{0,1,2,\dots\}.$$

Proof. Denote by J_k, $k\in\{0,1,2,\dots\}$, the matrix of $(k+1)$th order given by

$$J_0:=1\ \ \textit{and}\ \ J_k:=\begin{pmatrix}0 & \dots & 1\\ \vdots & \iddots & \vdots\\ 1 & \dots & 0\end{pmatrix}\ ,\ \ \textit{if}\ \ k\in\mathbb{N}\ .$$

If we set $\widehat{C}_k=J_kC^*_kJ_k$, $k\in\mathbb{N}$, then for proving (2.20) it suffices to verify the identity

$$\Gamma(\phi'_1,\dots,\phi'_k)=\det(I-\widehat{C}_{k-1}\widehat{C}^*_{k-1})\ ,k\in\mathbb{N}\ . \tag{2.21}$$

From (1.13) we obtain $\theta(\zeta)=S+\sum\limits_{k=1}^{\infty}\zeta^kGT^{k-1}F$, $\zeta\in\mathbb{D}$. Hence,

$$c_0=S\ ,\ c_k=GT^{k-1}F\ ,\ k\in\{1,2,\dots\}\ . \tag{2.22}$$

Thus,

$$\begin{aligned}\widehat{C}^*_k &= \begin{pmatrix}c_0 & c_1 & \dots & c_k\\ 0 & c_0 & \dots & c_{k-1}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \dots & c_0\end{pmatrix}=\begin{pmatrix}S & GF & GTF & \dots & GT^{k-1}F\\ 0 & S & GF & \dots & GT^{k-2}F\\ \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & 0 & \dots & GF\\ 0 & 0 & 0 & \dots & S\end{pmatrix}\\ &= \operatorname{diag}_{k+1}(S)+\operatorname{diag}_{k+1}(G)\operatorname{triang}_{k+1}(I_{\mathfrak{H}},T)\operatorname{diag}_{k+1}(F)\ ,\end{aligned} \tag{2.23}$$

where we use the following matrices: $\operatorname{diag}_{k+1}(S) := \operatorname{diag}(\underbrace{S, S, \ldots, S}_{k+1})$,

$$\operatorname{triang}_{k+1}(Q,T) := \begin{pmatrix} 0 & Q & QT & \ldots & QT^{k-1} \\ 0 & 0 & Q & \ldots & QT^{k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & Q \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, Q \in [\mathfrak{H}] \ .$$

From this taking into account the colligation conditions (1.3), we obtain

$$I - \widehat{C}_k \widehat{C}_k^* = \operatorname{diag}_{k+1}(F^*) \left\{ \operatorname{diag}_{k+1}(I_{\mathfrak{H}}) + \operatorname{triang}_{k+1}(T) + (\operatorname{triang}_{k+1}(T))^* \right.$$
$$\left. -(\operatorname{triang}_{k+1}(I_{\mathfrak{H}}, T))^* \operatorname{diag}_{k+1}(I_{\mathfrak{H}} - T^*T) \operatorname{triang}_{k+1}(I_{\mathfrak{H}}, T) \right\} \operatorname{diag}_{k+1}(F) \ ,$$

where $\operatorname{triang}_{k+1}(T) := \operatorname{triang}_{k+1}(T,T)$. After some simple manipulations the expression in braces takes the form

$$\begin{pmatrix} I & T & \ldots & T^k \\ T^* & T^*T & \ldots & T^*T^k \\ \vdots & \vdots & & \vdots \\ T^{*k} & T^{*k}T & \ldots & T^{*k}T^k \end{pmatrix} = \begin{pmatrix} I \\ T^* \\ \vdots \\ T^{*k} \end{pmatrix} (I, T, \ldots, T^k) \ .$$

Thus,

$$I - \widehat{C}_k \widehat{C}_k^* = \begin{pmatrix} F^* \\ F^*T^* \\ \vdots \\ F^*T^{*k} \end{pmatrix} (F, TF, \ldots, T^k F) \ , \ k \in \{0, 1, \ldots\} \ . \tag{2.24}$$

Taking into account the identities $F^*T^{*j}T^iF = (T^iF(1), T^jF(1)) = (\phi'_{i+1}, \phi'_{j+1})$ we obtain (2.21). $\square$

It should be mentioned that analogous expressions were obtained in [19, part III] for $(I - \widehat{C}_k \widehat{C}_k^*)^n$, $n \in \mathbb{Z}$.

The determinants $\det(I - C_k^* C_k)$, $k \in \{0, 1, 2, \ldots\}$ were introduced by I. Schur in [31] and it is known (see, e.g., Bertin et al. [7, Ch.3]) that

$$\det(I - C_k^* C_k) = (1 - |\gamma_0|^2)^{k+1}(1 - |\gamma_1|^2)^k \ldots (1 - |\gamma_k|^2) \ , k \in \{0, 1, 2, \ldots\} \tag{2.25}$$

where $(\gamma_k)_{k=0}^{\omega}$ are the Schur parameters of the function θ.

Lemma 2.4. *For $k \in \{1, 2, \ldots\}$ the identities $\|\widehat{\phi}_k\| = \prod_{j=0}^{k-1} \sqrt{1 - |\gamma_j|^2}$ hold true.*

Proof. For $k = 1$, from (2.5) and (2.17) we infer $\|\widehat{\phi}_1\| = \|\phi'_1\| = \sqrt{1 - |\gamma_0|^2}$. For $k \in \{2, 3, \ldots\}$ the assertion follows from (2.18), (2.20) and (2.25). $\square$

From (2.7) and Lemma 2.4 we get the following result.

Corollary 2.5. *The identities*

$$t_{k+1,k} = \sqrt{1-|\gamma_k|^2}\ ,\ k \in \{1,2,\dots\} \tag{2.26}$$

hold true.

Corollary 2.6. *The sequence* $(T^n\phi_1')_{n=0}^{\infty}$ *consists of linearly independent vectors if and only if* $|\gamma_k| < 1$, $k \in \{0,1,\dots\}$.

The proof follows from Lemma 2.4 and the observation that the sequence $(T^n\phi_1')_{n=0}^{\infty}$ consists of linearly independent elements if and only if $\|\widehat{\phi}_k\| > 0$, $k \in \{1,2,\dots\}$.

2.4. Schur determinants and contractive operators again. Computation of g_n

We return to the Schur algorithm and set $\theta_0(\zeta) := \theta(\zeta)$. We assume that $|\gamma_k| < 1$, $k \in \{0,1,2,\dots\}$. Then $\theta_1(\zeta) = \frac{1}{\zeta}\frac{\theta_0(\zeta)\ -\gamma_0}{1-\overline{\gamma}_0\theta_0(\zeta)}$. Using the representation (1.13) we get

$$\begin{aligned}\frac{1}{\zeta}[\theta_0(\zeta)-\gamma_0] &= G(I-\zeta I)^{-1}F = (G(I-\zeta T)^{-1}F(1),1)\\ &= \big((I-\zeta T)^{-1}F(1),G^*(1)\big) = \big((I-\zeta T)^{-1}\phi_1',\widetilde{\phi}_1'\big)\ ,\end{aligned}$$

where ϕ_1' and $\widetilde{\phi}_1'$ are defined in (2.2). Taking into account the series representation (2.19), (2.22) and the colligation conditions (1.3) we obtain

$$\begin{aligned}1-\overline{\gamma}_0\theta_0(\zeta) &= 1-\overline{S}\Big(S+\zeta\sum_{n=0}^{\infty}\zeta^n GT^nF\Big) = (I-S^*S)-\zeta\sum_{n=0}^{\infty}\zeta^n S^*GT^nF\\ &= F^*F+\zeta\sum_{n=0}^{\infty}\zeta^nF^*T^{n+1}F = F^*(I-\zeta T)^{-1}F\\ &= ((I-\zeta T)^{-1}F(1),F(1)) = ((I-\zeta T)^{-1}\phi_1',\phi_1')\end{aligned}$$

Thus, $\theta_1(\zeta) = \frac{((I-\zeta T)^{-1}\phi_1',\widetilde{\phi}_1')}{((I-\zeta T)^{-1}\phi_1',\phi_1')}$. In other words,

$$\theta_1(\zeta) \quad = \quad \frac{a_0+a_1\zeta+a_2\zeta^2+\dots}{b_0+b_1\zeta+b_2\zeta^2+\dots}\ , \tag{2.27}$$

where taking into account (2.4) we get for $n \in \{0,1,2,\dots\}$

$$a_n = (T^n\phi_1',\widetilde{\phi}_1') = (\phi_{n+1}',\widetilde{\phi}_1') \tag{2.28}$$

and

$$b_n = (T^n\phi_1',\phi_1') = (\phi_{n+1}',\phi_1')\ . \tag{2.29}$$

Exactly for the functions represented via (2.27), I. Schur [31, part I,§4] derived the following representation for the γ_k's:

$$\gamma_1 = \frac{a_0}{b_0}\ ,\gamma_k = -\frac{d_{k-1}}{\delta_{k-1}}\ ,k \in \{2,3,\dots\} \tag{2.30}$$

where

$$d_1 = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \quad , \quad \delta_1 = \begin{vmatrix} \overline{b}_0 & a_0 \\ \overline{a}_0 & b_0 \end{vmatrix}$$

and

$$d_k = \begin{vmatrix} 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{k-1} & a_k \\ \overline{b}_0 & 0 & \dots & 0 & 0 & a_0 & \dots & a_{k-2} & a_{k-1} \\ \overline{b}_1 & \overline{b}_0 & \dots & 0 & 0 & 0 & \dots & a_{k-3} & a_{k-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \overline{b}_{k-2} & \overline{b}_{k-3} & \dots & \overline{b}_0 & 0 & 0 & \dots & a_0 & a_1 \\ 0 & 0 & \dots & 0 & b_0 & b_1 & \dots & b_{k-1} & b_k \\ \overline{a}_0 & 0 & \dots & 0 & 0 & b_0 & \dots & b_{k-2} & b_{k-1} \\ \overline{a}_1 & \overline{a}_0 & \dots & 0 & 0 & 0 & \dots & b_{k-3} & b_{k-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \overline{a}_{k-2} & \overline{a}_{k-3} & \dots & \overline{a}_0 & 0 & 0 & \dots & b_0 & b_1 \end{vmatrix} , k \in \mathbb{N} \setminus \{1\} , \quad (2.31)$$

$$\delta_k = \begin{vmatrix} \overline{b}_0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{k-2} & a_{k-1} \\ \overline{b}_1 & \overline{b}_0 & \dots & 0 & 0 & a_0 & \dots & a_{k-3} & a_{k-2} \\ \overline{b}_2 & \overline{b}_1 & \dots & 0 & 0 & 0 & \dots & a_{k-4} & a_{k-3} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \overline{b}_{k-1} & \overline{b}_{k-2} & \dots & \overline{b}_0 & 0 & 0 & \dots & 0 & a_0 \\ \overline{a}_0 & 0 & \dots & 0 & b_0 & b_1 & \dots & b_{k-2} & b_{k-1} \\ \overline{a}_1 & \overline{a}_0 & \dots & 0 & 0 & b_0 & \dots & b_{k-3} & b_{k-2} \\ \overline{a}_2 & \overline{a}_1 & \dots & 0 & 0 & 0 & \dots & b_{k-4} & b_{k-3} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \overline{a}_{k-1} & \overline{a}_{k-2} & \dots & \overline{a}_0 & 0 & 0 & \dots & 0 & b_0 \end{vmatrix} , k \in \mathbb{N} \setminus \{1\} . \quad (2.32)$$

Observe that in (2.30) the index associated with γ_k is shifted for one unit in comparison with [31]. This is related to the fact that I. Schur had obtained these formulas under the assumption that $\theta_0(\zeta)$ had the form (2.27).

We denote the columns of the determinant d_k by $l_{k1}, l_{k2}, \dots, l_{k,2k}$. The value of d_k does not change if $l_{k,2k}$ is replaced by the linear combination

$$\widehat{l}_{k,2k} = l_{k,2k} - \lambda_{k+1,1} l_{kk} - \lambda_{k+1,2} l_{k,k+1} - \dots - \lambda_{k+1,k} l_{k,2k-1} \quad (2.33)$$

where $(\lambda_{k+1,s})_{s=1}^{k}$ is taken from (2.5). Using coordinates we get

$$\widehat{l}_{k,2k} = \operatorname{col}(p_{kk}, p_{k,k-1}, \dots, p_{k1}, q_{kk}, q_{k,k-1}, \dots, q_{k1}) .$$

Taking into account (2.28) and (2.29) for $j \in \{1, \dots, k\}$ we obtain

$$p_{kj} = (w_{kj}, \widetilde{\phi}'_1) , \quad q_{kj} = (w_{kj}, \phi'_1) \quad (2.34)$$

where $w_{kj} = \phi'_{j+1} - \lambda_{k+1,k-j+1}\phi'_1$ - $\dots - \lambda_{k+1,k}\phi'_j$. Observe that for $j \in \{1, \dots, k-1\}$

$$T w_{kj} = w_{k,j+1} + \lambda_{k+1,k-j}\phi'_1 . \quad (2.35)$$

From (2.5) it follows $w_{kk} = \widehat{\phi}_{k+1}$, $k \in \{1, 2, \dots\}$. Thus, taking into account Lemma 2.4, (2.2) and (2.16), we obtain for $k \in \{1, 2, \dots\}$

$$\begin{aligned} p_{kk} &= (\widehat{\phi}_{k+1}, \widetilde{\phi}_1') = \|\widehat{\phi}_{k+1}\|(\phi_{k+1}, \widetilde{\phi}_1') = \|\widehat{\phi}_{k+1}\|(\phi_{k+1}, G^*(1)) \\ &= \|\widehat{\phi}_{k+1}\| G\phi_{k+1} = g_{k+1} \prod_{j=0}^{k} \sqrt{1 - |\gamma_j|^2} \ . \end{aligned} \tag{2.36}$$

Moreover, in view of $\widehat{\phi}_k \perp \phi_1'$, $k \in \{2, 3, \dots\}$ we infer $q_{kk} = (\widehat{\phi}_{k+1}, \phi_1') = 0$, $k \in \{1, 2, \dots\}$. Consequently, we get the additive decomposition

$$\widehat{l}_{k,2k} = \widehat{l}^{(1)}_{k,2k} + \widehat{l}^{(2)}_{k,2k} \tag{2.37}$$

where $\widehat{l}^{(1)}_{k,2k} = \operatorname{col}(p_{kk}, 0, \dots, 0, 0, 0, \dots, 0)$ and

$$\widehat{l}^{(2)}_{k,2k} = \operatorname{col}(0, p_{k,k-1}, \dots, p_{k1}, 0, q_{k,k-1}, \dots, q_{k1}) \ .$$

Lemma 2.7. *The determinant obtained by replacing the last column in* d_k *by* $\widehat{l}^{(2)}_{k,2k}$ *vanishes, i.e.,* $\det(l_{k1}, \dots, l_{k,2k-1}, \widehat{l}^{(2)}_{k,2k}) = 0$, $k \in \{1, 2, \dots\}$.

Proof. We will show that the column $\widehat{l}^{(2)}_{k,2k}$ is a linear combination of the vectors $l_{k1}, \dots, l_{k,k-1}$, i.e., there exists a vector $x = \operatorname{col}(x_1, \dots, x_n)$ which satisfies

$$\sum_{j=1}^{k-1} x_j l_{kj} = \widehat{l}^{(2)}_{k,2k} \ . \tag{2.38}$$

Let

$$A_k = \begin{pmatrix} \overline{a}_0 & \dots & 0 \\ \vdots & & \vdots \\ \overline{a}_k & \dots & \overline{a}_0 \end{pmatrix}, B_k = \begin{pmatrix} \overline{b}_0 & \dots & 0 \\ \vdots & & \vdots \\ \overline{b}_k & \dots & \overline{b}_0 \end{pmatrix} .$$

Then from the form (2.31) of the determinant d_k it can be seen that the system (2.38) can be rewritten in the form

$$B_{k-2} x = p_k \ , \tag{2.39}$$

$$A_{k-2} x = q_k \ , \tag{2.40}$$

where $p_k = \operatorname{col}(p_{k,k-1}, \dots, p_{k1})$, $q_k = \operatorname{col}(q_{k,k-1}, \dots, q_{k1})$. Because of $b_0 = \|\phi_1'\|^2 = (1 - |\gamma_0|^2) > 0$ the matrix B_{k-2} is invertible and the system (2.39) has a unique solution. To complete the proof it suffices to show that this solution also solves (2.40). Let $Z_{k-2} := A_{k-2} B_{k-2}^{-1}$. Then

$$Z_{k-2} B_{k-2} = A_{k-2} \ . \tag{2.41}$$

From the structure of the matrices A_{k-2} and B_{k-2} we obtain that Z_{k-2} has the same structure, namely

$$Z_{k-2} = \begin{pmatrix} z_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ z_{k-2} & \dots & z_0 \end{pmatrix} .$$

Hence, formula (2.41) can be rewritten in coordinate form as

$$z_j\overline{b}_0 + z_{j-1}\overline{b}_1 + \cdots + z_0\overline{b}_j = \overline{a}_j \ , j \in \{0,1,\ldots,k-2\} \ .$$

Taking into account (2.28) and (2.29) these equations can be rewritten for $j \in \{0,1,\ldots,k-2\}$ in the form

$$z_j(\phi_1', \phi_1') + z_{j-1}(\phi_1', T\phi_1') + \cdots + z_0(\phi_1', T^j\phi_1') = (\widetilde{\phi}_1', \phi_{j+1}') \ . \tag{2.42}$$

From the colligation condition $GG^* + SS^* = 1$ and (2.17) we get

$$(\phi_1', \phi_1') = 1 - |\gamma_0|^2 = 1 - SS^* = GG^* = (\widetilde{\phi}_1', \widetilde{\phi}_1') \tag{2.43}$$

and for $r \in \{1,2,\ldots\}$

$$\begin{aligned} & (\phi_1', T^r\phi_1') = (T^*F(1), T^{r-1}\phi_1') \\ = \quad & -(G^*S(1), T^{r-1}\phi_1') = -\gamma_0(\widetilde{\phi}_1', \phi_r') = -(\widetilde{\phi}_1', \overline{\gamma}_0\phi_r') \ . \end{aligned}$$

Hence, the system (2.42) can be rewritten in the form

$$z_j(\widetilde{\phi}_1', \widetilde{\phi}_1') - z_{j-1}(\widetilde{\phi}_1', \overline{\gamma}_0\phi_1') - \cdots - z_0(\widetilde{\phi}_1', \overline{\gamma}_0\phi_j') = (\widetilde{\phi}_1', \phi_{j+1}') \ , j \in \{0,1,\ldots,k-2\} \ .$$

This is equivalent to

$$h_j \perp \widetilde{\phi}_1' \ , j \in \{0,1,\ldots,k-2\} \tag{2.44}$$

where

$$h_j = \overline{z}_j\widetilde{\phi}_1' - \overline{\gamma}_0(\overline{z}_{j-1}\phi_1' + \overline{z}_{j-2}\phi_2' + \cdots + \overline{z}_0\phi_j') - \phi_{j+1}' \ . \tag{2.45}$$

From (2.2) and (1.4) it follows that (2.44) is equivalent to

$$h_j \in \ker(I - T^*T) \ , j \in \{0,1,\ldots,k-2\} \ . \tag{2.46}$$

Thus, equation (2.41) is equivalent to the conditions (2.44) and also to (2.46). Note that for $j \in \{0,1,\ldots,k-3\}$

$$h_{j+1} = \overline{z}_{j+1}\widetilde{\phi}_1' + Th_j \ . \tag{2.47}$$

Let x be the unique solution of the system (2.39). Then from (2.41) it follows $A_{k-2}x = Z_{k-2}B_{k-2}x = Z_{k-2}p_k$. To complete the proof it suffices to verify that

$$Z_{k-2}p_k = q_k \ . \tag{2.48}$$

Using coordinates the system (2.48) can be rewritten in the form

$$z_jp_{k,k-1} + z_{j-1}p_{k,k-2} + \cdots + z_0p_{k,k-j-1} = q_{k,k-j-1} \ , j \in \{0,1,\ldots,k-2\} \ .$$

Taking into account (2.34) these equations can be rewritten again as

$$(w_{k,k-1}, \overline{z}_j\widetilde{\phi}_1') + (w_{k,k-2}, \overline{z}_{j-1}\widetilde{\phi}_1') + \cdots + (w_{k,k-j-1}, \overline{z}_0\widetilde{\phi}_1') = (w_{k,k-j-1}, \phi_1')$$

or equivalently for $j \in \{0,1,\ldots,k-2\}$

$$\begin{aligned} & (w_{k,k-1}, \overline{z}_j\widetilde{\phi}_1') + (w_{k,k-2}, \overline{z}_{j-1}\widetilde{\phi}_1') + \cdots + \\ & +(w_{k,k-j}, \overline{z}_1\widetilde{\phi}_1') + (w_{k,k-j-1}, \overline{z}_0\widetilde{\phi}_1' - \phi_1') = 0 \ . \end{aligned} \tag{2.49}$$

From (2.45) and (2.46) we infer for $j = 0$ now $\overline{z}_0\widetilde{\phi}_1' - \phi_1' = h_0 \in \ker(I - T^*T)$. Consequently, taking into account (2.35) the last term in (2.49) can be rewritten as

$$\begin{aligned}
& (w_{k,k-j-1}, \overline{z}_0\widetilde{\phi}_1' - \phi_1') \\
= \ & (Tw_{k,k-j-1}, T(\overline{z}_0\widetilde{\phi}_1' - \phi_1')) = (w_{k,k-j} + \lambda_{k+1,j+1}\phi_1', T(\overline{z}_0\widetilde{\phi}_1' - \phi_1')) \\
= \ & (w_{k,k-j}, T(\overline{z}_0\widetilde{\phi}_1' - \phi_1')) = (w_{k,k-j}, Th_0) \ .
\end{aligned}$$

Hereby, we have taken into account that from the colligation condition $T^*F + G^*S = 0$ and (2.44) it follows

$$\begin{aligned}
& (\phi_1', T(\overline{z}_0\widetilde{\phi}_1' - \phi_1)) = (F(1), Th_0) = (T^*F(1), h_0) \\
= \ & (G^*S(1), h_0) = -\gamma_0(G^*(1), h_0) = -\gamma_0(\widetilde{\phi}_1', h_0) = 0 \ .
\end{aligned}$$

Combining now the last two terms in (2.49) and taking into account formula (2.47) for $j = 0$ we rewrite (2.49) for $j \in \{0, 1, \dots, k-2\}$ in the form

$$(w_{k,k-1}, \overline{z}_j\widetilde{\phi}_1') + (w_{k,k-2}, \overline{z}_{j-1}\widetilde{\phi}_1') + \cdots + (w_{k,k-j+1}, \overline{z}_2\widetilde{\phi}_1') + (w_{k,k-j}, h_1) = 0.$$

Taking into account now that h_1 belongs to $\ker(I - T^*T)$ the above considerations can be repeated. After the kth step the system (2.49) has the form $(w_{kk}, Th_j) = 0$, $j \in \{0, 1, \dots, k-2\}$. The validity of these conditions follows from the fact that according to (2.45) and the colligation condition $TG^* + FS^* = 0$ the relations $Th_j \in \bigvee\limits_{r=1}^{k} \phi_r'$, $j \in \{0, 1, \dots, k-2\}$ are satisfied, but $w_{kk} = \widehat{\phi}_{k+1}$. Hereby, keeping in mind the orthogonalization procedure, we have $\widehat{\phi}_{k+1} \perp \bigvee\limits_{r=1}^{k} \phi_r'$. □

Corollary 2.8. *For $k \in \{1, 2, \dots\}$ the identities*

$$d_k = -g_{k+1}\delta_{k-1}(1 - |\gamma_0|^2) \prod_{j=0}^{k} \sqrt{1 - |\gamma_j|^2} \ , \quad \delta_0 = 1 \tag{2.50}$$

hold true.

Proof. From (2.33), (2.37) and Lemma 2.7 we get

$$\begin{aligned}
d_k &= \det(l_{k1}, \dots, l_{k,2k-1}, l_{k,2k}) = \det(l_{k1}, \dots, l_{k,2k-1}, \widehat{l}_{k,2k}) \\
&= \det(l_{k1}, \dots, l_{k,2k-1}, \widehat{l}_{k,2k}^{(1)}) = -p_{kk}M_{1,2k}
\end{aligned}$$

where $M_{1,2k}$ is the minor of the element at position $(1, 2k)$ in the determinant (2.31). Computing $M_{1,2k}$ with the aid of the Laplace formula for the kth column and taking into account (2.29), (2.17) and (2.32), we obtain $M_{1,2k} = b_0\delta_{k-1} = (1 - |\gamma_0|^2)\delta_{k-1}$, $k \in \{1, 2, \dots\}$. From this and (2.36) we infer $d_k = -p_{kk}M_{1,2k} = -g_{k+1}\delta_{k-1}(1 - |\gamma_0|^2) \prod\limits_{j=0}^{k} \sqrt{1 - |\gamma_j|^2}$, $k \in \{1, 2, \dots\}$. □

Lemma 2.9. *For $k \in \{1, 2, \dots\}$ the identities*

$$g_k = \gamma_k \prod_{j=0}^{k-1} \sqrt{1 - |\gamma_j|^2} \tag{2.51}$$

hold true.

Proof. From (2.16), (2.17), (2.28) and (2.30) we infer

$$g_1 = (\phi_1, \widetilde{\phi}_1') = \frac{1}{\|\phi_1'\|}(\phi_1', \widetilde{\phi}_1') = \frac{1}{\|\phi_1'\|} a_0 = \gamma_1 \frac{b_0}{\|\phi_1'\|} = \gamma_1 \sqrt{1 - |\gamma_0|^2}\ ,$$

i.e., formula (2.51) is proved for $k = 1$. For $k \in \{2, 3, \dots\}$, using (2.30) and (2.50), we get

$$\gamma_k \frac{\delta_{k-1}}{\delta_{k-2}} = g_k (1 - |\gamma_0|^2) \prod_{j=0}^{k-1} \sqrt{1 - |\gamma_j|^2}\ . \tag{2.52}$$

It is known (see Schur[31, part I, §4]) that $1 - |\gamma_{j+1}|^2 = \frac{\delta_{j-1}\delta_{j+1}}{\delta_j^2}$, $j \in \{0, 1, 2, \dots\}$, $\delta_{-1} = \frac{1}{b_0^2}$. Hereby, comparing with [31] one has to take into account that we have shifted the index associated with γ_j for one unit. Thus, $\prod_{j=0}^{k-2}(1 - |\gamma_{j+1}|^2) = \frac{\delta_{-1}\delta_{k-1}}{\delta_0 \delta_{k-2}}$, $k \in \{2, 3, \dots\}$. Taking into account the identity $b_0 = 1 - |\gamma_0|^2$, we obtain

$$\frac{\delta_{k-1}}{\delta_{k-2}} = (1 - |\gamma_0|^2) \prod_{j=0}^{k-1} (1 - |\gamma_j|^2)\ .$$

Substituting this expression in (2.52) we obtain (2.51) for $k \in \{2, 3, \dots\}$. □

Corollary 2.10. *The vector system $(\phi_k')_{k=1}^{\infty}$ (see (2.4)) is not total in $\mathfrak{H}$ if and only if the product*

$$\prod_{j=0}^{\infty} (1 - |\gamma_j|^2) \tag{2.53}$$

converges. If this condition is satisfied then

$$g_\infty = \prod_{j=0}^{\infty} \sqrt{1 - |\gamma_j|^2}\ . \tag{2.54}$$

Proof. Because of the Corollary 1.10 the vector system (2.4) is not total in $\mathfrak{H}$ if and only if the vector $\widetilde{\phi}_1' = G^*(1)$ does not belong to $\mathfrak{H}_{\mathfrak{F}}$, i.e.,

$$\|\widetilde{\phi}_1'\|^2 - \|P_{\mathfrak{H}_{\mathfrak{F}}} \widetilde{\phi}_1'\|^2 > 0 \tag{2.55}$$

where $P_{\mathfrak{H}_{\mathfrak{F}}}$ is the orthogonal projection from $\mathfrak{H}$ onto $\mathfrak{H}_{\mathfrak{F}}$. From the coordinate representation (2.16) and (2.43) we get

$$|g_\infty|^2 = \|\widetilde{\phi}_1'\|^2 - \|P_{\mathfrak{H}_{\mathfrak{F}}} \widetilde{\phi}_1'\|^2 = (1 - |\gamma_0|^2) - \sum_{k=1}^{\infty} |g_k|^2\ .$$

Using (2.51) we obtain

$$\begin{aligned}|g_\infty|^2 &= \lim_{n\to\infty}\{(1-|\gamma_0|^2)-\sum_{k=1}^{n}|\gamma_k|^2\prod_{j=0}^{k-1}(1-|\gamma_j|^2)\} \\ &= \lim_{n\to\infty}\prod_{j=0}^{n}(1-|\gamma_j|^2)=\prod_{j=0}^{\infty}(1-|\gamma_j|^2)\ . \end{aligned}\tag{2.56}$$

Hence, the inequality (2.55) is satisfied if and only if the infinite product (2.53) converges. Using now the normalization condition (2.11) we obtain $g_\infty = (\widetilde{\phi}_1', \psi_1) > 0$. From this and (2.56) we get (2.54). □

If the vector system (2.4) is not total in $\mathfrak{H}$ then we obtain $\mathfrak{H}_{\mathfrak{F}}^{\perp} = \mathfrak{H}\ominus\mathfrak{H}_{\mathfrak{F}} \neq \{0\}$. In view of Theorem 1.6 this implies that T^* contains a nonzero maximal shift V_{T^*}. Because of $\delta_T = 1$, from Theorem 1.9 we obtain that the multiplicity of V_{T^*} is equal to 1, too. In view of Remark 1.11 this is equivalent to the fact that T contains a nonzero maximal shift V_T of multiplicity 1. Thus, we have obtained the following result.

Lemma 2.11. *Let $\theta(\zeta) \in \mathcal{S}$ and let Δ be a simple unitary colligation of the form* (2.1) *which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Then the contraction T* (*resp.* T^*) *contains a nonzero maximal shift if and only if the infinite product* (2.53) *converges. If this condition is satisfied the multiplicities of the maximal shifts V_T and V_{T^*} are both equal to* 1.

Remark 2.12. *It is known* (*see, e.g., Bertin et al.* [7, Chapter 3]), *that*

$$\prod_{j=0}^{\infty}(1-|\gamma_j|^2) = \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln(1-|\theta(e^{i\alpha})|^2)d\alpha\}\ ,$$

where $\theta(e^{i\alpha})$ denotes the nontangential boundary values of $\theta(\zeta)$ which exist and are finite almost everywhere in view of a theorem due to Fatou. Hence, the convergence of the product (2.53) *means that* $\ln(1-|\theta(e^{i\alpha})|^2) \in L^1[-\pi,\pi]$.

2.5. Description of the model of a unitary colligation if $\prod_{j=0}^{\infty}(1-|\gamma_j|^2)$ converges

In this case we have in particular $|\gamma_k| < 1$ for all $k \in \{0,1,2,\dots\}$. Therefore, in view of Corollary 2.6 the sequence (2.4) does not contain linearly dependent elements whereas the Corollary 2.10 states us that this vector system is not total in $\mathfrak{H}$. This means that the canonical basis in $\mathfrak{H}$ has the form (2.12). The operators T, F, G and S have with respect to this basis the matrix representations (2.13)-(2.16). In view of the above results in order to reach a complete description of the model of a unitary colligation it is sufficient to find the elements $(t_{kj})_{j=1}^{k}$, $k \in \{1,2,\dots\}$ in the matrix representation (2.8), (2.13) of $T_{\mathfrak{F}}$ and $(r_k')_{k=1}^{\infty}$ in the matrix representation (2.14) of $\widetilde{R}$.

From the colligation condition $T^*F+G^*S=0$ and (2.15) we conclude $T^*\phi_1 = \frac{1}{\|\phi_1'\|}T^*F(1) = -\frac{1}{\|\phi_1'\|}G^*S$. From this in view of (2.16) and (2.17) we get

$$T^*\phi_1 \quad = \quad -\frac{\gamma_0}{\sqrt{1-|\gamma_0|^2}}G^*(1) = -\frac{\gamma_0}{\sqrt{1-|\gamma_0|^2}}\{\sum_{k=1}^{\infty}\overline{g}_k\phi_k + \overline{g}_\infty\psi_1\} \ . \quad (2.57)$$

On the other side, the matrix representations (2.8), (2.13) and (2.14) yield $T^*\phi_1 = \sum\limits_{k=1}^{\infty}\overline{t}_{1k}\phi_k + \overline{r}_1'\psi_1$. Comparing this series representation with (2.57) and taking into account (2.51) and (2.54), we obtain

$$t_{11} = -\frac{\overline{\gamma}_0}{\sqrt{1-|\gamma_0|^2}}g_1 = -\overline{\gamma}_0\gamma_1 \ , \quad (2.58)$$

$$t_{1k} = -\frac{\overline{\gamma}_0}{\sqrt{1-|\gamma_0|^2}}g_k = -\overline{\gamma}_0\gamma_k\prod_{j=1}^{k-1}\sqrt{1-|\gamma_j|^2} \ , k\in\{2,3,\dots\} \quad (2.59)$$

and

$$r_1' = -\frac{\overline{\gamma}_0}{\sqrt{1-|\gamma_0|^2}}g_\infty = -\overline{\gamma}_0\prod_{j=1}^{\infty}\sqrt{1-|\gamma_j|^2} \ . \quad (2.60)$$

Thus, the elements in the first row of the matrix representation (2.13) of T are determined.

We consider the colligation condition $T^*T + G^*G = I$. Using the matrix representation (2.13), we get

$$\begin{pmatrix} T_{\mathfrak{F}}^* & 0 \\ \widetilde{R}^* & \widetilde{V}_T^* \end{pmatrix}\begin{pmatrix} T_{\mathfrak{F}} & \widetilde{R} \\ 0 & \widetilde{V}_T \end{pmatrix} + G^*G = I \ . \quad (2.61)$$

Postmultiplying in this identity the kth row with the first column and taking into account formulas (2.8), (2.14) and (2.16) we get for $k\in\{2,3,\dots\}$ the equations $\overline{t}_{1k}t_{11}+\overline{t}_{2k}t_{21}+\overline{g}_kg_1 = 0$. Substituting in this identity the expressions (2.58) for t_{11}, (2.59) for t_{1k}, (2.26) for t_{21} and (2.51) for g_k after straightforward computations we obtain $t_{22} = -\overline{\gamma}_1\gamma_2 t_{2k} = -\overline{\gamma}_1\gamma_k\prod\limits_{j=2}^{k-1}\sqrt{1-|\gamma_j|^2}$, $k\in\{3,4,\dots\}$. Multiplying in (2.61) the row with elements $(\overline{r}_j')_{j=1}^{\infty}$ with the first column we get $\overline{r}_1't_{11}+\overline{r}_2't_{21}+\overline{g}_\infty g_1 = 0$. Inserting in this identity the expressions (2.58) for t_{11}, (2.26) for t_{21}, (2.60) for r_1' and (2.54) for g_∞ as above we obtain $r_2' = -\overline{\gamma}_1\prod\limits_{j=2}^{\infty}\sqrt{1-|\gamma_j|^2}$. Thus, we have determined the elements in the second row of the matrix representation (2.13) of the operator T.

Postmultiplying now in (2.61) the rows with the second column, as above, we obtain $t_{33} = -\overline{\gamma}_2\gamma_3$, $t_{3k} = -\overline{\gamma}_2\gamma_k\prod\limits_{j=3}^{k-1}\sqrt{1-|\gamma_j|^2}$, $k\in\{4,5,\dots\}$, and $r_3' =$

$-\gamma_2 \prod\limits_{j=3}^{\infty} \sqrt{1-|\gamma_j|^2}$. Thus, the elements in the third row of the matrix representation (2.13) of the operator T are determined. Applying the method of mathematical induction we assume that the first n rows in the matrix representation (2.13) of T are determined. Then postmultiplying in (2.61) the rows with the nth column as above we obtain for $n \in \{1, 2, \dots\}$ the formulas $t_{nn} = -\overline{\gamma}_{n-1}\gamma_n$, $t_{nk} = -\overline{\gamma}_{n-1}\gamma_k \prod\limits_{j=n}^{k-1} \sqrt{1-|\gamma_j|^2}$, $k \geq n+1$, and $r'_n = -\gamma_{n-1} \prod\limits_{j=n}^{\infty} \sqrt{1-|\gamma_j|^2}$.

Let us set $D_{\gamma_j} := \sqrt{1-|\gamma_j|^2}$, $j \in \{0, 1, 2, \dots\}$. Thus, we obtain

Theorem 2.13. *Let $\theta(\zeta) \in \mathcal{S}$ and let Δ be a simple unitary colligation of the form* (2.1) *which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Assume that for the Schur parameter sequence of the function $\theta(\zeta)$ the product* (2.53) *converges. Then the canonical basis of the space $\mathfrak{H}$ has the form* (2.12). *The operators T, F, G and S have with respect to this basis the following matrix representations:*

$$T = \begin{pmatrix} T_{\mathfrak{F}} & \widetilde{R} \\ 0 & \widetilde{V}_T \end{pmatrix}, \tag{2.62}$$

where the operators in (2.62) are given by

$$T_{\mathfrak{F}} = \begin{pmatrix} -\overline{\gamma}_0\gamma_1 & -\overline{\gamma}_0 D_{\gamma_1}\gamma_2 & \dots & -\overline{\gamma}_0 \prod\limits_{j=1}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ D_{\gamma_1} & -\overline{\gamma}_1\gamma_2 & \dots & -\overline{\gamma}_1 \prod\limits_{j=2}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ 0 & D_{\gamma_2} & \dots & -\overline{\gamma}_2 \prod\limits_{j=3}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \dots & -\overline{\gamma}_{n-1}\gamma_n & \dots \\ 0 & 0 & \dots & D_{\gamma_n} & \dots \\ \vdots & \vdots & & \vdots & \end{pmatrix}, \tag{2.63}$$

$$\widetilde{R} = \begin{pmatrix} -\overline{\gamma}_0 \prod\limits_{j=1}^{\infty} D_{\gamma_j} & 0 & 0 & \dots \\ -\overline{\gamma}_1 \prod\limits_{j=2}^{\infty} D_{\gamma_j} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \\ -\overline{\gamma}_n \prod\limits_{j=n+1}^{\infty} D_{\gamma_j} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \quad \widetilde{V}_T = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

$$F = \operatorname{col}(D_{\gamma_0}, 0, 0, \dots; 0, 0, 0, \dots), \tag{2.64}$$

$$G = (\gamma_1 D_{\gamma_0}, \gamma_2 \prod_{j=0}^{1} D_{\gamma_j}, \dots, \gamma_n \prod_{j=0}^{n-1} D_{\gamma_j}, \dots ; \prod_{j=0}^{\infty} D_{\gamma_j}, 0, 0, \dots) \tag{2.65}$$

and $S = \gamma_0$.

We consider the model space

$$\widetilde{\mathfrak{H}} = l_2 \oplus l_2 = \{[(x_k)_{k=1}^{\infty}, (y_k)_{k=1}^{\infty}] : x_k, y_k \in \mathbb{C}; \sum_{k=1}^{\infty} |x_k|^2 < \infty, \sum_{k=1}^{\infty} |y_k|^2 < \infty\} \tag{2.66}$$

For $h_j = [(x_{jk})_{k=1}^{\infty}, (y_{jk})_{k=1}^{\infty}] \in \widetilde{\mathfrak{H}}$, $j = 1, 2$ we define

$h_1 + h_2 := [(x_{1k} + x_{2k})_{k=1}^{\infty}, (y_{1k} + y_{2k})_{k=1}^{\infty}]$, $\lambda h_1 := [(\lambda x_{1k})_{k=1}^{\infty}, (\lambda y_{1k})_{k=1}^{\infty}]$, $\lambda \in \mathbb{C}$,

$$(h_1, h_2) := \sum_{k=1}^{\infty} x_{1k} \overline{x}_{2k} + \sum_{k=1}^{\infty} y_{1k} \overline{y}_{2k}.$$

Equipped with these operations $\widetilde{\mathfrak{H}}$ becomes a Hilbert space. By the *canonical* basis in $\widetilde{\mathfrak{H}}$ we mean the orthonormal basis

$$e_1, e_2, \dots, e_n, \dots ; e'_1, e'_2, \dots, e'_n, \dots \tag{2.67}$$

where $e_j = [(\delta_{jk})_{k=1}^{\infty}, (\delta_{0k})_{k=1}^{\infty}]$, $e'_j = [(\delta_{0k})_{k=1}^{\infty}, (\delta_{jk})_{k=1}^{\infty}]$, $j \in \{1, 2, \dots\}$, and, as usual, for $j, k \in \{0, 1, 2, \dots\}$ $\delta_{jk} = \begin{cases} 1, & k = j, \\ 0, & k \neq j \end{cases}$. We suppose that the elements of the canonical basis are ordered as in (2.67).

Corollary 2.14. (*Description of the model*) *Let* $\theta(\zeta) \in \mathcal{S}$ *and let* Δ *be a simple unitary colligation of the form* (2.1) *which satisfies* $\theta_\Delta(\zeta) = \theta(\zeta)$. *Assume that the product* (2.53) *formed from the Schur parameter sequence of the function* $\theta(\zeta)$ *converges. Let us consider the model space* (2.66) *and let* $\widetilde{T}$ *be the operator in* $\widetilde{\mathfrak{H}}$ *which has the matrix representation* (2.62) *with respect to the canonical basis* (2.67). *Moreover, let* $\widetilde{F} : \mathbb{C} \to \widetilde{\mathfrak{H}}$, $\widetilde{G} : \widetilde{\mathfrak{H}} \to \mathbb{C}$ *be those operators which have the matrix representations* (2.64) *and* (2.65) *with respect to the canonical basis in* $\widetilde{\mathfrak{H}}$, *respectively. Furthermore, let* $\widetilde{S} := \gamma_0$. *Then the tuple* $\widetilde{\Delta} = (\widetilde{\mathfrak{H}}, \widetilde{\mathfrak{F}}, \widetilde{\mathfrak{G}}; \widetilde{T}, \widetilde{F}, \widetilde{G}, \widetilde{S})$ *where* $\widetilde{\mathfrak{F}} = \widetilde{\mathfrak{G}} = \mathbb{C}$, *is a simple unitary colligation which is unitarily equivalent to* Δ *and, thus,* $\theta_{\widetilde{\Delta}}(\zeta) = \theta(\zeta)$.

Proof. From Theorem 2.13 it is obvious that the unitary operator $Z : \mathfrak{H} \to \widetilde{\mathfrak{H}}$ which maps the canonical basis (2.12) of $\mathfrak{H}$ to the canonical basis (2.67) of $\widetilde{\mathfrak{H}}$ via $Z\phi_k = e_k$, $Z\psi_k = \widetilde{e}_k$, $k = 1, 2$, satisfies the conditions

$$ZT = \widetilde{T}Z \ , ZF = \widetilde{F} \ , \widetilde{G}Z = G \ . \tag{2.68}$$

Thus, the tuple $\widetilde{\Delta}$ is a simple unitary colligation which is unitarily equivalent to Δ. □

2.6. Description of the model of a unitary colligation in the case of divergence of the series $\sum_{j=0}^{\infty} |\gamma_j|^2$

In the case considered now the sequence of Schur parameters does not terminate. Thus, $|\gamma_j| < 1$ for all $j \in \{0, 1, 2, \dots\}$. From Corollary 2.6 we obtain that in this case the sequence (2.4) does not contain linearly dependent vectors. On the other hand, the infinite product (2.53) diverges in this case. Thus, in view of Corollary 2.10, we have $\mathfrak{H}_{\mathfrak{F}} = \mathfrak{H}$. This means that in this case the canonical basis of the space $\mathfrak{H}$ consists of the sequence

$$\phi_1, \phi_2, \dots, \phi_n, \dots \ . \tag{2.69}$$

Hence, in the case considered now we have $T = T_{\mathfrak{F}}$. So we obtain the following statement.

Theorem 2.15. *Let $\theta(\zeta) \in \mathcal{S}$ and let Δ be a simple unitary colligation of the form* (2.1) *which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Assume that the Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ of the function $\theta(\zeta)$ satisfies*

$$\sum_{j=0}^{\infty} |\gamma_j|^2 = +\infty \ . \tag{2.70}$$

Then the canonical basis of $\mathfrak{H}$ has the shape (2.69)*. The operator T has the matrix representation* (2.63) *with respect to this basis, whereas the matrix representation of the operators F, G and S with respect to this basis are given by*

$$F = \operatorname{col}\left(D_{\gamma_0}, 0, 0, \dots\right) \ , \tag{2.71}$$

$$G = \left(\gamma_1 D_{\gamma_0}, \gamma_2 \prod_{j=0}^{1} D_{\gamma_j}, \dots, \gamma_n \prod_{j=0}^{n-1} D_{\gamma_j}, \dots\right) \tag{2.72}$$

and $S = \gamma_0$.

In the case considered now as model space $\widetilde{\mathfrak{H}}$ we choose the space l_2 equipped with the above defined operations. By the canonical basis in $\widetilde{\mathfrak{H}}$ we mean the orthonormal basis $e_j = (\delta_{jk})_{k=1}^{\infty}$, $j \in \{1, 2, 3, \dots\}$. Hereby, the elements of this basis are supposed to be naturally ordered via

$$e_1, e_2, \dots, e_n, \dots \ . \tag{2.73}$$

Corollary 2.16. (*Description of the model*) *Let $\theta(\zeta) \in \mathcal{S}$ and let Δ be a simple unitary colligation of the form* (2.1) *which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Assume that the Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ of the function $\theta(\zeta)$ satisfies the divergence condition* (2.70)*. Let us consider the model space $\widetilde{\mathfrak{H}} = l_2$ and let $\widetilde{T}$ be the operator in $\widetilde{\mathfrak{H}}$ which has the matrix representation* (2.63) *with respect to the canonical basis* (2.73)*. Moreover, let $\widetilde{F} : \mathbb{C} \to \widetilde{\mathfrak{H}}$, $\widetilde{G} : \widetilde{\mathfrak{H}} \to \mathbb{C}$ be those operators which have the matrix representations* (2.71) *and* (2.72) *with respect to the canonical basis in $\widetilde{\mathfrak{H}}$, respectively. Furthermore, let $\widetilde{S} := \gamma_0$. Then the tuple $\widetilde{\Delta} = (\widetilde{\mathfrak{H}}, \widetilde{\mathfrak{F}}, \widetilde{\mathfrak{G}}; \widetilde{T}, \widetilde{F}, \widetilde{G}, \widetilde{S})$*

where $\widetilde{\mathfrak{F}} = \widetilde{\mathfrak{G}} = \mathbb{C}$, *is a simple unitary colligation which is unitarily equivalent to* Δ *and, thus,* $\theta_{\widetilde{\Delta}}(\zeta) = \theta(\zeta)$.

Proof. It suffices to mention that the unitary operator $Z : \mathfrak{H} \to \widetilde{\mathfrak{H}}$ which maps the canonical basis (2.69) of $\mathfrak{H}$ to the canonical basis (2.73) of $\widetilde{\mathfrak{H}}$ satisfies the conditions (2.68). □

2.7. Description of the model in the case that the function θ is a finite Blaschke product

Now we consider the case when the product (2.53) diverges whereas the series (2.70) converges. Obviously, this can only occur, if there exists a number n such that $|\gamma_k| < 1$, $k = 0, 1, \dots, n-1$; $|\gamma_n| = 1$. As already mentioned this means that the function $\theta(\zeta)$ is a finite Blaschke product of degrees n. From (2.26) it follows that in this case $t_{k+1,k} > 0$, $k = 1, 2, \dots, n-1$; $t_{n+1,n} = 0$. Then from (2.7) we see that this is equivalent to the fact that the vectors $(\phi'_k)_{k=1}^n$ are linearly dependent whereas the vector ϕ'_{n+1} is a linear combination of them. This means that in the case considered now we have $\mathfrak{H} = \mathfrak{H}_{\mathfrak{F}} = \bigvee_{k=1}^{\infty} \phi'_k = \bigvee_{k=1}^{n} \phi'_k$. Hence, $\dim \mathfrak{H} = n$ and the canonical basis in $\mathfrak{H}$ has the form

$$\phi_1, \phi_2, \dots, \phi_n \, . \tag{2.74}$$

As above we obtain the following result.

Theorem 2.17. *Let* $\theta(\zeta)$ *be a finite Blaschke product of degree* n *and let* Δ *be a simple unitary colligation of the form* (2.1) *which satisfies* $\theta_\Delta(\zeta) = \theta(\zeta)$. *Then the canonical basis of the space* $\mathfrak{H}$ *has the form* (2.74). *The operators* T, F, G *and* S *have the following matrix representations with respect to this basis:*

$$T = \begin{pmatrix} -\overline{\gamma}_0\gamma_1 & -\overline{\gamma}_0 D_{\gamma_1}\gamma_2 & \dots & -\overline{\gamma}_0 \prod\limits_{j=1}^{n-1} D_{\gamma_j}\gamma_n \\ D_{\gamma_1} & -\overline{\gamma}_1\gamma_2 & \dots & -\overline{\gamma}_1 \prod\limits_{j=2}^{n-1} D_{\gamma_j}\gamma_n \\ 0 & D_{\gamma_2} & \dots & -\overline{\gamma}_2 \prod\limits_{j=3}^{n-1} D_{\gamma_j}\gamma_n \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & -\overline{\gamma}_{n-1}\gamma_n \end{pmatrix}, \tag{2.75}$$

$$F = \operatorname{col}(D_{\gamma_0}, 0, \dots, 0) \, , \tag{2.76}$$

$$G = (\gamma_1 D_{\gamma_0}, \gamma_2 \prod_{j=0}^{1} D_{\gamma_j}, \dots, \gamma_n \prod_{j=0}^{n-1} D_{\gamma_j}), \tag{2.77}$$

and $S = \gamma_0$.

In the case considered now we choose the n-dimensional Hilbert space $\mathbb{C}^n$ as model space $\widetilde{\mathfrak{H}}$. By the canonical basis in $\widetilde{\mathfrak{H}}$ we mean the orthonormal basis $e_j = (\delta_{jk})_{k=1}^n$, $j \in \{1, \dots, n\}$. Hereby we assume that the elements of this basis are naturally ordered via

$$e_1, e_2, \dots, e_n \ . \tag{2.78}$$

As above the following result can be verified:

Corollary 2.18. (*Description of the model*) *Let $\theta(\zeta)$ be a finite Blaschke product of degree n and let Δ be a simple unitary colligation of the form* (2.1) *which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Let $(\gamma_j)_{j=0}^n$ be the Schur parameter sequence of the function θ. Let us consider the model space $\widetilde{\mathfrak{H}} = \mathbb{C}^n$ and let $\widetilde{T}$ be the operator in $\widetilde{\mathfrak{H}}$ which has the matrix representation* (2.75) *with respect to the canonical basis* (2.78)*. Moreover, let $\widetilde{F} : \mathbb{C} \to \widetilde{\mathfrak{H}}$, $\widetilde{G} : \widetilde{\mathfrak{H}} \to \mathbb{C}$ be those operators which have the matrix representations* (2.76) *and* (2.77) *with respect to the canonical basis in $\widetilde{\mathfrak{H}}$, respectively. Furthermore, let $\widetilde{S} := \gamma_0$. Then the tuple $\widetilde{\Delta} = (\widetilde{\mathfrak{H}}, \widetilde{\mathfrak{F}}, \widetilde{\mathfrak{G}}; \widetilde{T}, \widetilde{F}, \widetilde{G}, \widetilde{S})$ where $\widetilde{\mathfrak{F}} = \widetilde{\mathfrak{G}} = \mathbb{C}$, is a simple unitary colligation which is unitarily equivalent to Δ and, thus, $\theta_{\widetilde{\Delta}}(\zeta) = \theta(\zeta)$.*

2.8. Comments

A. Let $\mu \in \mathfrak{M}(\mathbb{T}, \mathbb{C})$ and assume $\mu(\mathbb{T}) = 1$, i.e., the measure μ is a scalar, normalized Borel measure on $\mathbb{T}$. Let us define the usual Hilbert space of square integrable complex-valued functions on $\mathbb{T}$ with respect to μ by

$$L^2(\mu) = L^2(\mu, \mathbb{T}) = \{f : f \ is \ \mu\text{-measurable and } \int_{\mathbb{T}} |f(t)|^2 \mu(dt) < \infty\}.$$

On the space $L^2(\mu)$ we consider the unitary operator $U^\times$ which is generated by multiplication by $\overline{t}$ where $t \in \mathbb{T}$ is the independent variable : $(U^\times f)(t) = \overline{t} f(t)$, $f \in L^2(\mu)$.

