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TWO-DIMENSIONAL GENERALIZED MOMENT REPRESENTATIONS AND RATIONAL APPROXIMATIONS OF FUNCTIONS OF TWO VARIABLES

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The Dzyadyk method of generalized moment representations is extended to the case of two-dimensional sequences and used to construct Padé approximants for functions of two variables.

Generalized moment representations introduced by Dzyadyk [1] in 1981 are very convenient for the construction and study of Padé approximations and their generalizations (see [2]).

Definition 1. *We say that a sequence of complex numbers* $\{s_k\}_{k=0}^{\infty}$ *has a generalized moment representation on the product of linear spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *for the bilinear form* $\langle .\,,.\rangle$ *defined on this product if the sequence of elements* $\{x_k\}_{k=0}^{\infty}$ *is defined in the space* $\mathcal{X}$ *and the sequence of elements* $\{y_j\}_{j=0}^{\infty}$ *is defined in the space* $\mathcal{Y}$ *so that*

$$s_{k+j} = \langle x_k, y_j \rangle, \quad k, j \in \mathbb{Z}_+. \tag{1}$$

By analogy with (1), we can define generalized moment representations of two-dimensional number sequences.

Definition 2. *We say that the two-dimensional number sequence* $\{s_{k,m}\}_{k,m=0}^{\infty}$ *has a generalized moment representation on the product of the linear spaces* $\mathcal{X}$ *and* $\mathcal{Y}$ *for the bilinear form* $\langle .\,,.\rangle$ *defined on this product if the two-dimensional sequence of elements* $\{x_{k,m}\}_{k,m=0}^{\infty}$ *is defined in the space* $\mathcal{X}$ *and the two-dimensional sequence of elements* $\{y_{j,n}\}_{j,n=0}^{\infty}$ *is defined in the space* $\mathcal{Y}$ *so that*

$$s_{k+j,m+n} = \langle x_{k,m}, y_{j,n} \rangle, \quad k,\, j,\, m,\, n \in \mathbb{Z}_+. \tag{2}$$

By analogy with the correspondence between the number sequence $\{s_k\}_{k=0}^{\infty}$ and the formal power series

$$f(z) = \sum_{k=0}^{\infty} s_k z^k,$$

we can associate the number sequence $\{s_{k,m}\}_{k,m=0}^{\infty}$ with the formal power series of two variables

$$f(z,w) = \sum_{k,m=0}^{\infty} s_{k,m} z^k w^m. \tag{3}$$

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Rational approximants for series of the form (3) can be determined by using various schemes (see [3, p. 323]) as generalized one-dimensional Padé approximants. To this end, it is necessary to fix certain bounded domains $\mathcal{N}$ and $\mathcal{D}$ from $\mathbb{Z}_+^2$ and construct algebraic polynomials

$$P_{\mathcal{N}}(z,w) = \sum_{(k,m)\in\mathcal{N}} p_{k,m} z^k w^m,$$

$$Q_{\mathcal{D}}(z,w) = \sum_{(k,m)\in\mathcal{D}} q_{k,m} z^k w^m$$

for which the maximum possible number of coefficients $e_{k,m}$ in the expansion

$$f(z,w) - \frac{P_{\mathcal{N}}(z,w)}{Q_{\mathcal{D}}(z,w)} = \sum_{(k,m)\in\mathbb{Z}_+^2} e_{k,m} z^k w^m$$

is equal to zero. As in the case of one-dimensional Padé approximations, the construction of these polynomials is reduced to the solution of a system of linear algebraic equations. Thus, if we demand that $e_{k,m} = 0$ for $(k,m) \in \mathcal{E} \subset \mathbb{Z}_+^2$, then the following equality must be true in the general case:

$$\dim \mathcal{E} = \dim \mathcal{N} + \dim \mathcal{D} - 1.$$

In fact, in each nondegenerate case, it is possible to guarantee the validity of the inequality

$$\dim \mathcal{E} \geqslant \dim \mathcal{N} + \dim \mathcal{D} - 1.$$

Various modifications of many-dimensional (and, in particular, two-dimensional) Padé approximations are studied in [4–12].

The following result is an analog of Dzyadyk's theorem [1] for functions of two variables:

Theorem 1. *Assume that a formal power series of two variables has the form (3) and that a generalized moment representation of the form (2) is true for a two-dimensional sequence* $\{s_{k,m}\}_{k,m=0}^{\infty}$. *If, in addition, for some* $N_1, N_2 \in \mathbb{N}$, *there exists a nontrivial generalized polynomial*

$$Y_{N_1,N_2} = \sum_{j=0}^{N_1} \sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} y_{j,n} \tag{4}$$

satisfying the conditions of biorthogonality

$$\langle x_{k,m}, Y_{N_1,N_2} \rangle = 0 \tag{5}$$

for

$$(k,m) \in ([0,N_1] \times [0,N_2]) \setminus \{(N_1,N_2)\}$$

and $c_{N_1,N_2}^{(N_1,N_2)} \neq 0,$ *then the rational function*

$$\frac{1}{Q_{N_1,N_2}(z,w)} \left\{ \sum_{k=0}^{N_1-1} \sum_{m=0}^{N_2-1} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{m} c_{N_1-j,N_2-n}^{(N_1,N_2)} s_{k-j,m-n} \right.$$

$$+ z^{N_1} \sum_{k=0}^{N_1} \sum_{m=0}^{N_2-1} z^k w^m \sum_{j=0}^{N_1} \sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

$$\left. + w^{N_2} \sum_{k=0}^{N_1-1} \sum_{m=0}^{N_2} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n} \right\},$$

where

$$Q_{N_1,N_2}(z,w) = \sum_{j=0}^{N_1} \sum_{n=0}^{N_2} c_{N_1-j,N_2-n}^{(N_1,N_2)} z^j w^n,$$

admits an expansion in a power series whose coefficients coincide with the coefficients of series (3) for all

$$(j,n) \in ([0,2N_1] \times [0,2N_2]) \setminus \{(2N_1,2N_2)\}.$$

Proof. We multiply equality (2) by $z^k w^m$ and find the sum over k and m from 0 to sufficiently large numbers $\tilde{k}$ and $\tilde{m}$, respectively. This yields

$$\left\langle \sum_{k=0}^{\tilde{k}} \sum_{m=0}^{\tilde{m}} z^k w^m x_{k,m}, y_{j,n} \right\rangle$$

from the right and

$$\sum_{k=0}^{\tilde{k}} \sum_{m=0}^{\tilde{m}} s_{k+j,m+n} z^k w^m = \sum_{k=j}^{\tilde{k}+j} \sum_{m=n}^{\tilde{m}+n} s_{k,m} z^{k-j} w^{m-n}$$

$$= \frac{1}{z^j w^n} \left\{ f(z,w) - \sum_{k=j}^{\tilde{k}+j} \sum_{m=0}^{n-1} s_{k,m} z^k w^m - \sum_{k=0}^{j-1} \sum_{m=n}^{\tilde{m}+n} s_{k,m} z^k w^m \right.$$

$$- \sum_{k=0}^{j-1} \sum