Let τ be the embedding operator of $\mathbb{C}$ into $L^2(\mu)$, i.e., $\tau : \mathbb{C} \to L^2(\mu)$ and for each $c \in \mathbb{C}$ the value τc is the constant function with the value c. It is obvious that the triple $(L^2(\mu), U^\times, \tau)$ is the minimal Naimark dilation of the measure μ.

Consider the subspace $\mathfrak{H}_\mu := L^2(\mu) \ominus \tau(\mathbb{C})$. According to the decomposition $L^2(\mu) = \mathfrak{H}_\mu \oplus \tau(\mathbb{C})$ the operator $U^\times$ is given by the block matrix

$$U^\times = \begin{pmatrix} T^\times & F^\times \\ G^\times & S^\times \end{pmatrix}.$$

Then from Theorem 1.20, statement (b), it follows that the set

$$\Delta_\mu := (\mathfrak{H}_\mu, \mathbb{C}, \mathbb{C}; T^\times, F^\times \tau, \tau^* G^\times, \tau^* S^\times \tau)$$

is a simple unitary colligation. Moreover, the characteristic function $\theta_{\Delta_\mu}(\zeta)$ is associated with the measure μ. Thus, if the function $\Phi(\zeta)$ has the form (1.17) then from (1.19) it follows that $\zeta \theta_{\Delta_\mu}(\zeta) = (\Phi(\zeta) - I)(\Phi(\zeta) + I)^{-1}$.

It is important that the canonical basis (2.12) for the colligation Δ_μ is generated by the system of orthogonal polynomials in $L^2(\mu)$. Hence (see Theorem 1.20 and Remark 1.22), Theorem 2.13, Theorem 2.15 and Theorem 2.17 give the matrix representation of the operator $U^{\times}$ in this basis. The first appearance (1944) of this matrix is in Geronimus [27]. Ya.L. Geronimus considered the case when the sequence of the orthogonal polynomials is basis in $L^2(\mu)$, i.e., when the series $\sum_{j=0}^{\infty} |\gamma_j|^2$ diverges (see Theorem 2.15). W.B. Gragg [27] in 1982 rediscovered this matrix representation and used it for calculations. A.V. Teplyaev [34] (1991) seems to be first to use it for spectral purposes. What concerns the role of this matrix representation in theory of orthogonal polynomials on the unit circle we refer the reader to B. Simon [32, Chapter 4].

B. The full matrix representation (see Theorem 2.13) appeared in Constantinescu [14] in 1984 (see also Bakonyi/Constantinescu [6]). He finds it as the Naimark dilation. Let us establish some connections with results in [14]. We note that from Remark 1.22 it follows that the above constructed models of unitary colligations are also models of Naimark dilations of corresponding Borel measures on $\mathbb{T}$. Under this aspect we consider in more detail the model described in Corollary 2.14. In this case the model space $\mathfrak{K}$ for the Naimark dilation (1.22) has the form

$$\mathfrak{K} = \widetilde{\mathfrak{H}} \oplus \mathbb{C} = (l_2 \oplus l_2) \oplus \mathbb{C} \ . \tag{2.79}$$

Moreover, $\mathcal{U} = \begin{pmatrix} \widetilde{T} & \widetilde{F} \\ \widetilde{G} & \widetilde{S} \end{pmatrix} : (l_2 \oplus l_2) \oplus \mathbb{C} \to (l_2 \oplus l_2) \oplus \mathbb{C}$. In accordance to (2.79), the vectors $k \in \mathfrak{K}$ have the form

$$k = (x_1, x_2, \ldots, x_n, \ldots; y_1, y_2, \ldots, y_n, \ldots; c) \ , \tag{2.80}$$

where $(x_k)_{k\in\mathbb{N}} \in l_2, (y_k)_{k\in\mathbb{N}} \in l_2$, $c \in \mathbb{C}$. The operator τ embeds $\mathbb{C}$ into $\mathfrak{K}$ in the following way: $\tau c = (0, 0, \ldots, 0, \ldots; 0, 0, \ldots, 0, \ldots; c)$, $c \in \mathbb{C}$. We change the order in the considered orthonormal base of $\mathfrak{K}$ in such way that the vector k which has the form (2.80) is given in the following way

$$k = (\ldots, y_n, \ldots, y_2, y_1, c, x_1, x_2, \ldots, x_n, \ldots) \ ,$$

In this case, it is convenient to set $c = x_0$, $y_k = x_{-k}$, $k \in \{1, 2, \ldots\}$, i.e.,

$$k = (\ldots, x_{-n}, \ldots, x_{-2}, x_{-1}, \boxed{x_0}, x_1, x_2, \ldots, x_n, \ldots) \tag{2.81}$$

where we have drawn a square around the central entry with index 0. Now

$$\tau x_0 = (\ldots, 0, \ldots, 0, \boxed{x_0}, 0, \ldots, 0, \ldots) \ .$$

We associate with the representation (2.81) the following orthogonal decomposition

$$\mathfrak{K} = l_2^- \oplus l_2^+ \tag{2.82}$$

where $l_2^- = \{k \in \mathfrak{K} : x_k = 0,\ k \geq 0\}$ *and* $l_2^+ = \{k \in \mathfrak{K} : x_k = 0,\ k < 0\}$. From the form of the operators $\widetilde{T}, \widetilde{F}, \widetilde{G}$ and $\widetilde{S}$ it follows that the operator $\mathcal{U}$ has the following matrix representation with respect to the new basis and with respect to the orthogonal decomposition (2.82): $\begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{21} & \mathcal{U}_{22} \end{pmatrix}$, where $\mathcal{U}_{12} = 0$,

$$\mathcal{U}_{11} = \begin{pmatrix} \ddots & & & \\ & & & \\ & \ddots & 0 & \\ & & 1 & 0 & \\ & & & 1 & 0 \end{pmatrix}, \quad \mathcal{U}_{21} = \begin{pmatrix} \dots & 0 & 0 & \prod\limits_{j=0}^{\infty} D_{\gamma_j} \\ \dots & 0 & 0 & -\overline{\gamma}_1 \prod\limits_{j=1}^{\infty} D_{\gamma_j} \\ & \vdots & \vdots & \vdots \\ \dots & 0 & 0 & -\overline{\gamma}_n \prod\limits_{j=n+1}^{\infty} D_{\gamma_j} \end{pmatrix}$$

and

$$\mathcal{U}_{22} = \begin{pmatrix} \gamma_0 & D_{\gamma_0}\gamma_1 & \dots & \prod\limits_{j=0}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ D_{\gamma_0} & -\overline{\gamma}_0\gamma_1 & \dots & -\overline{\gamma}_0 \prod\limits_{j=1}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ 0 & D_{\gamma_1} & \dots & -\overline{\gamma}_1 \prod\limits_{j=2}^{n-1} D_{\gamma_j}\gamma_n & \dots \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \dots & -\overline{\gamma}_{n-1}\gamma_n & \dots \\ 0 & 0 & \dots & D_{\gamma_n} & \dots \\ \vdots & \vdots & & \vdots & \end{pmatrix}. \tag{2.83}$$

In this form but using different methods a Naimark dilation is constructed in Constantinescu [14] and Bakonyi/Constantinescu [6, Chapter 2].

C. We consider a simple unitary colligation Δ of the form (2.1). If we choose in $\mathfrak{H}$ the canonical basis in accordance with (2.15) we obtain for the contraction (S, G) the matrix representation $(S, G) = (S, g_1, g_2, \dots, g_\infty, 0, 0, \dots)$. The above results show that if we parametrize the contractive block row $(S, g_1, g_2, \dots, g_n, \dots)$ by the method proposed in Constantinescu [15, Chapter 1] we will obtain all blocks described in Theorem 2.13 with exception of the coshift $\widetilde{V}_T$.

D. Because of $\mathcal{U}_{12} = 0$ the operator $\mathcal{U}_{22}$ is an isometry acting in l_2^+. We mention that the representation of an isometry in the form (2.83) plays an important role in Foias/Frazho [25, Chapter 13] in connection with the construction of Schur representations for the commutant lifting theorem.

E. If $|\gamma_k| < 1$, $k \in \{0, 1, 2, \dots\}$ and the product (2.53) diverges then $\mathfrak{K} = l_2^+$, i.e., $\mathcal{U} = \mathcal{U}_{22}$. In this case the layered form of the model is particularly clear. For example, if we pass in the Schur algorithm from the Schur function $\theta_0(\zeta) = \theta(\zeta)$ to the function $\theta_1(\zeta) = \frac{\theta_0(\zeta) - \gamma_0}{\zeta(1 - \overline{\gamma}_0\theta_0(\zeta))}$ the Schur parameter sequence changes from $(\gamma_k)_{k=0}^{\infty}$ to $(\gamma_k)_{k=1}^{\infty}$. This is expressed in the model representation (2.83) in the following way. One has to cancel the first column and the first row. After that one has to divide the second row by $-\overline{\gamma}_0$. This "layered form" finds its expression in the following multiplicative representation of $\mathcal{U}_{22}$ which can be immediately checked (see also Foias/Frazho [25], Constantinescu [15]): $\mathcal{U}_{22} = V_0 V_1 V_2 \dots V_n \dots = s - \lim_{n\to\infty} V_0 V_1 \dots V_n$ where $V_0 = R_{\gamma_0} \oplus 1 \oplus 1 \oplus \dots$, $V_1 = 1 \oplus R_{\gamma_1} \oplus 1 \oplus \dots$, $V_2 = 1 \oplus 1 \oplus R_{\gamma_2} \oplus \dots$ and R_{γ_j} is the elementary rotation matrix associated with γ_j, i.e., $R_{\gamma_j} = \begin{pmatrix} \gamma_j & D_{\gamma_j} \\ D_{\gamma_j} & -\overline{\gamma}_j \end{pmatrix}$, $j \in \{0, 1, 2, \dots\}$.

3. A model representation of the maximal shift V_T contained in a contraction T

3.1. The conjugate canonical basis

Let $\theta(\zeta) \in \mathcal{S}$. Assume that

$$\Delta = (\mathfrak{H}, \mathfrak{F}, \mathfrak{G}; T, F, G, S) \tag{3.1}$$

is a simple unitary colligation satisfying $\theta(\zeta) = \theta_\Delta(\zeta)$. As above we consider the case $\mathfrak{F} = \mathfrak{G} = \mathbb{C}$. Moreover, we choose the complex number 1 as basis vector of the one-dimensional complex vector space $\mathbb{C}$. We assume that the sequence $\gamma = (\gamma_j)_{j=0}^{\omega}$ of Schur parameters of the function $\theta(\zeta)$ is infinite (i.e., $\omega = \infty$) and that the infinite product (2.53) converges. In this case, as it follows from Theorem 2.13, the canonical basis of the space $\mathfrak{H}$ has the form (2.12). Hereby, the matrix representation of the operators of the colligation Δ with respect to this basis are given by formulas (2.62)–(2.65).

We consider the function $\widetilde{\theta}(\zeta)$ which is associate to $\theta(\zeta)$, i.e., $\widetilde{\theta}(\zeta) = \overline{\theta(\overline{\zeta})}$, $\zeta \in \mathbb{D}$. Clearly, that $\widetilde{\theta}(\zeta) \in \mathcal{S}$ and

$$\widetilde{\Delta} := (\mathfrak{H}, \mathfrak{G}, \mathfrak{F}; T^*, G^*, F^*, S^*) \tag{3.2}$$

is a simple unitary colligation satisfying $\widetilde{\theta}(\zeta) = \theta_{\widetilde{\Delta}}(\zeta)$(see Brodskii[12]). The unitary colligation (3.2) is called adjoint to the colligation (3.1). Hence, the function $\widetilde{\theta}(\zeta)$ is the c.o.f. of the contraction T^*. It can be easily seen that the Schur parameter sequence $(\widetilde{\gamma}_j)_{j=0}^{\infty}$ of the function $\widetilde{\theta}(\zeta)$ is given by $\widetilde{\gamma}_j = \overline{\gamma}_j, j \in \{0, 1, 2, \dots\}$ and, consequently, the product (2.53) converges for $(\widetilde{\gamma}_j)_{j=0}^{\infty}$, too. This means that the canonical basis of the space $\mathfrak{H}$ which is constructed for the colligation $\widetilde{\Delta}$ will also consist of two sequences of vectors

$$\widetilde{\phi}_1, \widetilde{\phi}_2, \dots; \widetilde{\psi}_1, \widetilde{\psi}_2, \dots \ . \tag{3.3}$$

From the considerations in Section 2.2 it follows that this basis can be uniquely characterized by the following conditions:

(1) The sequence $(\widetilde{\phi}_k)_{k=1}^{\infty}$ arises in the result of the Gram-Schmidt orthogonalization procedure of the sequence $(T^{*k-1}G^*(1))_{k=1}^{\infty}$ taking into account the normalization conditions $(T^{*k-1}G^*(1), \widetilde{\phi}_k) > 0$, $k \in \{1, 2, 3, \dots\}$.

(2) The vector $\widetilde{\psi}_1$ is that basis vector of the one-dimensional generating wandering subspace of the maximal unilateral shift V_T acting in $\mathfrak{H}_{\mathfrak{G}}^{\perp} = \mathfrak{H} \ominus \mathfrak{H}_{\mathfrak{G}}$ which satisfies the inequality $(\phi_1, \widetilde{\psi}_1) > 0$ and, moreover,

$$\widetilde{\psi}_{k+1} = T\widetilde{\psi}_k, \ k \in \{1, 2, 3, \ldots\}. \tag{3.4}$$

Definition 3.1. *The canonical basis* (3.3) *which is constructed for the adjoint colligation* (3.2) *is called conjugated to the canonical basis* (2.12) *constructed for the colligation* (3.1).

Remark 3.2. *In view of* $\theta(\zeta) = (\widetilde{\widetilde{\theta}}(\zeta))$ *and* $\Delta = (\widetilde{\widetilde{\Delta}})$ *the canonical basis* (2.12) *is conjugated to the canonical basis* (3.3).

Our approach is based on the study of interrelations between the canonical basis (2.12) and the basis (3.3) which is conjugated to it. For this reason, we introduce the unitary operator $\mathcal{U}(\gamma) : \mathfrak{H} \to \mathfrak{H}$ which maps the first basis onto the second one:

$$\mathcal{U}(\gamma)\phi_k = \widetilde{\phi}_k, \ \ \mathcal{U}(\gamma)\psi_k = \widetilde{\psi}_k, \quad k \in \{1, 2, 3, \ldots\}. \tag{3.5}$$

The orthonormal systems $(\phi_k)_{k=1}^{\infty}$ and $(\psi_k)_{k=1}^{\infty}$ are bases of the subspaces $\mathfrak{H}_{\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{F}}^{\perp}$, respectively, whereas the orthonormal systems $(\widetilde{\phi}_k)_{k=1}^{\infty}$ and $(\widetilde{\psi}_k)_{k=1}^{\infty}$ are bases of the subspaces $\mathfrak{H}_{\mathfrak{G}}$ and $\mathfrak{H}_{\mathfrak{G}}^{\perp}$, respectively. Therefore, the operator $U(\gamma)$ transfers the decomposition $\mathfrak{H} = \mathfrak{H}_{\mathfrak{F}} \oplus \mathfrak{H}_{\mathfrak{F}}^{\perp}$ into the decomposition $\mathfrak{H} = \mathfrak{H}_{\mathfrak{G}} \oplus \mathfrak{H}_{\mathfrak{G}}^{\perp}$ taking into account the structures of the canonical bases. Consequently, the knowledge of the operator $\mathcal{U}(\gamma)$ enables us to describe the position of each of the subspaces $\mathfrak{H}_{\mathfrak{G}}$ and $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ in relation to $\mathfrak{H}_{\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{F}}^{\perp}$. We emphasize that many properties of the function $\theta(\zeta)$ and the corresponding contraction T depend on the mutual position of these subspaces.

In view of $\widetilde{\gamma}_j = \overline{\gamma}_j, j \in \{0, 1, 2, \ldots\}$, the replacement of the canonical basis (2.12) by its conjugated basis (3.3) requires that in corresponding matrix representations we have to replace γ_j by $\overline{\gamma}_j$. In particular, the following result holds:

Theorem 3.3. *The matrix representation of the operator* T^* *with respect to the canonical basis* (3.3) *is obtained from the matrix representation of the operator* T *with respect to the canonical basis* (2.12) *by replacing* γ_j *by* $\overline{\gamma}_j, \ j \in \{0, 1, 2, \ldots\}$.

3.2. A model representation of the maximal unilateral shift V_T contained in a contraction T

Let $\theta(\zeta) \in \mathcal{S}$ and assume that Δ is a simple unitary colligation of the form (3.1) which satisfies $\theta(\zeta) = \theta_{\Delta}(\zeta)$. We assume that the sequence of Schur parameters of the function $\theta(\zeta)$ is infinite and that the infinite product (2.53) converges. Then it follows from Lemma 2.11 that in this and only this case the contraction T (resp. T^*) contains a nontrivial maximal shift V_T (resp. V_{T^*}). Hereby, the multiplicity of the shift V_T (resp. V_{T^*}) equals 1. The shift V_{T^*} in the model representation associated with the canonical basis (2.12) is immediately determined by the sequence of basis vectors $(\psi_k)_{k=1}^{\infty}$ since ψ_1 is a basis vector of the one-dimensional generating wandering subspace of V_{T^*} and $\psi_k = V_{T^*}^{k-1}\psi_1,\ k \in \{2,3,4,\dots\}$. Analogously (see property (2) of the conjugate canonical basis (3.3)) the sequence $(\widetilde{\psi}_k)_{k=1}^{\infty}$ of the basis (3.3) determines the maximal shift V_T. Thus, representing the vectors $(\widetilde{\psi}_k)_{k=1}^{\infty}$ in terms of the vectors of the basis (2.12) we obtain a model representation of the maximal shift V_T with the aid of the canonical basis (2.12). The main goal of this paragraph is the detailed description of this model. In the following we use the same symbol for an operator and its matrix with respect to the canonical basis (2.12).

The unitary operator (3.5) has the matrix representation

$$\mathcal{U}(\gamma) = \begin{pmatrix} \mathcal{R}(\gamma) & \mathfrak{L}(\gamma) \\ \mathcal{P}(\gamma) & \mathcal{Q}(\gamma) \end{pmatrix} \tag{3.6}$$

where $\mathcal{R}, \mathcal{P}, \mathfrak{L}$ and $\mathcal{Q}$ are the matrices of the operators

$$P_{\mathfrak{H}_{\mathfrak{F}}}\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}}\mathcal{U} : \mathfrak{H}_{\mathfrak{F}} \to \mathfrak{H}_{\mathfrak{F}}\ ,\ P_{\mathfrak{H}_{\mathfrak{F}}^{\perp}}\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}}\mathcal{U} : \mathfrak{H}_{\mathfrak{F}} \to \mathfrak{H}_{\mathfrak{F}}^{\perp},$$

$$P_{\mathfrak{H}_{\mathfrak{F}}}\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}}\mathcal{U} : \mathfrak{H}_{\mathfrak{F}}^{\perp} \to \mathfrak{H}_{\mathfrak{F}}\quad \textit{and}\quad P_{\mathfrak{H}_{\mathfrak{F}}^{\perp}}\mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{F}}^{\perp}}\mathcal{U} : \mathfrak{H}_{\mathfrak{F}}^{\perp} \to \mathfrak{H}_{\mathfrak{F}}^{\perp},$$

respectively. Hereby, if $\mathfrak{K}$ is a closed subspace of $\mathfrak{H}$, the operator $P_{\mathfrak{K}}$ denotes the orthoprojection from $\mathfrak{H}$ onto $\mathfrak{K}$.

From (3.5) we see that the columns of the matrix

$$\begin{pmatrix} \mathfrak{L}(\gamma) \\ \mathcal{Q}(\gamma) \end{pmatrix} \tag{3.7}$$

provide the coefficients in the representation of the vectors $(\widetilde{\psi}_k)_{k=1}^{\infty}$ with respect to the canonical basis (2.12). Thus, the model description of the shift V_T leads to the determination of the matrix (3.7). We note that the matrix (3.7) shows how the subspace $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ is located relatively to the subspaces $\mathfrak{H}_{\mathfrak{F}}$ and $\mathfrak{H}_{\mathfrak{F}}^{\perp}$.

Theorem 3.4. *The identities*

$$(\widetilde{\psi}_1, \phi_1) = \prod_{j=1}^{\infty}(1-|\gamma_j|^2)^{\frac{1}{2}} \tag{3.8}$$

and

$$(\widetilde{\psi}_j, \phi_1) = 0,\ j \in \{2,3,\dots\} \tag{3.9}$$

hold true.

Proof. In view of $\widetilde{\phi}_1 = \frac{1}{\sqrt{1-|\gamma_0|^2}}G^*(1)$ from the matrix representation (2.65) of the operator G it follows

$$(\psi_1, \widetilde{\phi}_1) = \frac{1}{\sqrt{1-|\gamma_0|^2}}(\psi_1, G^*(1)) = \prod_{j=1}^{\infty}(1-|\gamma_j|^2)^{\frac{1}{2}}. \tag{3.10}$$

Since changing from $(\psi_1, \widetilde{\phi}_1)$ to $(\widetilde{\psi}_1, \phi_1)$ is realized by replacing γ_j by $\widetilde{\gamma}_j = \overline{\gamma}_j$, $j \in \{0,1,2,\dots,\}$, formula (3.8) follows from (3.10).

For $j \in \{2, 3, \ldots,\}$ we obtain

$$(\widetilde{\psi}_j, \phi_1) = (T\widetilde{\psi}_{j-1}, \frac{1}{\sqrt{1-|\gamma_0|^2}}F(1)) = \frac{1}{\sqrt{1-|\gamma_0|^2}}(\widetilde{\psi}_{j-1}, T^*F(1)).$$

From the colligation condition (1.3) we infer

$$T^*F(1) = -G^*S(1) = -\gamma_0 G^*(1) = -\gamma_0\sqrt{1-|\gamma_0|^2}\;\widetilde{\phi}_1.$$

Thus, $(\widetilde{\psi}_j, \phi_1) = -\overline{\gamma}_0(\widetilde{\psi}_{j-1}, \widetilde{\phi}_1) = 0.$ □

Definition 3.5. *Denote by* Γ *the set of all sequences* $\gamma = (\gamma_j)_{j=0}^{\omega}$ *which occur as Schur parameters of Schur functions. Furthermore, denote* Γl_2 *the subset of all sequences belonging to* Γ *for which the product* (2.53) *converges. Thus,*

$$\Gamma l_2 := \{\gamma = (\gamma_j)_{j=0}^{\infty} : \gamma_j \in \mathbb{C}, |\gamma_j| < 1, j \in \{0,1,2,\ldots\} \;\; \textit{and} \;\; \sum_{j=0}^{\infty} |\gamma_j|^2 < \infty\}.$$

We define the coshift $W : l_2 \to l_2$ via

$$(\gamma_0, \gamma_1, \gamma_2, \ldots) \mapsto (\gamma_1, \gamma_2, \gamma_3, \ldots)\ ,\ \gamma = (\gamma_j)_{j=0}^{\infty} \in\ l_2. \tag{3.11}$$

In the sequel, the system of functions $(L_n(\gamma))_{n=0}^{\infty}$, which was introduced for $\gamma \in \Gamma l_2$ in [21] will play an important role. For $\gamma \in \Gamma l_2$ we set

$$L_0(\gamma) := 1, \quad L_n(\gamma) = L_n(\gamma_0, \gamma_1, \gamma_2, \ldots) := \tag{3.12}$$
$$\sum_{r=1}^{n}(-1)^r \sum_{s_1+\ldots+s_r=n} \sum_{j_1=n-s_1}^{\infty} \sum_{j_2=j_1-s_2}^{\infty} \cdots \sum_{j_r=j_{r-1}-s_r}^{\infty} \gamma_{j_1}\overline{\gamma}_{j_1+s_1}\gamma_{j_2}\overline{\gamma}_{j_2+s_2}\cdots\gamma_{j_r}\overline{\gamma}_{j_r+s_r}.$$

Here the summation runs over all ordered r-tuples $(s_1, \ldots, s_r)$ of positive integers which satisfy $s_1 + s_2 + \ldots + s_r = n$. For example,

$$L_1(\gamma) = -\sum_{j=0}^{\infty}\gamma_j\overline{\gamma}_{j+1}, \quad L_2(\gamma) = -\sum_{j=0}^{\infty}\gamma_j\overline{\gamma}_{j+2} + \sum_{j_1=1}^{\infty}\sum_{j_2=j_1-1}^{\infty}\gamma_{j_1}\overline{\gamma}_{j_1+1}\gamma_{j_2}\overline{\gamma}_{j_2+1}.$$

In view of $\gamma \in \Gamma l_2$ the series in (3.12) converges absolutely.

Theorem 3.6. (*Model representation of the maximal shift* V_T *with respect to the canonical basis* (2.12)) *Let* $\theta(\zeta)$ *be a function of class* $\mathcal{S}$ *the Schur parameter sequence* $\gamma = (\gamma_j)_{j=0}^{\infty}$ *of which belongs to* Γl_2. *Further, let* Δ *be a simple unitary colligation of the form* (3.1) *which satisfies* $\theta(\zeta) = \theta_\Delta(\zeta)$. *Then the vectors* $(\widetilde{\psi}_j)_{j=1}^{\infty}$ *of the conjugate canonical basis* (3.3) *admit the following representations in terms of the vectors of the canonical basis* (2.12)*:*

$$\widetilde{\psi}_j = \sum_{k=j}^{\infty}\Pi_k L_{k-j}(W^j\gamma)\phi_k + \sum_{k=1}^{\infty} Q(W^{k+j-1}\gamma)\psi_k \tag{3.13}$$

where

$$\Pi_k = \prod_{j=k}^{\infty}\sqrt{1-|\gamma_j|^2},\ k \in \{0,1,2,\ldots\} \tag{3.14}$$

and

$$Q(\gamma) = -\sum_{j=0}^{\infty} \gamma_j L_j(\gamma). \tag{3.15}$$

Hereby, the sequence $(L_n(\gamma))_{n=0}^{\infty}$ *is defined by* (3.12) *whereas the coshift* W *is given via* (3.11). *The series in* (3.15) *converges absolutely.*

Proof. Clearly, we have $\widetilde{\psi}_1 = \sum\limits_{k=1}^{\infty} (\widetilde{\psi}_1, \phi_k)\phi_k + \sum\limits_{k=1}^{\infty} (\widetilde{\psi}_1, \psi_k)\psi_k$. If $k \in \mathbb{N}$, then taking into account $\Pi_k \neq 0$ we define $\Phi_{k-1}(\gamma) := \frac{(\widetilde{\psi}_1, \phi_k)}{\Pi_k}$ and $Q_k(\gamma) := (\widetilde{\psi}_1, \psi_k)$. Thus,

$$\widetilde{\psi}_1 = \sum_{k=1}^{\infty} \Pi_k \Phi_{k-1}(\gamma)\phi_k + \sum_{k=1}^{\infty} Q_k(\gamma)\psi_k. \tag{3.16}$$

Then from (3.8) it follows $\Phi_0(\gamma) = 1$. Thus, in view of (3.12) we have

$$\Phi_0(\gamma) = L_0(W\gamma). \tag{3.17}$$

As well the vectors $(\phi_j)_{j=1}^{\infty}$ and $(\psi_j)_{j=1}^{\infty}$ from the canonical basis (2.12) as the vectors $(\widetilde{\phi}_j)_{j=1}^{\infty}$ and $(\widetilde{\psi}_j)_{j=1}^{\infty}$ from the conjugate canonical basis (3.3) clearly depend on γ. For this reason, we will mark this dependence on γ in the following consideration by the notations $\phi_j(\gamma), \psi_j(\gamma), \widetilde{\phi}_j(\gamma)$ and $\widetilde{\psi}_j(\gamma),\ j \in \mathbb{N}$. The identity

$$\widetilde{\psi}_{j+1}(\gamma) = \begin{pmatrix} 0 \\ \widetilde{\psi}_j(W\gamma) \end{pmatrix}, \ j \in \mathbb{N} \tag{3.18}$$

will turn out to be essential in the sequel. In order to prove (3.18) we will mainly use the layered structure of the model of the colligation Δ (see Theorem 2.13). Namely, the matrix representation (2.62) implies

$$T(\gamma) = \begin{pmatrix} -\overline{\gamma}_0\gamma_1 & -\overline{\gamma}_0 G(W\gamma) \\ F(W\gamma) & T(W\gamma) \end{pmatrix} \tag{3.19}$$

where

$$T(W\gamma) = \begin{pmatrix} T_{\mathfrak{F}}(W\gamma) & \widetilde{R}(W\gamma) \\ 0 & \widetilde{V}_T(W\gamma) \end{pmatrix} \tag{3.20}$$

and where $T(\gamma)$, $F(\gamma)$ and $G(\gamma)$ are given via (2.62), (2.64) and (2.65). Hereby, we have $\widetilde{V}_T(W\gamma) = \widetilde{V}_T(\gamma) = \widetilde{V}_T$. From (3.19) we infer

$$T(\gamma)\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix} = \begin{pmatrix} -\overline{\gamma}_0(\widetilde{\psi}_1(W\gamma), G^*(W\gamma)(1)) \\ T(W\gamma)\widetilde{\psi}_1(W\gamma) \end{pmatrix} \tag{3.21}$$

In view of $\widetilde{\phi}_1(W\gamma) = \frac{1}{\sqrt{1-|\gamma_1|^2}} G^*(W\gamma)(1)$ we get

$$(\widetilde{\psi}_1(W\gamma)\ ,\ G^*(W\gamma)(1)) = \sqrt{1-|\gamma_1|^2}(\widetilde{\psi}_1(W\gamma)\ ,\ \widetilde{\phi}_1(W\gamma)) = 0.$$

Taking into account the identity $T(W\gamma)\widetilde{\psi}_1(W\gamma) = \widetilde{\psi}_2(W\gamma)$ from (3.21) it follows that $T(\gamma)\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\psi}_2(W\gamma) \end{pmatrix}$. Analogously, the identity

$$T^n(\gamma)\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\psi}_{n+1}(W\gamma) \end{pmatrix}, \ n \in \mathbb{N},$$

can be obtained. Thus, for $n \in \mathbb{N}$ we have

$$\|T^n(\gamma)\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix}\| = \|\begin{pmatrix} 0 \\ \widetilde{\psi}_{n+1}(W\gamma) \end{pmatrix}\| = 1 = \|\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix}\|.$$

Using Theorem 1.2, we obtain $\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix} \in \mathfrak{H}_{\mathfrak{G}}^{\perp}$. This implies

$$\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix} = \sum_{j=1}^{\infty} y_j \widetilde{\psi}_j(\gamma) \tag{3.22}$$

where $y_j = (\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix}, \widetilde{\psi}_j(\gamma))$, $j \in \mathbb{N}$. Combining formula (3.22) with Theorem 3.4 we infer

$$\begin{aligned} 0 &= (\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = (\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix}, \phi_1(\gamma)) \\ &= \sum_{j=1}^{\infty} y_j(\widetilde{\psi}_j(\gamma), \phi_1(\gamma)) = y_1(\widetilde{\psi}_1(\gamma), \phi_1(\gamma)) = y_1\Pi_1. \end{aligned}$$

Hence $y_1 = 0$. Thus, from (3.22) we get

$$(\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix}, \phi_2(\gamma)) = \sum_{j=2}^{\infty} y_j(\widetilde{\psi}_j(\gamma), \phi_2(\gamma)). \tag{3.23}$$

From the matrix representation (2.62) we find

$$T(\gamma)\phi_1(\gamma) = -\overline{\gamma}_0\gamma_1\phi_1(\gamma) + D_{\gamma_1}\phi_2(\gamma).$$

This implies

$$\phi_2(\gamma) = \frac{1}{D_{\gamma_1}}\overline{\gamma}_0\gamma_1\phi_1(\gamma) + \frac{1}{D_{\gamma_1}}T(\gamma)\phi_1(\gamma). \tag{3.24}$$

Therefore, taking into account Theorem 3.4 for $j \geq 3$ we obtain

$$\begin{aligned} (\widetilde{\psi}_j(\gamma), \phi_2(\gamma)) &= \frac{1}{D_{\gamma_1}}\gamma_0\overline{\gamma}_1(\widetilde{\psi}_j(\gamma), \phi_1(\gamma)) + \frac{1}{D_{\gamma_1}}(\widetilde{\psi}_j(\gamma), T(\gamma)\phi_1(\gamma)) \\ &= \frac{1}{D_{\gamma_1}}(\widetilde{\psi}_j(\gamma), T(\gamma)\phi_1(\gamma)) = \frac{1}{D_{\gamma_1}}(T^*(\gamma)\widetilde{\psi}_j(\gamma), \phi_1(\gamma)) \\ &= \frac{1}{D_{\gamma_1}}(\widetilde{\psi}_{j-1}(\gamma), \phi_1(\gamma)) = 0. \end{aligned}$$

From this and (3.23) we get

$$\left(\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix},\ \phi_2(\gamma)\right) = y_2(\widetilde{\psi}_2(\gamma)\ ,\ \phi_2(\gamma)). \tag{3.25}$$

From (3.8) it follows that

$$\left(\begin{pmatrix} 0 \\ \widetilde{\psi}_1(W\gamma) \end{pmatrix},\ \phi_2(\gamma)\right) = \left(\begin{pmatrix} 0 \\ \Pi_2 \\ * \\ \vdots \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}\right) = \Pi_2. \tag{3.26}$$

On the other hand, using (3.24) and Theorem 3.4 we obtain

$$\begin{aligned}
(\widetilde{\psi}_2(\gamma)\ ,\ \phi_2(\gamma)) &= \frac{1}{D_{\gamma_1}}\gamma_0\overline{\gamma}_1(\widetilde{\psi}_2(\gamma)\ ,\ \phi_1(\gamma)) + \frac{1}{D_{\gamma_1}}(\widetilde{\psi}_2(\gamma)\ ,\ T(\gamma)\phi_1(\gamma)) \\
&= \frac{1}{D_{\gamma_1}}(\widetilde{\psi}_2(\gamma)\ ,\ T(\gamma)\phi_1(\gamma)) = \frac{1}{D_{\gamma_1}}(T^*(\gamma)\widetilde{\psi}_2(\gamma)\ ,\ \phi_1(\gamma)) \\
&= \frac{1}{D_{\gamma_1}}(\widetilde{\psi}_1(\gamma)\ ,\ \phi_1(\gamma)) = \Pi_2.
\end{aligned}$$

Combining this with (3.26) and (3.25) we infer $y_2 = 1$. Comparing now the norms of the vectors of both sides of identity (3.22) we obtain formula (3.18) for $j = 1$. Assume now that formula (3.18) holds true for some $j \in \mathbb{N}$. Then using (3.19) we get

$$\begin{aligned}
&\widetilde{\psi}_{j+2}(\gamma) = T(\gamma)\widetilde{\psi}_{j+1}(\gamma) = T(\gamma)\begin{pmatrix} 0 \\ \widetilde{\psi}_j(W\gamma) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ T(W\gamma)\widetilde{\psi}_j(W\gamma) \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{\psi}_{j+1}(W\gamma) \end{pmatrix}.
\end{aligned}$$

Thus, formula (3.18) is proved by mathematical induction. Consequently,

$$\widetilde{\psi}_j(\gamma) = \begin{pmatrix} 0 \\ \widetilde{\psi}_{j-1}(W\gamma) \end{pmatrix} = \begin{pmatrix} 0_{2\times 1} \\ \widetilde{\psi}_{j-2}(W^2\gamma) \end{pmatrix} = \cdots = \begin{pmatrix} 0_{(j-1)\times 1} \\ \widetilde{\psi}_1(W^{j-1}\gamma) \end{pmatrix}.$$

Hence, from (3.16) and (3.18) it follows that for $j \in \{2, 3, 4, \ldots\}$

$$\widetilde{\psi}_j = \sum_{k=j}^{\infty} \Pi_k \Phi_{k-j}(W^{j-1}\gamma)\phi_k + \sum_{k=1}^{\infty} Q_k(W^{j-1}\gamma)\psi_k. \tag{3.27}$$

Thus, the matrices $\mathfrak{L}(\gamma)$ and $\mathcal{Q}(\gamma)$ in (3.7) have the form

$$\mathfrak{L}(\gamma) = \begin{pmatrix}
\Pi_1 & 0 & 0 & \ldots \\
\Pi_2\Phi_1(\gamma) & \Pi_2 & 0 & \ldots \\
\Pi_3\Phi_2(\gamma) & \Pi_3\Phi_1(W\gamma) & \Pi_3 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\Pi_n\Phi_{n-1}(\gamma) & \Pi_n\Phi_{n-2}(W\gamma) & \Pi_n\Phi_{n-3}(W^2\gamma) & \ldots \\
\vdots & \vdots & \vdots &
\end{pmatrix} \tag{3.28}$$

and

$$\mathcal{Q}(\gamma) = \begin{pmatrix} Q_1(\gamma) & Q_1(W\gamma) & Q_1(W^2\gamma) & \dots \\ Q_2(\gamma) & Q_2(W\gamma) & Q_2(W^2\gamma) & \dots \\ Q_3(\gamma) & Q_3(W\gamma) & Q_3(W^2\gamma) & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}, \tag{3.29}$$

respectively.

Since $\lim\limits_{n\to\infty} \Pi_n = 1$ and since $\mathfrak{L}(\gamma)$ is a block of a unitary operator matrix, from (3.28) it follows that

$$\lim_{n\to\infty} \Phi_k(W^n\gamma) = 0,\ k \in \{1,2,3,\dots\} \tag{3.30}$$

and

$$\sum_{k=1}^{\infty} |\Phi_k(W^j\gamma)|^2 < \infty,\ j \in \{0,1,2,\dots\}. \tag{3.31}$$

Analogously, from (3.29) we infer

$$\sum_{k=1}^{\infty} |Q_k(W^j\gamma)|^2 < \infty,\ j \in \{0,1,2,\dots\}. \tag{3.32}$$

Taking into account the identities $T\widetilde{\psi}_k = \widetilde{\psi}_{k+1},\ T^*\psi_k = \psi_{k+1},\ k \in \mathbb{N}$ for $j \in \mathbb{N}$ we get $Q_j(W\gamma) = (\widetilde{\psi}_2, \psi_j) = (T\widetilde{\psi}_1, \psi_j) = (\widetilde{\psi}_1, T^*\psi_j) = (\widetilde{\psi}_1, \psi_{j+1}) = Q_{j+1}(\gamma)$. Thus, for $j \in \{2,3,\dots\}$ we obtain

$$Q_j(\gamma) = Q_{j-1}(W\gamma) = \dots = Q_1(W^{j-1}\gamma). \tag{3.33}$$

The identities (3.33) show that the matrix $\mathcal{Q}(\gamma)$ has Hankel structure

$$\begin{pmatrix} Q_1(\gamma) & Q_1(W\gamma) & Q_1(W^2\gamma) & \dots \\ Q_1(W\gamma) & Q_1(W^2\gamma) & Q_1(W^3\gamma) & \dots \\ Q_1(W^2\gamma) & Q_1(W^3\gamma) & Q_1(W^4\gamma) & \dots \\ \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.34}$$

Hereby from (3.32) we infer

$$\sum_{k=0}^{\infty} |Q_1(W^k\gamma)|^2 < \infty \tag{3.35}$$

whereas the representations (3.16) and (3.27) take the form

$$\widetilde{\psi}_j = \sum_{k=j}^{\infty} \Pi_k \Phi_{k-j}(W^{j-1}\gamma)\phi_k + \sum_{k=1}^{\infty} Q_1(W^{k+j-2}\gamma)\psi_k,\ j \in \mathbb{N}. \tag{3.36}$$

From (3.36) and the matrix representation (2.62) we find

$$(T\widetilde{\psi}_1, \phi_1) = -\overline{\gamma}_0 \Pi_1 (\sum_{k=0}^{\infty} \gamma_{k+1}\Phi_k(\gamma) + Q_1(\gamma)). \tag{3.37}$$

On the other side, in view of (3.9) we have

$$(T\widetilde{\psi}_1, \phi_1) = (\widetilde{\psi}_2, \phi_1) = 0. \tag{3.38}$$

Assume that $\gamma_0 \neq 0$. Then the identities (3.37) and (3.38) provide

$$Q_1(\gamma) = -\sum_{k=0}^{\infty} \gamma_{k+1}\Phi_k(\gamma). \tag{3.39}$$

Now we show that the identities

$$\Phi_k(\gamma) = L_k(W\gamma),\ k \in \{0, 1, 2, \ldots\} \tag{3.40}$$

are satisfied, herein the sequence $(L_k(\gamma))_{k=0}^{\infty}$ is given via (3.12). In view of (3.17) the identity (3.40) holds true for $k = 0$. As above from formula (3.36) and the matrix representation (2.62) we find

$$(T\widetilde{\psi}_1, \phi_3) = \sqrt{1 - |\gamma_2|^2}\Pi_2\Phi_1(\gamma) - \overline{\gamma}_2\Pi_3(\sum_{k=2}^{\infty} \gamma_{k+1}\Phi_k(\gamma) + Q_1(\gamma)).$$

Taking into account (3.39) we get $(T\widetilde{\psi}_1, \phi_3) = \Pi_3(\Phi_1(\gamma)+\gamma_1\overline{\gamma}_2)$. On the other side, from (3.36) we infer $(T\widetilde{\psi}_1, \phi_3) = (\widetilde{\psi}_2, \phi_3) = \Pi_3\Phi_1(W\gamma)$. The last two relations imply $\Phi_1(\gamma) = -\gamma_1\overline{\gamma}_2 + \Phi_1(W\gamma)$. Thus, $\Phi_1(\gamma) = -\sum\limits_{k=1}^{n} \gamma_k\overline{\gamma}_{k+1} + \Phi_1(W^n\gamma)$. Using the limit process $n \to \infty$ and (3.30) we obtain $\Phi_1(\gamma) = -\sum\limits_{k=1}^{\infty} \gamma_k\overline{\gamma}_{k+1} = L_1(W\gamma)$ and the identity (3.40) is proved for $k = 1$. Starting from $(T\widetilde{\psi}_1, \phi_4)$ one can analogously verify that formula (3.40) is also true for $k = 2$. We assume that (3.40) holds true for $0 \leq k \leq n-1$. Then, as above, we obtain

$$\begin{aligned}
(T\widetilde{\psi}_1, \phi_{n+2}) &= \sqrt{1 - |\gamma_{n+1}|^2}\Pi_{n+1}\Phi_n(\gamma) - \overline{\gamma}_{n+1}\Pi_{n+2}(\sum_{j=n+1}^{\infty} \gamma_{j+1}\Phi_j(\gamma) + Q_1(\gamma)) \\
&= \sqrt{1 - |\gamma_{n+1}|^2}\Pi_{n+1}\Phi_n(\gamma) + \overline{\gamma}_{n+1}\Pi_{n+2}\sum_{p=0}^{n} \gamma_{p+1}\Phi_p(\gamma) \\
&= \Pi_{n+2}[\Phi_n(\gamma) + \overline{\gamma}_{n+1}\sum_{p=0}^{n-1} \gamma_{p+1}\Phi_p(\gamma)] = \Pi_{n+2}[\Phi_n(\gamma) + \overline{\gamma}_{n+1}\sum_{p=0}^{n-1} \gamma_{p+1}L_p(W\gamma)].
\end{aligned}$$

On the other hand, $(T\widetilde{\psi}_1, \phi_{n+2}) = (\widetilde{\psi}_2, \phi_{n+2}) = \Pi_{n+2}\Phi_n(W\gamma)$. Thus, we obtain the recurrent formula

$$\Phi_n(\gamma) = -\overline{\gamma}_{n+1}\sum_{p=0}^{n-1} \gamma_{p+1}L_p(W\gamma) + \Phi_n(W\gamma). \tag{3.41}$$

This implies $\Phi_n(\gamma) = -\sum\limits_{k=0}^{m} \overline{\gamma}_{n+k+1} \sum\limits_{p=0}^{n-1} \gamma_{p+k+1} L_p(W^{k+1}\gamma) + \Phi_n(W^{m+1}\gamma)$. Applying the limit process $m \to \infty$ and taking into account (3.30) we get

$$\Phi_n(\gamma) = -\sum_{k=0}^{\infty}\sum_{p=0}^{n-1} \gamma_{p+k+1}\overline{\gamma}_{n+k+1} L_p(W^{k+1}\gamma).$$

Changing the order of summation we find

$$\Phi_n(\gamma) = -\sum_{p=1}^{n}\sum_{k=0}^{\infty} \gamma_{p+k}\overline{\gamma}_{n+k+1} L_{p-1}(W^{k+1}\gamma).$$

Substituting new variables of summation via $s_1 = n-(p-1)$, $j_1 = n - s_1 + k$ we have $\Phi_n(\gamma) = -\sum\limits_{s_1=1}^{n} \sum\limits_{j_1=n-s_1}^{\infty} \gamma_{j_1+1}\overline{\gamma}_{j_1+1+s_1} L_{n-s_1}(W^{j_1+1-n+s_1}\gamma)$. Thus,

$$\Phi_n(\gamma) = -\sum_{j_1=0}^{\infty} \gamma_{j_1+1}\overline{\gamma}_{j_1+1+n} - \sum_{s_1=1}^{n-1}\sum_{j_1=n-s_1}^{\infty} \gamma_{j_1+1}\overline{\gamma}_{j_1+1+s_1} L_{n-s_1}(W^{j_1+1-n+s_1}\gamma) \quad (3.42)$$

Taking into account for simplicity that $k = j_1 - n + s_1$ from (3.12) we find

$$L_{n-s_1}(W^{j_1+1-n+s_1}\gamma) = \sum_{r=1}^{n-s_1} (-1)^r \sum_{s_2+s_3+\ldots+s_{r+1}=n-s_1}$$

$$\sum_{k_2=n-s_1-s_2}^{\infty} \cdots \sum_{k_{r+1}=k_r-s_{r+1}}^{\infty} \gamma_{k_2+1+k}\overline{\gamma}_{k_2+1+k+s_2} \cdots \gamma_{k_{r+1}+1+k}\overline{\gamma}_{k_{r+1}+1+k+s_{r+1}}.$$

Inserting this expression into (3.42) and introducing in the second sum new indices of summation via $j_l = k_l + k$, $l \in \{2, 3, \ldots, r+1\}$ we obtain

$$\Phi_n(\gamma) = -\sum_{j_1=0}^{\infty} \gamma_{j_1+1}\overline{\gamma}_{j_1+1+n} + \sum_{s_1=1}^{n-1}\sum_{r=1}^{n-s_1} (-1)^{r+1} \sum_{s_1+s_2+\ldots+s_{r+1}=n}$$

$$\sum_{j_1=n-s_1}^{\infty}\sum_{j_2=j_1-s_2}^{\infty} \cdots \sum_{j_{r+1}=j_r-s_{r+1}}^{\infty} \gamma_{j_1+1}\overline{\gamma}_{j_1+1+s_1}\gamma_{j_2+1}\overline{\gamma}_{j_2+1+s_2} \cdots \gamma_{j_{r+1}+1}\overline{\gamma}_{j_{r+1}+1+s_{r+1}}.$$

This implies

$$\Phi_n(\gamma) = -\sum_{j_1=0}^{\infty} \gamma_{j_1+1}\overline{\gamma}_{j_1+1+n} + \sum_{r=1}^{n-1}(-1)^{r+1} \sum_{\substack{s_1+s_2+\ldots+s_{r+1}=n \\ 1\le s_1\le n-1}} \sum_{j_1=n-s_1}^{\infty}$$

$$\sum_{j_2=j_1-s_2}^{\infty} \cdots \sum_{j_{r+1}=j_r-s_{r+1}}^{\infty} \gamma_{j_1+1}\overline{\gamma}_{j_1+1+s_1}\gamma_{j_2+1}\overline{\gamma}_{j_2+1+s_2} \cdots \gamma_{j_{r+1}+1}\overline{\gamma}_{j_{r+1}+1+s_{r+1}}. \quad (3.43)$$

Hereby, the sum runs over all ordered $(r+1)$-tuples $(s_1, s_2, \ldots, s_{r+1})$ of positive integers satisfying $s_1 + s_2 + \ldots + s_{r+1} = n$ and $1 \le s_1 \le n-1$. This means that the sum

$$\sum_{r=1}^{n-1}(-1)^{r+1}\sum_{\substack{s_1+s_2+\ldots+s_{r+1}=n\\ 1\le s_1\le n-1}}\cdots$$

can be replaced by an analogous sum of the type

$$\sum_{r=2}^{n}(-1)^{r}\sum_{\substack{s_1+s_2+\ldots+s_{r+1}=n\\ 1\le s_1\le n-1}}\cdots. \tag{3.44}$$

Taking into account that the first term in (3.43) corresponds to the index $s_1 = n$ in the sum (3.44), i.e., $r = 1$, from (3.43) we find $\Phi_n(\gamma) = L_n(W\gamma)$. Thus, the identity (3.40) holds true for all $k \in \{0, 1, 2, \ldots, \}$.

From (3.39) and (3.40) it follows that

$$Q_1(\gamma) = -\sum_{k=0}^{\infty}\gamma_{k+1}L_k(W\gamma) = Q(W\gamma) \tag{3.45}$$

where $Q(\gamma)$ is given by (3.15). From (3.31) and (3.40) we get

$$\sum_{k=1}^{\infty}|L_k(W^j\gamma)|^2 < \infty,\ j \in \mathbb{N}. \tag{3.46}$$

Obviously (3.46) holds also true for $j = 0$. As $\gamma \in \Gamma l_2$ from (3.46) it follows the absolute convergence of the series in (3.15). From (3.36), (3.40) and (3.45) we obtain the representations (3.13).

Finally from (3.40) and (3.45) it is clear that $\Phi_k(\gamma),\ k \in \{0, 1, 2, \ldots\}$ and $Q_1(\gamma)$ do not depend on γ_0. Since γ_0 is an arbitrary number from $\mathbb{D}$ we see now that the assumption $\gamma_0 \neq 0$ can be omitted. □

Corollary 3.7. *The matrices $\mathfrak{L}(\gamma)$ and $\mathfrak{Q}(\gamma)$ introduced via* (3.6) *can be expressed as*

$$\mathfrak{L}(\gamma) = \begin{pmatrix} \Pi_1 & 0 & 0 & \ldots \\ \Pi_2L_1(W\gamma) & \Pi_2 & 0 & \ldots \\ \Pi_3L_2(W\gamma) & \Pi_3L_1(W^2\gamma) & \Pi_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ \Pi_nL_{n-1}(W\gamma) & \Pi_nL_{n-2}(W^2\gamma) & \Pi_nL_{n-3}(W^3\gamma) & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.47}$$

and

$$\mathfrak{Q}(\gamma) = \begin{pmatrix} Q(W\gamma) & Q(W^2\gamma) & Q(W^3\gamma) & \ldots \\ Q(W^2\gamma) & Q(W^3\gamma) & Q(W^4\gamma) & \ldots \\ Q(W^3\gamma) & Q(W^4\gamma) & Q(W^5\gamma) & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.48}$$

where $(L_n(\gamma))_{n=0}^{\infty}$, $(\Pi_n)_{n=0}^{\infty}$ *and* $Q(\gamma)$ *are defined via formulas* (3.12), (3.14) *and* (3.15), *respectively, whereas* W *is the coshift introduced in* (3.11).

Proof. The representation formula (3.47) follows from (3.28) and (3.40), whereas formula (3.48) is an immediate consequence of (3.34) and (3.45). □

Corollary 3.8. ([21]) *Each sequence* $(\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$ *satisfies the following orthogonality relations:*

$$\sum_{n=0}^{\infty} \Pi_{n+k}^2 L_{n+k}(\gamma)\overline{L_n(W^k\gamma)} + \sum_{n=0}^{\infty} Q(W^n\gamma)\overline{Q(W^{n+k}\gamma)} = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \in \{1, 2, 3, \ldots\}. \end{cases}$$

Proof. It suffices to consider for the sequence $\gamma = (0, \gamma_0, \gamma_1, \ldots)$ the representations (3.13) and to substitute them into the orthogonality relations

$$(\widetilde{\psi}_1, \widetilde{\psi}_{k+1}) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{if } k \in \{1, 2, 3, \ldots\}. \end{cases}$$

□

Corollary 3.9. *The recurrent formulas*

$$L_0(\gamma) = L_0(W\gamma) \tag{3.49}$$

and

$$L_n(\gamma) = L_n(W\gamma) - \overline{\gamma}_n \sum_{j=0}^{n-1} \gamma_j L_j(\gamma), \ n \in \mathbb{N}, \tag{3.50}$$

hold true.

Proof. The relation (3.49) is obvious whereas the formulas (3.50) follow by combining (3.40) and (3.41). □

The Hankel matrix (3.48) is the matrix of the Hankel operator which describes the mutual position of the subspaces $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ and $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ in which the maximal shifts V_T and V_{T^*} are acting, respectively. As it was already mentioned (see Introduction) the subspaces $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ and $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ are interpreted as inner channels of scattering in the scattering system associated with the contraction T. In this connection we introduce the following notion.

Definition 3.10. *The Hankel matrix* (3.48) *will be called the Hankel matrix of the maximal shifts* V_T *and* V_{T^*} *or the Hankel matrix of the inner channels of scattering associated with* T.

We note that the unitarity of the operator matrix given via (3.6) implies

$$I - \mathcal{Q}^*(\gamma)\mathcal{Q}(\gamma) = \mathfrak{L}^*(\gamma)\mathfrak{L}(\gamma).$$

This means the matrix $\mathfrak{L}(\gamma)$ plays the role of a defect operator for $\mathcal{Q}(\gamma)$. Taking into account (3.15) and (3.12) from (3.48) we infer $\mathcal{Q}^*(\gamma) = \mathcal{Q}(\overline{\gamma})$.

From the form (3.47) we get immediately the following observation.

Lemma 3.11. *The block representation*

$$\mathfrak{L}(\gamma) = \begin{pmatrix} \Pi_1 & 0 \\ B(\gamma) & \mathfrak{L}(W\gamma) \end{pmatrix} \tag{3.51}$$

with $B(\gamma) = \operatorname{col}(\Pi_2 L_1(W\gamma), \Pi_3 L_2(W\gamma), \ldots, \Pi_n L_{n-1}(W\gamma), \ldots)$ *holds true.*

Theorem 3.12. *It holds*

$$\mathfrak{L}(\gamma) = \mathfrak{M}(\gamma)\mathfrak{L}(W\gamma) \tag{3.52}$$

where

$$\mathfrak{M}(\gamma) = \begin{pmatrix} D_{\gamma_1} & 0 & 0 & \ldots \\ -\gamma_1\overline{\gamma}_2 & D_{\gamma_2} & 0 & \ldots \\ -\gamma_1 D_{\gamma_2}\overline{\gamma}_3 & -\gamma_2\overline{\gamma}_3 & D_{\gamma_3} & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ -\gamma_1 \prod\limits_{j=2}^{n-1} D_{\gamma_j}\overline{\gamma}_n & -\gamma_2 \prod\limits_{j=3}^{n-1} D_{\gamma_j}\overline{\gamma}_n & -\gamma_3 \prod\limits_{j=4}^{n-1} D_{\gamma_j}\overline{\gamma}_n & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.53}$$

and $D_{\gamma_j} = \sqrt{1-|\gamma_j|^2}$, $j \in \{0,1,2,\ldots\}$.

Proof. From (3.4) we infer $T^*\widetilde{\psi}_{k+1} = \widetilde{\psi}_k$, $k \in \{1,2,3,\ldots\}$. Thus, T^* maps the sequence $(\widetilde{\psi}_2, \widetilde{\psi}_3, \widetilde{\psi}_4, \ldots)$ to the sequence $(\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3, \ldots)$, i.e.,

$$(\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3, \ldots) = (T^*\widetilde{\psi}_2, T^*\widetilde{\psi}_3, T^*\widetilde{\psi}_4, \ldots). \tag{3.54}$$

From (2.62) it follows that the matrix representation of the operator T^* with respect to the canonical basis (2.12) has the shape

$$T^* = \begin{pmatrix} T^*_{\mathfrak{F}} & 0 \\ \widetilde{R}^* & \widetilde{V}^*_T \end{pmatrix}. \tag{3.55}$$

Hereby, as it can be seen from (2.66) and (3.53), we have

$$T^*_{\mathfrak{F}} = (-\gamma_0\eta(\gamma)\ ,\ \mathfrak{M}(\gamma)\ ) \tag{3.56}$$

where

$$\eta(\gamma) := \operatorname{col}(\overline{\gamma}_1, \overline{\gamma}_2 D_{\gamma_1}, \ldots, \overline{\gamma}_n \prod_{j=1}^{n-1} D_{\gamma_j}, \ldots). \tag{3.57}$$

Taking into account (3.7), (3.48) and (3.51) we get the representations

$$(\widetilde{\psi}_1, \widetilde{\psi}_2, \widetilde{\psi}_3, \ldots) = \begin{pmatrix} \mathfrak{L}(\gamma) \\ \mathcal{Q}(\gamma) \end{pmatrix} \tag{3.58}$$

and

$$(\widetilde{\psi}_2, \widetilde{\psi}_3, \widetilde{\psi}_4, \ldots) = \begin{pmatrix} \begin{pmatrix} 0 \\ \mathfrak{L}(W\gamma) \end{pmatrix} \\ \mathcal{Q}(W\gamma) \end{pmatrix} \tag{3.59}$$

with respect to the canonical basis (2.12). Inserting the matrix representations (3.55), (3.58) and (3.59) in formula (3.54) we find in particular

$$\mathfrak{L}(\gamma) = T^*_{\mathfrak{F}} \begin{pmatrix} 0 \\ \mathfrak{L}(W\gamma) \end{pmatrix}.$$

Combining this with (3.56) we obtain (3.52). □

Corollary 3.13. *It holds*

$$I - \mathfrak{M}(\gamma)\mathfrak{M}^*(\gamma) = \eta(\gamma)\eta^*(\gamma) \tag{3.60}$$

where $\eta(\gamma)$ *is given via* (3.57).

Proof. From (2.65) and (3.57) we obtain

$$G = D_{\gamma_0}(\ \eta^*(\gamma)\ ;\ \prod_{j=1}^{\infty} D_{\gamma_j}\ , 0, 0, \ldots). \tag{3.61}$$

Substituting now the matrix representations (2.62) and (3.61) in the colligation condition $I - T^*T = G^*G$ we infer in particular $I_{\mathfrak{H}_{\mathfrak{F}}} - T^*_{\mathfrak{F}}T_{\mathfrak{F}} = (1-|\gamma_0|^2)\eta(\gamma)\eta^*(\gamma)$. Substituting the block representation (3.56) in this representation we get (3.60). □

Lemma 3.14. *The matrices* $\mathcal{P}(\gamma)$ *and* $\mathfrak{L}(\gamma)$ *introduced via* (3.6) *are linked by the formula* $\mathcal{P}(\gamma) = \mathfrak{L}(\overline{\gamma})^*$.

Proof. Let $\mathcal{P}(\gamma) = (p_{kj}(\gamma))_{k,j=1}^{\infty}$ and $\mathfrak{L}(\gamma) = (l_{kj}(\gamma))_{k,j=1}^{\infty}$. Since the change from the canonical basis (2.12) to the conjugate canonical basis (3.3) is connected via the replacement of γ_j by $\overline{\gamma}_j$, $j \in \{0,1,2,\ldots\}$ and taking into account matrix representation (3.47) we get $p_{kj}(\gamma) = (\widetilde{\phi}_j, \psi_k) = \overline{(\phi_j, \widetilde{\psi}_k)} = (\widetilde{\psi}_k, \phi_j) = l_{jk}(\gamma) = \overline{l_{jk}(\overline{\gamma})}, j, k \in \mathbb{N}$. □

4. The connection of the maximal shifts V_T and V_{T^*} with the pseudocontinuability of the corresponding c.o.f. θ

4.1. Pseudocontinuability of Schur functions

Let f be a function which is meromorphic in $\mathbb{D}$ and which has nontangential boundary limit values a.e. with respect to the Lebesgue measure on $\mathbb{T} := \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Denote by $\mathbb{D}_e := \{\zeta : |\zeta| > 1\}$ the exterior of the unit circle including the point infinity. The function f is said to admit a pseudocontinuation of bounded type into $\mathbb{D}_e$ if there exist functions $\alpha(\zeta)$ and $\beta(\zeta) \not\equiv 0$ which are bounded and holomorphic in $\mathbb{D}_e$ such that the boundary values of f and $\widehat{f} := \frac{\alpha}{\beta}$ coincide a.e. on $\mathbb{T}$. From the Theorem of Luzin–Privalov (see, e.g., Koosis [28]) it follows that there is at most one pseudocontinuation.

The study of the phenomenon of pseudocontinuability is important in many questions of analysis. We draw our attention to two of them. For more detailed information we refer the reader to Douglas/Shapiro/Shields [17], Ross/Shapiro [30], Arov [3], Nikolskii [29], Cima/Ross [13].

In the Hardy space $H^2(\mathbb{D})$ we consider the unilateral shift $U^\times$ which is generated by multiplication by the independent variable $\zeta \in \mathbb{D}$, i.e., $(U^\times f)(\zeta) = \zeta f(\zeta),\ f \in H^2(\mathbb{D})$. The operator which is adjoint to $U^\times$ is given by

$$(Wf)(\zeta) = \frac{f(\zeta) - f(0)}{\zeta},\ f \in H^2(\mathbb{D}).$$

If we represent a function $f \in H^2(\mathbb{D})$ as Taylor series via

$$f(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_n\zeta^n + \ldots,\ \zeta \in \mathbb{D},$$

and identify f with the sequence $(a_k)_{k=0}^\infty \in l^2$ then the actions of the operators $U^\times$ and W (by preserving the notations) are given by

$$U^\times : (a_0, a_1, a_2, a_3, \ldots) \mapsto (0, a_0, a_1, a_2, \ldots)$$

and

$$W : (a_0, a_1, a_2, a_3, \ldots) \mapsto (a_1, a_2, a_3, a_4, \ldots).$$

In view of the Beurling theorem (see, e.g., Koosis [28]) the invariant subspaces of the shift U in $H^2(\mathbb{D})$ are described by inner functions whereas a function $f \in H^2(\mathbb{D})$ is cyclic for $U^\times$ if and only if f is outer. In this connection we note that in view of a theorem due to Douglas, Shapiro and Shields [17] a function $f \in H^2(\mathbb{D})$ is not cyclic for the backward shift W if and only if it admits a pseudocontinuation of bounded type in $\mathbb{D}_e$.

Following D.Z. Arov [4] we denote by $\mathcal{S}\Pi$ the subset of all functions belonging to $\mathcal{S}$ which admit a pseudocontinuation of bounded type in $\mathbb{D}_e$. We note that the set J of all inner functions in $\mathbb{D}$ is a subset of $\mathcal{S}\Pi$. Indeed, if $\theta \in J$ then the function $\widehat{\theta}(\zeta) = \overline{\theta^{-1}(\frac{1}{\bar{\zeta}})},\ \zeta \in \mathbb{D}_e$ is the pseudocontinuation of θ.

It is known (see Adamjan/Arov [1], Arov [4]) that each function of the Schur class $\mathcal{S}$ is realized as a scattering suboperator (Heisenberg scattering function) of a corresponding unitary coupling. D.Z. Arov indicated the important role of the class $\mathcal{S}\Pi$ in the theory of scattering with loss (see Arov [3], [4], [5]). In this connection the following result is essential for our subsequent considerations.

Theorem 4.1. (*Arov* [3], *De Wilde* [16], *Douglas/Helton* [18]) *A function θ belongs to the class $\mathcal{S}\Pi$ if and only if there exists a 2×2 inner (in $\mathbb{D}$) matrix function $\Omega(\zeta)$ which satisfies*

$$\Omega(\zeta) = \begin{pmatrix} \chi(\zeta) & \phi(\zeta) \\ \psi(\zeta) & \theta(\zeta) \end{pmatrix},\ \zeta \in \mathbb{D}. \tag{4.1}$$

The fact that the function $\Omega(\zeta)$ has unitary boundary limit values a.e. on $\mathbb{T}$ means that $\Omega(\zeta)$ is the scattering suboperator of an orthogonal coupling without loss.

Definition 4.2. *Let*

$$w_a(\zeta) := \begin{cases} \frac{|a|}{a}\frac{a-\zeta}{1-\overline{a}\zeta}, & \text{if } a \in \mathbb{D}\setminus\{0\}, \\ \zeta, & \text{if } a = 0 \end{cases}$$

denote the elementary Blaschke factor associated with a. By an elementary 2×2-Blaschke–Potapov factor we mean a 2×2-inner (in $\mathbb{D}$) matrix function of the form

$$b(\zeta) := I_2 + (w_a(\zeta) - 1)P \tag{4.2}$$

where $w_a(\zeta)$ is an elementary Blaschke factor whereas P is an orthoprojection in $\mathbb{C}^2$ of rank one, i.e., $P^2 = P$, $P^* = P$ and $\operatorname{rank} P = 1$. A 2×2-matrix function $B(\zeta)$ which is inner in $\mathbb{D}$ is called a finite Blaschke–Potapov product if B admits a representation of the form

$$B(\zeta) = u b_1(\zeta) b_2(\zeta) \cdot \dots \cdot b_n(\zeta) \tag{4.3}$$

where u is a constant unitary matrix and $(b_k(\zeta))_{k=1}^n$ is a sequence of elementary 2×2-Blaschke–Potapov factors.

It follows easily from a result due to D.Z. Arov [3] that a function $\theta \in \mathcal{S}$ is rational if and only if there exists a finite Blaschke–Potapov product $\Omega(\zeta)$ of the form (4.1). Thus, a function $\theta \in \mathcal{S}$ is rational if and only if it can be represented as a block of a finite product of elementary 2×2-Blaschke–Potapov factors.

The following statement shows the principal difference between the properties of Schur parameters of inner functions and the properties of Schur parameters of pseudocontinuable Schur functions which are not inner.

Theorem 4.3. ([21]) *Let $\theta \in \mathcal{S}\Pi$ and denote $(\gamma_j)_{j=0}^{\omega}$ the sequence of Schur parameters of θ. If θ is not inner then $\omega = \infty$ and the product* (2.53) *converges. If θ is inner then the product* (2.53) *diverges.*

Proof. If $\theta \in \mathcal{S}\Pi \setminus J$ then the function ϕ in the representation (4.1) does not identically vanish. Hence, $\ln(1 - |\theta(e^{i\alpha})|^2) = 2\ln|\phi(e^{i\alpha})| \in L^1[-\pi,\pi]$ and in view of Remark 2.12 the product (2.53) converges. If $\theta \in J$ then from Remark 2.12 we infer that the product (2.53) diverges. □

Corollary 4.4. *Let $\theta \in \mathcal{S}\Pi\setminus J$. Then the sequence of Schur parameters of θ belongs to Γl_2.*

4.2. On some connections between the maximal shifts V_T and V_{T^*} and the pseudocontinuability of the corresponding c.o.f. θ

Let $\theta \in \mathcal{S}$. Assume that Δ is a simple unitary colligation of type (3.1) which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. We suppose that the Schur parameter sequence of θ belongs to Γl_2. Then from Lemma 2.11 it follows that in this and only in this case the contraction T (resp. T^*) contains a nontrivial maximal shift V_T (resp. V_{T^*}).

Hereby, the multiplicities of the shifts V_T and V_{T^*} coincide and are equal to one. We consider the decompositions (1.6). Let

$$\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} := \mathfrak{H}_{\mathfrak{G}} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp} \,, \qquad \mathfrak{N}_{\mathfrak{F}\mathfrak{G}} := \mathfrak{H}_{\mathfrak{F}} \cap \mathfrak{H}_{\mathfrak{G}}^{\perp}, \tag{4.4}$$

$$\mathfrak{H}_{\mathfrak{G}\mathfrak{F}} := \mathfrak{H}_{\mathfrak{G}} \ominus \mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \,, \quad \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} := \mathfrak{H}_{\mathfrak{F}} \ominus \mathfrak{N}_{\mathfrak{F}\mathfrak{G}}. \tag{4.5}$$

Then

$$\mathfrak{H} = \mathfrak{H}_{\mathfrak{G}}^{\perp} \oplus \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} \oplus \mathfrak{N}_{\mathfrak{G}\mathfrak{F}}, \tag{4.6}$$

$$\mathfrak{H} = \mathfrak{N}_{\mathfrak{F}\mathfrak{G}} \oplus \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} \oplus \mathfrak{H}_{\mathfrak{F}}^{\perp}. \tag{4.7}$$

From (4.4) and (4.5) it follows that

$$\mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = \overline{P_{\mathfrak{H}_{\mathfrak{G}}}\mathfrak{H}_{\mathfrak{F}}} \,, \ \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} = \overline{P_{\mathfrak{H}_{\mathfrak{F}}}\mathfrak{H}_{\mathfrak{G}}}. \tag{4.8}$$

Thus,

$$\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = \dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}}, \tag{4.9}$$

The following criterion of pseudocontinuability of a noninner Schur function (see, e.g., [10, Theorem 3.17]) plays an important role in our subsequent investigations.

Theorem 4.5. *Let $\theta \in \mathcal{S}$ and assume that Δ is a simple unitary colligation of the form* (3.1) *which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Then the conditions $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \neq \{0\}$ and $\mathfrak{N}_{\mathfrak{F}\mathfrak{G}} \neq \{0\}$ are equivalent. They are satisfied if and only if $\theta \in \mathcal{S}\Pi \setminus J$.*

Theorem 4.5 will be complemented by the following result (see Arov [5]) which is obtained here in another way.

Theorem 4.6. (*Arov* [5]) *Let θ be a function of class $\mathcal{S}$ such that its Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ belongs to Γl_2. Assume that Δ is a simple unitary colligation of the form* (3.1) *which satisfies $\theta_\Delta(\zeta) = \theta(\zeta)$. Then θ is a rational function if and only if* $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} < \infty$ *(resp.* $\dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} < \infty$*). If* $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} < \infty$ *then* $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ *(resp.* $\dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}}$*) is the smallest number of elementary 2×2-Blaschke–Potapov factors in a finite Blaschke–Potapov product of the form* (4.3) *with block θ.*

Proof. From Lemma 2.11 it follows that in the given case we have $\mathfrak{H}_{\mathfrak{G}}^{\perp} \neq \{0\}$ and $\mathfrak{H}_{\mathfrak{F}}^{\perp} \neq \{0\}$. Hereby the multiplicities of the shifts V_T and V_{T^*} are equal to one.

Assume that $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} < \infty$. Taking into account that $\mathfrak{H}_{\mathfrak{G}}^{\perp} \cap \mathfrak{H}_{\mathfrak{F}}^{\perp} = \{0\}$ we see that in the decomposition (4.6) the relation $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \neq \{0\}$ holds true. Since $\mathfrak{H}_{\mathfrak{G}}$ and $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ are invariant with respect to T^* then $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$ is also invariant with respect to T^*. Hereby it is easily seen that the operator $\widetilde{V}_{T_{\mathfrak{G}}} = \mathrm{Rstr.}_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}} T^*$ is the maximal shift which is contained in $T_{\mathfrak{G}}^*$ (see matrix representation (1.9)). Thus, with respect to the decomposition (4.6) the operator T has the matrix representation

$$T = \begin{pmatrix} V_T & * & * \\ 0 & T_{\mathfrak{G}\mathfrak{F}} & * \\ 0 & 0 & \widetilde{V}_{T_{\mathfrak{G}}} \end{pmatrix} \tag{4.10}$$

where $T_{\mathfrak{G}\mathfrak{F}} := \mathrm{Rstr.}_{\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}}(P_{\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}}T) : \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} \to \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. From (4.10) it follows in view of [10, Theorem 3.3] that θ admits the factorization

$$\theta(\zeta) = (0,1)\theta_{\mathfrak{G}\mathfrak{F}}(\zeta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.11}$$

where $\theta_{\mathfrak{G}\mathfrak{F}}$ is the c.o.f. of the contraction $T_{\mathfrak{G}\mathfrak{F}}$.

Let $n = \dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. For the contraction $T_{\mathfrak{G}\mathfrak{F}}$ we consider the nested chain of invariant subspaces $(\mathfrak{H}^{(k)}_{\mathfrak{G}\mathfrak{F}})_{k=1}^{n}$ where $\dim \mathfrak{H}^{(k)}_{\mathfrak{G}\mathfrak{F}} = k$. This chain generates a representation of the function $\theta_{\mathfrak{G}\mathfrak{F}}(\zeta)$ as product of n elementary 2×2-Blaschke–Potapov factors of the form (4.2) (see Brodskii [12], Sz.-Nagy/Foias [33]).

Suppose that in addition to (4.11) the function $\theta(\zeta)$ admits the factorization

$$\theta(\zeta) = (0,1)B(\zeta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.12}$$

where $B(\zeta)$ is a finite 2×2-Blaschke–Potapov product of the form (4.3) with m factors. Then we will show that $m \geq n$.

We proceed by contradiction. Assume $m < n$. Then using [10, Theorem 3.19] from (4.12) it follows that the space $\mathfrak{H}$ admits the decomposition

$$\mathfrak{H} = \mathfrak{N} \oplus \widehat{\mathfrak{H}} \oplus \widetilde{\mathfrak{N}}. \tag{4.13}$$

With respect to the decomposition (4.13) of $\mathfrak{H}$ the operator T has the matrix representation

$$\begin{pmatrix} V & * & * \\ 0 & \widehat{T} & * \\ 0 & 0 & \widetilde{V} \end{pmatrix}.$$

Hereby, $(0,1)$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are the characteristic function's of the shift V and the coshift $\widetilde{V}$, respectively, whereas $B(\zeta)$ is the c.o.f. of the contraction $\widehat{T}$.

Obviously, an elementary 2×2-Blaschke–Potapov factor is the c.o.f. of a completely nonunitary one-dimensional contraction. Since every of these factors is an inner function then (see Brodskii [12]) their product is regular. In the case of regular factorizations the inner space will be summed up. Hence, it is $\dim \widehat{\mathfrak{H}} = m$. Thus,

$$\dim \widehat{\mathfrak{H}} < \dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}. \tag{4.14}$$

Let $\mathfrak{H}_0 := \mathfrak{H} \ominus (\mathfrak{H}^{\perp}_{\mathfrak{F}} \bigvee \mathfrak{H}^{\perp}_{\mathfrak{G}})$. Obviously, $\mathfrak{H}_0 \subseteq \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ and $\mathfrak{H}_0 \subseteq \widehat{\mathfrak{H}}$. Further, let $\mathfrak{H}^{(0)}_{\mathfrak{G}\mathfrak{F}} := \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} \ominus \mathfrak{H}_0$, $\widehat{\mathfrak{H}}_0 := \widehat{\mathfrak{H}} \ominus \mathfrak{H}_0$. From (4.14) it follows $\dim \widehat{\mathfrak{H}}_0 < \dim \mathfrak{H}^{(0)}_{\mathfrak{G}\mathfrak{F}}$. Hence, there exists a vector $h \neq 0$ with the properties

$$h \in \mathfrak{H}^{(0)}_{\mathfrak{G}\mathfrak{F}}\,,\ h \perp \widehat{\mathfrak{H}}_0. \tag{4.15}$$

It can be easily seen that $\mathfrak{H}^{(0)}_{\mathfrak{G}\mathfrak{F}} \subseteq \overline{P_{\mathfrak{H}_{\mathfrak{G}}}\mathfrak{H}^{\perp}_{\mathfrak{F}}}$. Hereby, $\mathfrak{H}^{(0)}_{\mathfrak{G}\mathfrak{F}} \cap \mathfrak{H}^{\perp}_{\mathfrak{F}} = \{0\}$. Thus, there exists a vector $f_1 \in \mathfrak{H}^{\perp}_{\mathfrak{F}}$ which satisfies $h = P_{\mathfrak{H}_{\mathfrak{G}}} f_1$. Hereby, $g_1 = f_1 - h \neq 0$, i.e.,

$$h = f_1 - g_1,\ f_1 \in \mathfrak{H}^{\perp}_{\mathfrak{F}},\ g_1 \in \mathfrak{H}^{\perp}_{\mathfrak{G}},\ \|h\| < \|f_1\|. \tag{4.16}$$

On the other hand, from (4.15) we infer $h \perp \widehat{\mathfrak{H}}$. Using (4.13) this implies $h \in \mathfrak{N} \oplus \widetilde{\mathfrak{N}}$. Hereby, we have $\mathfrak{N} \subseteq \mathfrak{H}_{\mathfrak{G}}^{\perp}$ and $\widetilde{\mathfrak{N}} \subseteq \mathfrak{H}_{\mathfrak{F}}^{\perp}$. Consequently, there exist vectors $f_2 \in \mathfrak{H}_{\mathfrak{F}}^{\perp}$ and $g_2 \in \mathfrak{H}_{\mathfrak{G}}^{\perp}$ satisfying $f_2 \perp g_2$ and $h = f_2 - g_2$. Thus,

$$h = f_2 - g_2,\ f_2 \in \mathfrak{H}_{\mathfrak{F}}^{\perp},\ g_2 \in \mathfrak{H}_{\mathfrak{G}}^{\perp},\ \|h\| \geq \|f_2\|. \tag{4.17}$$

Because of $\mathfrak{H}_{\mathfrak{F}}^{\perp} \cap \mathfrak{H}_{\mathfrak{G}}^{\perp} = \{0\}$ from (4.16) and (4.17) we get $f_1 = f_2$ and $g_1 = g_2$. Hence, $\|h\| < \|f_1\| = \|f_2\| \leq \|h\|$. This contradiction shows that the assumption $m < n$ was wrong. Hence, $m \geq \dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$.

Now assume that $\theta(\zeta)$ is rational. We represent θ in the form

$$\theta(\zeta) = (0,1)\widetilde{B}(\zeta)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $\widetilde{B}(\zeta)$ is a finite Blaschke–Potapov-product of the form (4.3). We assume that the number of elementary 2×2-Blaschke–Potapov factors satisfies the minimality condition. Denote by $\widetilde{m}$ this minimal number of elementary 2×2-Blaschke–Potapov factors. Using now the factorization (4.11) of $\theta(\zeta)$ we obtain, as in the above considered case of the factorization (4.12), that $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} \leq \widetilde{m} < \infty$. Since m satisfies the minimality condition we obtain the equality $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = \widetilde{m}$. □

Lemma 4.7. *It holds*

$$\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} = \ker \mathcal{Q}^*(\gamma) \tag{4.18}$$

where $\mathcal{Q}(\gamma)$ is that Hankel operator in $\mathfrak{H}_{\mathfrak{F}}^{\perp}$ the matrix representation of which with respect to the basis $(\psi_j)_{j=1}^{\infty}$ has the form (3.48).

Proof. From (4.4) it follows that $h \in \mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$ if and only if $h \in \mathfrak{H}_{\mathfrak{F}}^{\perp}$ and $h \perp \mathfrak{H}_{\mathfrak{G}}^{\perp}$. Combining this with the fact that the vectors (3.13) form an orthonormal basis in $\mathfrak{H}_{\mathfrak{G}}^{\perp}$ we obtain (4.18). □

5. Some criteria for the pseudocontinuability of a Schur function in terms of its Schur parameters

5.1. Construction of a countable closed vector system in $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ and investigation of the properties of the sequence $(\sigma_n)_{n=1}^{\infty}$ of Gram determinants of this system

Let $\theta(\zeta) \in \mathcal{S}$ and assume that Δ is a simple unitary colligation of the form (3.1) which satisfies $\theta_{\Delta}(\zeta) = \theta(\zeta)$. As in the preceding chapter it is assumed that the Schur parameter sequence $(\gamma_j)_{j=0}^{\infty}$ of $\theta(\zeta)$ belongs to Γl_2.

Theorem 5.1. *The linear span of vectors*

$$h_n := \phi_n - \Pi_n \sum_{j=1}^{n} \overline{L_{n-j}(W^j\gamma)}\widetilde{\psi}_j,\ n \in \mathbb{N} \tag{5.1}$$

is dense in $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. Here $(\phi_k)_{k=1}^{\infty}$ and $(\widetilde{\psi}_k)_{k=1}^{\infty}$ denote the orthonormal systems taken from the canonical basis (2.12) *and the conjugate canonical basis* (3.3)*, respectively, whereas $W, (L_k(\gamma))_{k=1}^{\infty}$ and $(\Pi_k)_{k=1}^{\infty}$ are given via* (3.11)*,* (3.12) *and* (3.14).

Proof. Since $(\phi_k)_{k=1}^{\infty}$ is an orthonormal basis in $\mathfrak{H}_{\mathfrak{F}}$, from (4.8) it follows that the vectors $h_n = P_{\mathfrak{H}_{\mathfrak{G}}}\phi_n,\ n \in \mathbb{N}$, form a closed system in $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. Since $(\widetilde{\psi}_k)_{k=1}^{\infty}$ is an orthonormal basis in $\mathfrak{H}_{\mathfrak{G}}^{\perp}$, the identities $h_n = P_{\mathfrak{H}_{\mathfrak{G}}}\phi_n = \phi_n - \sum\limits_{j=1}^{\infty}(\phi_n, \widetilde{\psi}_j)\widetilde{\psi}_j,\ n \in \mathbb{N}$, hold true. It remains to note that from the decompositions (3.13) we obtain

$$(\phi_n, \widetilde{\psi}_j) = \begin{cases} \Pi_n \overline{L_{n-j}(W^j\gamma)}, & if\ j \leq n, \\ 0, & if\ j > n. \end{cases} \qquad \square$$

Corollary 5.2. *It holds*

$$\begin{pmatrix} (h_1, h_1) & (h_2, h_1) & \dots & (h_n, h_1) \\ (h_1, h_2) & (h_2, h_2) & \dots & (h_n, h_2) \\ \vdots & \vdots & & \vdots \\ (h_1, h_n) & (h_2, h_n) & \dots & (h_n, h_n) \end{pmatrix} = I - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma),\ n \in \mathbb{N} \tag{5.2}$$

where

$$\mathfrak{L}_n(\gamma) = \begin{pmatrix} \Pi_1 & 0 & 0 & \dots & 0 \\ \Pi_2 L_1(W\gamma) & \Pi_2 & 0 & \dots & 0 \\ \Pi_3 L_2(W\gamma) & \Pi_3 L_1(W^2\gamma) & \Pi_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \Pi_n L_{n-1}(W\gamma) & \Pi_n L_{n-2}(W^2\gamma) & \Pi_n L_{n-3}(W^3\gamma) & \dots & \Pi_n \end{pmatrix} \tag{5.3}$$

is the nth order principal submatrix of the matrix $\mathfrak{L}(\gamma)$ given in (3.47).

Proof. The identities (5.2) are an immediate consequence of (5.1). $\square$

In the sequel, the matrices

$$\mathcal{A}_n(\gamma) := I_n - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma),\ n \in \mathbb{N} \tag{5.4}$$

and their determinants

$$\sigma_n(\gamma) := \begin{cases} 1, & if\ n = 0, \\ \det \mathcal{A}_n(\gamma), & if\ n \in \mathbb{N} \end{cases} \tag{5.5}$$

will play an important role. They have a lot of remarkable properties. In order to prove these properties we need the following result which follows from Theorem 3.12 and Corollary 3.13.

Lemma 5.3. *It holds*

$$\mathfrak{L}_n(\gamma) = \mathfrak{M}_n(\gamma)\mathfrak{L}_n(W\gamma) \tag{5.6}$$

where $\mathfrak{L}_n(\gamma)$ *is given via* (5.3) *whereas*

$$\mathfrak{M}_n(\gamma)=\begin{pmatrix} D_{\gamma_1} & 0 & 0 & \dots & 0 \\ -\gamma_1\overline{\gamma}_2 & D_{\gamma_2} & 0 & \dots & 0 \\ -\gamma_1 D_{\gamma_2}\overline{\gamma}_3 & -\gamma_2\overline{\gamma}_3 & D_{\gamma_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_1\prod\limits_{j=2}^{n-1} D_{\gamma_j}\overline{\gamma}_n & -\gamma_2\prod\limits_{j=3}^{n-1} D_{\gamma_j}\overline{\gamma}_n & -\gamma_3\prod\limits_{j=4}^{n-1} D_{\gamma_j}\overline{\gamma}_n & \dots & D_{\gamma_n} \end{pmatrix} \tag{5.7}$$

is the nth order principal submatrix of the matrix $\mathfrak{M}(\gamma)$ *given in* (3.53). *Hereby,*

$$I_n-\mathfrak{M}_n(\gamma)\mathfrak{M}_n^*(\gamma)=\eta_n(\gamma)\eta_n^*(\gamma),\ \ n\in\mathbb{N}, \tag{5.8}$$

where

$$\eta_n(\gamma)=\operatorname{col}(\overline{\gamma}_1,\overline{\gamma}_2 D_{\gamma_1},\dots,\overline{\gamma}_n\prod_{j=1}^{n-1}D_{\gamma_j}). \tag{5.9}$$

Corollary 5.4. *The multiplicative decompositions*

$$\mathfrak{L}_n(\gamma)=\mathfrak{M}_n(\gamma)\cdot\mathfrak{M}_n(W\gamma)\cdot\mathfrak{M}_n(W^2\gamma)\cdot\ldots,\ \ n\in\mathbb{N}$$

hold true.

Proof. From the form (5.3) of the matrices $\mathfrak{L}_n(\gamma)$ it can be seen that

$$\lim_{m\to\infty}\mathfrak{L}_n(W^m\gamma)=I_n\quad\text{for all } n\in\mathbb{N}.$$

Now using (5.6) we obtain the assertion. □

Theorem 5.5. ([23]) *Let* $\theta(\zeta)$ *be a function from* $\mathcal{S}$ *the sequence* $(\gamma_j)_{j=0}^{\infty}$ *of Schur parameters of which belongs to* Γl_2. *Assume that* Δ *is a simple unitary colligation of the form* (3.1) *which satisfies* $\theta_\Delta(\zeta)=\theta(\zeta)$. *Then the matrices* $\mathcal{A}_n(\gamma)$ (*see* (5.4)) *and their determinants* $(\sigma_n(\gamma))_{n=1}^{\infty}$ *have the following properties:*

(1) *For* $n\in\mathbb{N}$, *it hold* $0\le\sigma_n(\gamma)<1$ *and* $\sigma_n(\gamma)\ge\sigma_{n+1}(\gamma)$.
Moreover, $\lim\limits_{n\to\infty}\sigma_n(\gamma)=0$.

(2) *If there exists some* $n_0\in\{0,1,2,\dots\}$ *which satisfies* $\sigma_{n_0}(\gamma)>0$ *and* $\sigma_{n_0+1}(\gamma)=0$, *then* $\operatorname{rank}\mathcal{A}_n(\gamma)=n_0$ *for* $n\ge n_0$ *holds true. Hereby,* $n_0=\dim\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}(=\dim\mathfrak{H}_{\mathfrak{F}\mathfrak{G}})$ *where* $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ *and* $\mathfrak{H}_{\mathfrak{F}\mathfrak{G}}$ *are given via* (4.5).
Conversely, if $\dim\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}(=\dim\mathfrak{H}_{\mathfrak{F}\mathfrak{G}})$ *is a finite number* n_0 *then* $\sigma_{n_0}(\gamma)>0$ *and* $\sigma_{n_0+1}(\gamma)=0$.

(3) *It holds*

$$\mathcal{A}_n(\gamma)=\eta_n(\gamma)\eta_n^*(\gamma)+\mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma),\ \ n\in\mathbb{N} \tag{5.10}$$

where $\mathfrak{M}_n(\gamma)$ *and* $\eta_n(\gamma)$ *are defined via* (5.7) *and* (5.9), *respectively.*

(4) *Let* $(\lambda_{n,j}(\gamma))_{j=1}^{n}$ *denote the increasingly ordered sequence of eigenvalues of the matrix* $\mathcal{A}_n(\gamma)$, $n \in \mathbb{N}$, *where each eigenvalue is counted with its multiplicity, then*

$$\begin{aligned} 0 \ &\le \ \lambda_{n,1}(W\gamma) \le \lambda_{n,1}(\gamma) \le \lambda_{n,2}(W\gamma) \le \lambda_{n,2}(\gamma) \le \dots \\ &\le \ \lambda_{n,n}(W\gamma) \le \lambda_{n,n}(\gamma) < 1. \end{aligned} \tag{5.11}$$

Thus, the eigenvalues of the matrices $\mathcal{A}_n(\gamma)$ *and* $\mathcal{A}_n(W\gamma)$ *interlace.*

(5) *For* $n \in \mathbb{N}$, *it holds*

$$\Gamma(h_1, h_2, \dots, h_n, G^*(1)) = \sigma_n(W\gamma) \prod_{j=0}^{n} (1 - |\gamma_j|^2) \tag{5.12}$$

where $\Gamma(h_1,h_2,\dots,h_n,G^*(1))$ *is the Gram determinant of the vectors* $(h_k)_{k=1}^{n}$ *given by* (5.1) *and the vector* $G^*(1)$ *defined by* (2.2). *Hereby, the rank of the Gram matrix of the vectors* $(h_k)_{k=1}^{n}$ *and* $G^*(1)$ *is equal to* $\operatorname{rank}\mathcal{A}_n(W\gamma) + 1$.

(6) *If* $\sigma_n(\gamma) > 0$ *for every* $n \in \mathbb{N}$ *then the sequence*

$$\left(\prod_{j=0}^{n} (1 - |\gamma_j|^2) \frac{\sigma_n(W\gamma)}{\sigma_n(\gamma)} \right)_{n=1}^{\infty}$$

monotonically decreases. Moreover,

$$\lim_{n\to\infty} \frac{\sigma_n(W\gamma)}{\sigma_n(\gamma)} = \frac{1}{\Pi_0^2} \|P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}} G^*(1)\|^2 \tag{5.13}$$

where Π_0 *and* $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$ *are defined via formulas* (3.14) *and* (4.4), *respectively.*

(7) *Assume that* $\sigma_n(\gamma) > 0$ *for every* $n \in \mathbb{N}$. *Then* $\sigma_n(W^m\gamma) > 0$ *for every* $m, n \in \mathbb{N}$. *Moreover, if the limit* (5.13) *is positive, then*

$$\lim_{n\to\infty} \frac{\sigma_n(W^{m+1}\gamma)}{\sigma_n(W^m\gamma)} > 0 \tag{5.14}$$

for every $m \in \mathbb{N}$.

Proof. (1) Since by Corollary 5.2 $\mathcal{A}_n(\gamma)$ is a Gram matrix then $\sigma_n(\gamma) \ge 0$, $n \in \mathbb{N}$. On the other side, in view of $\gamma \in \Gamma l_2$ the matrix $\mathfrak{L}_n(\gamma)$ is invertible. Thus, from (5.4) we infer $\sigma_n(\gamma) < 1$, $n \in \mathbb{N}$. From (5.3) it follows that

$$\mathcal{A}_{n+1}(\gamma) = \begin{pmatrix} \mathcal{A}_n(\gamma) & -\mathfrak{L}_n(\gamma) b_n(\gamma) \\ -b_n^*(\gamma)\mathfrak{L}_n^*(\gamma) & 1 - \Pi_{n+1}^2 - b_n^*(\gamma) b_n(\gamma) \end{pmatrix} \tag{5.15}$$

where

$$b_n(\gamma) = \Pi_{n+1} \mathrm{col}\,(\overline{L_n(W\gamma)}, \overline{L_{n-1}(W^2\gamma)}, \dots, \overline{L_1(W^n\gamma)}). \tag{5.16}$$

From (5.15) we find

$$\mathcal{A}_{n+1}(\gamma) = F_{n,1}(\gamma) \begin{pmatrix} \mathcal{A}_n(\gamma) & 0 \\ 0 & \mathcal{A}_n^{[c]}(\gamma) \end{pmatrix} F_{n,1}^*(\gamma) \tag{5.17}$$

where $F_{n,1}(\gamma) = \begin{pmatrix} I_n & 0 \\ X_{n,1}(\gamma) & 1 \end{pmatrix}$, $X_{n,1}(\gamma) = -b_n^*(\gamma)\mathfrak{L}_n^*(\gamma)\mathcal{A}_n^{-1}(\gamma)$, and

$$\mathcal{A}_n^{[c]}(\gamma) = 1 - \Pi_{n+1}^2 - b_n^*(\gamma)b_n(\gamma) - b_n^*(\gamma)\mathfrak{L}_n(\gamma)\mathcal{A}_n^+(\gamma)\mathfrak{L}_n(\gamma)b_n(\gamma) \tag{5.18}$$

is the Schur complement of the matrix $\mathcal{A}_n(\gamma)$ in the matrix $\mathcal{A}_{n+1}(\gamma)$. The symbol $\mathcal{A}_n^+(\gamma)$ stands for the Moore–Penrose inverse of the matrix $\mathcal{A}_n(\gamma)$ (see, e.g., [24, part 1.1]). Thus,

$$\sigma_{n+1}(\gamma) = \sigma_n(\gamma)\mathcal{A}_n^{[c]}(\gamma). \tag{5.19}$$

In view of $\mathcal{A}_{n+1}(\gamma) \geq 0$ we have $\mathcal{A}_n^{[c]}(\gamma) \geq 0$. Taking into account $\Pi_n > 0$ and $\lim_{n\to\infty} \Pi_n = 1$ from this and (5.18) we obtain $0 \leq \mathcal{A}_n^{[c]}(\gamma) < 1$ and $\lim_{n\to\infty} \mathcal{A}_n^{[c]}(\gamma) = 0$. Now (1) follows from (5.19).

(3) Using (5.6) we get

$$\begin{aligned} \mathcal{A}_n(\gamma) &= I_n - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma) = I_n - \mathfrak{M}_n(\gamma)\mathfrak{L}_n(W\gamma)\mathfrak{L}_n^*(W\gamma)\mathfrak{M}_n^*(\gamma) \\ &= I_n - \mathfrak{M}_n(\gamma)\mathfrak{M}_n^*(\gamma) + \mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma). \end{aligned}$$

Combining this with (5.8) we obtain (5.10).

(4) Since $\mathfrak{L}_n(\gamma), n \in \mathbb{N}$, is a contractive invertible matrix, from (5.4) we get

$$0 \leq \lambda_{n,j}(\gamma) < 1 \ , \ n \in \mathbb{N} \ , \ j \in \{1, 2, \ldots, n\}. \tag{5.20}$$

From (5.8) we see that the matrix $\mathfrak{M}_n(\gamma)$ is contractive. Therefore, using (5.6) we find

$$\begin{aligned} & I_n - \mathfrak{L}_n^*(\gamma)\mathfrak{L}_n(\gamma) = I_n - \mathfrak{L}_n^*(W\gamma)\mathfrak{M}_n^*(\gamma)\mathfrak{M}_n(\gamma)\mathfrak{L}_n(W\gamma) \\ = \ & I_n - \mathfrak{L}_n^*(W\gamma)\mathfrak{L}_n(W\gamma) + \mathfrak{L}_n^*(W\gamma)[I - \mathfrak{M}_n^*(\gamma)\mathfrak{M}_n(\gamma)]\mathfrak{L}_n(W\gamma) \\ \geq \ & I_n - \mathfrak{L}_n^*(W\gamma)\mathfrak{L}_n(W\gamma). \end{aligned}$$

Thus, taking into account that the eigenvalues of the matrices $I_n - \mathfrak{L}_n(\gamma)\mathfrak{L}_n^*(\gamma)$ and $I_n - \mathfrak{L}_n^*(\gamma)\mathfrak{L}_n(\gamma)$ coincide and using minimax principles for the eigenvalues of Hermitian matrices we get

$$\lambda_{n,k}(W\gamma) \leq \lambda_{n,k}(\gamma) \ , \ n \in \mathbb{N} \ , \ k \in \{1, 2, \ldots, n\}. \tag{5.21}$$

On the other side, applying (5.10) for $x \in \mathbb{C}^n$ and $n \in \mathbb{N}$ we obtain

$$(\mathcal{A}_n(\gamma)x, x) = |(x, \eta_n(\gamma))|^2 + (\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma)x, \mathfrak{M}_n^*(\gamma)x). \tag{5.22}$$

In the case $n = 1$ the inequality (5.11) follows from (5.21). Let $n \geq 2, k \in \{1, 2, \ldots, n\}$ and assume that $(w_j)_{j=1}^k$ is an arbitrary sequence of vectors from $\mathbb{C}^n$. In view of $\gamma \in \Gamma l_2$ from (5.7) it follows that the matrix $\mathfrak{M}_n(\gamma)$ is invertible. Let $\widetilde{w}_j := \mathfrak{M}_n^{-1}(\gamma)w_j, \ j \in \{1, \ldots, k\}$ and $\widetilde{\eta}_n(\gamma) := \mathfrak{M}_n^{-1}(\gamma)\eta_n(\gamma)$. For $x \in \mathbb{C}^n$, we set $y := \mathfrak{M}_n^*(\gamma)x$. From (5.8) it can be seen that the conditions $\|x\| = 1$

and $(x, \eta_n(\gamma)) = 0$ imply $\|y\| = 1$. Hereby, $(x, w_j) = (\mathfrak{M}_n^{*-1}(\gamma)y, w_j) = (y, \widetilde{w}_j)$. Therefore, using (5.22) and the minimax principle, for $k \in \{1, 2, \ldots, n-1\}$ we find

$$\begin{aligned}
\lambda_{n,k}(\gamma) &= \max_{w_1,\ldots,w_{k-1}} \min_{\|x\|=1,(x,w_j)=0,j\in\{1,\ldots,k-1\}} (\mathcal{A}_n(\gamma)x, x) \\
&\leq \max_{w_1,\ldots,w_{k-1}} \min_{\|x\|=1,(x,\eta_n(\gamma))=0,(x,w_j)=0,j\in\{1,\ldots,k-1\}} (\mathcal{A}_n(\gamma)x, x) \\
&= \max_{\widetilde{w}_1,\ldots,\widetilde{w}_{k-1}} \min_{\|y\|=1,(y,\widetilde{\eta}_n(\gamma))=0,(y,\widetilde{w}_j)=0,j\in\{1,\ldots,k-1\}} (\mathcal{A}_n(W\gamma)y, y) \\
&\leq \max_{\widetilde{w}_1,\ldots,\widetilde{w}_k} \min_{\|y\|=1,(y,\widetilde{w}_j)=0,j\in\{1,\ldots,k\}} (\mathcal{A}_n(W\gamma)y, y) \\
&= \lambda_{n,k+1}(W\gamma).
\end{aligned}$$

Combining this with (5.20) and (5.21) we get (5.11).

(2) Assume that $n_0 \in \{0, 1, 2, 3, \ldots\}$ satisfies $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$. If $n_0 = 0$ then using $\sigma_1(\gamma) = 1 - \prod_{j=1}^{\infty}(1 - |\gamma_j|^2)$, we infer $\gamma_j = 0, j \in \mathbb{N}$. Thus, (5.3) implies $\mathfrak{L}_n(\gamma) = I_n, n \in \mathbb{N}$ and $\mathcal{A}_n(\gamma) = 0, n \in \mathbb{N}$. Consequently, from (5.2) it follows that $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = 0$. In view of (4.9) this means $\dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} = 0$.

Let $n_0 \in \mathbb{N}$. From (5.3) we get the block partition

$$\mathfrak{L}_{n+1}(\gamma) = \begin{pmatrix} \Pi_1 & 0 \\ B_{n+1}(\gamma) & \mathfrak{L}_n(W\gamma) \end{pmatrix} \tag{5.23}$$

where

$$B_{n+1}(\gamma) = \operatorname{col}(\Pi_2 L_1(W\gamma), \Pi_3 L_2(W\gamma), \ldots, \Pi_{n+1} L_n(W\gamma)). \tag{5.24}$$

From this, we obtain the block representation

$$\mathcal{A}_{n+1}(\gamma) = \begin{pmatrix} 1 - \Pi_1^2 & -\Pi_1 B_{n+1}^*(\gamma) \\ -\Pi_1 B_{n+1}(\gamma) & \mathcal{A}_n(W\gamma) - B_{n+1}(\gamma)B_{n+1}^*(\gamma) \end{pmatrix}. \tag{5.25}$$

We consider this block representation for $n = n_0 + 1$. Since $\det \mathcal{A}_{n_0+1}(\gamma) = 0$, in view of (5.11), we have $\det \mathcal{A}_{n_0+1}(W\gamma) = 0$. Assume that $x \in \mathbb{C}^{n_0+1}, x \neq 0$ and $x \in \ker \mathcal{A}_{n_0+1}(W\gamma)$. Then from (5.25) we see that the vector $\widetilde{x} := \begin{pmatrix} 0 \\ x \end{pmatrix}$ belongs to $\ker \mathcal{A}_{n_0+2}(\gamma)$.

Now we consider the block representation (5.15) for $n = n_0 + 1$. Let $y \in \mathbb{C}^{n_0+1}, y \neq 0$ and $y \in \ker \mathcal{A}_{n_0+1}(\gamma)$. Then (5.15) implies that the vector $\widetilde{y} := \begin{pmatrix} y \\ 0 \end{pmatrix}$ belongs to $\ker \mathcal{A}_{n_0+2}(\gamma)$. If the vectors $\widetilde{x}$ and $\widetilde{y}$ are collinear then from their construction we get that $\ker \mathcal{A}_{n_0+2}(\gamma)$ contains the vector $w = \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix}$ where $z \in \mathbb{C}^{n_0}$ and $z \neq 0$. Then the representation (5.15) for $n = n_0 + 1$ implies that $\begin{pmatrix} 0 \\ z \end{pmatrix} \in \ker \mathcal{A}_{n_0+1}(\gamma)$. Now using representation (5.25) for $n = n_0 + 1$ we obtain $z \in \ker \mathcal{A}_{n_0}(\gamma)$. However, $\sigma_{n_0}(\gamma) > 0$ and consequently $\ker \mathcal{A}_{n_0}(\gamma) = \{0\}$. From this contradiction we infer that the vectors $\widetilde{x}$ and $\widetilde{y}$ are not collinear. This means $\dim\ \ker \mathcal{A}_{n_0+2}(\gamma) \geq 2$. Thus $\operatorname{rank} \mathcal{A}_{n_0+2}(\gamma) \leq n_0$. On the other side, using (5.15) we obtain $\operatorname{rank} \mathcal{A}_{n_0+2}(\gamma) \geq \operatorname{rank} \mathcal{A}_{n_0}(\gamma) = n_0$. Hence, $\operatorname{rank} \mathcal{A}_{n_0+2}(\gamma) = n_0$.

Applying the method of mathematical induction to the matrices $\mathcal{A}_{n_0+m}(\gamma)$ by analogous considerations we get $\operatorname{rank} \mathcal{A}_{n_0+m}(\gamma) = n_0$ for $m \in \mathbb{N}$. Now using (5.2), (4.9) and the fact that $(h_n)_{n=1}^{\infty}$ is a system of vectors which is total in $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ we find $\dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}} = \dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} = n_0$. The converse statement follows immediately from (5.2) and the above considerations.

(5) Because of $G^*(1) \in \mathfrak{H}_{\mathfrak{G}}$ and $\widetilde{\psi}_j \in \mathfrak{H}_{\mathfrak{G}}^{\perp}$, $j \in \mathbb{N}$, from (2.65) and (5.1) we get

$$(G^*(1), h_k) = (G^*(1), \phi_k) = \overline{\gamma}_k \prod_{j=0}^{k-1} D_{\gamma_j},\ \ k \in \mathbb{N}.$$

Combining this with (5.9) it follows that

$$\begin{aligned} & \operatorname{col}((G^*(1), h_1), (G^*(1), h_2), \ldots, (G^*(1), h_n)) \\ = \quad & D_{\gamma_0} \operatorname{col}(\overline{\gamma}_1, \overline{\gamma}_2 D_{\gamma_1}, \ldots, \overline{\gamma}_n \prod_{j=1}^{n-1} D_{\gamma_j}) = D_{\gamma_0} \eta_n(\gamma). \end{aligned}$$

Thus, taking into account $(G^*(1), G^*(1)) = 1 - |\gamma_0|^2$ and using (5.2) and (5.10) we get

$$\begin{aligned} \Gamma(h_1, h_2, \ldots, h_n, G^*(1)) &= \begin{vmatrix} \mathcal{A}_n(\gamma) & D_{\gamma_0}\eta_n(\gamma) \\ D_{\gamma_0}\eta_n^*(\gamma) & 1 - |\gamma_0|^2 \end{vmatrix} \\ = \quad (1 - |\gamma_0|^2) &\begin{vmatrix} \eta_n(\gamma)\eta_n^*(\gamma) + \mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma) & \eta_n(\gamma) \\ \eta_n^*(\gamma) & 1 \end{vmatrix}. \end{aligned}$$

Subtracting now the $(n+1)$th column multiplied by γ_1 from the first column and, moreover for $k \in \{2, \ldots, n\}$, subtracting the $(n+1)$th column multiplied by $\gamma_k \prod\limits_{j=1}^{k-1} D_{\gamma_j}$ from the kth column, we obtain

$$\begin{aligned} \Gamma(h_1, h_2, \ldots, h_n, G^*(1)) &= (1 - |\gamma_0|^2) \begin{vmatrix} \mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma) & \eta_n(\gamma) \\ 0 & 1 \end{vmatrix} \\ &= (1 - |\gamma_0|^2)\sigma_n(W\gamma)|\det \mathfrak{M}_n(\gamma)|^2. \end{aligned} \tag{5.26}$$

From (5.7) we see $\det \mathfrak{M}_n(\gamma) = \prod\limits_{j=1}^{n} D_{\gamma_j}$. Thus, (5.12) follows from (5.26). From the concrete form of the matrix

$$\begin{pmatrix} \mathfrak{M}_n(\gamma)\mathcal{A}_n(W\gamma)\mathfrak{M}_n^*(\gamma) & \eta_n(\gamma) \\ 0 & 1 \end{pmatrix}$$

it is clear that the rank of the Gram matrix of the vectors $(h_k)_{k=1}^{n}$ and $G^*(1)$ is equal to $\operatorname{rank} \mathcal{A}_n(W\gamma) + 1$.

(6) From (5.12) we get

$$\prod_{j=1}^{n}(1-|\gamma_j|^2)\frac{\sigma_n(W\gamma)}{\sigma_n(\gamma)} = \frac{\Gamma(h_1,h_2,\ldots,h_n,G^*(1))}{\Gamma(h_1,h_2,\ldots,h_n)},\ n\in\mathbb{N}. \tag{5.27}$$

Denote by P_n the orthoprojection from $\mathfrak{H}$ onto $\mathfrak{H}_{\mathfrak{G}}\ominus \mathrm{Lin}\,\{h_1,h_2,\ldots,h_n\}, n\in\mathbb{N}$. Because of $G^*(1)\in\mathfrak{H}_{\mathfrak{G}}$ using well-known properties of Gram determinants (see, e.g., Akhiezer/Glasman [2, Chapter I]) we see

$$\frac{\Gamma(h_1,h_2,\ldots,h_n,G^*(1))}{\Gamma(h_1,h_2,\ldots,h_n)} = \|P_nG^*(1)\|^2. \tag{5.28}$$

This implies that the sequence on the left-hand side of formula (5.27) is monotonically decreasing. Since the sequence $(h_n)_{n=1}^{\infty}$ is total in $\mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$ the decomposition (4.6) shows that $P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}}$ is the strong limit of the sequence $(P_n)_{n\in\mathbb{N}}$. Therefore, (5.13) follows from (5.27) and (5.28).

(7) Assume that $\sigma_n(\gamma)>0$ for every $n\in\mathbb{N}$. Then the block representation (5.25) shows that $\sigma_n(W\gamma)>0$ for every $n\in\mathbb{N}$. From this by induction we get $\sigma_n(W^m\gamma)>0$ for all $n,m\in\mathbb{N}$. Assume now that the limit (5.13) is positive. This means that $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}\neq\{0\}$ and $P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}}G^*(1)\neq 0$ are satisfied. As already mentioned, the operator $\mathrm{Rstr.}_{\,\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}}T^*$ is the maximal unilateral shift contained in $T^*_{\mathfrak{G}}$. Denote by $\tau, \|\tau\|=1$, a basis vector of the generating wandering subspace of this shift. Then the sequence $(T^{*(n-1)}\tau)_{n\in\mathbb{N}}$ is an orthonormal basis in $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$. Since $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}\subseteq\mathfrak{H}_{\mathfrak{F}}^{\perp}$ (see (4.4)) and since the part $(\psi_k)_{k=1}^{\infty}$ of the canonical basis (2.12) is an orthonormal basis in $\mathfrak{H}_{\mathfrak{F}}^{\perp}$, we obtain the representation

$$\tau = \beta_1\psi_1+\beta_2\psi_2+\cdots+\beta_n\psi_n+\cdots \tag{5.29}$$

where $\beta_j=(\tau,\phi_j), j\in\mathbb{N}$. Because of $T^*\psi_j=\psi_{j+1}, j\in\{1,2,\ldots\}$, we get

$$T^{*k}\tau = \beta_1\psi_{k+1}+\beta_2\psi_{k+2}+\cdots+\beta_n\psi_{k+n}+\cdots,\ k\in\mathbb{N}. \tag{5.30}$$

From (2.68) we see

$$(G^*(1),\psi_1) = \prod_{j=0}^{\infty}D_{\gamma_j}\ ,\ (G^*(1),\psi_k)=0\ ,\ k\in\{2,3,\ldots\}. \tag{5.31}$$

Combining (5.30) and (5.31) it follows that $(G^*(1),T^{*k}\tau)=0,\ k\in\mathbb{N}$. Thus,

$$P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}}G^*(1) = \sum_{k=0}^{\infty}(G^*(1),T^{*k}\tau)T^{*k}\tau = (G^*(1),\tau)\tau = \overline{\beta}_1\prod_{j=0}^{\infty}D_{\gamma_j}\tau.$$

This means

$$\|P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}}G^*(1)\| = |\beta_1|\prod_{j=0}^{\infty}D_{\gamma_j}. \tag{5.32}$$

Thus, the condition $\|P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}}G^*(1)\|\neq 0$ is equivalent to $(\tau,\psi_1)\neq 0$. This is equivalent to the fact that ψ_1 is not orthogonal to $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$. Now we pass to the model based on the sequence $W\gamma=(\gamma_1,\gamma_2,\gamma_3,\ldots)$. We will denote the corresponding objects associated with this model by a lower index 1. For example, $G_1, \mathfrak{N}_{\mathfrak{G}\mathfrak{F},1}, \psi_{j1}$ etc. The identity (4.18) takes now the form $\mathfrak{N}_{\mathfrak{G}\mathfrak{F},1} = \ker \mathcal{Q}_1^*(\gamma)$ where the matrix of the operator $\mathcal{Q}_1(\gamma)$ is obtained by deleting the first row (or first column) in the matrix of $\mathcal{Q}(\gamma)$. Therefore, if the vector τ with coordinate sequence $(\beta_j)_{j=1}^{\infty}$ (see (5.29)) belongs to $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$ then in view of (4.18) it belongs to $\ker \mathcal{Q}^*(\gamma)$. Thus, in view of the Hankel structure of $\mathcal{Q}^*(\gamma)$ it follows that the vector with these coordinates also belongs to $\ker \mathcal{Q}_1^*(\gamma)$. Hence, in view of (4.18) this vector belongs to $\mathfrak{N}_{\mathfrak{G}\mathfrak{F},1}$. Thus, the condition $\beta_1 \neq 0$ implies that $\psi_{1,1}$ is not orthogonal to $\mathfrak{N}_{\mathfrak{G}\mathfrak{F};1}$. This is equivalent to $\|P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F},1}} G_1^*(1)\| \neq 0$. Hence, if the limit (5.13) is positive then the limit (5.14) is positive for $m = 1$. The case $m \in \{2, 3, 4, \ldots\}$ is handled by induction. □

Using considerations as in the proof of statement (7) of the preceding Theorem and taking into account the "layered" structure of the model (see Theorem 2.13 and Corollary 3.7), we obtain the following result.

Corollary 5.6. *Suppose that the assumptions of Theorem* 5.5 *are fulfilled. Moreover, assume that $\sigma_n(\gamma) > 0$ for all $n \in \mathbb{N}$. Suppose that there exists an index $m \in \{0, 1, 2, \ldots\}$ for which* (5.14) *is satisfied and denote by $m_0(\gamma)$ the smallest index with this property. Then for $m \geq m_0(\gamma)$ the limit* (5.14) *is positive. The number $m_0(\gamma)$ is characterized by the following condition. If τ is a normalized basis vector of the generating wandering subspace of $V_{T_{\mathfrak{G}}^*}$ then*

$$\tau = \beta_{m_0(\gamma)+1}\psi_{m_0(\gamma)+1} + \beta_{m_0(\gamma)+2}\psi_{m_0(\gamma)+2} + \cdots \quad \text{and} \quad \beta_{m_0(\gamma)+1} \neq 0. \tag{5.33}$$

Hereby, the relations

$$m_0(W\gamma) = \begin{cases} m_0(\gamma), & \text{if } m_0(\gamma) = 0, \\ m_0(\gamma) - 1, & \text{if } m_0(\gamma) \geq 1 \end{cases} \tag{5.34}$$

hold true.

Definition 5.7. *Assume that $\gamma \in \Gamma l_2$. Let $\theta(\zeta)$ be the Schur function associated with γ and let Δ be a simple unitary colligation of the form* (3.1) *which satisfies $\theta(\zeta) = \theta_\Delta(\zeta)$. If $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \neq \{0\}$ then the number $m_0(\gamma)$ characterized by condition* (5.33) *is called the level of the subspace $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$ or also the level of the sequence γ. If $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} = \{0\}$ we set $m_0(\gamma) := \infty$*

Thus, it is convenient to consider the vectors $\psi_1, \psi_2, \psi_3, \ldots$ as levels of the subspace $\mathfrak{H}_{\mathfrak{F}}^{\perp}$. Hereby, we will say that the vector ψ_k, $k \in \{1, 2, 3, \ldots\}$ determines the kth level. Then the number $m_0(\gamma)$ expresses the number of levels which have to be overcome in order to "reach" the subspace $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$.

Theorem 4.5 implies that a function $\theta(\zeta)$ belongs to $\mathcal{S}\Pi \setminus J$ if and only if $m_0(\gamma) < \infty$. Hereby, as (5.32) shows, the verification of the statement $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \neq \{0\}$ with the aid of the vector $G^*(1)$ is only possible in the case $m_0(\gamma) = 0$, this means that $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$ "begins" at the first level. Therefore, if $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}} \neq \{0\}$ but $P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}} G^*(1) = 0$ then it is necessary to pass from the sequence γ to the sequence $W\gamma$. Then from (5.34) it follows that the subspace $\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}$ will be "found" after a finite number of such steps.

5.2. Some criteria of pseudocontinuability of Schur functions

Theorem 5.8. ([21]) *Let $\theta(\zeta)$ be a function from $\mathcal{S}$ whose the sequence $(\gamma_j)_{j=0}^{\infty}$ of Schur parameters of which belongs to Γl_2. Then the vector*

$$\xi(\gamma) := (Q(W\gamma), Q(W^2\gamma), \ldots, Q(W^n\gamma), \ldots)$$

where $Q(\gamma)$ is given in (3.15)*, belongs to l_2. The function $\theta(\zeta)$ admits a pseudocontinuation into $\mathbb{D}_e$ if and only if $\xi(\gamma)$ is not cyclic for the coshift W (see* (3.11)*) in l_2.*

Proof. The vector $\xi(\gamma)$ is not cyclic for W in l_2 if and only if $\ker \mathcal{Q}(\gamma) \neq \{0\}$, where $\mathcal{Q}(\gamma)$ is defined via (3.48). The Hankel structure of $\mathcal{Q}(\gamma)$ implies that $\ker \mathcal{Q}(\gamma) \neq \{0\}$ if and only if $\ker \mathcal{Q}^*(\gamma) \neq \{0\}$. Now the assertion of the Theorem follows from Lemma 4.7 and Theorem 4.5. □

The following series of quantitative criteria starts with a criterion which characterizes the Schur parameters of a rational function of the Schur class $\mathcal{S}$.

Theorem 5.9. ([23]) *Let $\theta(\zeta) \in \mathcal{S}$ and denote $\gamma = (\gamma_j)_{j=0}^{\omega}$ the sequence of its Schur parameters. Then the function $\theta(\zeta)$ is rational if and only if one of the following two conditions is satisfied:*

(1) *$\omega < \infty$, i.e., $|\gamma_\omega| = 1$.*
(2) *$\gamma \in \Gamma l_2$ and there exists an index $n \in \mathbb{N}$ such that $\sigma_n(\gamma) = 0$, where $\sigma_n(\gamma)$ is defined via* (5.5)*.*

Hereby:

(1a) *$\omega = 0$ if and only if $\theta(\zeta) \equiv \gamma_0$, $|\gamma_0| = 1$.*
(1b) *$\omega \in \mathbb{N}$ if and only if $\theta(\zeta)$ is a finite Blaschke product of degree ω.*

Let $\gamma \in \Gamma l_2$. If $n_0 \in \{0, 1, 2, \ldots\}$ satisfies $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$ then:

(2a) *$n_0 = 0$ if and only if $\theta(\zeta) \equiv \gamma_0$, $|\gamma_0| < 1$, i.e., if and only if $\theta(\zeta)$ is not a constant function with unitary value but a block of a constant unitary 2×2 matrix.*
(2b) *$n_0 \in \mathbb{N}$ if and only if $\theta(\zeta)$ is not a finite Blaschke product, but a block of a finite 2×2-matrix-valued Blaschke–Potapov product of the form* (4.3) *where n_0 is the smallest number of elementary Blaschke–Potapov factors forming such a 2×2-Blaschke–Potapov product.*

Proof. All what concerns condition (1) is the well-known criterion of Schur [31, part I] who described the Schur parameters of finite Blaschke products. Condition (2) follows from the corresponding assertions (2) of Theorems 5.5 and 4.6. □

Theorem 5.10. ([23]) *Let $\theta(\zeta) \in \mathcal{S}$ and denote by $\gamma = (\gamma_j)_{j=0}^{\infty}$ the sequence of its Schur parameters. Let $\sigma_n(\gamma)$, $n \in \{0, 1, 2, \ldots\}$, be the determinants defined via* (5.5)*. Then $\theta(\zeta) \in \mathcal{S}\Pi \backslash J$ if and only if $\gamma \in \Gamma l_2$ and one of the following conditions is satisfied:*

(a) *There exists an index* $n \in \mathbb{N}$ *such that* $\sigma_n(\gamma) = 0$.
(b) *If* $\sigma_n(\gamma) > 0$ *for all* $n \in \mathbb{N}$ *then there exists a number* $m \in \{0, 1, 2, \ldots\}$ *such that*

$$\lim_{n\to\infty} \frac{\sigma_n(W^{m+1}\gamma)}{\sigma_n(W^m\gamma)} > 0. \tag{5.35}$$

Suppose that there exists an index m *for which* (5.35) *is satisfied and denote* $m_0(\gamma)$ *the smallest index with this property. Then* (5.35) *is satisfied for all* $m \geq m_0(\gamma)$. *The number* $m_0(\gamma)$ *is characterized by condition* (5.33), *i.e.*, $m_0(\gamma)$ *is the level of the sequence* γ.

Proof. Theorem 5.9 implies that condition (a) is satisfied if and only if the function $\theta(\zeta)$ is rational and therefore belongs to $\mathcal{S}\Pi \backslash J$. In the case that $\theta(\zeta)$ is not rational the assertions of the Theorem follow from the assertions (6) and (7) of Theorem 5.5, Corollary 5.6 and Theorem 4.5. $\square$

For the proof of the next criterion we need additional facts about the matrices $\mathcal{A}_n(\gamma)$, $n \in \mathbb{N}$, and their determinants.

Lemma 5.11. *Assume that* $\gamma \in \Gamma l_2$ *and* $\sigma_{n+1}(\gamma) > 0$ *for some* $n \in \mathbb{N}$. *Then*

$$\prod_{j=1}^{n}(1-|\gamma_j|^2)\frac{\sigma_n(W\gamma)}{\sigma_n(\gamma)} = (1 + \frac{1}{1-|\gamma_1|^2}\Lambda_n^*(\gamma)\mathcal{A}_n^{-1}(W\gamma)\Lambda_n(\gamma))^{-1} \tag{5.36}$$

where

$$\Lambda_n(\gamma) = \operatorname{col}(\overline{\gamma}_1\ ,\ \overline{\gamma}_2 D_{\gamma_2}^{-1}\ ,\ \overline{\gamma}_3 D_{\gamma_2}^{-1} D_{\gamma_3}^{-1}\ , \ldots ,\ \overline{\gamma}_n \prod_{j=2}^{n} D_{\gamma_j}^{-1}). \tag{5.37}$$

Proof. From formula (5.25) it follows that in the considered case the matrix $\mathcal{A}_n(W\gamma)$ is invertible. Therefore, taking into account the invertibility of $\mathfrak{M}_n(\gamma)$ and using (5.10) we get

$$\mathcal{A}_n(\gamma) = \mathfrak{M}_n(\gamma)\mathcal{A}_n^{\frac{1}{2}}(W\gamma)[X_n(\gamma)X_n^*(\gamma) + I_n]\mathcal{A}_n^{\frac{1}{2}}(W\gamma)\mathfrak{M}_n^*(\gamma) \tag{5.38}$$

where $X_n(\gamma) = \mathcal{A}_n^{-\frac{1}{2}}(W\gamma)\mathfrak{M}_n^{-1}(\gamma)\eta_n(\gamma)$. By direct computation it is checked that

$$\mathfrak{M}_n(\gamma)\Lambda_n(\gamma) = D_{\gamma_1}\eta_n(\gamma). \tag{5.39}$$

Thus, $X_n(\gamma) = D_{\gamma_1}^{-1}\mathcal{A}_n^{-\frac{1}{2}}(W\gamma)\Lambda_n(\gamma)$. Taking the determinant in (5.38) and using the form (5.7) of the matrix $\mathfrak{M}_n(\gamma)$ we obtain

$$\sigma_n(\gamma) = \prod_{j=1}^{n}(1-|\gamma_j|^2)\sigma_n(W\gamma)\det(I_n + X_n(\gamma)X_n^*(\gamma)).$$

From this and the identity $\det(I_n + X_n(\gamma)X_n^*(\gamma)) = 1 + X_n^*(\gamma)X_n(\gamma)$ (see, e.g., [24, Lemma 1.1.8]) we obtain (5.36). $\square$

Lemma 5.12. *Assume that $\gamma \in \Gamma l_2$ and that $\sigma_n(\gamma) = 0$ for some $n \in \mathbb{N}$. Denote by $m_0(\gamma)$ the level of the sequence γ, i.e., $m_0(\gamma)$ is characterized by condition* (5.33). *Then:*

(a) *For $m \geq m_0(\gamma)$ it holds* $\operatorname{rank} \mathcal{A}_n(W^m\gamma) = \operatorname{rank} \mathcal{A}_n(W^{m+1}\gamma)$, $n \in \mathbb{N}$.
(b) *If $m_0(\gamma) \geq 1$, $m \in \{0, 1, \ldots, m_0(\gamma) - 1\}$ and $n_0(m)$ is such that $\sigma_{n_0(m)}(W^m\gamma) > 0$ and $\sigma_{n_0(m)+1}(W^m\gamma) = 0$ then*

$$\operatorname{rank} \mathcal{A}_n(W^m\gamma) \geq \operatorname{rank} \mathcal{A}_n(W^{m+1}\gamma),\ n \in \{1, 2, \ldots, n_0(m) - 1\},$$
$$\operatorname{rank} \mathcal{A}_n(W^m\gamma) = \operatorname{rank} \mathcal{A}_n(W^{m+1}\gamma) + 1,\ n \geq n_0(m).$$

Proof. (a) It suffices to consider the case $m_0(\gamma) = 0$. In the opposite case it is necessary to change from γ to $W^{m_0(\gamma)}\gamma$. Thus, assume that $m_0(\gamma) = 0$. Because of $\beta_1 \neq 0$ from (5.32) we get $P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}} G^*(1) \neq 0$. This means, for arbitrary $n \in \mathbb{N}$ the rank of the Gram matrices of the vectors $(h_j)_{j=1}^n$ is one smaller than the rank of the Gram matrix of the vectors $(h_j)_{j=1}^n$ and $G^*(1)$. Now for the case $m = 0$ the assertion follows from statement (5) of Theorem 5.5. The case $m > 0$ can be treated analogously.
(b) It suffices to consider the case $m = 0$. The other cases can be considered analogously. Thus, let $m = 0$. In the considered case we proceed as above and take into account that now $\beta_1 = 0$. Hence, $P_{\mathfrak{N}_{\mathfrak{G}\mathfrak{F}}} G^*(1) = 0$ and $G^*(1) \in \mathfrak{H}_{\mathfrak{G}\mathfrak{F}}$. □

Theorem 5.13. ([23]) *Let $\theta(\zeta) \in \mathcal{S}$ and denote by γ the sequence of its Schur parameters. Then $\theta(\zeta) \in \mathcal{S}\Pi \setminus J$ if and only if $\gamma \in \Gamma l_2$ and there exist numbers $m \in \{0, 1, 2, \ldots\}$ and $c > 0$, which depends on m, such that*

$$\begin{pmatrix} \mathcal{A}(W^{m+1}\gamma) & \Lambda(W^m\gamma) \\ \Lambda^*(W^m\gamma) & c \end{pmatrix} \geq 0 \tag{5.40}$$

where

$$\mathcal{A}(\gamma) = I - \mathfrak{L}(\gamma)\mathfrak{L}^*(\gamma),$$

$$\Lambda(\gamma) = \operatorname{col}\left(\overline{\gamma}_1\ ,\ \overline{\gamma}_2 D_{\gamma_2}^{-1}\ ,\ \overline{\gamma}_3 D_{\gamma_2}^{-1} D_{\gamma_3}^{-1}\ , \ldots,\ \overline{\gamma}_n \prod_{j=2}^n D_{\gamma_j}^{-1}\ , \ldots\right) \tag{5.41}$$

and $\mathfrak{L}(\gamma)$ is given via (3.47). *Suppose that there exists an index m for which* (5.40) *is satisfied and denote by $m_0(\gamma)$ the smallest index with this property. Then* (5.40) *is satisfied for all $m \geq m_0(\gamma)$. The number $m_0(\gamma)$ is characterized by condition* (5.33), *i.e., $m_0(\gamma)$ is the level of the sequence γ.*

Proof. We suppose first that $\sigma_n(\gamma) > 0$ for all $n \in \mathbb{N}$. Then from statement (6) of Theorem 5.5 and Lemma 5.11 it follows that the sequence $(\Lambda_n^*(\gamma)\mathcal{A}_n^{-1}(\gamma)\Lambda_n(\gamma))_{n=1}^\infty$ is monotonically increasing and bounded from above if and only if the limit (5.13) is positive. Thus, the existence of a number $c > 0$ such that for all $n \in \mathbb{N}$ the inequality

$$\Lambda_n^*(\gamma)\mathcal{A}_n^{-1}(\gamma)\Lambda_n(\gamma) \leq c \tag{5.42}$$

is satisfied is equivalent to the positivity of the limit (5.13). On the other hand, in view of $\mathcal{A}_n(\gamma) > 0,\ n \in \mathbb{N}$, the condition (5.42) is equivalent (see, e.g., [24, Lemma 1.1.9]) to the inequality

$$\begin{pmatrix} \mathcal{A}_n(W\gamma) & \Lambda_n(\gamma) \\ \Lambda_n^*(\gamma) & c \end{pmatrix} \geq 0,\ n \in \mathbb{N}. \tag{5.43}$$

But the conditions (5.40) and (5.43) are equivalent in the case $m = 0$.

Passing from the sequence γ to the sequence $W^m\gamma$ and using analogous considerations we obtain that the limit (5.35) is positive if and only if the condition (5.40) is satisfied. Thus, the application of Theorem 5.10 shows that the assertion is proved if $\sigma_n(\gamma) > 0$ for all $n \in \mathbb{N}$.

Assume now that $\sigma_n(\gamma) = 0$ for some $n \in \mathbb{N}$. We suppose that $m_0(\gamma) = 0$. In the opposite case, we pass from the sequence γ to the sequence $V^{m_0(\gamma)}\gamma$. Assume that $n_0 \in \{0, 1, 2, \ldots\}$ satisfies $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$. Then, if the constant c fulfills condition (5.42) for $n = n_0$, condition (5.43) will also be satisfied for $n = n_0$. We will show that for a constant c chosen in this way the inequality (5.43) will also be satisfied for all $n > n_0$.

Let $k \in \mathbb{N}$ and $n = n_0 + k$. The matrix $\mathcal{A}_{n_0+k}(\gamma)$ admits the block representation

$$\mathcal{A}_{n_0+k}(\gamma) = \begin{pmatrix} \mathcal{A}_{n_0}(\gamma) & B_{n_0,k}(\gamma) \\ B_{n_0,k}^*(\gamma) & C_{n_0,k}(\gamma) \end{pmatrix}.$$

Statement (2) of Theorem 5.5 implies (see [24, Lemma 1.1.7]) that this block representation leads to the factorization

$$\mathcal{A}_{n_0+k}(\gamma) = F_{n_0,k}(\gamma) \begin{pmatrix} \mathcal{A}_{n_0}(\gamma) & 0 \\ 0 & 0 \end{pmatrix} F_{n_0,k}^*(\gamma) \tag{5.44}$$

where $F_{n_0,k}(\gamma) = \begin{pmatrix} I_{n_0} & 0 \\ X_{n_0,k}(\gamma) & I_k \end{pmatrix}$, $X_{n_0,k}(\gamma) = B_{n_0,k}^*(\gamma)\mathcal{A}_{n_0}^{-1}(\gamma)$. Using (5.39) we rewrite (5.10) in the form

$$\mathcal{A}_n(\gamma) = \mathfrak{M}_n(\gamma)\ \big(\ \frac{1}{1-|\gamma_1|^2}\Lambda_n(\gamma)\Lambda_n^*(\gamma) + \mathcal{A}_n(W\gamma)\ \big)\ \mathfrak{M}_n^*(\gamma),\ n \in \mathbb{N}, \tag{5.45}$$

On the one hand, statement (a) of Lemma 5.12 implies that $\operatorname{rank}\mathcal{A}_{n_0+k}(\gamma) = \operatorname{rank}\mathcal{A}_{n_0+k}(W\gamma)$. Then from (5.45) for $n = n_0 + k$ it follows that $\Lambda_{n_0+k}(\gamma)$ is contained in the range of $\mathcal{A}_{n_0+k}(W\gamma)$. Thus, we have the representation

$$\Lambda_{n_0+k}(\gamma) = \begin{pmatrix} I_{n_0} & 0 \\ X_{n_0,k}(W\gamma) & I_k \end{pmatrix} \begin{pmatrix} \Lambda_{n_0}(\gamma) \\ 0 \end{pmatrix}. \tag{5.46}$$

We consider the matrix (5.43) for $n = n_0 + k$ and multiply it from the left by the matrix $\begin{pmatrix} F_{n_0,k}^{-1}(W\gamma) & 0 \\ 0 & 1 \end{pmatrix}$ and from the right by the adjoint of this matrix. Taking into account (5.44) and (5.46) this gives us the nonnegative Hermitian matrix

$$\left(\begin{array}{cc} \begin{pmatrix} \mathcal{A}_{n_0}(W\gamma) & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \Lambda_{n_0}(\gamma) \\ 0 \end{pmatrix} \\ \begin{pmatrix} \Lambda^*_{n_0}(\gamma), & 0 \end{pmatrix} & c \end{array} \right).$$

Thus, the matrix (5.43) is nonnegative Hermitian for $n = n_0 + k$, too.

Hence, in the considered case the Theorem is proved for the case $m = 0$. The case of an arbitrary $m \in \{1, 2, \ldots\}$ is treated as above. One has only to pass from the sequence γ to the sequence $W^m\gamma$. □

Corollary 5.14. *Let $\theta(\zeta) \in \mathcal{S}$ and denote by γ the sequence of its Schur parameters. Then $\theta(\zeta) \in \mathcal{S}\Pi\backslash J$ if and only if $\gamma \in \Gamma l_2$ and there exists an index $m \in \{0, 1, 2, \ldots\}$ for which the vector $\Lambda(W^m\gamma)$ belongs to the range of the operator $\mathcal{A}^{\frac{1}{2}}(W^{m+1}\gamma)$. Suppose that there exists such an index m and denote by $m_0(\gamma)$ the smallest one. Then for all $m \geq m_0(\gamma)$ the vector $\Lambda(W^m\gamma)$ belongs to the range of the operator $\mathcal{A}^{\frac{1}{2}}(W^{m+1}\gamma)$. The number $m_0(\gamma)$ is characterized by condition* (5.33). *This means that $m_0(\gamma)$ is the level of the sequence γ.*

Proof. Because of $\mathcal{A}(W^m\gamma) \geq 0$, $m \in \{0, 1, 2, \ldots\}$, the assertion follows from Theorem 5.13 and the well-known criterion for nonnegative Hermitian block matrices (see, e.g., [11, Lemma 2.1]). □

Remark 5.15. *The matrix representation* (3.6) *implies*

$$\mathcal{A}(\gamma) = I - \mathfrak{L}(\gamma)\mathfrak{L}^*(\gamma) = \mathcal{R}(\gamma)\mathcal{R}^*(\gamma).$$

Therefore, Corollary 5.14 *remains true if the range of the operator $\mathcal{A}^{\frac{1}{2}}(W^{m+1}\gamma)$ is replaced by the range of the operator $\mathcal{R}(W^{m+1}\gamma)$.*

5.3. On some properties of the Schur parameter sequences of pseudocontinuable Schur functions

In the term $\Lambda^*_n(\gamma)\mathcal{A}^{-1}_n(W\gamma)\Lambda_n(\gamma)$ (see Lemma 5.11) the parameter γ_1 is only contained in $\Lambda_n(\gamma)$. This enables us to give a more concrete description of the dependence of this expression on γ_1. For this we consider the representation (5.25). We assume that for $n \in \mathbb{N}$ the matrix $\mathcal{A}_{n+1}(\gamma)$ is invertible and introduce the notations

$$H_n(\gamma) := \mathcal{A}_n(W\gamma) - B_{n+1}(\gamma)B^*_{n+1}(\gamma) \tag{5.47}$$

and

$$H^{[c]}_n(\gamma) := 1 - \Pi^2_1 - \Pi^2_1 B^*_{n+1}(\gamma)H^{-1}_n(\gamma)B_{n+1}(\gamma). \tag{5.48}$$

Then from (5.25) it follows

$$\mathcal{A}_{n+1}(\gamma) = \begin{pmatrix} 1 & -\Pi_1 B^*_{n+1}(\gamma)H^{-1}_n(\gamma) \\ 0 & I_n \end{pmatrix} \begin{pmatrix} H^{[c]}_n(\gamma) & 0 \\ 0 & H_n(\gamma) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Pi_1 H^{-1}_n(\gamma)B_{n+1}(\gamma) & I_n \end{pmatrix}.$$

Thus,

$$\begin{aligned}
&\mathcal{A}_{n+1}^{-1}(\gamma)\\
&= \begin{pmatrix} 1 & 0 \\ \Pi_1 H_n^{-1}(\gamma)B_{n+1}(\gamma) & I_n \end{pmatrix} \begin{pmatrix} \frac{1}{H_n^{[c]}(\gamma)} & 0 \\ 0 & H_n^{-1}(\gamma) \end{pmatrix} \begin{pmatrix} 1 & \Pi_1 B_{n+1}^*(\gamma)H_n^{-1}(\gamma) \\ 0 & I_n \end{pmatrix}\\
&= \frac{1}{H_n^{[c]}(\gamma)} \begin{pmatrix} 1 \\ \Pi_1 H_n^{-1}(\gamma)B_{n+1}(\gamma) \end{pmatrix} (1\ ,\ \Pi_1 B_{n+1}^*(\gamma)H_n^{-1}(\gamma)) + \begin{pmatrix} 0 & 0 \\ 0 & H_n^{-1}(\gamma) \end{pmatrix}.
\end{aligned}$$

Using this product representation and the equality $\Lambda_{n+1}(\gamma) = \begin{pmatrix} \overline{\gamma}_1 \\ D_{\gamma_2}^{-1}\Lambda_n(W\gamma) \end{pmatrix}$ we find

$$\begin{aligned}
&\Lambda_{n+1}^*(\gamma)\mathcal{A}_{n+1}^{-1}(W\gamma)\Lambda_{n+1}(\gamma) \qquad (5.49)\\
=\ & \frac{1}{H_n^{[c]}(W\gamma)}|\gamma_1 + \Pi_3\Lambda_n^*(W\gamma)H_n^{-1}(W\gamma)B_{n+1}(W\gamma)|^2\\
&+\frac{1}{1-|\gamma_2|^2}\Lambda_n^*(W\gamma)H_n^{-1}(W\gamma)\Lambda_n(W\gamma).
\end{aligned}$$

Hereby, γ_1 occurs only in the expression in the modules.

Definition 5.16. *Denote* $\Pi\Gamma$ (*resp.* $\Pi\Gamma l_2$) *the set of all* $\gamma \in \Gamma$ *for which the associated Schur function belongs to* $\mathcal{S}\Pi$ (*resp.* $\mathcal{S}\Pi \setminus J$).

Lemma 5.17. *Let* $\gamma \in \Pi\Gamma l_2$. *Assume that* $\sigma_n(\gamma) > 0$ *for all* $n \in \mathbb{N}$ *and* $m_0(\gamma) = 0$. *Then*

$$\lim_{n\to\infty} H_n^{[c]}(\gamma) = 0 \qquad (5.50)$$

where $H_n^{[c]}(\gamma)$ *is given via* (5.48).

Proof. In view of (5.47) for $n \in \mathbb{N}$ we get

$$\begin{aligned}
&B_{n+1}^*(\gamma)H_n^{-1}(\gamma)B_{n+1}(\gamma)\\
=\ & B_{n+1}^*(\gamma)\mathcal{A}_n^{-\frac{1}{2}}(W\gamma)(I_n - \mathcal{A}_n^{-\frac{1}{2}}(W\gamma)B_{n+1}(\gamma)B_{n+1}^*(\gamma)\mathcal{A}_n^{-\frac{1}{2}}(W\gamma))^{-1}\cdot\\
&\cdot\mathcal{A}_n^{-\frac{1}{2}}(W\gamma)B_{n+1}(\gamma) = \sum_{k=1}^{\infty} q_n^k(\gamma) = \frac{q_n(\gamma)}{1-q_n(\gamma)}
\end{aligned}$$

where $q_n(\gamma) = B_{n+1}^*(\gamma)\mathcal{A}_n^{-1}(W\gamma)B_{n+1}(\gamma)$, $n \in \mathbb{N}$. Thus,

$$H_n^{[c]}(\gamma) = 1 - \Pi_1^2 - \Pi_1^2\frac{q_n(\gamma)}{1-q_n(\gamma)}\ ,\ n \in \mathbb{N}. \qquad (5.51)$$

Using the block partition (5.23) of the matrix $\mathfrak{L}_{n+1}(\gamma)$ we obtain for $n \in \mathbb{N}$ by analogy with the derivation of the formulas (5.19) (see, e.g., [24, Lemma 1.1.7])

$$\begin{aligned}
\sigma_{n+1}(\gamma) &= \det(I_{n+1} - \mathfrak{L}^*_{n+1}(\gamma)\mathfrak{L}_{n+1}(\gamma)) \\
&= \det \begin{pmatrix} 1 - \Pi_1^2 - B^*_{n+1}(\gamma)B_{n+1}(\gamma) & -B^*_{n+1}(\gamma)\mathfrak{L}_n(W\gamma) \\ -\mathfrak{L}^*_n(W\gamma)B_{n+1}(\gamma) & I_n - \mathfrak{L}^*_n(W\gamma)\mathfrak{L}_n(W\gamma) \end{pmatrix} \\
&= \sigma_n(W\gamma)\{\ 1-\Pi_1^2- B^*_{n+1}(\gamma)(\ I_n + \mathfrak{L}_n(W\gamma)\mathcal{A}_n^{-1}(W\gamma)\mathfrak{L}^*_n(W\gamma)\ )B_{n+1}(\gamma)\} \\
&= \sigma_n(W\gamma)(1 - \Pi_1^2 - B^*_{n+1}(\gamma)\mathcal{A}_n^{-1}(W\gamma)B_{n+1}(\gamma)).
\end{aligned}$$

This means $\sigma_{n+1}(\gamma) = \sigma_n(W\gamma)(1-\Pi_1^2-q_n(\gamma))$, $n \in \mathbb{N}$. Comparing this expression with (5.19) we obtain

$$1 = \frac{\sigma_n(W\gamma)}{\sigma_n(\gamma)} \cdot \frac{1 - \Pi_1^2 - q_n(\gamma)}{\mathcal{A}_n^{[c]}(\gamma)}.$$

It holds $\lim\limits_{n\to\infty} \mathcal{A}_n^{[c]}(\gamma) = 0$. Hereby, in view of $m_0(\gamma) = 0$, the limit (5.13) is positive. Thus, $\lim\limits_{n\to\infty} q_n(\gamma) = 1 - \Pi_1^2$. Now (5.51) implies (5.50). □

Lemma 5.18. *Let $\gamma \in \Pi\Gamma l_2$. Assume that $m_0(\gamma) = 0$ and that there exists an index $n_0 \in \mathbb{N}$ such that $\sigma_{n_0}(\gamma) > 0$ and $\sigma_{n_0+1}(\gamma) = 0$ are satisfied. Then there exists a unique constant vector $a = \operatorname{col}(a_1, \ldots, a_{n_0})$ such that $a_1 \neq 0$ and for $j \in \{0, 1, 2, \ldots\}$ the relations*

$$(I_{n_0} - \mathfrak{L}^*_{n_0}(W^j\gamma)\mathfrak{L}_{n_0}(W^j\gamma))a = \frac{1}{\Pi_{n_0+j+1}} b_{n_0}(W^j\gamma), \tag{5.52}$$

$$\begin{pmatrix} \Pi_{n_0+j+1}\mathfrak{L}_{n_0}(W^j\gamma)a \\ 1 \end{pmatrix} \in \ker \mathcal{A}_{n_0+1}(W^j\gamma) \tag{5.53}$$

and

$$\mathfrak{M}^*_{n_0+1}(W^j\gamma)\begin{pmatrix} \Pi_{n_0+j+1}\mathfrak{L}_{n_0}(W^j\gamma)a \\ 1 \end{pmatrix} = D_{\gamma_{n_0+j+1}}\begin{pmatrix} \Pi_{n_0+j+2}\mathfrak{L}_{n_0}(W^{j+1}\gamma)a \\ 1 \end{pmatrix} \tag{5.54}$$

*are fulfilled where $\Pi_n, b_n(\gamma)$ and $\mathfrak{M}_n(\gamma)$ are defined via (*3.14*), (*5.16*) and (*5.7*), respectively.*

Proof. From the assumptions of the lemma we obtain analogously to (5.17)

$$\mathcal{A}_{n_0+1}(\gamma) = \begin{pmatrix} I_{n_0} & 0 \\ X_{n_0,1}(\gamma) & 1 \end{pmatrix}\begin{pmatrix} \mathcal{A}_{n_0}(\gamma) & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_{n_0} & X^*_{n_0,1}(\gamma) \\ 0 & 1 \end{pmatrix}, \tag{5.55}$$

where

$$X^*_{n_0,1}(\gamma) = -\mathcal{A}^{-1}_{n_0}(\gamma)\mathfrak{L}_{n_0}(\gamma)b_{n_0}(\gamma). \tag{5.56}$$

Let $a(\gamma) := (I_{n_0} - \mathfrak{L}^*_{n_0}(\gamma)\mathfrak{L}_{n_0}(\gamma))^{-1}b_{n_0}(\gamma)$. Thus,

$$(I_{n_0} - \mathfrak{L}^*_{n_0}(\gamma)\mathfrak{L}_{n_0}(\gamma))a(\gamma) = b_{n_0}(\gamma). \tag{5.57}$$

From (5.55) and (5.56) we see that the vector

$$\begin{pmatrix} \mathfrak{L}_{n_0}(\gamma)a(\gamma) \\ 1 \end{pmatrix} \tag{5.58}$$

belongs to $\ker \mathcal{A}_{n_0+1}(\gamma)$. Because of $m_0(\gamma)=0$ Lemma 5.12 implies that for arbitrary $j \in \mathbb{N}$ the relations

$$\sigma_{n_0}(W^j\gamma) > 0\ ,\ \sigma_{n_0+1}(W^{j+1}\gamma) = 0 \tag{5.59}$$

hold true. This means $\dim \ker \mathcal{A}_{n_0+1}(W^j\gamma) = 1$, $j \in \{0,1,2,\ldots\}$. For this reasons, all computations can be done in the same way if we replace γ by $W^j\gamma$, $j \in \mathbb{N}$. Thus, for $j \in \{0,1,2,\ldots\}$ we have

$$\begin{pmatrix} \mathfrak{L}_{n_0}(W^j\gamma)a(W^j\gamma) \\ 1 \end{pmatrix} \in \ker \mathcal{A}_{n_0+1}(W^j\gamma). \tag{5.60}$$

Using (5.10) and (5.59) we infer

$$\mathfrak{M}^*_{n_0+1}(W^j\gamma)(\ker \mathcal{A}_{n_0+1}(W^j\gamma)) = \ker \mathcal{A}_{n_0+1}(W^{j+1}\gamma)\ ,\ j \in \{0,1,2,\ldots\}.$$

This means for $j \in \{0,1,2,\ldots\}$

$$\mathfrak{M}^*_{n_0+1}(W^j\gamma)\begin{pmatrix} \mathfrak{L}_{n_0}(W^j\gamma)a(W^j\gamma) \\ 1 \end{pmatrix} = k_j \begin{pmatrix} \mathfrak{L}_{n_0}(W^{j+1}\gamma)a(W^{j+1}\gamma) \\ 1 \end{pmatrix}. \tag{5.61}$$

Hereby, from (5.7) we get

$$k_j = D_{\gamma_{n_0+j+1}}\ ,\ j \in \{0,1,2,\ldots\}. \tag{5.62}$$

Using (5.10) it follows $\eta_{n_0+1}(W^j\gamma) \perp \ker \mathcal{A}_{n_0+1}(W^j\gamma)$, $j \in \{0,1,2,\ldots\}$. Therefore, in view of (5.8), the operators $\mathfrak{M}^*_{n_0+1}(W^j\gamma)$ and $\mathfrak{M}^{-1}_{n_0+1}(W^j\gamma)$ coincide on the subspace $\ker \mathcal{A}_{n_0+1}(W^j\gamma)$. Combining this with (5.6) and (5.7) we find for $j \in \{0,1,2,\ldots\}$ the equations

$$\mathfrak{M}^*_{n_0+1}(W^j\gamma)\begin{pmatrix} \mathfrak{L}_0(W^j\gamma)a(W^j\gamma) \\ 1 \end{pmatrix} = \mathfrak{M}^{-1}_{n_0+1}(W^j\gamma)\begin{pmatrix} \mathfrak{L}_0(W^j\gamma)a(W^j\gamma) \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathfrak{M}^{-1}_{n_0}(W^j\gamma) & 0 \\ * & * \end{pmatrix}\begin{pmatrix} \mathfrak{L}_0(W^j\gamma)a(W^j\gamma) \\ 1 \end{pmatrix} = \begin{pmatrix} \mathfrak{L}_0(W^{j+1}\gamma)a(W^j\gamma) \\ * \end{pmatrix}.$$

Taking into account (5.61) and (5.62) from this we get

$$a(W^{j+1}\gamma) = D^{-1}_{\gamma_{n_0+j+1}}a(W^j\gamma)\ ,\ j \in \{0,1,2,\ldots\}.$$

This means

$$a(W^j\gamma) = \prod_{k=1}^{j} D^{-1}_{\gamma_{n_0+k}}a(\gamma)\ ,\ j \in \{0,1,2,\ldots\}. \tag{5.63}$$

If we set $a := \Pi^{-1}_{n_0+1}a(\gamma)$ then (5.63) implies $a(W^j\gamma) = \Pi_{n_0+j+1}a$, $j \in \{0,1,2,\ldots\}$. Substituting this expression into formulas (5.57) for $W^j\gamma$ instead of γ, (5.60) and (5.61) we obtain (5.52), (5.53) and (5.54), respectively.

If we assume that $a_1 = 0$ then representation (5.3) shows that the first component of the vector (5.58) is 0. Then representation (5.25) implies that

$\ker \mathcal{A}_{n_0}(W\gamma) \neq 0$. This contradiction shows that $a_1 \neq 0$. Finally, the uniqueness of the vector a follows from (5.57). $\square$

Before formulating the next result we note that all functions $\Lambda_n(\gamma), H_n(\gamma)$ and $B_n(\gamma)$ only depend on $(\gamma_1, \gamma_2, \ldots)$. This means that the functions $\Lambda_n(W^m\gamma)$, $H_n(W^m\gamma)$ and $B_n(W^m\gamma)$ only depend on $(\gamma_{m+1}, \gamma_{m+2}, \ldots)$.

Theorem 5.19. *Assume $\gamma \in \Pi\Gamma l_2$. Denote by $m_0(\gamma)$ the level of the sequence γ. Then for every $m \geq m_0(\gamma)+1$ the element γ_m is uniquely determined by the subsequent elements $\gamma_{m+1}, \gamma_{m+2}, \ldots$. Moreover, the following statements hold true:*

(1) *Assume that $\sigma_n(\gamma) > 0$ for all $n \in \mathbb{N}$. Then*

$$\gamma_m = -\Pi_{m+2} \cdot \lim_{n\to\infty} \Lambda_n^*(W^m\gamma) H_n^{-1}(W^m\gamma) B_{n+1}(W^m\gamma) \ , \qquad m \geq m_0(\gamma)+1 \qquad (5.64)$$

where $\Pi_n, \Lambda_n(\gamma), H_n(\gamma)$ and $B_{n+1}(\gamma)$ are defined via (3.14), (5.37), (5.47) *and* (5.24), *respectively.*

(2) *Assume that there exists an $n \in \mathbb{N}$ such that $\sigma_n(\gamma) = 0$ is satisfied. Let $n_0 \in \{0, 1, 2, \ldots\}$ be chosen such that $\sigma_{n_0}(W^{m_0(\gamma)}\gamma) > 0$ and $\sigma_{n_0+1}(W^{m_0(\gamma)}\gamma) = 0$. Then there exists a function $w(\gamma) = w(\gamma_1, \gamma_2, \ldots)$ such that the identities*

$$\gamma_m = w(W^m\gamma) \ , \ m \geq m_0(\gamma)+1 \qquad (5.65)$$

are fulfilled. Hereby, we have the following cases:

(2a) *If $n_0 = 0$ then $w(\gamma_1, \gamma_2, \ldots) \equiv 0$, i.e., $\gamma_m = 0$ for $m \geq m_0(\gamma)+1$.*

(2b) *If $n_0 \in \mathbb{N}$ then*

$$w(\gamma) = -\frac{1}{w_1(\gamma)} \sum_{k=1}^{n_0} \gamma_k w_{k+1}(\gamma) \prod_{j=1}^{k} D_{\gamma_j}^{-1}, \qquad (5.66)$$

where for $k \in \{1, 2, \ldots, n_0\}$

$$w_k(\gamma) = \Pi_{n_0+1}\Pi_k \sum_{j=1}^{k} a_j L_{k-j}(W^j\gamma) \quad \text{and} \quad w_{n_0+1}(\gamma) \equiv 1. \qquad (5.67)$$

Hereby, the constant vector $a = \operatorname{col}(a_1, a_2, \ldots, a_{n_0})$ satisfies (5.52) *for $j \geq m_0(\gamma)$.*

Proof. Without loss of generality we assume that $m_0(\gamma) = 0$. If $\sigma_n(\gamma) > 0$ for all $n \in \mathbb{N}$ then Lemma 5.11 implies that the expression (5.49) has to be bounded if $n \to \infty$. Thus, in the case $m = 1$ formula (5.64) follows from the boundedness of the expressions (5.49) and (5.50). For arbitrary $m \geq 2$ formula (5.64) is verified analogously by passing from the sequence γ to the sequence $W^{m-1}\gamma$.

Assume now that there exists an $n \in \mathbb{N}$ such that $\sigma_n(\gamma) = 0$ is satisfied. Without loss of generality, as above, we assume that $m_0(\gamma) = 0$. If $n_0 = 0$ then $\sigma_1(\gamma) = 0$, i.e., $1 - \prod_{j=1}^{\infty}(1 - |\gamma_j|^2) = 0$. This implies (2a).

Suppose now that $n_0 \in \mathbb{N}$. Then (see the proof of Theorem 5.13) there exists $m \geq 0$ and a constant $c > 0$ such that the inequality

$$\begin{pmatrix} \mathcal{A}_{n_0+1}(W^{m+1}\gamma) & \Lambda_{n_0+1}(W^m\gamma) \\ \Lambda^*_{n_0+1}(W^m\gamma) & c \end{pmatrix} \geq 0 \tag{5.68}$$

holds true. Let

$$Y(\gamma) = \begin{pmatrix} \Pi_{n_0+1}\mathfrak{L}_{n_0}(\gamma)a \\ 1 \end{pmatrix}, \tag{5.69}$$

where the vector a satisfies (5.52). Then (5.53) implies $Y(W\gamma) \in \ker \mathcal{A}_{n_0+1}(W\gamma)$. From this and (5.68) for $m = 0$ we infer

$$\Lambda^*_{n_0+1}(\gamma)Y(W\gamma) = 0. \tag{5.70}$$

Using (5.69) and (5.3) we see that $Y(\gamma)$ has the form

$$Y(\gamma) = \operatorname{col}(w_1(\gamma), w_2(\gamma), \ldots, w_{n_0+1}(\gamma))$$

where the sequence $(w_j(\gamma))_{j=1}^{n_0+1}$ is defined via (5.67). Taking into account (5.37) and substituting the coordinates of $Y(\gamma)$ in (5.70) we obtain the identity (5.65) for $m = 1$. Hereby, $w(\gamma)$ has the form (5.66). Passing now from γ to $W^{m-1}\gamma$ and repeating the above considerations we obtain from Lemma 5.18 the formulas (5.65) for $m \in \{2, 3, 4, \ldots\}$. □

The theorems proved above motivate the introduction of the following notation

Definition 5.20. *The elements γ of the set $\Pi\Gamma l_2$ are called Π-sequences. A Π-sequence γ is called pure if $m_0(\gamma) = 0$. If $\gamma, \gamma' \in \Gamma l_2$ then γ' is called a extension of γ if there exists an $n \in \mathbb{N}$ such that $W^n\gamma' = \gamma$ is satisfied. If γ is a pure Π-sequence and γ' is a extension of γ then γ' is called a regular extension of γ if γ' is also a pure Π-sequence. Assume that $\gamma \in \Pi\Gamma l_2$. Let $\theta(\zeta)$ be the Schur function associated with γ and let Δ be a simple unitary colligation of type* (3.1) *which satisfies $\theta(\zeta) = \theta_\Delta(\zeta)$. Then the number* $\dim \mathfrak{H}_{\mathfrak{G}\mathfrak{F}} (= \dim \mathfrak{H}_{\mathfrak{F}\mathfrak{G}})$ *is called the rank of the Π-sequence γ.*

Theorem 5.19 shows that in the case of a pure Π-sequence $\gamma = (\gamma_j)_{j=0}^\infty$ every element γ_n, $n \in \mathbb{N}$, is uniquely determined by the sequence $\gamma = (\gamma_j)_{j=n+1}^\infty$. Therefore, every Π-sequence γ is a extension of a pure Π-sequence $W^{m_0(\gamma)}\gamma$.

Let us consider an arbitrary Π-sequence $\gamma = (\gamma_j)_{j=0}^\infty$. Then obviously the sequences $(\gamma_j)_{j=1}^\infty$ and $\gamma = (\gamma_j)_{j=-1}^\infty$ where $|\gamma_{-1}| < 1$ are Π-sequences. This means that as well deleting an arbitrary finite number of first elements of a Π-sequence as finite extension of a Π-sequence gives us again a Π-sequence. However, if $\gamma = (\gamma_j)_{j=0}^\infty$ is a Π-sequence then the freedom of choice is restricted only to the first $m_0(\gamma) + 1$ elements $(\gamma_j)_{j=0}^{m_0(\gamma)}$. Beginning with the element $\gamma_{m_0(\gamma)+1}$ all the following elements of the sequence γ are uniquely determined by the corresponding subsequent ones. Namely, the existence of a determinate chain $(\gamma_j)_{m_0(\gamma)+1}^\infty$ ensures the pseudocontinuability of the corresponding function $\theta(\zeta) \in \mathcal{S}$. Therefore, in order to understand the phenomenon of pseudocontinuability it will be necessary to study the structure of pure Π-sequences. Theorem 5.19 shows that a regular extension of a pure Π-sequence is always unique and preserves this structure.

Let γ be a pure Π-sequence and γ' one of its nonregular one-step extensions. Then, as it follows from Theorem 4.5, Lemma 4.7 and the structure of the kernel of a Hankel matrix, an arbitrary extension of γ' can never be a pure Π-sequence.

The combination of statement (2) of Theorem 5.5 and Theorem 5.9 shows that a Π-sequence γ has finite rank if and only if its associated function $\theta(\zeta)$ is rational. Hereby, this rank coincides with the smallest number of elementary 2×2-Blaschke–Potapov factors of type (4.2) occurring in a finite Blaschke–Potapov product which has the block θ.

Lemma 5.12 shows that a regular extension of a pure Π-sequence of finite rank has the same rank. On the other hand, the rank of every nonregular one-step extension of a pure Π-sequence is one larger. Since every Π-sequence γ is a nonregular $m_0(\gamma)$-steps extension of a pure Π-sequence $V^{m_0(\gamma)}\gamma$ we have

$$\operatorname{rank}\gamma = m_0(\gamma) + \operatorname{rank} W^{m_0(\gamma)}\gamma. \tag{5.71}$$

Hereby, $\operatorname{rank} W^{m_0(\gamma)}\gamma = \operatorname{rank} W^{m_0(\gamma)+n}\gamma$, $n \in \{1,2,3,\ldots\}$.

5.4. The structure of pure Π-sequences of rank 0 or 1

Lemma 5.21. *Every Π-sequence γ of rank 0 is pure and has the form*

$$\gamma = (\gamma_0, 0, 0, 0, \ldots)\ ,\ |\gamma_0| < 1. \tag{5.72}$$

Conversely, every sequence of type (5.72) *is a pure Π-sequence of rank* 0.

Proof. Indeed, if $\operatorname{rank}\gamma = 0$ then $\sigma_1(\gamma) = 0$, i.e., $1 - \prod_{j=1}^{\infty}(1-|\gamma_j|^2) = 0$. This implies $\gamma_j = 0$, $j \in \{1,2,3,\ldots\}$. The converse statement is obvious. □

Thus, Π-sequences of type $(\gamma_0, \gamma_1, \ldots, \gamma_n, 0, 0, \ldots)$, $|\gamma_n| > 0$, $n \in \mathbb{N}$ are never pure. They are n-step extensions of a pure Π-sequence of type (5.72) where $|\gamma_0| > 0$. Obviously, every such sequence has rank n.

Theorem 5.22. ([23]) *A sequence $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma$ is a pure Π-sequence of first rank if and only if $\gamma_1 \neq 0$ and there exists a complex number λ such that the conditions*

$$0 < |\lambda| \leq 1 - |\gamma_1| \tag{5.73}$$

and

$$\gamma_{m+1} = \lambda \frac{\gamma_m}{\prod_{j=1}^{m}(1-|\gamma_j|^2)}\ ,\ m \in \mathbb{N} \tag{5.74}$$

are satisfied.

Proof. Assume that γ is a pure Π-sequence of first rank. Using Theorem 5.19 we see that in the case (2b) for $n_0 = 1$ the function $w(\gamma)$ has the form

$$w(\gamma) = -\frac{1}{w_1(\gamma)}\gamma_1 D_{\gamma_1}^{-1} w_2(\gamma).$$

Hereby, we have $w_1(\gamma) = \Pi_2\Pi_1 a_1$ and $w_2(\gamma) = w_{n_0+1}(\gamma) = 1$. Thus, $w(\gamma) = -\frac{\gamma_1}{a_1\Pi_1^2}$. From this and (5.65) we see that the elements of the sequence γ are related by the identities $\gamma_m = -\frac{\gamma_{m+1}}{a_1\Pi_{m+1}^2}$, $m \geq 1$. This means

$$\gamma_{m+1} = -a_1\Pi_{m+1}^2\gamma_m = -a_1\Pi_1^2\frac{\gamma_m}{\prod\limits_{j=1}^{m}(1-|\gamma_j|^2)}.$$

Setting $\lambda := -a_1\Pi_1^2$ this gives us (5.74). Hereby, because of $a_1 \neq 0$ we have $\lambda \neq 0$. From (5.74) it follows $\gamma_1 \neq 0$ since otherwise we would have that γ has rank 0. Thus, $|\gamma_j| > 0$ for $j \in \mathbb{N}$.

From (5.74) we get

$$\frac{|\gamma_{m+1}|}{|\gamma_m|} = \frac{|\lambda|}{\prod\limits_{j=1}^{m}(1-|\gamma_j|^2)}\ ,\ m \in \mathbb{N}.$$

Thus, $\lim\limits_{n\to\infty}\frac{|\gamma_{m+1}|}{|\gamma_m|} = \frac{|\lambda|}{\Pi_1^2}$. In view of $\gamma \in \Gamma l_2$, this implies

$$|\lambda| \leq \Pi_1^2 < 1. \tag{5.75}$$

The identities (5.74) can be rewritten in the form

$$\Pi_1 D_{\gamma_1}D_{\gamma_2}\cdot\ldots\cdot D_{\gamma_m}\gamma_{m+1} = \lambda\gamma_m\Pi_{m+1}\ ,\ m \in \mathbb{N}. \tag{5.76}$$

Taking into account the equations

$$\sum_{m=1}^{\infty} D_{\gamma_1}^2 D_{\gamma_2}^2\cdot\ldots\cdot D_{\gamma_m}^2|\gamma_{m+1}|^2 = 1-|\gamma_1|^2-\Pi_1^2$$

and $\sum\limits_{m=1}^{\infty}|\gamma_m|^2\Pi_{m+1}^2 = 1-\Pi_1^2$, from (5.76) we get $\Pi_1^2(1-|\gamma_1|^2-\Pi_1^2) = |\lambda|^2(1-\Pi_1^2)$. Thus, Π_1^2 is a root of the equation

$$x^2 - x(1-|\gamma_1|^2+|\lambda|^2)+|\lambda|^2 = 0. \tag{5.77}$$

Hence, this equation has a root in the interval $(0,1)$. Consequently, taking into account (5.75) we obtain (5.73).

Conversely, assume that $0 < |\gamma_1| < 1$ and that the conditions (5.73) and (5.74) are satisfied. Then

$$|\gamma_2| = \frac{|\lambda|}{1-|\gamma_1|}\frac{|\gamma_1|}{1+|\gamma_1|} \leq \frac{|\gamma_1|}{1+|\gamma_1|}. \tag{5.78}$$

The identities (5.74) can be rewritten for $m \in \{2,3,4,\ldots\}$ in the form

$$\gamma_{m+1} = \lambda_1\frac{\gamma_m}{\prod\limits_{j=2}^{m}(1-|\gamma_j|^2)} \tag{5.79}$$

where $\lambda_1 = \frac{\lambda}{1-|\gamma_1|^2}$. From (5.78) we see $0 < |\gamma_2| < 1$. Hereby, it can be immediately checked that

$$0 < |\lambda_1| \leq 1 - |\gamma_2|. \tag{5.80}$$

Thus, after replacing λ by λ_1 and γ_j by $\gamma_{j+1}, j \in \{1, 2, 3, \ldots\}$ the conditions (5.73) and (5.74) are still in force and go over in the conditions (5.80) and (5.79). In particular, this implies

$$|\gamma_3| = \frac{|\lambda_1|}{1-|\gamma_2|} \frac{|\gamma_2|}{1+|\gamma_2|} \leq \frac{|\gamma_2|}{1+|\gamma_2|}.$$

Applying now the principle of mathematical induction we obtain

$$|\gamma_{m+1}| \leq \frac{|\gamma_m|}{1+|\gamma_m|}\ ,\ m \in \mathbb{N}.$$

This implies that the inequalities

$$|\gamma_{m+1}| \leq \frac{\frac{|\gamma_{m-1}|}{1+|\gamma_{m-1}|}}{1+\frac{|\gamma_{m-1}|}{1+|\gamma_{m-1}|}} = \frac{|\gamma_{m-1}|}{1+2|\gamma_{m-1}|} \leq \frac{|\gamma_{m-2}|}{1+3|\gamma_{m-2}|} \leq \ldots \leq \frac{|\gamma_1|}{1+m|\gamma_1|}\ ,\ m \in \mathbb{N}\,,$$

hold true. Hence, $\gamma \in \Gamma l_2$. Hereby, we have $\sigma_1(\gamma) > 0$.

Using (5.74) we find

$$\begin{aligned} L_1(\gamma_1, \gamma_2, \ldots) &= -\sum_{m=1}^{\infty} \gamma_m \overline{\gamma}_{m+1} = -\overline{\lambda} \sum_{m=1}^{\infty} \gamma_m \frac{\overline{\gamma}_m}{\prod\limits_{j=1}^{m}(1-|\gamma_j|^2)} \\ &= -\frac{\overline{\lambda}}{\Pi_1^2} \sum_{m=1}^{\infty} |\gamma_m|^2 \Pi_{m+1}^2 = -\frac{\overline{\lambda}}{\Pi_1^2}(1-\Pi_1^2). \end{aligned} \tag{5.81}$$

On the other hand, rewriting (5.74) in the form

$$\gamma_m = \frac{\Pi_1^2}{\lambda} \frac{\gamma_{m+1}}{\prod\limits_{j=m+1}^{\infty}(1-|\gamma_j|^2)}\ ,\ m \in \mathbb{N}$$

we obtain

$$L_1(\gamma_1, \gamma_2, \ldots) = -\sum_{m=1}^{\infty} \gamma_m \overline{\gamma}_{m+1} = -\frac{\Pi_1^2}{\lambda} \sum_{m=1}^{\infty} \frac{\gamma_{m+1}}{\prod\limits_{j=m+1}^{\infty}(1-|\gamma_j|^2)} \overline{\gamma}_{m+1}$$

$$= -\frac{\Pi_1^2}{\lambda \Pi_2^2}(|\gamma_2|^2 + |\gamma_3|^2(1-|\gamma_2|^2) + \ldots + |\gamma_m|^2 \prod_{j=1}^{m-1}(1-|\gamma_j|^2) + \ldots)$$

$$= -\frac{\Pi_1^2}{\lambda \Pi_2^2}(1-\Pi_2^2).$$

Combining this with (5.81) we get $|L_1(W\gamma)|^2 = \frac{(1-\Pi_1^2)(1-\Pi_2^2)}{\Pi_2^2}$. Thus,

$$\begin{aligned} \sigma_2(\gamma) &= \left| \begin{array}{cc} 1-\Pi_1^2 & -\Pi_1\Pi_2\overline{L_1(W\gamma)} \\ -\Pi_1\Pi_2 L_1(W\gamma) & 1-\Pi_2^2(1+|L_1(W\gamma)|^2) \end{array} \right| \\ &= (1-\Pi_1^2)(1-\Pi_2^2) - \Pi_2^2|L_1(W\gamma)|^2 = 0. \end{aligned}$$

Hence, the sequence γ has rank 1. Since the sequence γ is not an extension of a sequence of rank 0, in view of (5.71), it is pure. □

Corollary 5.23. *Let* $\gamma = (\gamma_j)_{j=0}^{\infty} \in \Gamma l_2$. *Then it is*

$$|\sum_{j=1}^{\infty} \gamma_j \overline{\gamma}_{j+1}| \le \frac{(1-\prod_{j=1}^{\infty}(1-|\gamma_j|^2))(1-\prod_{j=2}^{\infty}(1-|\gamma_j|^2))}{\prod_{j=2}^{\infty}(1-|\gamma_j|^2)}. \tag{5.82}$$

Equality holds true if and only if there exists a complex number λ *such that* $0 \le |\lambda| \le 1-|\gamma_1|$ *and the conditions* (5.74) *are satisfied. In this case we have:*

(1) *If* $\gamma_1 = 0$ *then the sequence* γ *is a pure* Π*-sequence of rank* 0.
(2) *If* $\gamma_1 \ne 0$ *and* $\lambda = 0$ *then the sequence* γ *is a nonregular one-step extension of a pure* Π*-sequence of rank* 0.
(3) *If* $\gamma_1 \ne 0$ *and* $\lambda \ne 0$ *then the sequence* γ *is a pure* Π*-sequence of rank* 1.

Proof. The inequality (5.82) is equivalent to the condition $\sigma_2(\gamma) \ge 0$. For this reason equality holds if and only if $\sigma_2(\gamma) = 0$. However, this occurs if and only if we have one of the three cases mentioned in Corollary 5.23. □

As examples we consider the functions $\frac{1+\zeta}{2}$ and $\frac{1}{2-\zeta}$ which belong to $\mathcal{S}\Pi \setminus J$. As it was shown by I. Schur [31, part II], their Schur parameter sequences are $(\frac{1}{2}, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots)$ and $(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots)$, respectively. We note that both sequences fulfill the conditions of Theorem 5.22 with values $\lambda = \frac{1}{3}$ and $\frac{2}{3}$, respectively. Thus, both sequences are pure Π-sequences of rank 1. Furthermore, it can be easily checked that the functions

$$\theta(\zeta) = e^{i\sigma}\frac{w(1+\alpha)+e^{i\beta}\zeta(1-\alpha w)}{(1+\alpha)-e^{i\beta}\zeta(\alpha-\overline{w})}\ ,\ \sigma,\beta \in \mathbb{R}\ ,\ w \in \mathbb{D}\ ,\ \alpha > 0$$

belong to $\mathcal{S}\Pi \setminus J$ and that their Schur parameter sequence $(\gamma_k)_{k=0}^{\infty}$ is given by

$$\gamma_0 = e^{i\sigma}w\ ,\ \gamma_n = \frac{e^{i(\sigma+n\beta)}}{\alpha+n}\ ,\ n \in \mathbb{N}. \tag{5.83}$$

Using the identity $1-|\gamma_n|^2 = \frac{(\alpha+n-1)(\alpha+n+1)}{(\alpha+n)^2}$ it can be checked by straightforward computations that the sequence (5.83) also satisfies the conditions of Theorem 5.22 with $\lambda = e^{i\beta}\frac{\alpha}{\alpha+1}$. Hence, the sequence (5.83) is a pure Π-sequence of rank 1, too.

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The Matricial Carathéodory Problem in Both Nondegenerate and Degenerate Cases

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Abstract. The main goal of this paper is to present a new approach to both the nondegenerate and degenerate case of the matricial Carathéodory problem. This approach is based on the analysis of central matrix-valued Carathéodory functions which was started in [FK1] and then continued in [FK3]. In the nondegenerate situation we will see that the parametrization of the solution set obtained here coincides with the well-known formula of D.Z. Arov and M.G. Kreĭn for that case (see [AK]).

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0. Introduction

Interpolation problems have a rich history. Important results for the scalar case were already obtained in the first half of the 20th century. In the early 1950's a new period started, where interpolation problems for matrix-valued functions were considered. These investigations culminated in a series of monographs (see, e.g., [BGR], [DFK], [Dy], [FF], [FFGK], and [Sa]). An essential common feature of these monographs is that the considerations mainly concentrated on the so-called nondegenerate case which is connected with positive Hermitian block Pick matrices built from the interpolation data.

The study of the degenerate case (where the associated block Pick matrix is nonnegative Hermitian and singular) began with the pioneering work [Du] of V.K. Dubovoj in the framework of the matricial Schur problem. In the sequel, quite different approaches to handle degenerate cases of matrix interpolation were

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used (see, e.g., [BH], [BD], [Br], [CH1], [CH2], [DGK3], [Dy, Chapter 7], and [Sa, Chapter 5]).

The principal object of this paper is to present an approach to the matricial Carathéodory problem in both nondegenerate and degenerate cases. Our method is essentially based on the first and second authors former investigations [FK1] and [FK3] on the central matrix-valued Carathéodory function associated with a finite Carathéodory sequence of matrices. In particular, we will make frequently use of the matrix ball description of the elements of matricial Carathéodory sequences. The main results of this paper (see Theorems 1.1, 3.2, 3.7, and 4.1) contain descriptions of the solution set of a matricial Carathéodory problem in terms of a linear fractional transformation, the generating matrix-valued function of which is a matrix polynomial. The canonical blocks of this matrix polynomial will be constructed with the aid of those quadruple of matrix polynomials which were used in [FK3] to derive right and left quotient representations of central matrix-valued Carathéodory functions (see Theorem 1.3).

A different approach to the degenerate matricial Carathéodory problem was used in the paper [CH2] of Chen and Hu. Their method is based on an adaptation of the Schur-Potapov algorithm to the degenerate case along the line proposed in [DGK3, Section 3]. The descriptions of the solution set which were given in [CH2, Theorems 3.5 and 4.1] are quite different from our parametrizations given in Theorems 1.1, 3.2, 3.7, and 4.1. In fact, the parameters of the linear fractional transformations presented here are expressed more explicitly by the given data of the interpolation problem.

In the nondegenerate case, our approach provides quickly those parametrizations of the solution set of a matricial Carathéodory problem (see Theorem 5.6) which was stated (without proof) by D.Z. Arov and M.G. Kreĭn in [AK] for that case. The right and left Arov-Kreĭn resolvent matrices possess contractivity properties with respect to the signature matrices

$$j_{qq} := \begin{pmatrix} I_q & 0 \\ 0 & -I_q \end{pmatrix} \quad \text{and} \quad J_q := \begin{pmatrix} 0 & -I_q \\ -I_q & 0 \end{pmatrix}.$$

A main theme of Section 4 is to show that appropriate degenerate analogues of the Arov-Kreĭn resolvent matrices satisfy natural generalizations of the above-mentioned contractivity properties with respect to j_{qq} and J_q. Moreover, we will see that the recurrent formulas for the Arov-Kreĭn resolvent matrices (see [FK3, Section 5]) admit generalizations to the degenerate case as well.

Finally, we study in Section 6 the special case that the matricial Carathéodory problem has a unique solution. In particular, we shall obtain some characterizations of that case in terms of the central matrix-valued Carathéodory function corresponding to the given data by the problem. Roughly speaking, the central matrix-valued Carathéodory function has a simple structure in this situation. It is a finite sum of rational Carathéodory functions having exactly one pole (which is located at the unit circle). This result can be regarded as a matricial extension of a well-known fact for the scalar case.

1. Preliminaries

Throughout this paper, let p and q be positive integers. We will use $\mathbb{C}$, $\mathbb{N}_0$, and $\mathbb{N}$ to denote the set of all complex numbers, the set of all nonnegative integers, and the set of all positive integers, respectively. If $m \in \mathbb{N}_0$ and if $n \in \mathbb{N}_0$ or $n = \infty$, then we will write $\mathbb{N}_{m,n}$ for the set of all integers k satisfying $m \leq k \leq n$. The set of all complex $p \times q$ matrices will be designated by $\mathbb{C}^{p\times q}$. For each $A \in \mathbb{C}^{p\times q}$, let A^+ be the Moore-Penrose inverse of A, let $\mathcal{R}(A)$ be the range of A, and let $\|A\|$ be designate the operator norm of A. If $A \in \mathbb{C}^{q\times q}$, then $\det A$ stands for the determinant of A and $\operatorname{tr} A$ denotes the trace of A. Further, for each $A \in \mathbb{C}^{q\times q}$, let $\operatorname{Re} A$ be the real part of A, i.e., let $\operatorname{Re} A := \frac{1}{2}(A + A^*)$. The null matrix which belongs to $\mathbb{C}^{p\times q}$ will be denoted by $0_{p\times q}$. If the size of a null matrix is obvious, we will omit the index. For each $A \in \mathbb{C}^{p\times p}$ and each $B \in \mathbb{C}^{q\times q}$, let

$$\operatorname{diag}\bigl(A, B\bigr) := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If $n \in \mathbb{N}_0$ and if $(\Gamma_j)_{j=0}^n$ is a sequence of complex $q \times q$ matrices, then we associate with $(\Gamma_j)_{j=0}^n$ the block Toeplitz matrices S_n and T_n given by

$$S_n := \begin{pmatrix} \Gamma_0 & 0 & 0 & \dots & 0 \\ \Gamma_1 & \Gamma_0 & 0 & \dots & 0 \\ \Gamma_2 & \Gamma_1 & \Gamma_0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \Gamma_n & \Gamma_{n-1} & \Gamma_{n-2} & \dots & \Gamma_0 \end{pmatrix} \tag{1.1}$$

and

$$T_n := \operatorname{Re}\, S_n. \tag{1.2}$$

If $n \in \mathbb{N}_0$, then a sequence $(\Gamma_j)_{j=0}^n$ of complex $q \times q$ matrices is called $q \times q$ *Carathéodory sequence* if the matrix T_n is nonnegative Hermitian. Obviously, if $n \in \mathbb{N}_0$ and if $(\Gamma_j)_{j=0}^n$ is a $q \times q$ Carathéodory sequence, then for each $m \in \mathbb{N}_{0,n}$ the sequence $(\Gamma_j)_{j=0}^m$ is also a $q \times q$ Carathéodory sequence. A sequence $(\Gamma_k)_{k=0}^\infty$ from $\mathbb{C}^{q\times q}$ is said to be a $q \times q$ *Carathéodory sequence* if for every nonnegative integer n the sequence $(\Gamma_j)_{j=0}^n$ is a $q \times q$ Carathéodory sequence.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit disk and the unit circle of the complex plane, respectively. A $q \times q$ matrix-valued function $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ which is holomorphic in $\mathbb{D}$ and which has nonnegative Hermitian real part $\operatorname{Re}\, \Omega(z)$ for each $z \in \mathbb{D}$ is called $q \times q$ *Carathéodory function* (*in* $\mathbb{D}$). The set of all $q \times q$ Carathéodory functions (in $\mathbb{D}$) will be denoted by $\mathcal{C}_q(\mathbb{D})$.

The well-studied matricial version of the classical Carathéodory interpolation problem consists of the following:

Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a sequence of complex $q \times q$ matrices. Describe the set $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ of all $q \times q$ Carathéodory functions Ω (in $\mathbb{D}$) such that

$$\frac{1}{j!}\Omega^{(j)}(0) = \Gamma_j \tag{1.3}$$

holds for each $j \in \mathbb{N}_{0,n}$ where $\Omega^{(j)}(0)$ is the jth derivative of Ω at the point $z = 0$.

If $n \in \mathbb{N}_0$ and if $(\Gamma_j)_{j=0}^n$ is a sequence of complex $q \times q$ matrices, then the set $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ is nonempty if and only if $(\Gamma_j)_{j=0}^n$ is a $q \times q$ Carathéodory sequence (see, e.g., [Ko] or [FK1, Part I, Section 4]). In the case of a given nondegenerate $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$, i.e., that the block Toeplitz matrix T_n defined by (1.1) and (1.2) is positive Hermitian, there are various parametrizations of $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ via linear fractional transformations (see, e.g., [AK], [BGR], [FF], [Ko], or [FK1, Part V]). The main results of this paper present such parametrizations in the general case of an arbitrarily given $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$. To formulate a particular version, we introduce now some further terms.

If $m \in \mathbb{N}_0$, let $e_{m,q}$ and $\varepsilon_{m,q}$ be the matrix polynomials defined by

$$e_{m,q}(z) := \left(I_q, zI_q, z^2 I_q, \ldots, z^m I_q\right) \tag{1.4}$$

and

$$\varepsilon_{m,q}(z) := \begin{pmatrix} z^m I_q \\ z^{m-1} I_q \\ \vdots \\ zI_q \\ I_q \end{pmatrix} \tag{1.5}$$

for all $z \in \mathbb{C}$. Let e be a $q \times q$ matrix polynomial, i.e., there are a nonnegative integer m and a complex $mq \times q$ matrix E such that $e(z) = e_{m,q}(z)E$ for each $z \in \mathbb{C}$. Then the *reciprocal matrix polynomial* $\tilde{e}^{[m]}$ *of* e with respect to the unit circle $\mathbb{T}$ and the formal degree m is given, for all $z \in \mathbb{C}$, by

$$\tilde{e}^{[m]}(z) := E^* \varepsilon_{m,q}(z).$$

If $n \in \mathbb{N}_0$ and if $(\Gamma_j)_{j=0}^n$ is a sequence of complex $q \times q$ matrices, then let

$$L_1 := \operatorname{Re} \Gamma_0, \quad R_1 := \operatorname{Re} \Gamma_0, \tag{1.6}$$

and in the case $n \geq 1$ moreover

$$Z_n := \frac{1}{2}\left(\Gamma_n, \Gamma_{n-1}, \ldots, \Gamma_1\right), \quad Y_n := \frac{1}{2} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \\ \vdots \\ \Gamma_n \end{pmatrix}, \tag{1.7}$$

and

$$L_{n+1} := \operatorname{Re} \Gamma_0 - Z_n T_{n-1}^+ Z_n^*, \quad R_{n+1} := \operatorname{Re} \Gamma_0 - Y_n^* T_{n-1}^+ Y_n. \tag{1.8}$$

Observe that the matrices L_{n+1} and R_{n+1} are both nonnegative Hermitian if $(\Gamma_j)_{j=0}^n$ is a $q \times q$ Carathéodory sequence (see, e.g., [DFK, Lemma 1.1.9]).

Recall that a matrix-valued function $S : \mathbb{D} \to \mathbb{C}^{q \times q}$ which is holomorphic in $\mathbb{D}$ is called $q \times q$ *Schur function* (*in* $\mathbb{D}$) if, for each $z \in \mathbb{D}$, the value $S(z)$ of S at the point z is a contractive matrix, i.e., the matrix $I - \left(S(z)\right)^* S(z)$ is nonnegative Hermitian. The set of all $q \times q$ Schur functions (in $\mathbb{D}$) will be denoted by $\mathcal{S}_{q \times q}(\mathbb{D})$.

A main goal of this paper is to prove the following description of the solution set $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ of the matricial version of the classical Carathéodory problem.

Theorem 1.1. *Let n be a nonnegative integer and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Let the matrix polynomials a_n, b_n, c_n, and d_n be given by*

$$a_n(z) := \begin{cases} \Gamma_0 & \text{if } n = 0 \\ \Gamma_0 + z e_{n-1,q}(z) S_{n-1}^* T_{n-1}^+ Y_n & \text{if } n \geq 1\,, \end{cases} \tag{1.9}$$

$$b_n(z) := \begin{cases} I_q & \text{if } n = 0 \\ I_q - z e_{n-1,q}(z) T_{n-1}^+ Y_n & \text{if } n \geq 1\,, \end{cases} \tag{1.10}$$

$$c_n(z) := \begin{cases} \Gamma_0 & \text{if } n = 0 \\ Z_n T_{n-1}^+ S_{n-1}^* z \varepsilon_{n-1,q}(z) + \Gamma_0 & \text{if } n \geq 1\,, \end{cases} \tag{1.11}$$

and

$$d_n(z) := \begin{cases} I_q & \text{if } n = 0 \\ -Z_n T_{n-1}^+ z \varepsilon_{n-1,q}(z) + I_q & \text{if } n \geq 1 \end{cases} \tag{1.12}$$

for each $z \in \mathbb{C}$. Then the following statements hold:

(a) *For each $S \in \mathcal{S}_{q\times q}(\mathbb{D})$ and each $z \in \mathbb{D}$,*

$$\det\left(z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{\,+} S(z)\sqrt{R_{n+1}} + b_n(z)\right) \neq 0$$

and

$$\det\left(z\sqrt{L_{n+1}}S(z)\sqrt{R_{n+1}}^{\,+}\tilde{b}_n^{[n]}(z) + d_n(z)\right) \neq 0.$$

Moreover, for each $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, the matrix-valued function $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ given by

$$\Omega(z) := \left(-z\tilde{c}_n^{[n]}(z)F(z) + a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1}$$

belongs to $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$, where $F := \sqrt{L_{n+1}}^{\,+} S\sqrt{R_{n+1}}$, and admits the representation

$$\Omega(z) = \left(zG(z)\tilde{b}_n^{[n]}(z) + d_n(z)\right)^{-1}\left(-zG(z)\tilde{a}_n^{[n]}(z) + c_n(z)\right)$$

for each $z \in \mathbb{D}$, where $G := \sqrt{L_{n+1}}S\sqrt{R_{n+1}}^{\,+}$.

(b) *For each $\Omega \in \mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$, there is an $S \in \mathcal{S}_{q\times q}(\mathbb{D})$ such that the identity*

$$\Omega(z) = \left(-z\tilde{c}_n^{[n]}(z)F(z) + a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1}$$

is fulfilled for each $z \in \mathbb{D}$, where $F := \sqrt{L_{n+1}}^{\,+} S\sqrt{R_{n+1}}$.

In fact, we prove some results which include Theorem 1.1 as a special case (see Theorems 3.2 and 3.7 for the exact formulation). A key role in the proof of these results plays a comparison of possible candidates for solutions with a distinguished solution, namely with the so-called central $q \times q$ Carathéodory function corresponding to the given $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$. For this reason, it seems to be useful to give some preliminaries. Let us consider an arbitrary non-negative integer n and an arbitrary $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$. Using (1.2) and (1.7), let furthermore

$$M_{n+1} := \begin{cases} 0_{q\times q} & \text{if } n = 0 \\ Z_n T_{n-1}^+ Y_n & \text{if } n \geq 1 \,. \end{cases} \tag{1.13}$$

In view of (1.13), then [FK1, Part I, Theorem 1] leads to the notion of central $q \times q$ Carathéodory functions as follows. If we put $\Gamma_{n+1} := 2M_{n+1}$, then [FK1, Part I, Theorem 1] implies particularly that $(\Gamma_j)_{j=0}^{n+1}$ is a $q \times q$ Carathéodory sequence. Consequently, we can continue this procedure, i.e., similar as in (1.13) let $M_{n+2} := Z_{n+1} T_n^+ Y_{n+1}$, we put $\Gamma_{n+2} := 2M_{n+2}$, and [FK1, Part I, Theorem 1] provides that $(\Gamma_j)_{j=0}^{n+2}$ is a $q \times q$ Carathéodory sequence, and so on. Therefore, if $(\Gamma_j)_{j=0}^n$ is a given $q \times q$ Carathéodory sequence, then the choice

$$\Gamma_{n+1+k} := 2\, M_{n+1+k}, \quad k \in \mathbb{N}_0, \tag{1.14}$$

yields a particular $q\times q$ Carathéodory sequence $(\Gamma_k)_{k=0}^\infty$ and hence (see, e.g., [BGR] or [Ko]) a particular function which belongs to $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$, the so-called *central $q \times q$ Carathéodory function* $\Omega_{c,n}$ corresponding to $(\Gamma_j)_{j=0}^n$. If $(\Gamma_j)_{j=0}^n$ is a $q \times q$ Carathéodory sequence, then we call the sequence $(\Gamma_k)_{k=0}^\infty$ given by (1.14) also the *central $q \times q$ Carathéodory sequence corresponding to* $(\Gamma_j)_{j=0}^n$. Clearly, the central $q \times q$ Carathéodory function $\Omega_{c,n}$ admits the Taylor series representation

$$\Omega_{c,n}(z) = \sum_{k=0}^\infty \Gamma_k z^k$$

for each $z \in \mathbb{D}$, where $(\Gamma_k)_{k=0}^\infty$ is the central $q \times q$ Carathéodory sequence corresponding to $(\Gamma_j)_{j=0}^n$.

If $(\Gamma_j)_{j=0}^0$ is a $q \times q$ Carathéodory sequence, then the constant function (defined on $\mathbb{D}$) with value Γ_0 is the central $q\times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^0$ (see [FK3, Remark 1.1]). In the case that a positive integer n and a $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$ are given the central $q \times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^n$ is a rational matrix-valued function which can be explicitly constructed (see Theorem 1.3 below).

Remark 1.2. Let n be a positive integer and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Then the matrix $T_{n-1}^+ Y_n$ belongs to the set

$$\mathcal{Y}_n := \{V \in \mathbb{C}^{nq\times q} : T_{n-1} V = Y_n\}$$

and the matrix $Z_n T_{n-1}^+$ belongs to the set

$$\mathcal{Z}_n := \{W \in \mathbb{C}^{q\times nq} : W T_{n-1} = Z_n\}$$

(cf. [FK3, Remark 1.4]). Moreover, [FK3, Proposition 2.2] implies that $T_{n-1}^+ Y_n$ actually belongs to the set $\widetilde{\mathcal{Y}}_n$ of all $V_n \in \mathcal{Y}_n$ such that $\det b_n$ vanishes nowhere in $\mathbb{D}$, where b_n is the matrix polynomial defined by $b_n(z) := I_q - z e_{n-1,q}(z) V_n$. Furthermore, from [FK3, Theorem 2.3] one can see that $Z_n T_{n-1}^+$ actually belongs to the set $\widetilde{\mathcal{Z}}_n$ of all $W_n \in \mathcal{Z}_n$ such that $\det d_n$ vanishes nowhere in $\mathbb{D}$, where d_n is the matrix polynomial defined by $d_n(z) := -W_n z e_{n-1,q}(z) + I_q$.

Theorem 1.3. *Let n be a nonnegative integer and let $(\Gamma_j)_{j=0}^n$ be a $q\times q$ Carathéodory sequence. If $n \geq 1$, then let the matrix polynomials $e_{n-1,q}$ and $\varepsilon_{n-1,q}$ be defined by (1.4) and (1.5), let $V_n \in \mathcal{Y}_n$, and let $W_n \in \mathcal{Z}_n$. Then:*

(a) *The central $q \times q$ Carathéodory function $\Omega_{c,n}$ corresponding to $(\Gamma_j)_{j=0}^n$ is the restriction of the rational matrix function $a_n b_n^{-1}$ onto $\mathbb{D}$, where a_n and b_n are the $q \times q$ matrix polynomials which are defined, for each $z \in \mathbb{C}$, by*

$$a_n(z) := \begin{cases} \Gamma_0 & \text{if } n = 0 \\ \Gamma_0 + z e_{n-1,q}(z) S_{n-1}^* V_n & \text{if } n \geq 1 \end{cases} \tag{1.15}$$

and

$$b_n(z) := \begin{cases} I_q & \text{if } n = 0 \\ I_q - z e_{n-1,q}(z) V_n & \text{if } n \geq 1\,. \end{cases} \tag{1.16}$$

(b) *The function $\Omega_{c,n}$ is the restriction of the rational matrix function $d_n^{-1} c_n$ onto $\mathbb{D}$, where c_n and d_n are the $q \times q$ matrix polynomials which are given, for each $z \in \mathbb{C}$, by*

$$c_n(z) := \begin{cases} \Gamma_0 & \text{if } n = 0 \\ W_n S_{n-1}^* z \varepsilon_{n-1,q}(z) + \Gamma_0 & \text{if } n \geq 1 \end{cases} \tag{1.17}$$

and

$$d_n(z) := \begin{cases} I_q & \text{if } n = 0 \\ -W_n z \varepsilon_{n-1,q}(z) + I_q & \text{if } n \geq 1\,. \end{cases} \tag{1.18}$$

A proof of Theorem 1.3 is given in [FK3, Theorems 1.7 and 2.3, Remark 1.1].

2. On particular matrix polynomials

In this section we study the matrix polynomials realizing the representations of the central Carathéodory function $\Omega_{c,n}$ according to Theorem 1.3. In fact, we deduce certain formulas for these matrix polynomials which are useful in view of the proof of Theorem 1.1. Before, some further remarks on the matrices L_{n+1}, R_{n+1}, and M_{n+1} are stated which can be computed from a given $q\times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$ via (1.6), (1.8), and (1.13).

Remark 2.1. Let $n \in \mathbb{N}_0$ and $(\Gamma_j)_{j=0}^{n+1}$ be a $q \times q$ Carathéodory sequence. The matrix

$$K_{n+1} := \sqrt{L_{n+1}}^{\,+} \left(\tfrac{1}{2}\Gamma_{n+1} - M_{n+1}\right) \sqrt{R_{n+1}}^{\,+} \tag{2.1}$$

is contractive and the equation $\frac{1}{2}\Gamma_{n+1} - M_{n+1} = \sqrt{L_{n+1}} K_{n+1} \sqrt{R_{n+1}}$ holds (see [FK1, Part I, Theorem 1] and [DFK, Lemma 1.5.1]). Hence the matrices

$$f_{n+1} := L_{n+1}^+ \left(\tfrac{1}{2}\Gamma_{n+1} - M_{n+1}\right) \quad \text{and} \quad g_{n+1} := \left(\tfrac{1}{2}\Gamma_{n+1} - M_{n+1}\right) R_{n+1}^+ \tag{2.2}$$

fulfill the identities

$$L_{n+1} f_{n+1} = \tfrac{1}{2}\Gamma_{n+1} - M_{n+1} \quad \text{and} \quad g_{n+1} R_{n+1} = \tfrac{1}{2}\Gamma_{n+1} - M_{n+1}. \tag{2.3}$$

Let $\mathbb{C}_{\geq}^{q\times q}$ be the set of all nonnegative Hermitian $q\times q$ matrices and let $\mathbb{C}_{>}^{q\times q}$ be the set of all positive Hermitian $q\times q$ matrices. Further, we will write $A\geq B$ or $B\leq A$ to indicate that A and B are (quadratic) Hermitian matrices of the same size such that $A-B$ is a nonnegative Hermitian matrix. If A is a complex $p\times q$ matrix, then we will use $\mathcal{N}(A)$ to denote the null space of A.

Remark 2.2. Let $\tau\in\mathbb{N}$ or $\tau=+\infty$, and let $(\Gamma_j)_{j=0}^{\tau}$ be a $q\times q$ Carathéodory sequence. For each $n\in\mathbb{N}_{1,\tau}$, let the matrix K_n be defined by (2.1). Then

$$0\leq L_{n+1}=\sqrt{L_n}(I-K_nK_n^*)\sqrt{L_n}\leq L_n,$$

$$0\leq R_{n+1}=\sqrt{R_n}(I-K_n^*K_n)\sqrt{R_n}\leq R_n,$$

and, in particular, $\mathcal{N}(L_n)\subseteq\mathcal{N}(L_{n+1})$, $\mathcal{N}(R_n)\subseteq\mathcal{N}(R_{n+1})$,

$$\sqrt{L_n}\sqrt{L_n}^{+}\sqrt{L_{n+1}}=\sqrt{L_{n+1}},\quad\text{and}\quad\sqrt{R_{n+1}}\sqrt{R_n}^{+}\sqrt{R_n}=\sqrt{R_{n+1}}$$

hold for each $n\in\mathbb{N}_{1,\tau}$ (see [DFK, Remark 3.4.3]).

Remark 2.3. Let $n\in\mathbb{N}$ and let $(\Gamma_j)_{j=0}^{n}$ be a $q\times q$ Carathéodory sequence. Further, let $V_n\in\mathcal{Y}_n$ and $W_n\in\mathcal{Z}_n$. In view of the equations

$$S_{n-1}^*V_n=2Y_n-S_{n-1}V_n\quad\text{and}\quad W_nS_{n-1}^*=2Z_n-W_nS_{n-1}$$

it is readily checked that, for each $z\in\mathbb{C}$, the matrix polynomials a_n, b_n, c_n, and d_n given by (1.15), (1.16), (1.17), and (1.18) admit the representations

$$a_n(z)=e_{n,q}(z)S_nV_n^{\square},\quad b_n(z)=e_{n,q}(z)V_n^{\square},$$

$$c_n(z)=W_n^{\square}S_n\varepsilon_{n,q}(z),\quad\text{and}\quad d_n(z)=W_n^{\square}\varepsilon_{n,q}(z),$$

where the matrix polynomials $e_{n,q}$ and $\varepsilon_{n,q}$ are defined by (1.4) and (1.5),

$$V_n^{\square}:=\begin{pmatrix}I_q\\-V_n\end{pmatrix},\quad\text{and}\quad W_n^{\square}:=\begin{pmatrix}-W_n, & I_q\end{pmatrix}.$$

Let F_0 be the constant matrix-valued function with value $0_{q\times q}$. For each $n\in\mathbb{N}$, let $F_n:\mathbb{C}\to\mathbb{C}^{(n+1)q\times(n+1)q}$ be defined by

$$F_n(z):=\begin{pmatrix}0&0&0&\dots&0&0\\ zI_q&0&0&\dots&0&0\\ z^2I_q&zI_q&0&\dots&0&0\\ z^3I_q&z^2I_q&zI_q&\ddots&\vdots&\vdots\\ \vdots&\vdots&\vdots&\ddots&0&0\\ z^nI_q&z^{n-1}I_q&z^{n-2}I_q&\dots&zI_q&0\end{pmatrix}.$$

Proposition 2.4. *Let $n\in\mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^{n}$ be a $q\times q$ Carathéodory sequence. If $n\geq 1$, then let $V_n\in\mathcal{Y}_n$ and $W_n\in\mathcal{Z}_n$. Let the matrix polynomials a_n, b_n, c_n, and d_n be defined by* (1.15), (1.16), (1.17), *and* (1.18). *Let the matrices R_{n+1} and L_{n+1} be given by* (1.6) *and* (1.8). *Then:*

(a) *For every choice of z in $\mathbb{T}$,*

$$\operatorname{Re}\left(\left(a_n(z)\right)^* b_n(z)\right) = R_{n+1} \quad \textit{and} \quad \operatorname{Re}\left(c_n(z)\left(d_n(z)\right)^*\right) = L_{n+1}.$$

(b) *The identities*

$$\tilde{a}_n^{[n]}(z) b_n(z) + \tilde{b}_n^{[n]}(z) a_n(z) = 2\, z^n R_{n+1}$$

and

$$c_n(z) \tilde{d}_n^{[n]}(z) + d_n(z) \tilde{c}_n^{[n]}(z) = 2\, z^n L_{n+1}$$

hold for each $z \in \mathbb{C}$.

Proof. (a) The case $n=0$ is trivial. Suppose $n \geq 1$. Using [DFK, Lemma 4.2.1] we get

$$F_n(w) S_n = S_n F_n(w) \tag{2.4}$$

for each $w \in \mathbb{C}$. Moreover, for each $w \in \mathbb{C} \setminus \{0\}$, it is readily checked that

$$\left(e_{n,q}\left(\tfrac{1}{\overline{w}}\right)\right)^* e_{n,q}(w) = F_n\left(\tfrac{1}{w}\right) + I + \left(F_n(\overline{w})\right)^*$$

holds. Now let $z \in \mathbb{T}$. Then we have

$$\left(e_{n,q}(z)\right)^* e_{n,q}(z) = F_n(\overline{z}) + I + \left(F_n(\overline{z})\right)^*.$$

Taking into account Remark 2.3, (2.4), and (1.2) it follows

$$\begin{aligned}
2 \operatorname{Re}&\left(\left(a_n(z)\right)^* b_n(z)\right) = \left(a_n(z)\right)^* b_n(z) + \left(b_n(z)\right)^* a_n(z) \\
&= \left(V_n^{\square}\right)^* S_n^* \left(e_{n,q}(z)\right)^* e_{n,q}(z) V_n^{\square} + \left(V_n^{\square}\right)^* \left(e_{n,q}(z)\right)^* e_{n,q}(z) S_n V_n^{\square} \\
&= \left(V_n^{\square}\right)^* \left(S_n^*\left(F_n(\overline{z}) + I + \left(F_n(\overline{z})\right)^*\right) + \left(F_n(\overline{z}) + I + \left(F_n(\overline{z})\right)^*\right) S_n\right) V_n^{\square} \\
&= \left(V_n^{\square}\right)^* \left(S_n^* F_n(\overline{z}) + S_n^* + \left(F_n(\overline{z})\right)^* S_n^* + S_n F_n(\overline{z}) + S_n + \left(F_n(\overline{z})\right)^* S_n\right) V_n^{\square} \\
&= 2 \left(V_n^{\square}\right)^* \left(T_n F_n(\overline{z}) + T_n + \left(F_n(\overline{z})\right)^* T_n\right) V_n^{\square}.
\end{aligned} \tag{2.5}$$

The matrix T_n is nonnegative Hermitian and admits the block representation

$$T_n = \begin{pmatrix} \operatorname{Re}\Gamma_0 & Y_n^* \\ Y_n & T_{n-1} \end{pmatrix}.$$

This implies $T_{n-1} \in \mathbb{C}_{\geq}^{nq \times nq}$, $R_{n+1} \in \mathbb{C}_{\geq}^{q \times q}$, and

$$T_{n-1} T_{n-1}^{+} Y_n = Y_n$$

(see [Al], [EP], or [DFK, Lemma 1.1.9 and Theorem 1.1.1]). Thus

$$T_n V_n^{\square} = \begin{pmatrix} \operatorname{Re}\Gamma_0 - Y_n^* V_n \\ Y_n - T_{n-1} V_n \end{pmatrix} = \begin{pmatrix} \operatorname{Re}\Gamma_0 - Y_n^* T_{n-1}^{+} T_{n-1} V_n \\ 0_{nq \times q} \end{pmatrix} = \begin{pmatrix} R_{n+1} \\ 0_{nq \times q} \end{pmatrix}.$$

Consequently, from (2.5) we obtain then

$$\operatorname{Re}\left(\left(a_n(z)\right)^* b_n(z)\right) = \left(R_{n+1},\ 0_{q \times nq}\right) F_n(z) V_n^{\square} + R_{n+1} + \left(V_n^{\square}\right)^* \left(F_n(\overline{z})\right)^* \begin{pmatrix} R_{n+1} \\ 0_{nq \times q} \end{pmatrix}.$$

Because of $\Big(R_{n+1},\ 0_{q\times nq}\Big)F_n(z) = 0_{q\times(n+1)q}$ it follows the first equation in (a). The second one can be proved analogously.

(b) For each $z \in \mathbb{T}$, from [DFK, Lemma 1.2.2] and part (a) we obtain

$$\begin{aligned}\tilde{a}_n^{[n]}(z)b_n(z) + \tilde{b}_n^{[n]}(z)a_n(z) &= z^n\Big(a_n\left(\tfrac{1}{\bar{z}}\right)\Big)^* b_n(z) + z^n\Big(b_n\left(\tfrac{1}{\bar{z}}\right)\Big)^* a_n(z) \\ &= z^n\Big(\big(a_n(z)\big)^* b_n(z) + \big(b_n(z)\big)^* a_n(z)\Big) \\ &= 2\,z^n \mathrm{Re}\,\Big(\big(a_n(z)\big)^* b_n(z)\Big) \;=\; 2\,z^n R_{n+1}.\end{aligned}$$

Since the left-hand side and the right-hand side of this equation form matrix polynomials, one can conclude $\tilde{a}_n^{[n]}(z)b_n(z) + \tilde{b}_n^{[n]}(z)a_n(z) = 2\,z^n R_{n+1}$ for each $z \in \mathbb{C}$. Similarly, the second equality can be derived from part (a). □

For a $q \times q$ matrix polynomial e, we use in the following the notation

$$\mathcal{N}_e := \{w \in \mathbb{C} : \det e(w) = 0\}.$$

Corollary 2.5. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. If $n \geq 1$, then let $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$. Let the matrix polynomials a_n, b_n, c_n, and d_n be defined by* (1.15), (1.16), (1.17), *and* (1.18). *Furthermore, let the matrices R_{n+1} and L_{n+1} be given by* (1.6) *and* (1.8). *Then*

$$\mathrm{Re}\,\Big(b_n(z)\big(a_n(z)\big)^{-1}\Big) = \big(a_n(z)\big)^{-*}R_{n+1}\big(a_n(z)\big)^{-1} \geq 0, \quad z \in \mathbb{T}\setminus\mathcal{N}_{a_n},$$

$$\mathrm{Re}\,\Big(a_n(z)\big(b_n(z)\big)^{-1}\Big) = \big(b_n(z)\big)^{-*}R_{n+1}\big(b_n(z)\big)^{-1} \geq 0, \quad z \in \mathbb{T}\setminus\mathcal{N}_{b_n},$$

$$\mathrm{Re}\,\Big(\big(c_n(z)\big)^{-1}d_n(z)\Big) = \big(c_n(z)\big)^{-1}L_{n+1}\big(c_n(z)\big)^{-*} \geq 0, \quad z \in \mathbb{T}\setminus\mathcal{N}_{c_n},$$

and

$$\mathrm{Re}\,\Big(\big(d_n(z)\big)^{-1}c_n(z)\Big) = \big(d_n(z)\big)^{-1}L_{n+1}\big(d_n(z)\big)^{-*} \geq 0, \quad z \in \mathbb{T}\setminus\mathcal{N}_{d_n}.$$

The sets $\mathcal{N}_{b_n}$ and $\mathcal{N}_{d_n}$ consist of at most $n \cdot q$ elements (*and hence the sets $\mathbb{T}\setminus\mathcal{N}_{b_n}$ and $\mathbb{T}\setminus\mathcal{N}_{d_n}$ are nonempty*).

Proof. From part (a) of Proposition 2.4, it follows

$$\begin{aligned}\mathrm{Re}\,\Big(b_n(z)\big(a_n(z)\big)^{-1}\Big) &= \frac{1}{2}\big(a_n(z)\big)^{-*}\Big(\big(a_n(z)\big)^* b_n(z) + \big(b_n(z)\big)^* a_n(z)\Big)\big(a_n(z)\big)^{-1} \\ &= \big(a_n(z)\big)^{-*}\,\mathrm{Re}\,\Big(\big(a_n(z)\big)^* b_n(z)\Big)\big(a_n(z)\big)^{-1} \\ &= \big(a_n(z)\big)^{-*}R_{n+1}\big(a_n(z)\big)^{-1} \;\geq\; 0\end{aligned}$$

for each $z \in \mathbb{T}\setminus\mathcal{N}_{a_n}$. Analogously, the relations with respect to $z \in \mathbb{T}\setminus\mathcal{N}_{b_n}$, $z \in \mathbb{T}\setminus\mathcal{N}_{c_n}$, and $z \in \mathbb{T}\setminus\mathcal{N}_{d_n}$ are an easy consequence of part (a) of Proposition 2.4. Moreover, since b_n (respectively, d_n) is a $q \times q$ matrix polynomial of degree at most n such that $b_n(0) = I_q$ (respectively, $d_n(0) = I_q$), one can conclude that the set $\mathcal{N}_{b_n}$ (respectively, $\mathcal{N}_{d_n}$) consists of at most $n \cdot q$ elements. In particular, the set $\mathbb{T}\setminus\mathcal{N}_{b_n}$ (respectively, $\mathbb{T}\setminus\mathcal{N}_{d_n}$) is nonempty. □

Note that in view of (1.15) and (1.17) one can immediately see that the sets $\mathbb{T}\setminus\mathcal{N}_{a_n}$ and $\mathbb{T}\setminus\mathcal{N}_{c_n}$ can be empty. Otherwise, for the special situation that a $q\times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$ with nonsingular matrix Γ_0 is given, then the sets $\mathcal{N}_{a_n}$ and $\mathcal{N}_{c_n}$ consist of at most $n\cdot q$ elements (cf. [FK3, Section 3]). Hence the sets $\mathbb{T}\setminus\mathcal{N}_{a_n}$ and $\mathbb{T}\setminus\mathcal{N}_{c_n}$ are nonempty in that case.

Proposition 2.6. *Let $n\in\mathbb{N}_0$, $k\in\mathbb{N}$, and $(\Gamma_j)_{j=0}^{n+k}$ be a $q\times q$ Carathéodory sequence. If $n\geq 1$, then let $V_n\in\mathcal{Y}_n$ and $W_n\in\mathcal{Z}_n$. Further, let a_n, b_n, c_n, and d_n be defined by* (1.15), (1.16), (1.17), *and* (1.18). *For each $j\in\mathbb{N}_{0,k-1}$, let the matrix polynomials a_{n+j+1}, b_{n+j+1}, c_{n+j+1}, and d_{n+j+1} be defined by*

$$a_{n+j+1}(z):=a_{n+j}(z)+z\tilde{c}_{n+j}^{[n+j]}(z)f_{n+j+1}, \tag{2.6}$$

$$b_{n+j+1}(z):=b_{n+j}(z)-z\tilde{d}_{n+j}^{[n+j]}(z)f_{n+j+1}, \tag{2.7}$$

$$c_{n+j+1}(z):=c_{n+j}(z)+g_{n+j+1}z\tilde{a}_{n+j}^{[n+j]}(z), \tag{2.8}$$

and

$$d_{n+j+1}(z):=d_{n+j}(z)-g_{n+j+1}z\tilde{b}_{n+j}^{[n+j]}(z) \tag{2.9}$$

for each $z\in\mathbb{C}$, where f_{n+j+1} and g_{n+j+1} are the matrices given by (2.2). *For each $j\in\mathbb{N}_{0,k-1}$, the following statements hold:*

(a) *The central $q\times q$ Carathéodory function $\Omega_{c,n+j+1}$ corresponding to the $q\times q$ Carathéodory sequence $(\Gamma_\ell)_{\ell=0}^{n+j+1}$ admit the representations*

$$\Omega_{c,n+j+1}=a_{n+j+1}b_{n+j+1}^{-1}\quad\text{and}\quad\Omega_{c,n+j+1}=d_{n+j+1}^{-1}c_{n+j+1}.$$

(b) *If $n=0$ or in the case $n\geq 1$ both $V_n\in\widetilde{\mathcal{Y}}_n$ and $W_n\in\widetilde{\mathcal{Z}}_n$ are chosen, then the functions $\det b_{n+j+1}$ and $\det d_{n+j+1}$ vanish nowhere in $\mathbb{D}$.*

Proof. The assertion follows applying Theorem 1.3, Remark 2.1, and [FK3, Remark 4.2, Proposition 4.4, Remark 4.5, Lemma 4.6]. □

Corollary 2.7. *Let $n\in\mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q\times q$ Carathéodory sequence. Let the matrices L_{n+1},R_{n+1}, and M_{n+1} be defined by* (1.6), (1.8), *and* (1.13), *let K be a contractive $q\times q$ matrix, and let*

$$\Gamma_{n+1}:=2M_{n+1}+\sqrt{2L_{n+1}}(-K)\sqrt{2R_{n+1}}\,.$$

If $n\geq 1$, then let $V_n\in\widetilde{\mathcal{Y}}_n$ and $W_n\in\widetilde{\mathcal{Z}}_n$. Furthermore, let the matrix polynomials a_n, b_n, c_n, and d_n be given by (1.15), (1.16), (1.17), *and* (1.18). *Then $(\Gamma_j)_{j=0}^{n+1}$ is a $q\times q$ Carathéodory sequence. Moreover, for each $z\in\mathbb{D}$, the matrices $z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+b_n(z)$ and $z\sqrt{L_{n+1}}K\sqrt{R_{n+1}}^{+}\tilde{b}_n^{[n]}(z)+d_n(z)$ are nonsingular and the central $q\times q$ Carathéodory function $\Omega_{c,n+1}$ corresponding to the $q\times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^{n+1}$ admits the representations*

$$\Omega_{c,n+1}(z)=\Big(-z\tilde{c}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+a_n(z)\Big)\Big(z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+b_n(z)\Big)^{-1}$$

and

$$\Omega_{c,n+1}(z)=\Big(z\sqrt{L_{n+1}}K\sqrt{R_{n+1}}^{+}\tilde{b}_n^{[n]}(z)+d_n(z)\Big)^{-1}\Big(-z\sqrt{L_{n+1}}K\sqrt{R_{n+1}}^{+}\tilde{a}_n^{[n]}(z)+c_n(z)\Big).$$

Proof. Since $(\Gamma_j)_{j=0}^{n}$ is a $q\times q$ Carathéodory sequence, the matrices L_{n+1} and R_{n+1} are nonnegative Hermitian. According to [FK1, Part I, Theorem 1], by $(\Gamma_j)_{j=0}^{n+1}$ a $q \times q$ Carathéodory sequence is given. Moreover, the matrices f_{n+1} and g_{n+1} defined by (2.2) admit the representations

$$f_{n+1} = -\sqrt{L_{n+1}}^{+} K \sqrt{R_{n+1}} \quad \text{and} \quad g_{n+1} = -\sqrt{L_{n+1}} K \sqrt{R_{n+1}}^{+}.$$

Consequently, if the matrix polynomials a_{n+1}, b_{n+1}, c_{n+1}, and d_{n+1} are given as in (2.6), (2.7), (2.8), and (2.9) (with $j = 0$), then

$$a_{n+1}(z) = a_n(z) - z\tilde{c}_n^{[n]}(z)\sqrt{L_{n+1}}^{+} K \sqrt{R_{n+1}},$$

$$b_{n+1}(z) = b_n(z) + z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+} K \sqrt{R_{n+1}},$$

$$c_{n+1}(z) = c_n(z) - z\sqrt{L_{n+1}} K \sqrt{R_{n+1}}^{+} \tilde{a}_n^{[n]}(z),$$

and

$$d_{n+1}(z) = d_n(z) + z\sqrt{L_{n+1}} K \sqrt{R_{n+1}}^{+} \tilde{b}_n^{[n]}(z)$$

for each $z \in \mathbb{C}$. Application of Proposition 2.6 completes the proof. □

Corollary 2.8. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^{n+1}$ be a $q \times q$ Carathéodory sequence. If $n \geq 1$, then let $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$. Further, let f_{n+1} and g_{n+1} be defined by* (2.2)*. The $q \times q$ matrix polynomials a_n, b_n, c_n, and d_n given by* (1.15), (1.16), (1.17), *and* (1.18) *satisfy the identities*

$$g_{n+1}\big(\tilde{a}_n^{[n]} b_n + \tilde{b}_n^{[n]} a_n\big) = \big(c_n \tilde{d}_n^{[n]} + d_n \tilde{c}_n^{[n]}\big) f_{n+1}$$

and

$$\big(\tilde{a}_n^{[n]} b_n + \tilde{b}_n^{[n]} a_n\big) g_{n+1}^{*} = f_{n+1}^{*}\big(c_n \tilde{d}_n^{[n]} + d_n \tilde{c}_n^{[n]}\big).$$

Proof. In the case $n = 0$ the assertion is obviously satisfied. Now let $n \geq 1$. Further, let the matrix polynomials a_{n+1}, b_{n+1}, c_{n+1}, and d_{n+1} be defined by (2.6), (2.7), (2.8), and (2.9) (with $j = 0$). From Theorem 1.3 and Proposition 2.6 we get

$$d_n a_n = c_n b_n, \quad \tilde{a}_n^{[n]} \tilde{d}_n^{[n]} = \tilde{b}_n^{[n]} \tilde{c}_n^{[n]}, \tag{2.10}$$

and $d_{n+1} a_{n+1} = c_{n+1} b_{n+1}$. Hence, for each $z \in \mathbb{C}$, it follows

$$\begin{aligned}
d_n(z)a_n(z) &- g_{n+1} z \tilde{b}_n^{[n]}(z) a_n(z) + z d_n(z) \tilde{c}_n^{[n]}(z) f_{n+1} - g_{n+1} z^2 \tilde{b}_n^{[n]}(z) \tilde{c}_n^{[n]}(z) f_{n+1} \\
&= \big(d_n(z) - g_{n+1} z \tilde{b}_n^{[n]}(z)\big)\big(a_n(z) + z\tilde{c}_n^{[n]}(z) f_{n+1}\big) = d_{n+1}(z) a_{n+1}(z) \\
&= c_{n+1}(z) b_{n+1}(z) = \big(c_n(z) + g_{n+1} z \tilde{a}_n^{[n]}(z)\big)\big(b_n(z) - z\tilde{d}_n^{[n]}(z) f_{n+1}\big) \\
&= c_n(z) b_n(z) + g_{n+1} z \tilde{a}_n^{[n]}(z) b_n(z) - z c_n(z) \tilde{d}_n^{[n]}(z) f_{n+1} - g_{n+1} z^2 \tilde{a}_n^{[n]} \tilde{d}_n^{[n]} f_{n+1}
\end{aligned}$$

and consequently

$$-g_{n+1} z \tilde{b}_n^{[n]}(z) a_n(z) + z d_n(z) \tilde{c}_n^{[n]}(z) f_{n+1} = g_{n+1} z \tilde{a}_n^{[n]}(z) b_n(z) - z c_n(z) \tilde{d}_n^{[n]}(z) f_{n+1}.$$

Thus the first identity follows. The second identity is an immediate consequence of the first one. □

Corollary 2.9. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^{n+1}$ be a $q \times q$ Carathéodory sequence. If $n \geq 1$, then let $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$. Further, let the matrix polynomials a_n, b_n, c_n, and d_n be given by* (1.15), (1.16), (1.17), *and* (1.18), *let the matrix polynomials a_{n+1}, b_{n+1}, c_{n+1}, and d_{n+1} be defined by* (2.6), (2.7), (2.8), *and* (2.9) (*with $j = 0$*), *and let the matrices f_{n+1} and g_{n+1} be defined as in* (2.2). *For each $z \in \mathbb{C}$, then*

$$\tilde{a}_{n+1}^{[n+1]}(z)b_{n+1}(z)+\tilde{b}_{n+1}^{[n+1]}(z)a_{n+1}(z) = z\big(\tilde{a}_n^{[n]}(z)b_n(z)+\tilde{b}_n^{[n]}(z)a_n(z)\big)(I-g_{n+1}^*f_{n+1}),$$

$$\tilde{a}_{n+1}^{[n+1]}(z)b_{n+1}(z)+\tilde{b}_{n+1}^{[n+1]}(z)a_{n+1}(z) = z(I-f_{n+1}^*g_{n+1})\big(\tilde{a}_n^{[n]}(z)b_n(z)+\tilde{b}_n^{[n]}(z)a_n(z)\big),$$

$$c_{n+1}(z)\tilde{d}_{n+1}^{[n+1]}(z)+d_{n+1}(z)\tilde{c}_{n+1}^{[n+1]}(z) = z\big(c_n(z)\tilde{d}_n^{[n]}(z)+d_n(z)\tilde{c}_n^{[n]}(z)\big)(I-f_{n+1}g_{n+1}^*),$$

and

$$c_{n+1}(z)\tilde{d}_{n+1}^{[n+1]}(z)+d_{n+1}(z)\tilde{c}_{n+1}^{[n+1]}(z) = z(I-g_{n+1}f_{n+1}^*)\big(c_n(z)\tilde{d}_n^{[n]}(z)+d_n(z)\tilde{c}_n^{[n]}(z)\big).$$

Proof. Let $z \in \mathbb{C}$. Using (2.10), which follows from Theorem 1.3, we obtain

$$\begin{aligned}
&\tilde{a}_{n+1}^{[n+1]}(z)b_{n+1}(z) + \tilde{b}_{n+1}^{[n+1]}(z)a_{n+1}(z)\\
&\quad = \big(z\tilde{a}_n^{[n]}(z) + f_{n+1}^*c_n(z)\big)\big(b_n(z) - z\tilde{d}_n^{[n]}(z)f_{n+1}\big)\\
&\qquad +\big(z\tilde{b}_n^{[n]}(z) - f_{n+1}^*d_n(z)\big)\big(a_n(z) + z\tilde{c}_n^{[n]}(z)f_{n+1}\big)\\
&\quad = z\tilde{a}_n^{[n]}(z)b_n(z) - z^2\tilde{a}_n^{[n]}(z)\tilde{d}_n^{[n]}(z)f_{n+1} + f_{n+1}^*c_n(z)b_n(z)\\
&\qquad -zf_{n+1}^*c_n(z)\tilde{d}_n^{[n]}(z)f_{n+1} + z\tilde{b}_n^{[n]}(z)a_n(z) + z^2\tilde{b}_n^{[n]}(z)\tilde{c}_n^{[n]}(z)f_{n+1}\\
&\qquad -f_{n+1}^*d_n(z)a_n(z) - zf_{n+1}^*d_n(z)\tilde{c}_n^{[n]}(z)f_{n+1}\\
&\quad = z\big(\tilde{a}_n^{[n]}(z)b_n(z) + \tilde{b}_n^{[n]}(z)a_n(z)\big) - zf_{n+1}^*\big(c_n(z)\tilde{d}_n^{[n]}(z) + d_n(z)\tilde{c}_n^{[n]}(z)\big)f_{n+1}.
\end{aligned}$$

Hence, in view of Corollary 2.8, we get that the first and the second identities hold. The other identities can be verified analogously. □

Note that Corollary 2.8 and Corollary 2.9 can also be derived from part (b) of Proposition 2.4 in combination with Remark 2.1 and Remark 2.2.

3. Description of the set $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$

The main goal of this section is to prove Theorem 1.1. More precisely, combining Theorem 3.2 and Theorem 3.7 we will even verify a more general result which shows us that Theorem 1.1 corresponds to that particular case which is associated with a canonical choice of the matrix polynomials under consideration.

Lemma 3.1. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. If $n \geq 1$, then let $V_n \in \widetilde{\mathcal{Y}}_n$ and $W_n \in \widetilde{\mathcal{Z}}_n$. Let the matrix polynomials a_n, b_n, c_n, and d_n be given by* (1.15), (1.16), (1.17), *and* (1.18). *If K is a contractive $q \times q$*

matrix and if $z \in \mathbb{D}$, *then the matrices* $z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}} + b_n(z)$ *and* $z\sqrt{L_{n+1}}K\sqrt{R_{n+1}}^{+}\tilde{b}_n^{[n]}(z) + d_n(z)$ *are nonsingular, the equality*

$$\left(-z\tilde{c}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+b_n(z)\right)^{-1}$$
$$=\left(z\sqrt{L_{n+1}}K\sqrt{R_{n+1}}^{+}\tilde{b}_n^{[n]}(z)+d_n(z)\right)^{-1}\left(-z\sqrt{L_{n+1}}K\sqrt{R_{n+1}}^{+}\tilde{a}_n^{[n]}(z)+c_n(z)\right)$$

is satisfied, and

$$\operatorname{Re}\left(\left(-z\tilde{c}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+b_n(z)\right)^{-1}\right)$$

is a nonnegative Hermitian $q \times q$ *matrix, where the matrices* L_{n+1} *and* R_{n+1} *are defined by* (1.6) *and* (1.8). *Moreover, in the case* $n \geq 1$, *if* $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, *and* $\mathbf{d}_n$ *are further matrix polynomials which can be represented, for each* $z \in \mathbb{C}$, *via*

$$\mathbf{a}_n(z) = \Gamma_0 + ze_{n-1,q}(z)S_{n-1}^{*}\mathbf{V}_n, \quad \mathbf{b}_n(z) = I_q - ze_{n-1,q}(z)\mathbf{V}_n \tag{3.1}$$

and

$$\mathbf{c}_n(z) = \mathbf{W}_n S_{n-1}^{*} z\varepsilon_{n-1,q}(z) + \Gamma_0, \quad \mathbf{d}_n(z) = -\mathbf{W}_n z\varepsilon_{n-1,q}(z) + I_q \tag{3.2}$$

with some $\mathbf{V}_n \in \widetilde{\mathcal{Y}}_n$ *and* $\mathbf{W}_n \in \widetilde{\mathcal{Z}}_n$, *then the identity*

$$\left(-z\tilde{c}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+b_n(z)\right)^{-1}$$
$$=\left(-z\tilde{\mathbf{c}}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+\mathbf{a}_n(z)\right)\left(z\tilde{\mathbf{d}}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}K\sqrt{R_{n+1}}+\mathbf{b}_n(z)\right)^{-1}$$

is fulfilled for each $z \in \mathbb{D}$.

Proof. The assertion is an immediate consequence of Corollary 2.7. □

Now we are able to prove a result which includes the statement of part (a) of Theorem 1.1.

Theorem 3.2. *Let* $n \in \mathbb{N}_0$ *and let* $(\Gamma_j)_{j=0}^n$ *be a* $q \times q$ *Carathéodory sequence. If* $n \geq 1$, *then let* $V_n \in \widetilde{\mathcal{Y}}_n$ *and* $W_n \in \widetilde{\mathcal{Z}}_n$. *Let the matrix polynomials* a_n, b_n, c_n, *and* d_n *be given by* (1.15), (1.16), (1.17), *and* (1.18). *Further, let the matrices* L_{n+1} *and* R_{n+1} *be defined by* (1.6) *and* (1.8). *If* $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, *then*

$$\det\left(z\tilde{d}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}S(z)\sqrt{R_{n+1}} + b_n(z)\right) \neq 0 \tag{3.3}$$

and

$$\det\left(z\sqrt{L_{n+1}}S(z)\sqrt{R_{n+1}}^{+}\tilde{b}_n^{[n]}(z) + d_n(z)\right) \neq 0 \tag{3.4}$$

for each $z \in \mathbb{D}$ *and the function* $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ *given by*

$$\Omega(z) := \left(-z\tilde{c}_n^{[n]}(z)F(z) + a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1} \tag{3.5}$$

belongs to $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ *and satisfies, for each* $z \in \mathbb{D}$, *the representation*

$$\Omega(z) = \left(zG(z)\tilde{b}_n^{[n]}(z) + d_n(z)\right)^{-1}\left(-zG(z)\tilde{a}_n^{[n]}(z) + c_n(z)\right), \tag{3.6}$$

where

$$F := \sqrt{L_{n+1}}^{+} S \sqrt{R_{n+1}} \quad \textit{and} \quad G := \sqrt{L_{n+1}} S \sqrt{R_{n+1}}^{+}. \tag{3.7}$$

Moreover, in the case $n \geq 1$*, if* $S \in \mathcal{S}_{q\times q}(\mathbb{D})$ *and if* $\mathbf{a}_n$*,* $\mathbf{b}_n$*,* $\mathbf{c}_n$*, and* $\mathbf{d}_n$ *are further matrix polynomials which can be represented, for each* $z \in \mathbb{C}$*, via* (3.1) *and* (3.2) *with some* $\mathbf{V}_n \in \widetilde{\mathcal{Y}}_n$ *and* $\mathbf{W}_n \in \widetilde{\mathcal{Z}}_n$*, then*

$$\Omega(z) = \left(-z\tilde{\mathbf{c}}_n^{[n]}(z)F(z) + \mathbf{a}_n(z)\right)\left(z\tilde{\mathbf{d}}_n^{[n]}(z)F(z) + \mathbf{b}_n(z)\right)^{-1} \tag{3.8}$$

for each $z \in \mathbb{C}$*, where* F *is defined as in* (3.7)*.*

Proof. Let $S \in \mathcal{S}_{q\times q}(\mathbb{D})$ and let $z_0 \in \mathbb{D}$. Then $K := S(z_0)$ is a contractive $q \times q$ matrix. Consequently, from Lemma 3.1 we get that (3.3), (3.4), (3.6), and (3.8) hold for $z = z_0$ and that

$$\operatorname{Re}\Omega(z_0) \in \mathbb{C}_{\geq}^{q\times q}.$$

Therefore, since z_0 is arbitrarily chosen in $\mathbb{D}$, we get that (3.3), (3.4), (3.6), and (3.8) hold for each $z \in \mathbb{D}$ and that via (3.5) a $q \times q$ Carathéodory function Ω is given. It remains to prove that Ω fulfills the condition (1.3) for each $j \in \mathbb{N}_{0,n}$. In view of $W_n \in \widetilde{\mathcal{Z}}_n$, (3.3), and (3.7), we obtain that $\Upsilon_n : \mathbb{D} \to \mathbb{C}^{q\times q}$ defined by

$$\Upsilon_n(z) := -2\, z^{n+1}\big(d_n(z)\big)^{-1} L_{n+1} F(z) \left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1} \tag{3.9}$$

is a well-defined matrix-valued function which is holomorphic in $\mathbb{D}$. Because of $V_n \in \widetilde{\mathcal{Y}}_n$, $W_n \in \widetilde{\mathcal{Z}}_n$, and Theorem 1.3 the central $q \times q$ Carathéodory function $\Omega_{c,n}$ corresponding to $(\Gamma_j)_{j=0}^n$ admits, for each $z \in \mathbb{D}$, the representations

$$\Omega_{c,n}(z) = a_n(z)\big(b_n(z)\big)^{-1} \quad \text{and} \quad \Omega_{c,n}(z) = \big(d_n(z)\big)^{-1} c_n(z).$$

In particular, $d_n a_n = c_n b_n$. Thus using (3.5) and part (b) of Proposition 2.4 we get, for each $z \in \mathbb{D}$, the identity

$$\begin{aligned}
&\Omega(z) - \Omega_{c,n}(z) \\
&\ = \big(d_n(z)\big)^{-1}\Big(-z d_n(z)\tilde{c}_n^{[n]}(z)F(z) + d_n(z)a_n(z) - z c_n(z)\tilde{d}_n^{[n]}(z)F(z) - c_n(z)b_n(z)\Big) \cdot \\
&\qquad \cdot \left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1} \\
&\ = \big(d_n(z)\big)^{-1}\big(-2\, z^{n+1} L_{n+1} F(z)\big)\left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1} = \Upsilon_n(z).
\end{aligned}$$

Hence, because each entry of the matrix-valued function Υ_n forms a complex-valued function which is holomorphic in $\mathbb{D}$ and has a zero at least of order $n+1$ at the point 0 (see (3.9) and note $d_n(0) = I_q$ and $b_n(0) = I_q$), there is a sequence $(\triangle_j)_{j=n+1}^{\infty}$ of complex $q \times q$ matrices such that

$$\Omega(z) - \Omega_{c,n}(z) = \sum_{j=n+1}^{\infty} \triangle_j z^j$$

for each $z \in \mathbb{D}$. Consequently, since $\Omega_{c,n}$ belongs to $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ and we already know that Ω is a $q \times q$ Carathéodory function, the matrix-valued function Ω belongs to $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ as well. □

It should be mentioned that if $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, then the matrix-valued functions F and G defined by (3.7) do not belong to $\mathcal{S}_{q\times q}(\mathbb{D})$ in general. This will be emphasized by the following simple example.

Example 3.3. By setting

$$\Gamma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $(\Gamma_j)_{j=0}^0$ is a 2×2 Carathéodory sequence for which $L_1 = R_1 = \Gamma_0$ and S is a contractive 2×2 matrix. Moreover, $\sqrt{L_1} S \sqrt{R_1}^{\,+} = \left(\sqrt{L_1}^{\,+} S \sqrt{R_1}\right)^*$ and because of

$$I_2 - \left(\sqrt{L_1}^{\,+} S \sqrt{R_1}\right)^* \sqrt{L_1}^{\,+} S \sqrt{R_1} = \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & -3 \end{pmatrix}$$

the complex 2×2 matrices $\sqrt{L_1} S \sqrt{R_1}^{\,+}$ and $\sqrt{L_1}^{\,+} S \sqrt{R_1}$ are not contractive.

Now we are going to prove an inverse statement to Theorem 3.2, i.e., we will show that any solution $\Omega \in \mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ can be represented via (3.5) and (3.6) with some $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, where F and G are defined as in (3.7).

Remark 3.4. Let A and X be complex $p \times q$ matrices such that the following three conditions are satisfied:

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(X)$.
(ii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(X^*)$.
(iii) $\det(I + X^+A) \neq 0$ or $\det(I + AX^+) \neq 0$.

Then, in view of [DFK, Lemma 1.1.8, Theorem 1.1.1, and Corollary 1.1.2], it is readily checked that $\det(I + X^+A) \neq 0$, $\det(I + AX^+) \neq 0$, and

$$X(I + X^+A)^{-1} = (I + AX^+)^{-1}X.$$

Remark 3.5. Let $E \in \mathbb{C}_{\geq}^{(p+q)\times(p+q)}$ with block partition

$$E = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is a $p \times p$ block. Then one can easily see that $\|B\|^2 \leq \|A\| \cdot \|D\|$ holds.

Lemma 3.6. *Let $\Omega \in \mathcal{C}_q(\mathbb{D})$ and let*

$$\Omega(z) = \sum_{k=0}^{\infty} \Gamma_k z^k, \quad z \in \mathbb{D}, \tag{3.10}$$

be the Taylor series representation of Ω. For each nonnegative integer n, let $\Omega_{c,n}$ be the central $q\times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^n$. For each compact subset $\mathcal{K}$ of $\mathbb{D}$, the sequence $(\Omega_{c,n})_{n=0}^{\infty}$ converges uniformly on $\mathcal{K}$ to Ω.

Proof. For each $n \in \mathbb{N}$, let
$$\Omega_{c,n}(z) = \sum_{k=0}^{\infty} \Gamma_k^{(n)} z^k, \quad z \in \mathbb{D},$$
be the Taylor series representation of $\Omega_{c,n}$. Since Ω belongs to $\mathcal{C}_q(\mathbb{D})$, the sequence $(\Gamma_k)_{k=0}^{\infty}$ is a $q \times q$ Carathéodory sequence, i.e., for each $n \in \mathbb{N}_0$ the matrix T_n given by (1.1) and (1.2) is nonnegative Hermitian. Hence the block matrix
$$\begin{pmatrix} \operatorname{Re}\Gamma_0 & \frac{1}{2}\Gamma_k^* \\ \frac{1}{2}\Gamma_k & \operatorname{Re}\Gamma_0 \end{pmatrix}$$
is nonnegative Hermitian for each $k \in \mathbb{N}$. Consequently, Remark 3.5 yields that
$$\|\Gamma_k\| \leq \|2 \operatorname{Re}\Gamma_0\| = \|\Gamma_0 + \Gamma_0^*\| \leq 2\,\|\Gamma_0\|$$
holds for each $k \in \mathbb{N}$. Analogously, for each $n \in \mathbb{N}$ and each $k \in \mathbb{N}$, we get
$$\|\Gamma_k^{(n)}\| \leq 2\,\|\Gamma_0^{(n)}\| = 2\,\|\Gamma_0\|.$$
Thus, for each $n \in \mathbb{N}_0$ and each $z \in \mathbb{D}$, we obtain
$$\|\Omega_{c,n}(z) - \Omega(z)\| \leq \sum_{k=n+1}^{\infty} \|\Gamma_k^{(n)} - \Gamma_k\| \cdot |z|^k \leq 4\,\|\Gamma_0\| \sum_{k=n+1}^{\infty} |z|^k.$$
The assertion immediately follows. □

Theorem 3.7. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. If $n \geq 1$, then let $V_n \in \widetilde{\mathcal{Y}}_n$ and $W_n \in \widetilde{\mathcal{Z}}_n$. Let the matrix polynomials a_n, b_n, c_n, and d_n be given by* (1.15), (1.16), (1.17), *and* (1.18). *Further, let the matrices L_{n+1} and R_{n+1} be defined by* (1.6) *and* (1.8). *Let $\Omega \in \mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$. Then there is an $S \in \mathcal{S}_{q\times q}(\mathbb{D})$ such that the conditions* (3.3) *and* (3.4) *are fulfilled for each $z \in \mathbb{D}$ and that Ω admits, for each $z \in \mathbb{D}$, the representations*
$$\Omega(z) = \left(-z\tilde{c}_n^{[n]}(z)F(z) + a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1} \tag{3.11}$$
and
$$\Omega(z) = \left(zG(z)\tilde{b}_n^{[n]}(z) + d_n(z)\right)^{-1}\left(-zG(z)\tilde{a}_n^{[n]}(z) + c_n(z)\right),$$
where $F := \sqrt{L_{n+1}}^{\,+} S \sqrt{R_{n+1}}$ and $G := \sqrt{L_{n+1}} S \sqrt{R_{n+1}}^{\,+}$ as in (3.7).

Proof. In view of $\Omega \in \mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$, let $(\Gamma_k)_{k=n+1}^{\infty}$ be the sequence of complex $q\times q$ matrices such that (3.10) is satisfied. Since Ω is a $q\times q$ Carathéodory function, for all $k \in \mathbb{N}_0$, the sequence $(\Gamma_j)_{j=0}^k$ is a $q \times q$ Carathéodory sequence. For each $k \in \mathbb{N}_0$, let $\Omega_{c,k}$ be the central $q \times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^k$. Application of Lemma 3.6 provides, for each $z \in \mathbb{D}$, the relation
$$\lim_{k\to\infty} \Omega_{c,k}(z) = \Omega(z). \tag{3.12}$$
Since $V_n \in \widetilde{\mathcal{Y}}_n$, the function $\det b_n$ vanishes nowhere in $\mathbb{D}$. Theorem 1.3 yields that $\Omega_{c,n}$ admits, for each $z \in \mathbb{D}$, the representation $\Omega_{c,n}(z) = a_n(z)\big(b_n(z)\big)^{-1}$. For each $j \in \mathbb{N}_0$, let the $q \times q$ matrix polynomials a_{n+j+1}, b_{n+j+1}, c_{n+j+1}, and d_{n+j+1} be defined by (2.6), (2.7), (2.8), and (2.9), where f_{n+j+1} and g_{n+j+1} are the matrices given by (2.2). Since the function $\det b_n$ vanishes nowhere in $\mathbb{D}$, Proposition 2.6

implies that, for each $j \in \mathbb{N}_0$, the function $\det b_{n+j+1}$ vanishes nowhere in $\mathbb{D}$ as well. We are going to prove that, for each $k \in \mathbb{N}_0$ and each $j \in \mathbb{N}_0$, there is a $q \times q$ Schur function S_{jk} defined on $\mathbb{D}$ such that

$$\det\left(z\tilde{d}_{n+j}^{[n+j]}(z)\sqrt{L_{n+j+1}}^{+}S_{jk}(z)\sqrt{R_{n+j+1}}+b_{n+j}(z)\right)\neq 0 \tag{3.13}$$

and

$$\Omega_{c,n+j+k}(z)=\left(-z\tilde{c}_{n+j}^{[n+j]}(z)F_{jk}(z)+a_{n+j}(z)\right)\left(z\tilde{d}_{n+j}^{[n+j]}(z)F_{jk}(z)+b_{n+j}(z)\right)^{-1} \tag{3.14}$$

hold for each $z \in \mathbb{D}$, where $F_{jk} := \sqrt{L_{n+j+1}}^{+}S_{jk}\sqrt{R_{n+j+1}}$. If $k=0$, we choose the constant $q \times q$ Schur function S_{j0} (defined on $\mathbb{D}$) with value $0_{q\times q}$ for all $j \in \mathbb{N}_0$. For each $z \in \mathbb{D}$, then (3.13) holds and, moreover, Theorem 1.3 and Proposition 2.6 yield that (3.14) holds as well. Now we consider the case $k = 1$. Let $j \in \mathbb{N}_0$. From Remark 2.1 we see that $K_{n+j+1} := \sqrt{L_{n+j+1}}^{+}\left(\frac{1}{2}\Gamma_{n+j+1} - M_{n+j+1}\right)\sqrt{R_{n+j+1}}^{+}$ is a contractive matrix and that

$$\sqrt{L_{n+j+1}}^{+}K_{n+j+1}\sqrt{R_{n+j+1}} = L_{n+j+1}^{+}\left(\tfrac{1}{2}\Gamma_{n+j+1} - M_{n+j+1}\right) = f_{n+j+1}, \tag{3.15}$$

$$\sqrt{L_{n+j+1}}K_{n+j+1}\sqrt{R_{n+j+1}}^{+} = \left(\tfrac{1}{2}\Gamma_{n+j+1} - M_{n+j+1}\right)R_{n+j+1}^{+} = g_{n+j+1}. \tag{3.16}$$

Thus the constant function S_{j1} (defined on $\mathbb{D}$) with value $-K_{n+j+1}$ is a $q\times q$ Schur function and from (3.15), (2.6), (2.7), and Proposition 2.6 we obtain

$$\begin{aligned}&\det\left(z\tilde{d}_{n+j}^{[n+j]}(z)\sqrt{L_{n+j+1}}^{+}S_{j1}(z)\sqrt{R_{n+j+1}}+b_{n+j}(z)\right)\\&\quad=\det\left(-z\tilde{d}_{n+j}^{[n+j]}(z)f_{n+j+1}+b_{n+j}(z)\right)=\det b_{n+j+1}(z)\neq 0\end{aligned}$$

and, by setting $F_{j1} := \sqrt{L_{n+j+1}}^{+}S_{j1}\sqrt{R_{n+j+1}}$, moreover

$$\begin{aligned}\Omega_{c,n+j+1}(z) &= a_{n+j+1}(z)\big(b_{n+j+1}(z)\big)^{-1}\\&= \left(z\tilde{c}_{n+j}^{[n+j]}(z)f_{n+j+1}+a_{n+j}(z)\right)\left(-z\tilde{d}_{n+j}^{[n+j]}(z)f_{n+j+1}+b_{n+j}(z)\right)^{-1}\\&= \left(-z\tilde{c}_{n+j}^{[n+j]}(z)F_{j1}(z)+a_{n+j}(z)\right)\left(z\tilde{d}_{n+j}^{[n+j]}(z)F_{j1}(z)+b_{n+j}(z)\right)^{-1}\end{aligned}$$

for every choice of z in $\mathbb{D}$. Hence there exists a $\kappa \in \mathbb{N}$ such that, for each $k \in \mathbb{N}_{0,\kappa}$, there is a sequence $(S_{\ell k})_{\ell=0}^{\infty}$ from $\mathcal{S}_{q\times q}(\mathbb{D})$ such that

$$\det\left(z\tilde{d}_{n+\ell}^{[n+\ell]}(z)\sqrt{L_{n+\ell+1}}^{+}S_{\ell k}(z)\sqrt{R_{n+\ell+1}}+b_{n+\ell}(z)\right)\neq 0 \tag{3.17}$$

and

$$\Omega_{c,n+\ell+k}(z)=\left(-z\tilde{c}_{n+\ell}^{[n+\ell]}(z)F_{\ell k}(z)+a_{n+\ell}(z)\right)\left(z\tilde{d}_{n+\ell}^{[n+\ell]}(z)F_{\ell k}(z)+b_{n+\ell}(z)\right)^{-1} \tag{3.18}$$

hold for all $\ell \in \mathbb{N}_0$, and $z \in \mathbb{D}$, where $F_{\ell k} := \sqrt{L_{n+\ell+1}}^{+}S_{\ell k}\sqrt{R_{n+\ell+1}}$. Let $j \in \mathbb{N}_0$. The matrix-valued function

$$\Sigma_{j,\kappa+1} := \sqrt{L_{n+j+2}}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa} \tag{3.19}$$

obviously belongs to $\mathcal{S}_{q\times q}(\mathbb{D})$. The matrix-valued function $\Theta_{j,\kappa+1} : \mathbb{D} \to \mathbb{C}^{q\times q}$, $\Phi_{j,\kappa+1} : \mathbb{D} \to \mathbb{C}^{q\times q}$, and $\Psi_{j,\kappa+1} : \mathbb{D} \to \mathbb{C}^{q\times q}$ defined by

$$\Theta_{j,\kappa+1}(z) := \sqrt{L_{n+j+1}}\sqrt{L_{n+j+2}}^{+}\Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{+}, \tag{3.20}$$

$$\Phi_{j,\kappa+1}(z) := K_{n+j+1} - z\Theta_{j,\kappa+1}(z), \tag{3.21}$$

and

$$\Psi_{j,\kappa+1}(z) := I - zK^*_{n+j+1}\Theta_{j,\kappa+1}(z) \tag{3.22}$$

are holomorphic in $\mathbb{D}$. Because of $\det \Psi_{j,\kappa+1}(0) = \det I_q \neq 0$ there is a discrete subset $\mathcal{A}$ of $\mathbb{D}$ such that

$$\det \Psi_{j,\kappa+1}(z) \neq 0 \tag{3.23}$$

holds for each $z \in \mathbb{D} \setminus \mathcal{A}$. Hence $\check{S}_{j,\kappa+1} := \Phi_{j,\kappa+1}\Psi^{-1}_{j,\kappa+1}$ is a well-defined matrix-valued function which is meromorphic in $\mathbb{D}$. For each $z \in \mathbb{D} \setminus \mathcal{A}$ we have

$$\begin{aligned}
&\big(\Psi_{j,\kappa+1}(z)\big)^*\Psi_{j,\kappa+1}(z) - \big(\Phi_{j,\kappa+1}(z)\big)^*\Phi_{j,\kappa+1}(z)\\
&\quad = I - K^*_{n+j+1}K_{n+j+1} - |z|^2\big(\Theta_{j,\kappa+1}(z)\big)^*(I - K_{n+j+1}K^*_{n+j+1})\Theta_{j,\kappa+1}(z)\\
&\quad = I - K^*_{n+j+1}K_{n+j+1} - |z|^2\sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+2}}\big(\Sigma_{j,\kappa+1}(z)\big)^*\sqrt{L_{n+j+2}}^{+}\sqrt{L_{n+j+1}}\,\cdot\\
&\qquad \cdot\big(I - K_{n+j+1}K^*_{n+j+1}\big)\sqrt{L_{n+j+1}}\sqrt{L_{n+j+2}}^{+}\Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{+}.
\end{aligned} \tag{3.24}$$

From Remark 2.2 we obtain

$$\begin{aligned}
&\sqrt{L_{n+j+2}}^{+}\sqrt{L_{n+j+1}}\left(I - K_{n+j+1}K^*_{n+j+1}\right)\sqrt{L_{n+j+1}}\sqrt{L_{n+j+2}}^{+}\\
&\quad = \sqrt{L_{n+j+2}}^{+}L_{n+j+2}\sqrt{L_{n+j+2}}^{+} = \sqrt{L_{n+j+2}}\sqrt{L_{n+j+2}}^{+}.
\end{aligned}$$

Thus, in view of (3.19), for each $z \in \mathbb{D} \setminus \mathcal{A}$ it follows

$$\begin{aligned}
&\sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+2}}\big(\Sigma_{j,\kappa+1}(z)\big)^*\sqrt{L_{n+j+2}}^{+}\sqrt{L_{n+j+1}}\left(I - K_{n+j+1}K^*_{n+j+1}\right)\cdot\\
&\qquad \cdot\sqrt{L_{n+j+1}}\sqrt{L_{n+j+2}}^{+}\Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{+}\\
&\quad = \sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+2}}\big(\Sigma_{j,\kappa+1}(z)\big)^*\sqrt{L_{n+j+2}}\sqrt{L_{n+j+2}}^{+}\Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{+}\\
&\quad = \sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+2}}\big(\Sigma_{j,\kappa+1}(z)\big)^*\Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{+}.
\end{aligned} \tag{3.25}$$

From Remark 2.1 and Remark 2.2 we can conclude

$$\begin{aligned}
&I - K^*_{n+j+1}K_{n+j+1}\\
&\quad = I - \sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+1}}\\
&\qquad + \sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+1}}(I - K^*_{n+j+1}K_{n+j+1})\sqrt{R_{n+j+1}}\sqrt{R_{n+j+1}}^{+}\\
&\quad = I - \sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+1}} + \sqrt{R_{n+j+1}}^{+}R_{n+j+2}\sqrt{R_{n+j+1}}^{+}.
\end{aligned} \tag{3.26}$$

Then using (3.24), (3.25), and (3.26), for each $z \in \mathbb{D} \setminus \mathcal{A}$, we get

$$\begin{aligned}
&\big(\Psi_{j,\kappa+1}(z)\big)^*\Psi_{j,\kappa+1}(z) - \big(\Phi_{j,\kappa+1}(z)\big)^*\Phi_{j,\kappa+1}(z)\\
&\quad = I - \sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+1}}\\
&\qquad + \sqrt{R_{n+j+1}}^{+}\sqrt{R_{n+j+2}}\Big(I - |z|^2\big(\Sigma_{j,\kappa+1}(z)\big)^*\Sigma_{j,\kappa+1}(z)\Big)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{+}.
\end{aligned}$$

For every choice of z in $\mathbb{D} \setminus \mathcal{A}$, the right-hand side of this equation is nonnegative Hermitian. Consequently, in view of the identity

$$\begin{aligned} & I - \big(\check{S}_{j,\kappa+1}(z)\big)^* \check{S}_{j,\kappa+1}(z) \\ &\quad = \big(\Psi_{j,\kappa+1}(z)\big)^{-*}\Big(\big(\Psi_{j,\kappa+1}(z)\big)^*\Psi_{j,\kappa+1}(z) - \big(\Phi_{j,\kappa+1}(z)\big)^*\Phi_{j,\kappa+1}(z)\Big)\big(\Psi_{j,\kappa+1}(z)\big)^{-1} \end{aligned}$$

for all $z \in \mathbb{D}\setminus\mathcal{A}$, we see that $\check{S}_{j,\kappa+1}$ is a meromorphic matrix-valued function which is both holomorphic and contractive in $\mathbb{D} \setminus \mathcal{A}$. Since $\mathcal{A}$ is a discrete subset of $\mathbb{D}$, because of Riemann's theorem on removable singularities of bounded holomorphic functions there is a $q \times q$ Schur function $S_{j,\kappa+1}$ (defined on $\mathbb{D}$) such that $\check{S}_{j,\kappa+1}$ is the restriction of $S_{j,\kappa+1}$ onto $\mathbb{D} \setminus \mathcal{A}$. For each $z \in \mathbb{D} \setminus \mathcal{A}$, from (3.23), (3.22), and (3.20) we have

$$\begin{aligned} & \big(\Psi_{j,\kappa+1}(z)\big)^{-1}\sqrt{R_{n+j+1}} \\ &\quad = \Big(I - zK^*_{n+j+1}\Theta_{j,\kappa+1}(z)\sqrt{R_{n+j+1}}\sqrt{R_{n+j+1}}^{\,+}\Big)^{-1}\sqrt{R_{n+j+1}}. \end{aligned}$$

For each $z \in \mathbb{D} \setminus \mathcal{A}$, because of (2.1), (3.20), (3.22), and (3.23), from Remark 3.4 it follows $\det\Big(I - z\sqrt{R_{n+j+1}}^{\,+}K^*_{n+j+1}\Theta_{j,k+1}(z)\sqrt{R_{n+j+1}}\Big) \neq 0$ and

$$\begin{aligned} & \big(\Psi_{j,\kappa+1}(z)\big)^{-1}\sqrt{R_{n+j+1}} \\ &\quad = \sqrt{R_{n+j+1}}\Big(I - z\sqrt{R_{n+j+1}}^{\,+}K^*_{n+j+1}\Theta_{j,\kappa+1}(z)\sqrt{R_{n+j+1}}\Big)^{-1}. \end{aligned} \tag{3.27}$$

For each $z \in \mathbb{D} \setminus \mathcal{A}$, from (3.20), Remark 2.2, (3.16), and (3.19) we get

$$\begin{aligned} & I - z\sqrt{R_{n+j+1}}^{\,+}K^*_{n+j+1}\Theta_{j,\kappa+1}(z)\sqrt{R_{n+j+1}} \\ &\quad = I - z\sqrt{R_{n+j+1}}^{\,+}K^*_{n+j+1}\sqrt{L_{n+j+1}}\sqrt{L_{n+j+2}}^{\,+} \cdot \\ &\qquad \cdot \Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{\,+}\sqrt{R_{n+j+1}} \\ &\quad = I - z\sqrt{R_{n+j+1}}^{\,+}K^*_{n+j+1}\sqrt{L_{n+j+1}}\sqrt{L_{n+j+2}}^{\,+}\Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}} \\ &\quad = I - zg^*_{n+j+1}\sqrt{L_{n+j+2}}^{\,+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}. \end{aligned} \tag{3.28}$$

In particular, for each $z \in \mathbb{D} \setminus \mathcal{A}$, the matrix on the right-hand side of (3.28) is nonsingular. Thus, for each $z \in \mathbb{D} \setminus \mathcal{A}$, from (3.27) and (3.28) we get

$$\begin{aligned} & \big(\Psi_{j,\kappa+1}(z)\big)^{-1}\sqrt{R_{n+j+1}} \\ &\quad = \sqrt{R_{n+j+1}}\Big(I - zg^*_{n+j+1}\sqrt{L_{n+j+2}}^{\,+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}. \end{aligned} \tag{3.29}$$

Using (3.21), (3.20), (3.19), (3.15), and Remark 2.2, for each $z \in \mathbb{D} \setminus \mathcal{A}$, we obtain

$$\begin{aligned} & \sqrt{L_{n+j+1}}^{\,+}\Phi_{j,\kappa+1}(z)\sqrt{R_{n+j+1}} = \sqrt{L_{n+j+1}}^{\,+}\big(K_{n+j+1} - z\Theta_{j,\kappa+1}(z)\big)\sqrt{R_{n+j+1}} \\ &\quad = \sqrt{L_{n+j+1}}^{\,+}K_{n+j+1}\sqrt{R_{n+j+1}} \\ &\qquad -z\sqrt{L_{n+j+1}}^{\,+}\sqrt{L_{n+j+1}}\sqrt{L_{n+j+2}}^{\,+}\Sigma_{j,\kappa+1}(z)\sqrt{R_{n+j+2}}\sqrt{R_{n+j+1}}^{\,+}\sqrt{R_{n+j+1}} \\ &\quad = f_{n+j+1} - z\sqrt{L_{n+j+2}}^{\,+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}} \end{aligned} \tag{3.30}$$

and, moreover, by application of (3.29), (3.30), (2.7), and (2.9) then

$$\begin{aligned}
&-z\tilde{d}_{n+j}^{[n+j]}(z)\sqrt{L_{n+j+1}}^{+}S_{j,\kappa+1}(z)\sqrt{R_{n+j+1}}+b_{n+j}(z)\\
&\quad=-z\tilde{d}_{n+j}^{[n+j]}(z)\sqrt{L_{n+j+1}}^{+}\Phi_{j,\kappa+1}(z)\big(\Psi_{j,\kappa+1}(z)\big)^{-1}\sqrt{R_{n+j+1}}+b_{n+j}(z)\\
&\quad=-z\tilde{d}_{n+j}^{[n+j]}(z)\sqrt{L_{n+j+1}}^{+}\Phi_{j,\kappa+1}(z)\sqrt{R_{n+j+1}}\cdot\\
&\qquad\cdot\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}+b_{n+j}(z)\\
&\quad=\Big(b_{n+j}(z)\big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\,\big)-z\tilde{d}_{n+j}^{[n+j]}(z)\big(f_{n+j+1}\\
&\qquad-z\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\,\big)\Big)\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}\\
&\quad=\Big(b_{n+j}(z)-z\tilde{d}_{n+j}^{[n+j]}(z)f_{n+j+1}+z\big(z\tilde{d}_{n+j}^{[n+j]}(z)-b_{n+j}(z)g_{n+j+1}^{*}\big)\sqrt{L_{n+j+2}}^{+}\cdot\\
&\qquad\cdot S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}\\
&\quad=\Big(b_{n+j+1}(z)+z\tilde{d}_{n+j+1}^{[n+j+1]}(z)\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)\cdot\\
&\qquad\cdot\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}.
\end{aligned}\tag{3.31}$$

Taking into account (3.29), (3.30), (2.6), and (2.8), for each $z\in\mathbb{D}\setminus\mathcal{A}$, we get

$$\begin{aligned}
&z\tilde{c}_{n+j}^{[n+j]}(z)\sqrt{L_{n+j+1}}^{+}S_{j,\kappa+1}(z)\sqrt{R_{n+j+1}}+a_{n+j}(z)\\
&\quad=z\tilde{c}_{n+j}^{[n+j]}(z)\sqrt{L_{n+j+1}}^{+}\Phi_{j,\kappa+1}(z)\big(\Psi_{j,\kappa+1}(z)\big)^{-1}\sqrt{R_{n+j+1}}+a_{n+j}(z)\\
&\quad=z\tilde{c}_{n+j}^{[n+j]}(z)\,\Big(f_{n+j+1}-z\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)\cdot\\
&\qquad\cdot\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}+a_{n+j}(z)\\
&\quad=\Big(z\tilde{c}_{n+j}^{[n+j]}(z)\big(f_{n+j+1}-z\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\,\big)+a_{n+j}(z)\big(I-zg_{n+j+1}^{*}\cdot\\
&\qquad\cdot\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\,\big)\Big)\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}\\
&\quad=\Big(-z\big(z\tilde{c}_{n+j}^{[n+j]}(z)+a_{n+j}(z)g_{n+j+1}^{*}\big)\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}+a_{n+j}(z)\\
&\qquad+z\tilde{c}_{n+j}^{[n+j]}(z)f_{n+j+1}\Big)\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}\\
&\quad=\Big(-z\tilde{c}_{n+j+1}^{[n+j+1]}(z)\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}+a_{n+j+1}(z)\Big)\cdot\\
&\qquad\cdot\Big(I-zg_{n+j+1}^{*}\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}\Big)^{-1}.
\end{aligned}\tag{3.32}$$

Because of (3.17), for each $z\in\mathbb{D}\setminus\mathcal{A}$, the right-hand side of (3.31) is nonsingular. Hence, for each $z\in\mathbb{D}\setminus\mathcal{A}$, the left-hand side of (3.31) is nonsingular as well and, by setting $F_{j,\kappa+1}:=\sqrt{L_{n+j+1}}^{+}S_{j,\kappa+1}\sqrt{R_{n+j+1}}$, from (3.32), (3.31), and (3.18), we obtain then

$$\begin{aligned}
&\left(z\tilde{c}_{n+j}^{[n+j]}(z)F_{j,\kappa+1}(z)+a_{n+j}(z)\right)\left(-z\tilde{d}_{n+j}^{[n+j]}(z)F_{j,\kappa+1}(z)+b_{n+j}(z)\right)^{-1}\\
&\quad=\left(-z\tilde{c}_{n+j+1}^{[n+j+1]}(z)\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}+a_{n+j+1}(z)\right)\cdot\\
&\qquad\cdot\left(z\tilde{d}_{n+j+1}^{[n+j+1]}(z)\sqrt{L_{n+j+2}}^{+}S_{j+1,\kappa}(z)\sqrt{R_{n+j+2}}+b_{n+j+1}(z)\right)^{-1}\\
&\quad=\Omega_{c,n+j+1+\kappa}(z)=\Omega_{c,n+j+\kappa+1}(z).
\end{aligned}\tag{3.33}$$

Using Lemma 3.1 (note [FK3, Proposition 4.4, Lemma 4.6]) and a continuity argument we get that (3.33) holds for each $z\in\mathbb{D}$. Thus, for all nonnegative integers j and k, there is a $q\times q$ Schur function S_{jk} defined on $\mathbb{D}$ such that (3.13) and (3.14) hold for each $z\in\mathbb{D}$, where $F_{jk}:=\sqrt{L_{n+j+1}}^{+}S_{jk}\sqrt{R_{n+j+1}}$. The matricial version of Montel's theorem yields that there are a $q\times q$ Schur function S defined on $\mathbb{D}$ and a subsequence $(S_{0k_m})_{m=0}^{\infty}$ of $(S_{0k})_{k=0}^{\infty}$ such that

$$\lim_{m\to\infty}S_{0k_m}(z)=S(z)\tag{3.34}$$

holds for each $z\in\mathbb{D}$. From Theorem 3.2 we get (3.3) and (3.4) for each $z\in\mathbb{D}$. Using (3.12), (3.14), and (3.34) we obtain (3.11) for each $z\in\mathbb{D}$. Application of Theorem 3.2 completes the proof. □

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. Use Remark 1.2, Theorem 3.2, and Theorem 3.7. □

4. Resolvent matrices which are constructed recursively

A closer look at the construction of the matrix polynomials a_n, b_n, c_n, and d_n which realize via Theorem 3.2 and Theorem 3.7 a parametrization of the solution set of an arbitrary matricial Carathéodory problem shows that there is some freedom in building polynomials a_n, b_n, c_n, and d_n with the required properties. The main objective of this section is to present a recursive construction of a distinguished quadrupel $[\mathbf{a}_n,\mathbf{b}_n,\mathbf{c}_n,\mathbf{d}_n]$ of matrix polynomials which satisfy the assumptions of Theorem 3.2 and Theorem 3.7.

In the present section, if a nonnegative integer n and a $q\times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^{n}$ are given, then always L_{k+1} and R_{k+1} stand for the matrices defined by (1.6) and (1.8) for each $k\in\mathbb{N}_{0,n}$. Furthermore, let $\mathbf{a}_0$, $\mathbf{b}_0$, $\mathbf{c}_0$, and $\mathbf{d}_0$ be the constant matrix-valued functions defined, for each $z\in\mathbb{C}$, by

$$\mathbf{a}_0(z):=\Gamma_0,\quad\mathbf{b}_0(z):=I_q,\quad\mathbf{c}_0(z):=\Gamma_0,\quad\mathbf{d}_0(z):=I_q\tag{4.1}$$

and for all $m\in\mathbb{N}_{0,n-1}$ let the matrix polynomials $\mathbf{a}_{m+1}$, $\mathbf{b}_{m+1}$, $\mathbf{c}_{m+1}$, and $\mathbf{d}_{m+1}$ be recursively defined, for each $z\in\mathbb{C}$, by

$$\mathbf{a}_{m+1}(z):=\mathbf{a}_m(z)+z\tilde{\mathbf{c}}_m^{[m]}(z)f_{m+1},\quad\mathbf{b}_{m+1}(z):=\mathbf{b}_m(z)-z\tilde{\mathbf{d}}_m^{[m]}(z)f_{m+1},\tag{4.2}$$

$$\mathbf{c}_{m+1}(z):=\mathbf{c}_m(z)+g_{m+1}z\tilde{\mathbf{a}}_m^{[m]}(z),\quad\mathbf{d}_{m+1}(z):=\mathbf{d}_m(z)-g_{m+1}z\tilde{\mathbf{b}}_m^{[m]}(z),\tag{4.3}$$

where the matrices f_{m+1} and g_{m+1} are given as in (2.2) with respect to $(\Gamma_j)_{j=0}^{m+1}$. In the following, we point out some results on the special structure of these matrix polynomials. Note that from [FK3, Example 4.3] (see also Example 4.5 below) one can see that the matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ do not coincide, in general, with the matrix polynomials a_n, b_n, c_n, and d_n, respectively, which are defined by (1.9), (1.10), (1.11), and (1.12). Nevertheless, they comply with the requirements of Theorem 3.2 and Theorem 3.7.

Theorem 4.1. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Then:*

(a) *For each $S \in \mathcal{S}_{q\times q}(\mathbb{D})$ and each $z \in \mathbb{D}$, the inequalities*

$$\det\left(z\tilde{\mathbf{d}}_n^{[n]}(z)\sqrt{L_{n+1}}^{+}S(z)\sqrt{R_{n+1}} + \mathbf{b}_n(z)\right) \neq 0$$

and

$$\det\left(z\sqrt{L_{n+1}}S(z)\sqrt{R_{n+1}}^{+}\tilde{\mathbf{b}}_n^{[n]}(z) + \mathbf{d}_n(z)\right) \neq 0$$

are satisfied. Moreover, for each $S \in \mathcal{S}_{q\times q}(\mathbb{D})$, the matrix-valued function $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ defined by

$$\Omega(z) := \left(-z\tilde{\mathbf{c}}_n^{[n]}(z)F(z) + \mathbf{a}_n(z)\right)\left(z\tilde{\mathbf{d}}_n^{[n]}(z)F(z) + \mathbf{b}_n(z)\right)^{-1}$$

belongs to $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$ and admits, for each $z \in \mathbb{D}$, the representations

$$\Omega(z) = \left(zG(z)\tilde{\mathbf{b}}_n^{[n]}(z) + \mathbf{d}_n(z)\right)^{-1}\left(-zG(z)\tilde{\mathbf{a}}_n^{[n]}(z) + \mathbf{c}_n(z)\right) \tag{4.4}$$

and

$$\Omega(z) = \left(-z\tilde{c}_n^{[n]}(z)F(z) + a_n(z)\right)\left(z\tilde{d}_n^{[n]}(z)F(z) + b_n(z)\right)^{-1},$$

where $F := \sqrt{L_{n+1}}^{+}S\sqrt{R_{n+1}}$ as well as $G := \sqrt{L_{n+1}}S\sqrt{R_{n+1}}^{+}$ and where a_n, b_n, c_n, and d_n are defined by (1.9), (1.10), (1.11), *and* (1.12).

(b) *For each $\Omega \in \mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$, there is an $S \in \mathcal{S}_{q\times q}(\mathbb{D})$ such that for each $z \in \mathbb{D}$ the representations* (4.4) *and*

$$\Omega(z) = \left(-z\tilde{\mathbf{c}}_n^{[n]}(z)F(z) + \mathbf{a}_n(z)\right)\left(z\tilde{\mathbf{d}}_n^{[n]}(z)F(z) + \mathbf{b}_n(z)\right)^{-1}$$

of Ω hold, where $F := \sqrt{L_{n+1}}^{+}S\sqrt{R_{n+1}}$ and $G := \sqrt{L_{n+1}}S\sqrt{R_{n+1}}^{+}$.

Proof. In the case $n = 0$ the assertion follows immediately from Theorem 3.2 and Theorem 3.7. Now suppose $n \geq 1$. According to [FK3, Proposition 4.4, Remark 4.5, and Lemma 4.6] there are some matrices $\mathbf{V}_n \in \widetilde{\mathcal{Y}}_n$ and $\mathbf{W}_n \in \widetilde{\mathcal{Z}}_n$ such that the $q \times q$ matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ can be represented, for each $z \in \mathbb{C}$, via (3.1) and (3.2). Application of Remark 1.2, Theorem 3.2, and Theorem 3.7 completes the proof for that case. □

Remark 4.2. Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. By induction, one can see that $\mathbf{a}_n(0) = \Gamma_0$, $\mathbf{b}_n(0) = I_q$, $\mathbf{c}_n(0) = \Gamma_0$, and $\mathbf{d}_n(0) = I_q$. In particular, the $q \times q$ matrix polynomials $\tilde{\mathbf{b}}_n^{[n]}$ and $\tilde{\mathbf{d}}_n^{[n]}$ are both of degree n

with leading coefficient matrix I_q and the $q \times q$ matrix polynomials $\tilde{\mathbf{a}}_n^{[n]}$ and $\tilde{\mathbf{c}}_n^{[n]}$ are either both the constant function with value $0_{q\times q}$ or both of degree n with leading coefficient matrix Γ_0^*. Moreover, $\tilde{\mathbf{a}}_n^{[n]}(0)=\Gamma_0^*$, $\tilde{\mathbf{b}}_n^{[n]}(0)=I_q$, $\tilde{\mathbf{c}}_n^{[n]}(0)=\Gamma_0^*$, and $\tilde{\mathbf{d}}_n^{[n]}(0)=I_q$ in the case of $n=0$ and if $n \geq 1$ then in view of the recursions it is not hard to see that $\tilde{\mathbf{a}}_n^{[n]}(0)=f_n^*\Gamma_0$, $\tilde{\mathbf{b}}_n^{[n]}(0)=-f_n^*$, $\tilde{\mathbf{c}}_n^{[n]}(0)=\Gamma_0 g_n^*$, and $\tilde{\mathbf{d}}_n^{[n]}(0)=-g_n^*$.

Lemma 4.3. *Let $C \in \mathbb{C}^{q\times q}$ be such that $\operatorname{Re} C$ is a nonnegative Hermitian matrix.*
(a) *If A is a complex $p \times q$ matrix such that $AC = 0_{p\times q}$, then $A(\operatorname{Re} C) = 0_{p\times q}$.*
(b) *If B is a complex $q \times p$ matrix such that $CB = 0_{q\times p}$, then $(\operatorname{Re} C)B = 0_{q\times p}$.*

Proof. Let $A \in \mathbb{C}^{p\times q}$ be such that $AC = 0_{p\times q}$. Then

$$\left(A\sqrt{\operatorname{Re} C}\right)\left(A\sqrt{\operatorname{Re} C}\right)^* = \frac{1}{2}A(C+C^*)A^* = \frac{1}{2}\Big(ACA^* + A(AC)^*\Big) = 0_{p\times p}$$

and consequently

$$A(\operatorname{Re} C) = \left(A\sqrt{\operatorname{Re} C}\right)\sqrt{\operatorname{Re} C} = 0_{p\times q}\sqrt{\operatorname{Re} C} = 0_{p\times q}.$$

Part (a) is proved. Part (b) follows easily from part (a). □

Proposition 4.4. *Let $n \in \mathbb{N}$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Further, let $\Omega_{c,n}$ and $\Omega_{c,n-1}$ be the central $q \times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^n$ and $(\Gamma_j)_{j=0}^{n-1}$, respectively. Then the following statements are equivalent:*
(i) $\Omega_{c,n} = \Omega_{c,n-1}$.
(ii) $\mathbf{a}_n = \mathbf{a}_{n-1}$, $\mathbf{b}_n = \mathbf{b}_{n-1}$, $\mathbf{c}_n = \mathbf{c}_{n-1}$, *and* $\mathbf{d}_n = \mathbf{d}_{n-1}$.
(iii) *At least one of the matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ is of degree not greater than $n-1$.*

Proof. (i) ⇒ (ii): Because of (i) we have $\frac{1}{2}\Gamma_n = M_n$. From (2.2) we obtain then $f_n = 0_{q\times q}$ and $g_n = 0_{q\times q}$. Using (4.2) and (4.3) we get (ii).
(ii) ⇒ (iii): Since the relations (4.1), (4.2), and (4.3) yield that $\mathbf{a}_{n-1}$, $\mathbf{b}_{n-1}$, $\mathbf{c}_{n-1}$, and $\mathbf{d}_{n-1}$ are matrix polynomials of degree not greater than $n-1$, this implication follows obviously.
(iii) ⇒ (i): First suppose that the matrix polynomial b_n is of degree not greater than $n-1$. From Remark 4.2 we can conclude $f_n = 0_{q\times q}$ so that (2.3) provides

$$\frac{1}{2}\Gamma_n - M_n = L_n f_n = 0_{q\times q}.$$

Thus (i) holds. Analogously, one can check that if d_n is a matrix polynomial of degree not greater than $n-1$, then (i) follows. Now we suppose that a_n is a matrix polynomial of degree not greater than $n-1$. Then Remark 4.2 yields $f_n^*\Gamma_0 = 0_{q\times q}$. Hence Lemma 4.3 provides us

$$L_1 f_n = (\operatorname{Re}\Gamma_0)f_n = \left(f_n^*(\operatorname{Re}\Gamma_0)\right)^* = 0_{q\times q}.$$

Taking into account (2.3) and Remark 2.2 we get

$$\frac{1}{2}\Gamma_n - M_n = L_n f_n = L_1 f_n = 0_{q\times q}.$$

This implies (i). If $\mathbf{c}_n$ is a matrix polynomial of degree not greater than $n-1$, then one can similarly verify that (i) holds. $\square$

In view of (3.1) and (3.2) (cf. the proof of Theorem 4.1), the matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ are special choices of the matrix polynomials a_n, b_n, c_n, and d_n given by (1.15), (1.16), (1.17), and (1.18). The following example emphasizes that the statement of Proposition 4.4 depends on these special choices. In fact, $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ can not be replaced via a_n, b_n, c_n, and d_n, respectively, defined by (1.9), (1.10), (1.11), and (1.12) (note Remark 1.2).

Example 4.5. By setting $\Gamma_0 := \frac{1}{2}I_q$, $\Gamma_1 := I_q$, and $\Gamma_2 := I_q$, then $(\Gamma_j)_{j=0}^2$ is a $q\times q$ Carathéodory sequence for which $T_0^+Y_1 = I_q$, $Z_1T_0^+ = I_q$, $L_2 = 0$, $R_2 = 0$, $T_1^+Y_2 = \frac{1}{2}\begin{pmatrix} I_q, & I_q\end{pmatrix}^*$, and $Z_2T_1^+ = \frac{1}{2}\begin{pmatrix} I_q, & I_q\end{pmatrix}$. Hence, for each $z\in\mathbb{D}$, it follows

$$\Omega_{c,2}(z) = \frac{1+z}{2(1-z)}I_q = \Omega_{c,1}(z),$$

but

$$a_2(z) = \frac{1}{2}I_q + \frac{3}{4}zI_q + \frac{1}{4}z^2I_q, \quad b_2(z) = I_q - \frac{1}{2}zI_q - \frac{1}{2}z^2I_q,$$

$c_2(z) = a_2(z)$, and $d_2(z) = b_2(z)$ if a_2, b_2, c_2, and d_2 are defined as in (1.9), (1.10), (1.11), and (1.12) with $n=2$. In particular, the matrix polynomials a_2, b_2, c_2, and d_2 are of degree 2.

A complex $p\times p$ matrix J is said to be $p\times p$ signature matrix if $J^* = J$ and $J^2 = I$ hold. In particular, the matrices

$$j_{qq} := \begin{pmatrix} I_q & 0 \\ 0 & -I_q \end{pmatrix} \quad \text{and} \quad J_q := \begin{pmatrix} 0 & -I_q \\ -I_q & 0 \end{pmatrix}$$

are $2q\times 2q$ signature matrices. Now we are going to show that the resolvent matrices formed by the matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ fulfill similar formulas with respect to these $2q\times 2q$ signature matrices j_{qq} and J_q as the Arov-Kreĭn's resolvent matrices in the nondegenerate case (cf., e.g., [FK3, Section 5]).

Proposition 4.6. *Let $n\in\mathbb{N}$ and let $(\Gamma_j)_{j=0}^n$ be a $q\times q$ Carathéodory sequence. For each $m\in\mathbb{N}_{0,n}$, let $\mathbf{\Phi}_m^\bullet : \mathbb{C}\to\mathbb{C}^{2q\times 2q}$ and $\mathbf{\Psi}_m^\bullet : \mathbb{C}\to\mathbb{C}^{2q\times 2q}$ be defined by*

$$\mathbf{\Phi}_m^\bullet(z) := \begin{pmatrix} -z\tilde{\mathbf{c}}_m^{[m]}(z) & \mathbf{a}_m(z) \\ z\tilde{\mathbf{d}}_m^{[m]}(z) & \mathbf{b}_m(z) \end{pmatrix} \begin{pmatrix} \sqrt{L_{m+1}}^+ & 0 \\ 0 & \sqrt{R_{m+1}}^+ \end{pmatrix} \tag{4.5}$$

and

$$\mathbf{\Psi}_m^\bullet(z) := \begin{pmatrix} \sqrt{R_{m+1}}^+ & 0 \\ 0 & \sqrt{L_{m+1}}^+ \end{pmatrix} \begin{pmatrix} -z\tilde{\mathbf{a}}_m^{[m]}(z) & z\tilde{\mathbf{b}}_m^{[m]}(z) \\ \mathbf{c}_m(z) & \mathbf{d}_m(z) \end{pmatrix}. \tag{4.6}$$

Further, for each $m\in\mathbb{N}_{0,n-1}$, let $K_{m+1} := \sqrt{L_{m+1}}^+\left(\frac{1}{2}\Gamma_{m+1} - M_{m+1}\right)\sqrt{R_{m+1}}^+$. Then the following statements hold for each $m\in\mathbb{N}_{0,n-1}$:

(a) *For each $z \in \mathbb{C}$, the identities*

$$\mathbf{\Phi}^{\bullet}_{m+1}(z) = \mathbf{\Phi}^{\bullet}_{m}(z) G_{m+1}(z) \quad \textit{and} \quad \mathbf{\Psi}^{\bullet}_{m+1}(z) = H_{m+1}(z) \mathbf{\Psi}^{\bullet}_{m}(z) \tag{4.7}$$

are satisfied, where

$$G_{m+1}(z) := \begin{pmatrix} I & -K_{m+1} \\ -K^*_{m+1} & I \end{pmatrix} \begin{pmatrix} z\sqrt{L_{m+1}}\sqrt{L_{m+2}}^{+} & 0 \\ 0 & \sqrt{R_{m+1}}\sqrt{R_{m+2}}^{+} \end{pmatrix}$$

and

$$H_{m+1}(z) := \begin{pmatrix} z\sqrt{R_{m+2}}^{+}\sqrt{R_{m+1}} & 0 \\ 0 & \sqrt{L_{m+2}}^{+}\sqrt{L_{m+1}} \end{pmatrix} \begin{pmatrix} I & -K^*_{m+1} \\ -K_{m+1} & I \end{pmatrix}.$$

(b) $\mathbf{\Phi}^{\bullet}_{m+1} = \mathbf{\Phi}^{\bullet}_{0} G_1 G_2 \cdots G_{m+1}$ *and* $\mathbf{\Psi}^{\bullet}_{m+1} = H_{m+1} H_m \cdots H_1 \mathbf{\Psi}^{\bullet}_{0}$.

(c) *For each $z \in \mathbb{C}$,*

$$\begin{aligned} &\operatorname{diag}\left(L_{m+2}L^{+}_{m+2}, -R_{m+2}R^{+}_{m+2}\right) - \left(G_{m+1}(z)\right)^* j_{qq}\, G_{m+1}(z) \\ &\quad = \operatorname{diag}\left((1-|z|^2)L_{m+2}L^{+}_{m+2}, 0_{q\times q}\right), \end{aligned} \tag{4.8}$$

$$\begin{aligned} &\left(G_{m+1}(z)\right)^* \operatorname{diag}\left(L_{m+1}L^{+}_{m+1}, -R_{m+1}R^{+}_{m+1}\right) G_{m+1}(z) \\ &\quad = \left(G_{m+1}(z)\right)^* j_{qq}\, G_{m+1}(z), \end{aligned} \tag{4.9}$$

$$\begin{aligned} &\operatorname{diag}\left(R_{m+2}R^{+}_{m+2}, -L_{m+2}L^{+}_{m+2}\right) - H_{m+1}(z)\, j_{qq} \left(H_{m+1}(z)\right)^* \\ &\quad = \operatorname{diag}\left((1-|z|^2)R_{m+2}R^{+}_{m+2}, 0_{q\times q}\right), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &H_{m+1}(z) \operatorname{diag}\left(R_{m+1}R^{+}_{m+1}, -L_{m+1}L^{+}_{m+1}\right) \left(H_{m+1}(z)\right)^* \\ &\quad = H_{m+1}(z)\, j_{qq} \left(H_{m+1}(z)\right)^*. \end{aligned} \tag{4.11}$$

Proof. Let $m \in \mathbb{N}_{0,n-1}$ and let $z \in \mathbb{C}$. From (4.2) and (4.3) we obtain

$$\mathbf{\Phi}^{\bullet}_{m+1}(z) = \begin{pmatrix} -z\tilde{\mathbf{c}}^{[m]}_m(z) & \mathbf{a}_m(z) \\ z\tilde{\mathbf{d}}^{[m]}_m(z) & \mathbf{b}_m(z) \end{pmatrix} \begin{pmatrix} zI_q & -f_{m+1} \\ -zg^*_{m+1} & I_q \end{pmatrix} \begin{pmatrix} \sqrt{L_{m+2}}^{+} & 0 \\ 0 & \sqrt{R_{m+2}}^{+} \end{pmatrix}. \tag{4.12}$$

Since the matrices L_{m+1} and R_{m+1} are nonnegative Hermitian, the relations $\sqrt{L_{m+1}}\sqrt{L_{m+1}}^{+} = \sqrt{L_{m+1}}^{+}\sqrt{L_{m+1}}$ and $\sqrt{R_{m+1}}\sqrt{R_{m+1}}^{+} = \sqrt{R_{m+1}}^{+}\sqrt{R_{m+1}}$

hold. Because of Remarks 2.1 and 2.2 we have then

$$\begin{pmatrix} zI_q & -f_{m+1} \\ -zg^*_{m+1} & I_q \end{pmatrix} \begin{pmatrix} \sqrt{L_{m+2}}^+ & 0 \\ 0 & \sqrt{R_{m+2}}^+ \end{pmatrix}$$
$$= \begin{pmatrix} z\sqrt{L_{m+2}}^+ & -\sqrt{L_{m+1}}^+ K_{m+1}\sqrt{R_{m+1}}\sqrt{R_{m+2}}^+ \\ -z\sqrt{R_{m+1}}^+ K^*_{m+1}\sqrt{L_{m+1}}\sqrt{L_{m+2}}^+ & \sqrt{R_{m+2}}^+ \end{pmatrix}$$
$$= \begin{pmatrix} z\sqrt{L_{m+1}}^+ \sqrt{L_{m+1}}\sqrt{L_{m+2}}^+ & -\sqrt{L_{m+1}}^+ K_{m+1}\sqrt{R_{m+1}}\sqrt{R_{m+2}}^+ \\ -z\sqrt{R_{m+1}}^+ K^*_{m+1}\sqrt{L_{m+1}}\sqrt{L_{m+2}}^+ & \sqrt{R_{m+1}}^+ \sqrt{R_{m+1}}\sqrt{R_{m+2}}^+ \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{L_{m+1}}^+ & 0 \\ 0 & \sqrt{R_{m+1}}^+ \end{pmatrix} \begin{pmatrix} I & -K_{m+1} \\ -K^*_{m+1} & I \end{pmatrix} \begin{pmatrix} z\sqrt{L_{m+1}}\sqrt{L_{m+2}}^+ & 0 \\ 0 & \sqrt{R_{m+1}}\sqrt{R_{m+2}}^+ \end{pmatrix}.$$

A combination of this equation with (4.12) supplies the first equality in (4.7). The second one can be proved analogously. Part (b) follows immediately from part (a). A straightforward calculation yields

$$\big(G_{m+1}(z)\big)^* j_{qq}\, G_{m+1}(z)$$
$$= \begin{pmatrix} |z|^2\sqrt{L_{m+2}}^+ \sqrt{L_{m+1}}(I-K_{m+1}K^*_{m+1})\sqrt{L_{m+1}}\sqrt{L_{m+2}}^+ & 0 \\ 0 & \sqrt{R_{m+2}}^+ \sqrt{R_{m+1}}(K^*_{m+1}K_{m+1}-I)\sqrt{R_{m+1}}\sqrt{R_{m+2}}^+ \end{pmatrix}$$

and (note (2.1))

$$\big(G_{m+1}(z)\big)^* \operatorname{diag}\left(L_{m+1}L^+_{m+1}, -R_{m+1}R^+_{m+1}\right) G_{m+1}(z)$$
$$= \begin{pmatrix} |z|^2\sqrt{L_{m+2}}^+ \sqrt{L_{m+1}}L_{m+1}L^+_{m+1}(I-K_{m+1}K^*_{m+1})\sqrt{L_{m+1}}\sqrt{L_{m+2}}^+ & 0 \\ 0 & \sqrt{R_{m+2}}^+ \sqrt{R_{m+1}}(K^*_{m+1}K_{m+1}-I)R_{m+1}R^+_{m+1}\sqrt{R_{m+1}}\sqrt{R_{m+2}}^+ \end{pmatrix}.$$

Hence, it follows (4.9) and from Remark 2.2 we get furthermore

$$\big(G_{m+1}(z)\big)^* j_{qq}\, G_{m+1}(z) = \begin{pmatrix} |z|^2\sqrt{L_{m+2}}^+ L_{m+2}\sqrt{L_{m+2}}^+ & 0 \\ 0 & -\sqrt{R_{m+2}}^+ R_{m+2}\sqrt{R_{m+2}}^+ \end{pmatrix}$$

which implies (4.8). Equations (4.10) and (4.11) can be verified analogously. □

Corollary 4.7. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Furthermore, let the $2q \times 2q$ matrix-valued functions $\mathbf{\Phi}^\bullet_n$ and $\mathbf{\Psi}^\bullet_n$ be defined as in* Proposition 4.6, *let*

$$\Theta_n := \begin{pmatrix} L_{n+1}L^+_{n+1} & 0 \\ 0 & -R_{n+1}R^+_{n+1} \end{pmatrix} - \left(\tfrac{1}{\sqrt{2}}\mathbf{\Phi}^\bullet_n\right)^* J_q \left(\tfrac{1}{\sqrt{2}}\mathbf{\Phi}^\bullet_n\right),$$

and let

$$\Xi_n := \begin{pmatrix} R_{n+1}R^+_{n+1} & 0 \\ 0 & -L_{n+1}L^+_{n+1} \end{pmatrix} - \left(\tfrac{1}{\sqrt{2}}\mathbf{\Psi}^\bullet_n\right) J_q \left(\tfrac{1}{\sqrt{2}}\mathbf{\Psi}^\bullet_n\right)^*.$$

(a) *For each $z \in \mathbb{D}$, the matrices $\Theta_n(z)$ and $\Xi_n(z)$ are nonnegative Hermitian.*
(b) *For each $z \in \mathbb{T}$, the identities $\Theta_n(z) = 0$ and $\Xi_n(z) = 0$ are satisfied.*
(c) *For each $z \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$, the complex $2q \times 2q$ matrices $-\Theta_n(z)$ and $-\Xi_n(z)$ are both nonnegative Hermitian.*

Proof. For each $z \in \mathbb{C}$, a straightforward calculation yields

$$\Theta_0(z) = \operatorname{diag}\left((1-|z|^2)L_1L_1^+, 0_{q\times q}\right) \quad \text{and} \quad \Xi_0(z) = \operatorname{diag}\left((1-|z|^2)R_1R_1^+, 0_{q\times q}\right).$$

Application of parts (b) and (c) of Proposition 4.6 completes the proof. □

Corollary 4.8. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Then*

$$\mathbf{\Psi}_n^\bullet(z)\begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}\mathbf{\Phi}_n^\bullet(z) = -2z^{n+1}\begin{pmatrix} 0 & R_{n+1}R_{n+1}^+ \\ -L_{n+1}L_{n+1}^+ & 0 \end{pmatrix}$$

for each $z \in \mathbb{C}$, where the $2q \times 2q$ matrix-valued functions $\mathbf{\Phi}_n^\bullet$ and $\mathbf{\Psi}_n^\bullet$ are defined as in Proposition 4.6*.*

Proof. For each $z \in \mathbb{T}$, an application of [DFK, Lemma 1.2.2] and $|z|^2 = 1$ implies

$$\begin{aligned}
-z^{n+1}\begin{pmatrix} I_q & 0 \\ 0 & -I_q \end{pmatrix}\left(\mathbf{\Psi}_n^\bullet(z)\right)^* &= -z^{n+1}\begin{pmatrix} -\overline{z}\left(\tilde{\mathbf{a}}_n^{[n]}(z)\right)^*\sqrt{R_{n+1}}^+ & \left(\mathbf{c}_n(z)\right)^*\sqrt{L_{n+1}}^+ \\ -\overline{z}\left(\tilde{\mathbf{b}}_n^{[n]}(z)\right)^*\sqrt{R_{n+1}}^+ & -\left(\mathbf{d}_n(z)\right)^*\sqrt{L_{n+1}}^+ \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{a}_n(z)\sqrt{R_{n+1}}^+ & -z\tilde{\mathbf{c}}_n^{[n]}(z)\sqrt{L_{n+1}}^+ \\ \mathbf{b}_n(z)\sqrt{R_{n+1}}^+ & z\tilde{\mathbf{d}}_n^{[n]}(z)\sqrt{L_{n+1}}^+ \end{pmatrix} \\
&= \mathbf{\Phi}_n^\bullet(z)\begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix}.
\end{aligned}$$

Consequently, from

$$\begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}\begin{pmatrix} I_q & 0 \\ 0 & -I_q \end{pmatrix} = J_q, \quad \begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix},$$

and part (b) of Corollary 4.7 it follows

$$\begin{aligned}
\mathbf{\Psi}_n^\bullet(z)\begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}\mathbf{\Phi}_n^\bullet(z) &= -z^{n+1}\mathbf{\Psi}_n^\bullet(z)J_q\left(\mathbf{\Psi}_n^\bullet(z)\right)^*\begin{pmatrix} 0 & I_q \\ I_q & 0 \end{pmatrix} \\
&= -2z^{n+1}\begin{pmatrix} 0 & R_{n+1}R_{n+1}^+ \\ -L_{n+1}L_{n+1}^+ & 0 \end{pmatrix}
\end{aligned}$$

for each $z \in \mathbb{T}$. Since the left-hand side and the right-hand side of this equation form matrix polynomials, one can conclude the assertion. □

Observe that the formula in Corollary 4.8 is closely related to Proposition 2.4. In particular, using part (b) of Proposition 2.4 and (2.10) we obtain an alternative approach to Corollary 4.8. In fact, these considerations yield that the statement of Corollary 4.8 remains true if one replaces in the definitions of $\mathbf{\Phi}_n^\bullet$ and $\mathbf{\Psi}_n^\bullet$ according to Proposition 4.6 the special polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ there by the polynomials a_n, b_n, c_n, and d_n defined as in (1.15), (1.16), (1.17), and (1.18) with some $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$ if $n \geq 1$. Applying the arguments of the proof of Corollary 4.8 then in a slightly modified order, one can see that also part (b) of Corollary 4.7 remains true under these general settings.

5. The nondegenerate case

In this section, we will deal with the nondegenerate matricial Carathéodory problem. There can be found several approaches to this problem in the literature (see, e.g., [AK], [Ko], [BGR], [FK1], and [FKK]). The main goal of the following considerations is to demonstrate that Theorem 1.1 (respectively, Theorem 4.1) quickly leads us to Arov-Kreĭn's parametrization of the solution set for this case. Before, we give some general remarks on the nondegenerate case which are expressed in terms of the $q \times q$ matrix polynomials a_n, b_n, c_n, and d_n defined by (1.15), (1.16), (1.17), and (1.18).

Let $n \in \mathbb{N}_0$. A sequence $(\Gamma_j)_{j=0}^n$ of complex $q \times q$ matrices is said to be a *nondegenerate $q \times q$ Carathéodory sequence* if the block Toeplitz matrix T_n given by (1.1) and (1.2) is positive Hermitian.

Lemma 5.1. *Let $n \in \mathbb{N}_0$, let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence, and let the matrices L_k and R_k be defined as in* (1.6) *and* (1.8) *for each $k \in \mathbb{N}_{1,n+1}$. Then the following statements are equivalent:*

(i) *$(\Gamma_j)_{j=0}^n$ is a nondegenerate $q \times q$ Carathéodory sequence.*
(ii) *For each $k \in \mathbb{N}_{1,n+1}$, the matrices L_k and R_k are both positive Hermitian.*
(iii) *L_{n+1} or R_{n+1} is nonsingular.*

Proof. (i) $\Rightarrow$ (ii): From (i) and [DFK, Lemma 1.1.9] we obtain that the matrices L_{n+1} and R_{n+1} are positive Hermitian. Thus Remark 2.2 yields (ii).
(ii) $\Rightarrow$ (iii): This implication holds obviously.
(iii) $\Rightarrow$ (i): Since $(\Gamma_j)_{j=0}^n$ is a $q \times q$ Carathéodory sequence, the matrices L_{n+1} and R_{n+1} are nonnegative Hermitian. Because of (iii) and Remark 2.2, one of the sequences $(L_k)_{k=1}^{n+1}$ and $(R_k)_{k=1}^{n+1}$ consists of positive Hermitian matrices. Hence, an application of [DFK, Lemma 1.1.9] provides us (i). □

Remark 5.2. Let $n \in \mathbb{N}$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence such that the $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^{n-1}$ is nondegenerate. Then $\mathcal{Y}_n = \{T_{n-1}^{-1} Y_n\}$ and $\mathcal{Z}_n = \{Z_n T_{n-1}^{-1}\}$. Thus the matrix polynomials a_n, b_n, c_n, and d_n defined by (1.9), (1.10), (1.11), and (1.12) coincide with the matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ which admit, for each $z \in \mathbb{C}$, the representations (3.1) and (3.2). On the other hand, from [FK3, Proposition 4.4 and Remark 4.5] we obtain that $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ can be constructed recursively by (4.1), (4.2), and (4.3), where the sequences $(f_{m+1})_{m=0}^{n-1}$ and $(g_{m+1})_{m=0}^{n-1}$ of complex $q \times q$ matrices are given by (2.2).

Remark 5.3. Let $n \in \mathbb{N}$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Further, let the matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ be constructed recursively by (4.1), (4.2), and (4.3), where the sequences $(f_{m+1})_{m=0}^{n-1}$ and $(g_{m+1})_{m=0}^{n-1}$ of complex $q \times q$ matrices are given by (2.2). If at least one of the matrix polynomials $\mathbf{a}_n$, $\mathbf{b}_n$, $\mathbf{c}_n$, and $\mathbf{d}_n$ is of degree n with a nonsingular leading coefficient matrix, then an application of Lemma 5.1, Remark 4.2, and (2.2) implies that the $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^{n-1}$ is nondegenerate.

Proposition 5.4. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. If $n \geq 1$, then let $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$. Further, let a_n, b_n, c_n, and d_n be the matrix polynomials which are defined by* (1.15), (1.16), (1.17), *and* (1.18). *Then the following statements are equivalent:*

(i) $(\Gamma_j)_{j=0}^n$ *is a nondegenerate $q \times q$ Carathéodory sequence.*

(ii) *For each $z \in \mathbb{T}$, the matrices $a_n(z)$, $b_n(z)$, $c_n(z)$, and $d_n(z)$ are all nonsingular and, moreover, the matrices* $\operatorname{Re}\left(a_n(z)\big(b_n(z)\big)^{-1}\right)$, $\operatorname{Re}\left(\big(d_n(z)\big)^{-1}c_n(z)\right)$, $\operatorname{Re}\left(b_n(z)\big(a_n(z)\big)^{-1}\right)$, *and* $\operatorname{Re}\left(\big(c_n(z)\big)^{-1}d_n(z)\right)$ *are all positive Hermitian.*

(iii) *There is a $z \in \mathbb{T}$ such that* $\operatorname{Re}\left(\big(a_n(z)\big)^*b_n(z)\right)$ *or* $\operatorname{Re}\left(c_n(z)\big(d_n(z)\big)^*\right)$ *is a nonsingular matrix.*

Proof. (i) $\Rightarrow$ (ii): According to Lemma 5.1, (i) implies that the matrices L_{n+1} and R_{n+1} are both positive Hermitian. Thus part (a) of Proposition 2.4 yields that, for each $z \in \mathbb{T}$, the matrices $\operatorname{Re}\left(\big(a_n(z)\big)^*b_n(z)\right)$ and $\operatorname{Re}\left(c_n(z)\big(d_n(z)\big)^*\right)$ are both positive Hermitian. Consequently, for each $z \in \mathbb{T}$, the matrices $\big(a_n(z)\big)^*b_n(z)$ and $c_n(z)\big(d_n(z)\big)^*$ are both nonsingular (see, e.g., [DFK, part (c) of Lemma 1.1.13]). Therefore we obtain that, for each $z \in \mathbb{T}$, the matrices $a_n(z)$, $b_n(z)$, $c_n(z)$, and $d_n(z)$ are nonsingular. Moreover, for each $z \in \mathbb{T}$, we can conclude

$$\begin{aligned} 2\operatorname{Re}\left(a_n(z)\big(b_n(z)\big)^{-1}\right) &= \big(b_n(z)\big)^{-*}\left(\big(b_n(z)\big)^*a_n(z) + \big(a_n(z)\big)^*b_n(z)\right)\big(b_n(z)\big)^{-1} \\ &= 2\,\big(b_n(z)\big)^{-*}\operatorname{Re}\left(\big(a_n(z)\big)^*b_n(z)\right)\big(b_n(z)\big)^{-1} \in \mathbb{C}_{>}^{q\times q} \end{aligned}$$

and, analogously, that the matrices $\operatorname{Re}\left(\big(d_n(z)\big)^{-1}c_n(z)\right)$, $\operatorname{Re}\left(b_n(z)\big(a_n(z)\big)^{-1}\right)$, and $\operatorname{Re}\left(\big(c_n(z)\big)^{-1}d_n(z)\right)$ are positive Hermitian.

(ii) $\Rightarrow$ (iii): Because of (ii) we obtain

$$\operatorname{Re}\left(\big(a_n(z)\big)^*b_n(z)\right) = \big(b_n(z)\big)^*\operatorname{Re}\left(a_n(z)\big(b_n(z)\big)^{-1}\right)b_n(z)$$

and

$$\operatorname{Re}\left(c_n(z)\big(d_n(z)\big)^*\right) = d_n(z)\operatorname{Re}\left(\big(d_n(z)\big)^{-1}c_n(z)\right)\big(d_n(z)\big)^*$$

for each $z \in \mathbb{T}$. In particular, one can see that (ii) implicates (iii).

(iii) $\Rightarrow$ (i): From part (a) of Proposition 2.4 and (iii) we get that at least one of the matrices R_{n+1} and L_{n+1} is nonsingular. Hence Lemma 5.1 provides us (i). $\square$

Now we are going to prove that Theorem 1.1 quickly leads to the Arov-Kreĭn's parametrization of the solution set for the nondegenerate case.

Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a sequence of complex $q \times q$ matrices with nonsingular matrix Γ_0. Then the matrix S_n defined in (1.1) is nonsingular as well and there is a unique sequence $(\Gamma_j^{\#})_{j=0}^n$ of complex $q \times q$ matrices such that the block Toeplitz matrix

$$S_n^{\#} := \begin{pmatrix} \Gamma_0^{\#} & 0 & 0 & \dots & 0 \\ \Gamma_1^{\#} & \Gamma_0^{\#} & 0 & \dots & 0 \\ \Gamma_2^{\#} & \Gamma_1^{\#} & \Gamma_0^{\#} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \Gamma_n^{\#} & \Gamma_{n-1}^{\#} & \Gamma_{n-2}^{\#} & \dots & \Gamma_0^{\#} \end{pmatrix}$$

coincides with S_n^{-1}. This sequence $(\Gamma_j^{\#})_{j=0}^n$ is called the *reciprocal $q \times q$ sequence corresponding to* $(\Gamma_j)_{j=0}^n$. Setting $T_n^{\#} := \operatorname{Re} S_n^{\#}$ it follows

$$T_n^{\#} = S_n^{-1} T_n S_n^{-*} \quad \text{and} \quad T_n^{\#} = S_n^{-*} T_n S_n^{-1}. \tag{5.1}$$

Hence it is obvious that $(\Gamma_j)_{j=0}^n$ is a $q \times q$ Carathéodory sequence (respectively, a nondegenerate $q \times q$ Carathéodory sequence) if and only if $(\Gamma_j^{\#})_{j=0}^n$ is a $q \times q$ Carathéodory sequence (respectively, a nondegenerate $q \times q$ Carathéodory sequence).

Now we assume that $(\Gamma_j)_{j=0}^n$ is a nondegenerate $q \times q$ Carathéodory sequence. Furthermore, let the $q \times q$ matrix polynomials η_n, ζ_n, $\eta_n^{\#}$, and $\zeta_n^{\#}$ be defined by

$$\eta_n := e_{n,q} T_n^{-1} \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \quad \zeta_n := \begin{pmatrix} 0, & I_q \end{pmatrix} T_n^{-1} \varepsilon_{n,q}, \tag{5.2}$$

and

$$\eta_n^{\#} := e_{n,q} (T_n^{\#})^{-1} \begin{pmatrix} I_q \\ 0 \end{pmatrix}, \quad \zeta_n^{\#} := \begin{pmatrix} 0, & I_q \end{pmatrix} (T_n^{\#})^{-1} \varepsilon_{n,q}, \tag{5.3}$$

where $e_{n,q}$ and $\varepsilon_{n,q}$ are given as in (1.4) and (1.5). Therewith, let the $2q \times 2q$ matrix polynomials Φ_n and Ψ_n be defined, for each $z \in \mathbb{C}$, by

$$\Phi_n(z) := \begin{pmatrix} -z(\tilde{\zeta}_n^{\#})^{[n]}(z) \Gamma_0^{-1} \sqrt{L_{n+1}} & \eta_n^{\#}(z) \Gamma_0^{-*} \sqrt{R_{n+1}} \\ z \tilde{\zeta}_n^{[n]}(z) \sqrt{L_{n+1}} & \eta_n(z) \sqrt{R_{n+1}} \end{pmatrix} \tag{5.4}$$

and

$$\Psi_n(z) := \begin{pmatrix} -\sqrt{R_{n+1}} \Gamma_0^{-1} z (\tilde{\eta}_n^{\#})^{[n]}(z) & \sqrt{R_{n+1}} z \tilde{\eta}_n^{[n]}(z) \\ \sqrt{L_{n+1}} \Gamma_0^{-*} \zeta_n^{\#}(z) & \sqrt{L_{n+1}} \zeta_n(z) \end{pmatrix}. \tag{5.5}$$

We check now that the matrix polynomials Φ_n and Ψ_n introduced in (5.4) and (5.5) coincide with the matrix polynomials $\mathbf{\Phi}_n^{\bullet}$ and $\mathbf{\Psi}_n^{\bullet}$ defined by (4.5) and (4.6).

Lemma 5.5. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence. Then $\Phi_n = \mathbf{\Phi}_n^{\bullet}$ and $\Psi_n = \mathbf{\Psi}_n^{\bullet}$.*

Proof. From Lemma 5.1 we know that the matrices R_{n+1} and L_{n+1} defined by (1.6) and (1.8) are both positive Hermitian. In particular, $\sqrt{R_{n+1}}^{+} = \sqrt{R_{n+1}}^{-1}$ and $\sqrt{L_{n+1}}^{+} = \sqrt{L_{n+1}}^{-1}$. Consequently, in view of (5.4), (5.5), (4.5), (4.6), and Remark 5.2, it is sufficient to show that the identities

$$\eta_n \sqrt{R_{n+1}} = b_n \sqrt{R_{n+1}}^{-1}, \quad \sqrt{L_{n+1}} \zeta_n = \sqrt{L_{n+1}}^{-1} d_n \tag{5.6}$$

and

$$\eta_n^{\#}\Gamma_0^{-*}\sqrt{R_{n+1}} = a_n\sqrt{R_{n+1}}^{\,-1}, \quad \sqrt{L_{n+1}}\Gamma_0^{-*}\zeta_n^{\#} = \sqrt{L_{n+1}}^{\,-1}c_n \tag{5.7}$$

are satisfied, where the matrix polynomials η_n, ζ_n, $\eta_n^{\#}$, and $\zeta_n^{\#}$ are defined by (5.2) and (5.3) as well as the matrix polynomials a_n, b_n, c_n, and d_n are given as in (1.9), (1.10), (1.11), and (1.12). The case $n = 0$ is trivial. Now let $n \geq 1$. Obviously, the nonnegative Hermitian block Toeplitz matrix T_n given by (1.1) and (1.2) can be represented via

$$T_n = \begin{pmatrix} T_{n-1} & Z_n^* \\ Z_n & \operatorname{Re}\Gamma_0 \end{pmatrix} \quad \text{and} \quad T_n = \begin{pmatrix} \operatorname{Re}\Gamma_0 & Y_n^* \\ Y_n & T_{n-1} \end{pmatrix}, \tag{5.8}$$

where Z_n and Y_n are given as in (1.7). From the first block representation in (5.8) we get

$$\Big(0,\ I_q\Big)T_n^{-1} = L_{n+1}^{-1}\Big(-Z_nT_{n-1}^{-1},\ I_q\Big)$$

and consequently, by virtue of (1.12) and (5.2), the identity $\zeta_n = L_{n+1}^{-1}d_n$. This implies immediately the second formula in (5.6). Moreover, the block Toeplitz matrix S_n given by (1.1) admits the block representations

$$S_n = \begin{pmatrix} S_{n-1} & 0 \\ 2\,Z_n & \Gamma_0 \end{pmatrix} \quad \text{and} \quad S_n = \begin{pmatrix} \Gamma_0 & 0 \\ 2\,Y_n & S_{n-1} \end{pmatrix}. \tag{5.9}$$

The first formula in (5.9), the first formula in (5.1), and

$$2\,Z_n - Z_nT_{n-1}^{-1}S_{n-1} = Z_nT_{n-1}^{-1}S_{n-1}^*$$

yield

$$\Gamma_0^{-*}\Big(0,\ I_q\Big)(T_n^{\#})^{-1} = L_{n+1}^{-1}\Big(-Z_nT_{n-1}^{-1},\ I_q\Big)S_n = L_{n+1}^{-1}\Big(Z_nT_{n-1}^{-1}S_{n-1}^*,\ \Gamma_0\Big).$$

Using (1.11) and (5.3) then the second formula in (5.7) follows. From the second identities in (5.8), (5.9), and (5.1) one can similarly derive the first formulas in (5.6) and (5.7). □

In order to describe the Arov-Kreĭn representation of the solution set of the nondegenerate matricial Carathéodory problem we give some further notations. Let B be a complex $2q \times 2q$ matrix and let

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

be the $q \times q$ block partition of B. If the set $\mathcal{D} := \{X \in \mathbb{C}^{q\times q} : \det(B_{21}X + B_{22}) \neq 0\}$ (respectively, $\mathcal{E} := \{X \in \mathbb{C}^{q\times q} : \det(XB_{12} + B_{22}) \neq 0\}$) is nonempty, then the right (respectively, left) linear fractional transformation $\mathcal{S}_B : \mathcal{D} \to \mathbb{C}^{q\times q}$ is given by

$$\mathcal{S}_B(X) := (B_{11}X + B_{12})(B_{21}X + B_{22})^{-1}, \quad X \in \mathcal{D},$$

(respectively, $\mathcal{T}_B : \mathcal{E} \to \mathbb{C}^{q\times q}$ is given by

$$\mathcal{T}_B(X) := (XB_{12} + B_{22})^{-1}(XB_{11} + B_{21}), \quad X \in \mathcal{E}\ ).$$

Now we are able to prove the announced result due to Arov and Kreĭn.

Theorem 5.6. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a nondegenerate $q \times q$ Carathéodory sequence. Further, let Ω be a complex $q \times q$ matrix-valued function defined on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *Ω belongs to $\mathcal{C}_q[\mathbb{D}, (\Gamma_j)_{j=0}^n]$.*

(ii) *There is a $q \times q$ Schur function g in $\mathbb{D}$ such that Ω can be represented via*

$$\Omega(z) = \mathcal{S}_{\Phi_n(z)}\big(g(z)\big), \quad z \in \mathbb{D}.$$

(iii) *There is a $q \times q$ Schur function h in $\mathbb{D}$ such that Ω can be represented via*

$$\Omega(z) = \mathcal{T}_{\Psi_n(z)}\big(h(z)\big), \quad z \in \mathbb{D}.$$

If (i) *is fulfilled, then $g = h$ and $g(z) = \mathcal{S}_{(\Phi_n(z))^{-1}}\big(\Omega(z)\big)$ for each $z \in \mathbb{D} \setminus \{0\}$.*

Proof. Combine Theorem 1.1, Lemma 5.1, Remark 5.2 and Lemma 5.5. □

An alternative proof of Theorem 5.6 was given in [FK1, Part V]. This proof is based on the interrelation between the matricial Carathéodory problem and the matricial Schur problem and makes essentially use of the analysis of the Schur-Potapov algorithm for matrix-valued Schur functions which was done in [FK2] (see also [DFK, Section 3.8]) on the basis of the foregoing papers of Delsarte, Genin, and Kamp [DGK1] and [DGK2] on orthogonal matrix polynomials and related questions.

6. The case of a unique solution

In this section, we consider finally the case of a given $q \times q$ Carathéodory sequence $(\Gamma_j)_{j=0}^n$ for which the central $q \times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^n$ is the unique $q \times q$ Carathéodory function Ω fulfilling (1.3) for each $j \in \mathbb{N}_{0,n}$.

Lemma 6.1. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. Further, let $(\Gamma_k)_{k=0}^\infty$ be the central $q \times q$ Carathéodory sequence corresponding to $(\Gamma_j)_{j=0}^n$ and, for each $\ell \in \mathbb{N}_{n+1,\infty}$, let the matrices L_ℓ and R_ℓ be defined as in* (1.6) *and* (1.8) *with respect to $(\Gamma_j)_{j=0}^\ell$. Then the following statements are equivalent:*

(i) *There is a unique Carathéodory function Ω (in $\mathbb{D}$) such that the relation* (1.3) *is fulfilled for each $j \in \mathbb{N}_{0,n}$ (namely the central $q \times q$ Carathéodory function $\Omega = \Omega_{c,n}$ corresponding to $(\Gamma_j)_{j=0}^n$).*

(ii) *For each $\ell \in \mathbb{N}_{n+1,\infty}$, the identities $L_\ell = 0$ and $R_\ell = 0$ hold.*

(iii) *$L_{n+1} = 0$ or $R_{n+1} = 0$.*

Proof. In view of the definition of the involved parameters, the connection between $q \times q$ Carathéodory sequences and $q \times q$ Carathéodory functions (see, e.g., [BGR] or [Ko]), and the equality $\operatorname{rank} L_\ell = \operatorname{rank} R_\ell$ for each $\ell \in \mathbb{N}_{n+1,\infty}$ (cf. [FK3, Remark 2]), the assertion follows from [DFK, Theorem 3.4.1 and Remark 3.4.3]. □

Proposition 6.2. *Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence. If $n \geq 1$, then let $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$. Furthermore, let a_n, b_n, c_n, and d_n be the matrix polynomials which are defined by* (1.15), (1.16), (1.17), *and* (1.18). *The following statements are equivalent:*

(i) *There is a unique Carathéodory function Ω (in $\mathbb{D}$) such that the relation* (1.3) *is fulfilled for each $j \in \mathbb{N}_{0,n}$.*

(ii) *The identity $\tilde{a}_n^{[n]} b_n = -\tilde{b}_n^{[n]} a_n$ is satisfied.*

(iii) *The identity $c_n \tilde{d}_n^{[n]} = -d_n \tilde{c}_n^{[n]}$ is satisfied.*

(iv) *For each $z \in \mathbb{T}$, $\operatorname{Re}\Big(\big(a_n(z)\big)^* b_n(z)\Big) = 0$ and $\operatorname{Re}\Big(c_n(z)\big(d_n(z)\big)^*\Big) = 0$.*

(v) $\operatorname{Re}\Big(\big(a_n(z)\big)^* b_n(z)\Big) = 0$ *or* $\operatorname{Re}\Big(c_n(z)\big(d_n(z)\big)^*\Big) = 0$ *for some $z \in \mathbb{T}$.*

Proof. Use Lemma 6.1 in combination with Proposition 2.4. □

Corollary 6.3. *Let $n \in \mathbb{N}_0$, let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence, and let $\Omega_{c,n}$ be the central $q \times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^n$. If $n \geq 1$, then let $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$. Furthermore, let a_n, b_n, c_n, and d_n be the matrix polynomials which are defined by* (1.15), (1.16), (1.17), *and* (1.18). *The following statements are equivalent:*

(i) *There is a unique Carathéodory function Ω (in $\mathbb{D}$) such that the relation* (1.3) *is fulfilled for each $j \in \mathbb{N}_{0,n}$.*

(ii) *$\Omega_{c,n}$ is the restriction of the rational matrix function $-(\tilde{b}_n^{[n]})^{-1}\tilde{a}_n^{[n]}$ onto $\mathbb{D}$.*

(iii) *$\Omega_{c,n}$ is the restriction of the rational matrix function $-\tilde{c}_n^{[n]}(\tilde{d}_n^{[n]})^{-1}$ onto $\mathbb{D}$.*

(iv) *The equality* $\operatorname{Re}\Big(-\big(\tilde{b}_n^{[n]}(z)\big)^{-1}\tilde{a}_n^{[n]}(z)\Big) = 0$ *holds for each $z \in \mathbb{T} \setminus \mathcal{N}_{\tilde{b}_n^{[n]}}$ and the equality* $\operatorname{Re}\Big(-\tilde{c}_n^{[n]}(z)\big(\tilde{d}_n^{[n]}(z)\big)^{-1}\Big) = 0$ *holds for each $z \in \mathbb{T} \setminus \mathcal{N}_{\tilde{d}_n^{[n]}}$.*

(v) *The complex $q \times q$ matrix* $\operatorname{Re}\Big(-\big(\tilde{b}_n^{[n]}(z)\big)^{-1}\tilde{a}_n^{[n]}(z)\Big)$ *is nonnegative Hermitian for some $z \in \mathbb{T} \setminus \mathcal{N}_{\tilde{b}_n^{[n]}}$ or for some $z \in \mathbb{T} \setminus \mathcal{N}_{\tilde{d}_n^{[n]}}$ the complex $q \times q$ matrix* $\operatorname{Re}\Big(-\tilde{c}_n^{[n]}(z)\big(\tilde{d}_n^{[n]}(z)\big)^{-1}\Big)$ *is nonnegative Hermitian.*

Proof. Taking into account that Corollary 2.5 yields that the set $\mathcal{N}_{b_n}$ (respectively, $\mathcal{N}_{d_n}$) consists of at most $n \cdot q$ elements and hence the set $\mathcal{N}_{\tilde{b}_n^{[n]}}$ (respectively, $\mathcal{N}_{\tilde{d}_n^{[n]}}$) as well, the equivalence of (i), (ii), and (iii) follows from Proposition 6.2, Theorem 1.3, and a continuity argument. Furthermore, (iv) implicates immediately (v). Moreover, Corollary 2.5 and [DFK, Lemma 1.2.2] imply

$$\operatorname{Re}\Big(\big(\tilde{b}_n^{[n]}(z)\big)^{-1}\tilde{a}_n^{[n]}(z)\Big) = \big(\tilde{b}_n^{[n]}(z)\big)^{-1} R_{n+1}\big(\tilde{b}_n^{[n]}(z)\big)^{-*} \geq 0, \quad z \in \mathbb{T} \setminus \mathcal{N}_{\tilde{b}_n^{[n]}},$$

and

$$\operatorname{Re}\Big(\tilde{c}_n^{[n]}(z)\big(\tilde{d}_n^{[n]}(z)\big)^{-1}\Big) = \big(\tilde{d}_n^{[n]}(z)\big)^{-*} L_{n+1}\big(\tilde{d}_n^{[n]}(z)\big)^{-1} \geq 0, \quad z \in \mathbb{T} \setminus \mathcal{N}_{\tilde{d}_n^{[n]}},$$

where $\mathbb{T} \setminus \mathcal{N}_{\tilde{b}_n^{[n]}} = \mathbb{T} \setminus \mathcal{N}_{b_n}$ and $\mathbb{T} \setminus \mathcal{N}_{\tilde{d}_n^{[n]}} = \mathbb{T} \setminus \mathcal{N}_{d_n}$. Therefore (v) involves that $R_{n+1} = 0$ or $L_{n+1} = 0$ is fulfilled on the one hand and on the other hand if

$R_{n+1} = 0$ and $L_{n+1} = 0$ is satisfied then it follows (iv), so that Lemma 6.1 finally yields the equivalence of (i), (iv), and (v). □

Now we are going to give a further characterization of the case that the matricial Carathéodory problem has a unique solution. This characterization is also given in terms of the central matrix-valued Carathéodory function $\Omega_{c,n}$ corresponding to the given data. Recall that a $q \times q$ Carathéodory function Ω is called *degenerate* if the block Toeplitz matrix T_m given by (1.1) and (1.2) is singular for some nonnegative integer m, where $(\Gamma_k)_{k=0}^{\infty}$ is the sequence of complex $q \times q$ matrices fulfilling the Taylor series representation (3.10) of Ω. Clearly, if the matricial Carathéodory problem has a unique solution, then the function $\Omega_{c,n}$ corresponding to the given data has to be a degenerate Carathéodory function. The following considerations show that $\Omega_{c,n}$ has a very specialized structure in that case.

Remark 6.4. Let $r \in \mathbb{N}$, let $(z_s)_{s=1}^{r}$ be a sequence from the unit circle $\mathbb{T}$, let $(A_s)_{s=1}^{r}$ be a sequence of nonnegative Hermitian $q \times q$ matrices, and let H be a Hermitian $q \times q$ matrix. Then it is readily checked that the matrix-valued function $\Psi : \mathbb{C} \setminus \{z_1, z_2, \ldots, z_r\} \to \mathbb{C}^{q\times q}$ given by

$$\Psi(z) := \sum_{s=1}^{r} \frac{z_s + z}{z_s - z} A_s + iH$$

satisfies $\operatorname{Re}\Psi(z) \geq 0$ for each $z \in \mathbb{D}$ and $\operatorname{Re}\Psi(z) = 0$ for each $z \in \mathbb{T} \setminus \{z_1, z_2, \ldots, z_r\}$.

Remark 6.5. Let μ be a nonnegative Hermitian $q \times q$ Borel measure on $\mathbb{T}$ and let $\tau := \operatorname{tr}\mu$ be the trace measure of μ. Since a nonnegative Hermitian $q \times q$ matrix A is equal to $0_{q\times q}$ if and only if $\operatorname{tr} A = 0$, the following statements are equivalent:

(i) There is a finite subset $\mathfrak{F}$ of $\mathbb{T}$ such that $\mu(\mathbb{T} \setminus \mathfrak{F}) = 0_{q\times q}$.

(ii) There is a finite subset $\mathfrak{G}$ of $\mathbb{T}$ such that $\tau(\mathbb{T} \setminus \mathfrak{G}) = 0$.

If (i) or (ii) holds, one can choose $\mathfrak{F} = \mathfrak{G}$.

Lemma 6.6. *Let Ω be a $q \times q$ Carathéodory function (in $\mathbb{D}$) and let $\varphi := \operatorname{tr}\Omega$. The following statements are equivalent:*

(i) *There are an $\ell \in \mathbb{N}$, a sequence $(w_s)_{s=1}^{\ell}$ of points belonging to $\mathbb{T}$, a sequence $(a_s)_{s=1}^{\ell}$ of nonnegative numbers, and a real number h such that*

$$\varphi(z) = \sum_{s=1}^{\ell} \frac{w_s + z}{w_s - z} a_s + i\,h, \quad z \in \mathbb{D}. \tag{6.1}$$

(ii) *There are an $r \in \mathbb{N}$, a sequence $(z_s)_{s=1}^{r}$ of points belonging to $\mathbb{T}$, a sequence $(A_s)_{s=1}^{r}$ of nonnegative Hermitian $q \times q$ matrices, and a Hermitian $q \times q$ matrix H such that*

$$\Omega(z) = \sum_{s=1}^{r} \frac{z_s + z}{z_s - z} A_s + i\,H, \quad z \in \mathbb{D}. \tag{6.2}$$

(iii) *Ω is the restriction of a rational $q \times q$ matrix-valued function $\Omega^{\diamondsuit}$ which satisfies $\operatorname{Re}\Omega^{\diamondsuit}(z) = 0_{q\times q}$ for each $\mathbb{T} \setminus \mathfrak{F}$, where $\mathfrak{F}$ is some finite set.*

(iv) *φ is the restriction of a rational (complex-valued) function $\varphi^\diamondsuit$ which satisfies $\operatorname{Re}\varphi^\diamondsuit(z) = 0$ for each $\mathbb{T} \setminus \mathfrak{G}$, where $\mathfrak{G}$ is some finite set.*

In particular, one can choose $\ell = r$, $w_s = z_s$ for each $s \in \mathbb{N}_{1,r}$, $\mathfrak{F} = \mathfrak{G}$, and $\mathfrak{F} = \{z_1, z_2, \dots, z_r\}$.

Proof. (i) $\Rightarrow$ (ii): Clearly, since Ω is a $q \times q$ Carathéodory function, φ is a 1×1 Carathéodory function. For each $x \in \mathbb{T}$, let ε_x be denote the Dirac measure on the Borelian σ-algebra of $\mathbb{T}$ which has its unit mass at the point x. Because of (i) and the Riesz-Herglotz theorem (see, e.g., [DFK, Theorem 2.2.2]), by setting

$$\tau := \sum_{s=1}^{\ell} a_s \varepsilon_{w_s}$$

we obtain the Riesz-Herglotz measure of φ. In particular, we have $\tau(\mathbb{T} \setminus \mathfrak{G}) = 0$ for $\mathfrak{G} := \{w_1, w_2, \dots, w_\ell\}$. Thus if μ denotes the Riesz-Herglotz measure of Ω then $\varphi = \operatorname{tr}\Omega$ implies $\tau = \operatorname{tr}\mu$ and from Remark 6.5 we get that $\mu(\mathbb{T} \setminus \mathfrak{G}) = 0_{q\times q}$. Applying the matricial version of the Riesz-Herglotz theorem (see, e.g., [DFK, Theorem 2.2.2]) we can finally conclude that (6.2) is fulfilled with $r := \ell$, $z_s := w_s$ for each $s \in \mathbb{N}_{1,r}$, some sequence $(A_s)_{s=1}^r$ of nonnegative Hermitian $q \times q$ matrices, and some Hermitian $q \times q$ matrix H.
(ii) $\Rightarrow$ (iii): In view of Remark 6.4 we see that (ii) yields (iii) by the special choice $\mathfrak{F} := \{z_1, z_2, \dots, z_r\}$.
(iii) $\Rightarrow$ (iv): This implication is obviously fulfilled with $\mathfrak{G} := \mathfrak{F}$.
(iv) $\Rightarrow$ (i): Since $\varphi \in \mathcal{C}_1(\mathbb{D})$, from (iv) it follows that the function $1 + \varphi^\diamondsuit$ does not vanish in $\mathbb{D}$ and that

$$B := \frac{1 - \varphi^\diamondsuit}{1 + \varphi^\diamondsuit} \tag{6.3}$$

is a well-defined rational function such that its restriction onto $\mathbb{D}$ belongs to $\mathcal{S}_{1\times 1}(\mathbb{D})$. Furthermore, (iv) and (6.3) imply that $|B(z)| = 1$ for each $z \in \mathbb{T} \setminus \mathfrak{G}$. Consequently (see, e.g., [FFK, Lemma 36]), the function B is a finite Blaschke product. In view of (6.3), some well-known results on finite Blaschke products and the Cayley transform (see, e.g., [Sc] and use [DFK, Lemma 1.1.21]) one can conclude that φ is a degenerate 1×1 Carathéodory function. Finally, applying [FKL, Proposition 3.2] we obtain that (6.1) is fulfilled for some $\ell \in \mathbb{N}$, some sequence $(w_s)_{s=1}^\ell$ of points belonging to $\mathbb{T}$, some sequence $(a_s)_{s=1}^\ell$ of nonnegative numbers, and some real number h, whereby Remark 6.4 and (iv) supply that one can particularly choose $\{w_1, w_2, \dots, w_\ell\} = \mathfrak{G}$. $\square$

Note that Theorem 1.3 and Corollary 2.5 show particularly that there is a unique rational extension $\Omega_{c,n}^\diamondsuit$ of $\Omega_{c,n}$ to $\mathbb{C} \setminus \{z_1, z_2, \dots, z_r\}$ for some complex numbers $z_1, z_2, \dots, z_r$ with $r \le n \cdot q$.

Theorem 6.7. *Let $n \in \mathbb{N}_0$, let $(\Gamma_j)_{j=0}^n$ be a $q \times q$ Carathéodory sequence, let $\Omega_{c,n}$ be the central $q \times q$ Carathéodory function corresponding to $(\Gamma_j)_{j=0}^n$, and let $\Omega_{c,n}^\diamondsuit$ be the unique rational extension of $\Omega_{c,n}$. Then the following statements are equivalent:*

(i) *There is a unique Carathéodory function* Ω *(in* $\mathbb{D}$*) such that the condition* (1.3) *is fulfilled for each* $j \in \mathbb{N}_{0,n}$ *(namely* $\Omega = \Omega_{c,n}$*).*

(ii) *There are an* $r \in \mathbb{N}$*, a sequence* $(z_s)_{s=1}^r$ *of points belonging to the unit circle* $\mathbb{T}$*, a sequence* $(A_s)_{s=1}^r$ *of nonnegative Hermitian* $q\times q$ *matrices, and a Hermitian* $q \times q$ *matrix* H *such that*

$$\Omega_{c,n}(z) = \sum_{s=1}^{r} \frac{z_s + z}{z_s - z} A_s + i\,H, \quad z \in \mathbb{D}. \tag{6.4}$$

(iii) *There is a finite subset* $\mathfrak{F}$ *of* $\mathbb{T}$ *such that* $\operatorname{Re}\Omega^{\diamondsuit}_{c,n}(z) = 0_{q\times q}$ *for each* $\mathbb{T}\setminus\mathfrak{F}$*.*

(iv) $\operatorname{Re}\Omega^{\diamondsuit}_{c,n}(z) = 0_{q\times q}$ *for a* $z \in \mathbb{T}\setminus\mathcal{N}_{b_n}$ *or* $z \in \mathbb{T}\setminus\mathcal{N}_{d_n}$*, where* b_n *and* d_n *are defined as in* (1.16) *and* (1.18) *with some* $V_n \in \mathcal{Y}_n$ *and* $W_n \in \mathcal{Z}_n$ *if* $n \geq 1$*.*

Proof. (i) $\Rightarrow$ (iii): If $n \geq 1$ then let $V_n \in \mathcal{Y}_n$. Furthermore, let a_n and b_n be defined as in (1.15) and (1.16). Thus part (a) of Theorem 1.3 yields that the equality

$$\Omega^{\diamondsuit}_{c,n}(z) = a_n(z)\big(b_n(z)\big)^{-1} \tag{6.5}$$

is fulfilled for each $z \in \mathbb{C}\setminus\mathcal{N}_{b_n}$. From (6.5) and Corollary 2.5 we obtain then

$$\big(b_n(z)\big)^*\big(\operatorname{Re}\Omega^{\diamondsuit}_{c,n}(z)\big)b_n(z) = \big(b_n(z)\big)^*\operatorname{Re}\Big(a_n(z)\big(b_n(z)\big)^{-1}\Big)\,b_n(z) = R_{n+1} \tag{6.6}$$

for each $z \in \mathbb{T}\setminus\mathcal{N}_{b_n}$, where the matrix R_{n+1} is defined as in (1.6) and (1.8). In view of (i) and Lemma 6.1 we have $R_{n+1} = 0_{q\times q}$, so that (6.6) implies $\operatorname{Re}\Omega^{\diamondsuit}_{c,n}(z) = 0_{q\times q}$ for each $z \in \mathbb{T}\setminus\mathcal{N}_{b_n}$. Since Corollary 2.5 includes particularly that the set $\mathcal{N}_{b_n}$ consists of at most $n\cdot q$ elements, it follows (iii).
(ii) $\Leftrightarrow$ (iii): Taking into account Theorem 1.3, the equivalence of (ii) and (iii) is an easy consequence of Lemma 6.6.
(iii) $\Rightarrow$ (iv): Because $\mathfrak{F}$ is a finite set but Corollary 2.5 shows that $\mathbb{T}\setminus\mathcal{N}_{b_n}$ is an infinite set, this implication follows immediately.
(iv) $\Rightarrow$ (i): If $\operatorname{Re}\Omega^{\diamondsuit}_{c,n}(z) = 0_{q\times q}$ for some $z \in \mathbb{T}\setminus\mathcal{N}_{b_n}$ then by using (6.6) we get $R_{n+1} = 0_{q\times q}$. Similarly, in the case of $\operatorname{Re}\Omega^{\diamondsuit}_{c,n}(z) = 0_{q\times q}$ for a certain $z \in \mathbb{T}\setminus\mathcal{N}_{d_n}$, where d_n is the matrix polynomial defined as in (1.18) with some $W_n \in \mathcal{Z}_n$ if $n \geq 1$, by an application of part (b) of Theorem 1.3 and Corollary 2.5 one can conclude $L_{n+1} = 0_{q\times q}$, where the matrix L_{n+1} is defined as in (1.6) and (1.8). Hence (iv) and Lemma 6.1 supply (i). □

Remark 6.8. Let $n \in \mathbb{N}_0$ and let $(\Gamma_j)_{j=0}^n$ be a $q\times q$ Carathéodory sequence such that (i) of Theorem 6.7 is satisfied. If $n = 0$ then Lemma 6.1 implies $\operatorname{Re}\Gamma_0 = 0$ and hence the constant function (defined on $\mathbb{D}$) with value $i\operatorname{Im}\Gamma_0$ is the unique Carathéodory function Ω (in $\mathbb{D}$) such that $\Omega(0) = \Gamma_0$. If $n \in \mathbb{N}$, then by virtue of Theorem 1.3, Corollary 2.5, and Theorem 6.7 it is not hard to accept that one can choose an $r \in \mathbb{N}_{1,n\cdot q}$ and $z_1, \ldots, z_r$ belonging to

$$\{z \in \mathbb{T} : \det b_n(z) = 0\} \cap \{z \in \mathbb{T} : \det d_n(z) = 0\},$$

where b_n and d_n are defined as in (1.16) and (1.18) with some $V_n \in \mathcal{Y}_n$ and $W_n \in \mathcal{Z}_n$, such that the central $q\times q$ Carathéodory function $\Omega_{c,n}$ corresponding to $(\Gamma_j)_{j=0}^n$ admits, for each $z \in \mathbb{D}$, the representation (6.4) with some Hermitian $q \times q$ matrix H and sequence $(A_s)_{s=1}^r$ of nonnegative Hermitian $q \times q$ matrices.

Corollary 6.9. *Let $n \in \mathbb{N}$, let $(z_s)_{s=1}^n$ be a sequence of pairwise different points belonging to $\mathbb{T}$, let $(A_s)_{s=1}^n$ be a sequence of nonnegative Hermitian $q \times q$ matrices such that $\sum_{s=1}^n A_s$ is a positive Hermitian $q \times q$ matrix, and let H be a Hermitian $q \times q$ matrix. Further, let $\Omega : \mathbb{D} \to \mathbb{C}^{q\times q}$ be defined by*

$$\Omega(z) := \sum_{s=1}^n \frac{z_s + z}{z_s - z} A_s + i\, H.$$

Then Ω is a $q \times q$ Carathéodory function such that $\operatorname{Re}\Omega(0)$ is a positive Hermitian $q \times q$ matrix. In particular, $\Omega(z)$ is a nonsingular matrix for each $z \in \mathbb{D}$ and Ω^{-1} is a $q \times q$ Carathéodory function. Moreover, there are a sequence $(u_s)_{s=1}^{n\cdot q}$ of pairwise different points belonging to $\mathbb{T}$, a sequence $(B_s)_{s=1}^{n\cdot q}$ of nonnegative Hermitian $q \times q$ matrices, and a Hermitian $q \times q$ matrix K such that Ω^{-1} admits the representation

$$\big(\Omega(z)\big)^{-1} = \sum_{s=1}^{n\cdot q} \frac{u_s + z}{u_s - z} B_s + i\, K, \quad z \in \mathbb{D}.$$

Proof. Let

$$\Gamma_0 := \sum_{s=1}^n A_s + i\, H \qquad \text{and} \qquad \Gamma_j := 2 \sum_{s=1}^n z_s^{-j} A_s, \quad j \in \mathbb{N}_{1,n}.$$

From [FKL, Corollary 3.4] we know that Ω is the unique Carathéodory function Ω such that the condition (1.3) is fulfilled for each $j \in \mathbb{N}_{0,n}$. Furthermore, by virtue of $\operatorname{Re}\Omega(0) = \sum_{s=1}^n A_s$ we get that $\operatorname{Re}\Omega(0)$ is a positive Hermitian matrix. Consequently, since [DFK, Proposition 2.1.3, Lemma 1.1.13, Lemma 2.1.10 and Lemma 1.1.21] imply that if Ψ is a Carathéodory function fulfilling $\operatorname{Re}\Psi(0) \in \mathbb{C}_{>}^{q\times q}$ then $\Psi(z)$ is a nonsingular matrix for each $z \in \mathbb{D}$, that Ψ^{-1} is a $q \times q$ Carathéodory function as well, and that the first $n+1$ coefficients of the Taylor expansion of Ψ^{-1} at the point 0 are uniquely determined by the first $n+1$ coefficients of the Taylor expansion of Ψ at the point 0, an application of Theorem 6.7 and Remark 6.8 completes the proof. □

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A Gohberg-Heinig Type Inversion Formula Involving Hankel Operators

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Abstract. A Gohberg-Heinig type inversion formula is derived for operators $I - K_2K_1$, where K_1 and K_2 are Hankel integral operators acting between vector-valued L_1-spaces over $[0, \infty]$. The main result is first proved, by using linear algebra tools, for the case when the corresponding kernel functions have a finite dimensional stable exponential representation.

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0. Introduction

This paper deals with the inversion of the operator $I - K_2K_1$, where K_1 and K_2 are Hankel operators given by

$$(K_j f)(t) = \int_0^\infty k_j(t+s)f(s)\,ds, \quad t \geq 0, \quad j = 1, 2. \tag{0.1}$$

Here $k_1 \in L_1^{m\times p}(\mathbb{R}_+)$ and $k_2 \in L_1^{p\times m}(\mathbb{R}_+)$, that is, k_1 and k_2 are matrix functions on $\mathbb{R}_+ = [0, \infty)$, of sizes $m \times p$ and $p \times m$, respectively, and the entries of k_1 and k_2 are Lebesgue integrable over $\mathbb{R}_+$. We consider K_1 as an operator from $L_1^p(\mathbb{R}_+)$ into $L_1^m(\mathbb{R}_+)$, and K_2 as an operator from $L_1^m(\mathbb{R}_+)$ into $L_1^p(\mathbb{R}_+)$. The main result is the following theorem.

Theorem 0.1. *The following statements are equivalent:*

(i) *the operator $I - K_2K_1$ is invertible,*

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(ii) *there exist solutions* $a_1 \in L_1^{p\times m}(\mathbb{R}_+)$ *and* $a_2 \in L_1^{m\times p}(\mathbb{R}_+)$ *to the equations*

$$a_1(t) - \int_0^\infty \int_0^\infty k_2(t+s)k_1(s+r)a_1(r)\,dsdr = -k_2(t), \quad t \geq 0, \tag{0.2}$$

$$a_2(t) - \int_0^\infty \int_0^\infty k_1(t+s)k_2(s+r)a_2(r)\,dsdr = -k_1(t), \quad t \geq 0, \tag{0.3}$$

(iii) *there exist solutions* $\alpha_1 \in L_1^{p\times m}(\mathbb{R}_+)$ *and* $\alpha_2 \in L_1^{m\times p}(\mathbb{R}_+)$ *to the equations*

$$\alpha_1(t) - \int_0^\infty \int_0^\infty \alpha_1(r)k_1(r+s)k_2(s+t)\,dsdr = -k_2(t), \quad t \geq 0, \tag{0.4}$$

$$\alpha_2(t) - \int_0^\infty \int_0^\infty \alpha_2(r)k_2(r+s)k_1(s+t)\,dsdr = -k_1(t), \quad t \geq 0. \tag{0.5}$$

Moreover, in this case the inverse of $I - K_2K_1$ *is given by*

$$\begin{aligned}((I-K_2K_1)^{-1}f)(t) = f(t) &+ \int_0^\infty \int_0^\infty a(t+s)b(s+r)f(r)\,dsdr \\ &- \int_0^\infty \int_0^\infty c(t+s)d(s+r)f(r)\,dsdr, \quad t \geq 0,\end{aligned} \tag{0.6}$$

where $a = a_1,\ b = \alpha_2$, *and*

$$c(t) = \int_0^\infty k_2(t+s)a_2(s)\,ds, \quad d(t) = \int_0^\infty \alpha_1(s)k_1(t+s)\,ds \quad (t \geq 0). \tag{0.7}$$

Here $a_1, a_2, \alpha_1, \alpha_2$ *are the functions determined by* (0.2)–(0.5).

For the case when $p = m$ and $k_2(\cdot) = k_1(\cdot)^*$ the equivalence of the statements (i)–(iii) is proved in Chapter 12 of [3] (see also [1]). It turns out that with some modifications the proof of the equivalence of (i)–(iii) given in [3] carries over to the more general setting considered here (see the second part of Section 3). The inversion formula (0.6) is new. It can be viewed as an analogue of the Gohberg-Heinig formula for convolution operators on a finite interval, [4]. The discrete analogue of Theorem 0.1, with the operators K_1 and K_2 being replaced by Hankel operators with Wiener algebra symbols on ℓ_1^m, is known and can be found in [2].

Our approach is inspired by the proof of the Gohberg-Heinig inversion theorem given in [5]. We shall obtain the inversion formula (0.6) in two steps. In the first step k_1 and k_2 admit a stable exponential representation, that is,

$$k_1(t) = C_1e^{tA_1}B_1, \qquad k_2(t) = C_2e^{tA_2}B_2. \tag{0.8}$$

Here A_1 and A_2 are square matrices of sizes $n_1 \times n_1$ and $n_2 \times n_2$, respectively, and we require these matrices to be *stable*, that is, the eigenvalues of A_1 and A_2 are in the open left half plane. Furthermore, C_1 and C_2 are matrices of sizes $m \times n_1$ and $p \times n_2$, respectively, and B_1 and B_2 are matrices of sizes $n_1 \times p$ and $n_2 \times m$, respectively. In this case K_1 and K_2 are operators of finite rank, and we show that the inversion formula (0.6) can be obtained by inverting the matrix $M = I - PQ$, where P and Q are the unique matrix solutions of

$$A_1P + PA_2 = -B_1C_2, \qquad A_2Q + QA_1 = -B_2C_1. \tag{0.9}$$

Notice that P has size $n_1 \times n_2$ and Q has size $n_2 \times n_1$, and that these matrices are also given by

$$P = \int_0^\infty e^{sA_1} B_1 C_2 e^{sA_2}\, ds, \qquad Q = \int_0^\infty e^{sA_2} B_2 C_1 e^{sA_1}\, ds. \tag{0.10}$$

We refer to M as the *indicator* for $I - K_2K_1$ corresponding to representations (0.8). In the second step we use the fact that any L_1-kernel function is the limit in the L_1-norm of a sequence of kernels with a stable exponential representation. We derive the inversion formula (0.6) as a limit of the inverse formula for the case when k_1 and k_2 are given by (0.8).

This paper consists of three sections (not counting this introduction). In Section 1 we study the indicator, and we show that it can be inverted whenever the matrix equations $MZ = -PB_2$ and $M^{\#}U = -QB_1$ are solvable. Here $M^{\#} = I - QP$ is the *associate indicator*, which is equal to the indicator for $L^{\#} = I - K_1K_2$ corresponding to representations (0.8). In Section 2 we prove Theorem 0.1 for kernel functions of the type (0.8). In the final section we prove the equivalence of statements (i), (ii) and (iii) in Theorem 0.1, and we use the approximation argument referred to above to prove the inversion formula (0.6) for the general case.

Finally, we mention that in the sequel we shall often use the following fact. If $A : \mathcal{X} \to \mathcal{Y}$ and $B : \mathcal{Y} \to \mathcal{X}$ are bounded linear operators acting between Banach spaces, then $I_{\mathcal{Y}} - AB$ is invertible if and only if $I_{\mathcal{X}} - BA$ is invertible, and in this case

$$(I_{\mathcal{Y}} - AB)^{-1} = I_{\mathcal{Y}} + A(I_{\mathcal{X}} - BA)^{-1}B. \tag{0.11}$$

1. The indicator

Throughout this section M is the indicator corresponding to the representations (0.8), and $M^{\#}$ is the associate indicator. In other words,

$$M = I - PQ, \qquad M^{\#} = I - QP,$$

where P and Q are determined by (0.9) or, equivalently, by (0.10). From the remark made at the end of the previous section, it is clear that M is invertible if and only if $M^{\#}$ is invertible.

Proposition 1.1. *The indicator M is invertible if and only if the following matrix equations are solvable:*

$$MZ = -PB_2, \qquad M^{\#}U = -QB_1. \tag{1.1}$$

Moreover, if M is invertible, then $M^{\#}$ is invertible, and

$$QM^{-1} = \int_0^\infty e^{sA_2} X_1 Y_1 e^{sA_1}\, ds - \int_0^\infty e^{sA_2} X_2 Y_2 e^{sA_1}\, ds \tag{1.2}$$

where

$$X_1 = (M^{\#})^{-1}B_2,\ X_2 = QM^{-1}B_1,\ Y_1 = C_1M^{-1},\ Y_2 = C_2QM^{-1}. \tag{1.3}$$

Proof. We first show that

$$MA_1 - A_1M = PB_2C_1 - B_1C_2Q. \tag{1.4}$$

To do this we use $M = I - PQ$ and the formulas for P and Q in (0.10). Indeed,

$$\begin{aligned} MA_1 &= A_1 - PQA_1 = A_1 - P(-B_2C_1 - A_2Q) \\ &= A_1 + PB_2C_1 + PA_2Q \\ &= A_1 + PB_2C_1 + (-A_1P - B_1C_2)Q \\ &= A_1 + PB_2C_1 - A_1PQ - B_1C_2Q \\ &= A_1M + PB_2C_1 - B_1C_2Q. \end{aligned}$$

This yields (1.4).

If M is invertible, then the matrix equations in (1.1) are solvable. To prove the reverse implication, assume the matrix equations in (1.1) are solvable. Since M is a square matrix of order n_1, it suffices to show that $x \in \mathbb{C}^{n_1}$ and $x^*M = 0$, imply that $x = 0$. To do this, we use (1.1). The identity $x^*M = 0$ together with the first identity in (1.1), yields $x^*PB_2 = 0$. Using the second identity in (1.1) and $MP = PM^\#$, we also have $x^*PQB_1 = 0$.

Using $x^*M = 0$, $x^*PB_2 = 0$ and $x^*B_1 = 0$ in (1.4) yields $x^*A_1M = 0$. Repeating the above arguments with x^*A_1 in place of x^* we obtain $x^*A_1^2M = 0$. Continuing by induction we see that $x^*A_1^nM = 0$ for $n = 0, 1, 2, \ldots$. As we have seen, $x^*M = 0$ implies $x^*B_1 = 0$. Thus $x^*A_1^nB_1 = 0$ for $n = 0, 1, 2, \ldots$. Using the formula for P in (0.10), we see that the latter implies that $x^*P = 0$. Hence $x^*PQ = 0$. But then

$$x^* = x^*(I - PQ) + x^*PQ = x^*M + x^*PQ = 0.$$

Thus M is invertible.

We already know that the invertibility of M implies that of $M^\#$. Thus to complete the proof it remains to prove (1.2). Assume M is invertible. We first show that

$$A_2(QM^{-1}) + (QM^{-1})A_1 = -(M^\#)^{-1}B_2C_1M^{-1} + QM^{-1}B_1C_2QM^{-1}. \tag{1.5}$$

Indeed, note that the definitions of M and $M^\#$ imply

$$M^\#A_2Q = (I - QP)A_2Q = A_2Q - Q(PA_2)Q$$

and

$$QA_1M = QA_1(I - PQ) = QA_1 - Q(A_1P)Q.$$

The sum of the above equations and (0.9) gives

$$M^\#A_2Q + QA_1M = -B_2C_1 + QB_1C_2Q.$$

Next, we premultiply by $(M^\#)^{-1}$ and postmultiply by M^{-1} to obtain

$$A_2QM^{-1} + (M^\#)^{-1}QA_1 = -(M^\#)^{-1}B_2C_1M^{-1} + (M^\#)^{-1}QB_1C_2QM^{-1}.$$

Now use that $QM = M^{\#}Q$, and hence $(M^{\#})^{-1}Q = QM^{-1}$. This proves (1.5). Since A_1 and A_2 are stable, (1.5) shows

$$QM^{-1} = \int_0^\infty e^{sA_2}\Big(- (M^{\#})^{-1}B_2C_1M^{-1} + QM^{-1}B_1C_2QM^{-1}\Big)e^{sA_1}\,ds.$$

Finally, use the matrices defined by (1.3) to obtain formula (1.2). □

2. The main theorem for kernel functions of stable exponential type

Throughout this section K_1 and K_2 are the Hankel operators given by (0.1), and we assume that the kernel functions k_1 and k_2 are given by (0.8). As before M is the indicator of $I - K_2K_1$ corresponding to the representations (0.8).

To analyze $I - K_2K_1$ in terms of the representations (0.8) we introduce the following auxiliary operators:

$$\Lambda_1 : \mathbb{C}^{n_1} \to L_1^m(\mathbb{R}_+), \qquad (\Lambda_1 x)(t) = C_1e^{tA_1}x,$$

$$\Lambda_2 : \mathbb{C}^{n_2} \to L_1^p(\mathbb{R}_+), \qquad (\Lambda_2 x)(t) = C_2e^{tA_2}x,$$

$$\Gamma_1 : L_1^p(\mathbb{R}_+) \to \mathbb{C}^{n_1}, \qquad \Gamma_1 f = \int_0^\infty e^{sA_1}B_1f(s)\,ds,$$

$$\Gamma_2 : L_1^m(\mathbb{R}_+) \to \mathbb{C}^{n_2}, \qquad \Gamma_2 f = \int_0^\infty e^{sA_2}B_2f(s)\,ds.$$

Allowing for a slight abuse of notation we shall apply Λ_1 and Λ_2 also to matrices, and Γ_1 and Γ_2 also to matrix functions. For instance, when X is an $n_1 \times q$ matrix, then Λ_1X is the matrix function of which the k-th column is obtained by applying Λ_1 to the k-th column of X. Similarly, if $a \in L_1^{p\times q}(\mathbb{R}_+)$, then $\Gamma_1 a$ is the $n_1 \times q$ matrix of which the k-th column is obtained by applying Γ_1 to the function given by the k-th column of a.

Note that

$$P = \Gamma_1\Lambda_2, \qquad Q = \Gamma_2\Lambda_1,$$

and hence $M = I - \Gamma_1\Lambda_2\Gamma_2\Lambda_1$. Furthermore, $K_1 = \Lambda_1\Gamma_1$ and $K_2 = \Lambda_2\Gamma_2$. It follows that $I - K_2K_1 = I - \Lambda_2\Gamma_2\Lambda_1\Gamma_1$. Now put $A = \Lambda_2\Gamma_2\Lambda_1$, $B = \Gamma_1$, and apply the result mentioned in the final paragraph of the introduction. This shows that $I - K_2K_1$ is invertible if and only if M is invertible, and in that case

$$\begin{aligned}(I - K_2K_1)^{-1} &= I + \Lambda_2\Gamma_2\Lambda_1(I - \Gamma_1\Lambda_2\Gamma_2\Lambda_1)^{-1}\Gamma_1\\ &= I + \Lambda_2Q(I - PQ)^{-1}\Gamma_1\\ &= I + \Lambda_2QM^{-1}\Gamma_1.\end{aligned}$$

Since (1.2) provides a formula for QM^{-1}, we shall see that the above calculation will allow us to prove (0.6) for the case when the kernel functions are given by (0.8). For this purpose we also need the following lemma.

Lemma 2.1. *Equation* (0.2) *is solvable if and only if the following matrix equation is solvable*

$$MZ = -PB_2. \tag{2.1}$$

More precisely, if $a \in L_1^{p\times m}(\mathbb{R}_+)$ *satisfies*

$$a(t) - \int_0^\infty \int_0^\infty k_2(t+s)k_1(s+r)a(r)\, dsdr = -k_2(t), \quad t \geq 0, \tag{2.2}$$

then $Z = \Gamma_1 a$ *satisfies* (2.1). *Conversely, if* Z *is a solution of* (2.1), *then* $a = \Lambda_2(QZ - B_2)$ *satisfies* (2.2).

Proof. Equation (2.2) can be rewritten as

$$a - \Lambda_2\Gamma_2\Lambda_1\Gamma_1 a = -k_2 \tag{2.3}$$

Notice that now we consider Γ_1 as a map from $L_1^{p\times m}(\mathbb{R}_+)$ into $\mathbb{C}^{n_1\times m}$. Similar remarks apply to the other operators in (2.3). Put $Z = \Gamma_1 a$. Then

$$\begin{aligned} MZ &= (I-PQ)\Gamma_1 a = (I - \Gamma_1\Lambda_2\Gamma_2\Lambda_1)\Gamma_1 a \\ &= \Gamma_1 a - \Gamma_1(\Lambda_2\Gamma_2\Lambda_1\Gamma_1 a) \\ &= \Gamma_1 a - \Gamma_1(a + k_2) = -\Gamma_1 k_2 = -\Gamma_1\Lambda_2 k_2 = -PB_2. \end{aligned}$$

Conversely, assume Z is a solution of (2.1). Put $a = \Lambda_2(QZ - B_2)$. Then

$$\begin{aligned} a - \Lambda_2\Gamma_2\Lambda_1\Gamma_1 a &= \Lambda_2(QZ - B_2) - \Lambda_2\Gamma_2\Lambda_1\Gamma_1\Lambda_2(QZ - B_2) \\ &= \Lambda_2\Gamma_2\Lambda_1 Z - \Lambda_2 B_2 - \Lambda_2\Gamma_2\Lambda_1 PQZ + \Lambda_2\Gamma_2\Lambda_1 PB_2 \\ &= \Lambda_2\Gamma_2\Lambda_1 Z - \Lambda_2 B_2 - \Lambda_2\Gamma_2\Lambda_1(Z + PB_2) + \Lambda_2\Gamma_2\Lambda_1 PB_2 \\ &= -\Lambda_2 B_2 = -k_2. \end{aligned}$$

This proves that a is a solution of (2.2). □

Proof of Theorem 0.1 *with* k_1 *and* k_2 *given by* (0.8).
We divide the proof into five parts.

Part 1. In this part we show that (ii) implies (i). So assume equations (0.2) and (0.3) are solvable. Recall that $I - K_2K_1$ is invertible if and only if its indicator M is invertible. Therefore it suffices to prove the invertibility of M. Since equation (0.2) is solvable, we know from Lemma 2.1 that the first equation in (1.1) is solvable. Next, we apply Lemma 2.1 to $I - K_1K_2$ in place of $I - K_2K_1$. Note that $M^\#$ is the indicator of $I - K_1K_2$ corresponding to the representations (0.8). Moreover equation (2.1) transforms into $M^\# U = -QB_1$. Thus (0.3) is solvable if and only if the second equation in (1.1) is solvable. But then we can apply Proposition 1.1 to show that (ii) implies that M is invertible.

Part 2. We show that (iii) implies (i). Assume equations (0.4) and (0.5) are solvable. Again it suffices to show that the indicator M is invertible. By taking adjoints we can rewrite (0.4) and (0.5) in the following equivalent form:

$$\alpha_1^*(t) - \int_0^\infty \int_0^\infty k_2^*(t+s)k_1^*(s+r)\alpha_1^*(r)\, dsdr = -k_2^*(t), \quad t \geq 0, \tag{2.4}$$

$$\alpha_2^*(t) - \int_0^\infty \int_0^\infty k_1^*(t+s)k_2^*(s+r)\alpha_2^*(r)\,dsdr = -k_1^*(t), \quad t \geq 0. \tag{2.5}$$

Here for any matrix function g we use the convention that $g^*(t) = g(t)^*$. Now let $\tilde{K_1}$ and $\tilde{K_2}$ be the Hankel operators corresponding to the kernel functions k_1^* and k_2^*, respectively, that is,

$$(\tilde{K_1}f)(t) = \int_0^\infty k_1^*(t+s)f(s)\,ds, \quad t \geq 0, \tag{2.6}$$

$$(\tilde{K_2}g)(t) = \int_0^\infty k_2^*(t+s)g(s)\,ds, \quad t \geq 0. \tag{2.7}$$

Then applying the result of the first step to $I - \tilde{K_2}\tilde{K_1}$ in place of $I - K_2K_1$, we conclude that $I - \tilde{K_2}\tilde{K_1}$ is invertible. The kernel functions k_1^* and k_2^* have stable exponential representations, namely

$$k_1^*(t) = B_1^*e^{tA_1^*}C_1^*, \qquad k_2^*(t) = B_2^*e^{tA_2^*}C_2^*. \tag{2.8}$$

Notice that

$$Q^* = \int_0^\infty e^{sA_1^*}C_1^*B_2^*e^{sA_2^*}\,ds, \quad P^* = \int_0^\infty e^{sA_2^*}C_2^*B_1^*e^{sA_1^*}\,ds.$$

It follows that the indicator for $I - \tilde{K_2}\tilde{K_1}$ corresponding to the representations (2.8) is precisely equal to M^*. Since $I - \tilde{K_2}\tilde{K_1}$ is invertible, we conclude that M^* is invertible, and hence M is.

Part 3. In this part we show that (i) implies (ii) and (iii). The implication from (i) to (ii) is trivial. To prove (i) implies (iii), note that (i) is equivalent to the invertibility of M. Thus (i) implies M^* is invertible, which is the indicator of $I - \tilde{K_2}\tilde{K_1}$, and hence $I - \tilde{K_2}\tilde{K_1}$ is invertible. Here $\tilde{K_1}$ and $\tilde{K_2}$ are defined by (2.6) and (2.7). It follows that equations (2.4) and (2.5) are solvable. Taking adjoints, we see that (iii) holds.

Summarizing we have proved that statements (i), (ii) and (iii) in Theorem 0.1 are equivalent.

Part 4. In this part we assume that $I - K_2K_1$ is invertible and we derive the solutions of (0.2)–(0.5). Note that our assumption implies that all operators

$$I - K_2K_1, \quad I - K_1K_2, \quad I - \tilde{K_2}\tilde{K_1}, \quad I - \tilde{K_1}\tilde{K_2} \tag{2.9}$$

are invertible, and hence each of the equations (0.2), (0.3), (0.4), (0.5) is uniquely solvable. Here we used that (0.4) and (0.5) are equivalent to (2.4) and (2.5), respectively. Now we apply Lemma 2.1 to each of the operators in (2.9). This yields that the unique solutions of the equations (0.2), (0.3), (0.4), (0.5) are, respectively, given by

$$a_1(t) = -C_2e^{tA_2}X, \qquad X = (M^{\#})^{-1}B_2, \tag{2.10}$$

$$a_2(t) = -C_1e^{tA_1}\tilde{X}, \qquad \tilde{X} = M^{-1}B_1, \tag{2.11}$$

$$\alpha_1(t) = -Ye^{tA_2}B_2, \qquad Y = C_2(M^{\#})^{-1}, \tag{2.12}$$

$$\alpha_2(t) = -\tilde{Y}e^{tA_1}B_1, \qquad \tilde{Y} = C_1M^{-1}. \tag{2.13}$$

Let us prove (2.10). According to Lemma 2.1 we have

$$\begin{aligned} a_1 &= \Lambda_2(QZ - B_2) = -\Lambda_2(QM^{-1}P + I)B_2 \\ &= -\Lambda_2(I - QP)^{-1}B_2 = -\Lambda_2(M^{\#})^{-1}B_2 = -\Lambda_2 X. \end{aligned}$$

Using the definition of Λ_2, this yields (2.10).

Interchanging the roles of K_1 and K_2 transforms (2.10) into (2.11). Indeed, the indicator for $I - K_1K_2$ is equal to $M^{\#}$, and hence the associate indicator for $I - K_1K_2$ is M. In a similar way, replacing K_1 by $\tilde{K_1}$ and K_2 by $\tilde{K_2}$, and using the dual representations (2.8) in place of (0.8), we see that

$$\alpha_1^*(t) = -B_2^* e^{tA_2^*} Y^*, \quad Y^* = ((M^{\#})^{-1})^* C_2^*, \tag{2.14}$$

$$\alpha_2^*(t) = -B_1^* e^{tA_1^*} \tilde{Y}^*, \quad \tilde{Y}^* = (M^{-1})^* C_1^*. \tag{2.15}$$

Here we used that the indicator for $I - \tilde{K_2}\tilde{K_1}$ corresponding to the representations (2.8) is equal to M^*, and that the associate indicator is equal to $(M^{\#})^*$. Taking adjoints in (2.14) and (2.15) gives (2.12) and (2.13).

Part 5. In this part we derive the inversion formula (0.6). Thus $I - K_2K_1$ is assumed to be invertible. We claim that

$$a(t) = -C_2 e^{tA_2} X_1, \qquad X_1 = (M^{\#})^{-1} B_2, \tag{2.16}$$

$$b(t) = -Y_1 e^{tA_1} B_1, \qquad Y_1 = C_1 M^{-1}, \tag{2.17}$$

$$c(t) = -C_2 e^{tA_2} X_2, \qquad X_2 = QM^{-1} B_1, \tag{2.18}$$

$$d(t) = -Y_2 e^{tA_1} B_1, \qquad Y_2 = C_2 QM^{-1}. \tag{2.19}$$

Since $a = a_1$ and $b = \alpha_2$, formulas (2.16) and (2.17) follow directly from (2.10) and (2.13). To compute (2.18), we use that c is given by the first identity in (0.7). Together with (2.11), this yields

$$\begin{aligned} c(t) &= -\int_0^\infty C_2 e^{(t+s)A_2} B_2 C_1 e^{sA_1} \tilde{X}\, ds \\ &= -C_2 e^{tA_2} \Big(\int_0^\infty e^{sA_2} B_2 C_1 e^{sA_1}\, ds\Big)\tilde{X} = -C_2 e^{tA_2} Q\tilde{X}. \end{aligned}$$

But $\tilde{X} = M^{-1}B_1$. So c is given by (2.18). In a similar way, using (2.12), the second identity in (0.7), and $(M^{\#})^{-1}Q = QM^{-1}$, one obtains (2.19). Indeed,

$$\begin{aligned} d(t) &= -\int_0^\infty Y e^{sA_2} B_2 C_1 e^{(t+s)A_1} B_1\, ds \\ &= -Y\Big(\int_0^\infty e^{sA_2} B_2 C_1 e^{sA_1}\, ds\Big) e^{tA_1} B_1 = -C_2 (M^{\#})^{-1} Q e^{tA_1} B_1. \end{aligned}$$

Now to get (0.6), recall that $(I - K_2K_1)^{-1} = I + \Lambda_2 QM^{-1}\Gamma_1$. Hence

$$((I - K_2K_1)^{-1} f)(t) = f(t) + C_2 e^{tA_2} QM^{-1} \int_0^\infty e^{sA_1} B_1 f(s)\, ds.$$

Using formula (1.2) for QM^{-1}, together with (1.3) and the formulas (2.16)–(2.19), we obtain (0.6).

3. Proof of the main theorem (general case)

In this section we prove Theorem 0.1 for arbitrary kernel functions. The proof is split into two parts. In the first part we assume that $I - K_2K_1$ is invertible, and we derive the inversion formula (0.6) by an approximation argument using the result of the previous section. In the second part we prove the equivalence of the statements (i), (ii) and (iii).

Part 1. Assume $L = I - K_2K_1$ is invertible, and let us prove the inversion formula (0.6). To do this we choose for $j = 1, 2$ a sequence $k_{j,1}, k_{j,2}, k_{j,3}, \ldots$, consisting of kernel functions with a stable exponential representation, such that

$$\|k_j - k_{j,n}\|_{L_1} \to 0 \quad (n \to \infty). \tag{3.1}$$

Put $L_n = I - K_{2,n}K_{1,n}$, where

$$(K_{j,n}f)(t) = \int_0^\infty k_{j,n}(t+s)f(s)\,ds, \quad t \geq 0 \quad (j = 1, 2).$$

Then (3.1) implies that $\|L - L_n\| \to 0$ if $n \to \infty$, and hence L_n is invertible for n sufficiently large. By passing to a subsequence we can assume that L_n is invertible for each n, and

$$\|L_n^{-1} - L^{-1}\| \to 0 \quad (n \to \infty). \tag{3.2}$$

Now, let $a_{1,n}, a_{2,n}, \alpha_{1,n}, \alpha_{2,n}$ be the solutions of the equations (0.2) – (0.5) which one obtains with $k_{1,n}$ in place of k_1 and $k_{2,n}$ in place of k_2. Put

$$\begin{aligned} a_n(t) &= a_{1,n}(t), & b_n(t) &= \alpha_{2,n}(t), \\ c_n(t) &= \int_0^\infty k_{2,n}(t+s)a_{2,n}(s)\,ds, & d_n(t) &= \int_0^\infty \alpha_{1,n}k_{1,n}(t+s)\,ds. \end{aligned}$$

Then (3.1) implies that

$$\|a - a_n\|_{L_1} + \|b - b_n\|_{L_1} + \|c - c_n\|_{L_1} + \|d - d_n\|_{L_1} \to 0 \quad (n \to \infty). \tag{3.3}$$

Here a, b, c, d are the matrix functions defined in the second part of Theorem 0.1.

Consider the operators

$$\begin{aligned} (Af)(t) &= \int_0^\infty a(t+s)f(s)\,ds, & (Bf)(t) &= \int_0^\infty b(t+s)f(s)\,ds, \\ (Cf)(t) &= \int_0^\infty c(t+s)f(s)\,ds, & (Df)(t) &= \int_0^\infty d(t+s)f(s)\,ds, \end{aligned}$$

and let A_n, B_n, C_n, D_n be the operators which one obtains from A, B, C, D when the role of a is taken over by a_n, that of b by b_n, that of c by c_n, and that of d by d_n. From the result of the previous section we know that

$$L_n^{-1} = I + A_nB_n - C_nD_n,$$

and (3.3) implies that

$$\|A - A_n\| + \|B - B_n\| + \|C - C_n\| + \|D - D_n\| \to 0 \quad (n \to \infty).$$

It follows that

$$L^{-1} = \lim_{n\to\infty} L_n^{-1} = \lim_{n\to\infty} (I + A_nB_n - C_nD_n) = I + AB - CD,$$

which proves (0.6).

Part 2. In this part we prove the equivalence of (i), (ii), and (iii). For $k_2 = k_1^*$ this proof can be found in Chapter 12 of [3], pages 213–218. The general case requires some modifications of the arguments given in [3]. In what follows we concentrate on these modifications. We begin with some preparations.

By W^p we denote the linear space of all $\phi \in L_1^p(\mathbb{R}_+)$ that are absolutely continuous on compact intervals of $\mathbb{R}_+$ and such that ϕ' again belongs to $L_1^p(\mathbb{R}_+)$. Notice that for each $\phi \in W^p$ we have

$$\phi(t) = -\int_t^\infty \phi'(s)\,ds, \quad t \geq 0. \tag{3.4}$$

The space W^p endowed with the norm $\|\phi\|_W = \|\phi\|_{L_1} + \|\phi'\|_{L_1}$ is a Banach space. As a set W^p is dense in $L_1^p(\mathbb{R}_+)$.

Now, let $k \in L_1^{m\times p}(\mathbb{R}_+)$, and let K be the corresponding Hankel operator from $L_1^p(\mathbb{R}_+)$ into $L_1^m(\mathbb{R}_+)$. In [3], page 214, it is proved that K maps W^p into W^m and

$$(K\phi)' = K\phi' - k(\cdot)\phi(0), \quad \phi \in W^p. \tag{3.5}$$

From (3.4) and (3.5) it follows that K induces a bounded linear operator from W^p into W^q which we shall denote by K_W. The operator K_W is compact ([3], page 215).

From (3.4) it follows that $W^p \subset L_\infty^p(\mathbb{R}_+)$. Hence for $\phi \in W^p$ and $f \in L_\infty^p(\mathbb{R}_+)$ we can define

$$\langle \phi, f\rangle = \int_0^\infty f^*(t)\phi(t)\,dt.$$

Using Fubini's theorem it is straightforward to check that

$$\langle K_W\phi, f\rangle = \langle \phi, \tilde{K}f\rangle, \quad \phi \in W^p,\ f \in L_1^m(\mathbb{R}_+), \tag{3.6}$$

where $\tilde{K}$ is the Hankel operator from $L_1^m(\mathbb{R}_+)$ into $L_1^p(\mathbb{R}_+)$ corresponding to k^* (cf., (2.6), (2.7)).

Now, let K_1 and K_2 be the Hankel operators defined by (0.1), and consider the corresponding operators K_{1W} and K_{2W}. From (3.5) it follows that

$$K_2K_1D - DK_{2W}K_{1W} = k_2E_2K_{1W} + K_2k_1E_1. \tag{3.7}$$

Here D and E are the operators defined by

$$D\colon W^p \to L_1^p(\mathbb{R}_+), \quad D\phi = \phi',$$
$$E_1\colon W^p \to \mathbb{C}^p, \quad E_1\phi = \phi(0), \quad E_2\colon W^m \to \mathbb{C}^m, \quad E_2\phi = \phi(0).$$

Since K_2K_1 and $K_{2W}K_{1W}$ are compact, $I - K_2K_1$ and $I - K_{2W}K_{1W}$ are Fredholm operators of index zero. Furthermore, because W^p is dense in $L_1^p(\mathbb{R}_+)$, we have

$$\operatorname{Ker}(I - K_{2W}K_{1W}) = \operatorname{Ker}(I - K_2K_1). \tag{3.8}$$

To prove this one can use the same arguments as in [3], pages 215, 216.

Now, assume that (iii) is satisfied. We have to prove that $I - K_2K_1$ is invertible. Since $I - K_2K_1$ is a Fredholm operator of index zero, it suffices to show that $\mathrm{Ker}\,(I - K_2K_1) = \{0\}$.

Using (3.8) it suffices to show that $\mathrm{Ker}\,(I - K_{2W}K_{1W}) = \{0\}$.

So, take $\psi \in W^p$ and assume $K_{2W}K_{1W}\psi = \psi$. Using (3.6) and taking adjoints we see that

$$\begin{aligned} 0 &= \langle (I - K_{2W}K_{1W})\psi, \alpha_2^* \rangle = \langle \psi, (I - \tilde{K_1}\tilde{K_2})\alpha_2^* \rangle = \langle \psi, -k_1^* \rangle \\ &= -\int_0^\infty k_1(t)\psi(t)\,dt = -(K_{1W}\psi)(0) = -E_2K_{1W}\psi. \end{aligned}$$

Thus $E_2K_{1W}\psi = 0$.

Next, put $\phi = K_{1W}\psi$. Then $K_{2W}\phi = \psi$, and

$$(I - K_{1W}K_{2W})\phi = \phi - K_{1W}\psi = 0.$$

Repeating the arguments given in the previous paragraph with the roles of K_1 and K_2 interchanged, it follows that $E_1K_{2W}\phi = 0$. But $K_{2W}\phi = \psi$, and hence $E_1\psi = 0$.

Using that the vectors $E_2K_{1W}\psi$ and $E_1\psi$ are both zero in (3.7), we see that $K_2K_1\psi' = \psi'$, that is, $\psi' \in \mathrm{Ker}\,(I - K_2K_1)$. Now, use again (3.8). So we can use the same arguments with ψ' in place of ψ. This yields

$$E_2K_{1W}\psi' = 0, \quad E_1\psi' = 0, \quad K_2K_1\psi'' = \psi''.$$

Proceeding by induction we conclude that for each $n = 0, 1, 2, \ldots$ the function $\psi^{(n)} \in \mathrm{Ker}\,(I - K_2K_1)$ and $\psi^{(n)}(0) = 0$. Since $\mathrm{Ker}\,(I - K_2K_1)$ is finite-dimensional, this implies (see [3], page 218) that $\psi = 0$. Hence $\mathrm{Ker}\,(I - K_2K_1) = \{0\}$, and $I - K_2K_1$ is invertible.

In a similar way one shows that (ii) implies that $I - \tilde{K_2}\tilde{K_1}$ is invertible. Here $\tilde{K_1}$ and $\tilde{K_2}$ are given by (2.6) and (2.7), respectively. Using (3.6) and (3.8), it then follows that (ii) implies that $I - K_1K_2$ is invertible, which is equivalent to (ii) implies (i).

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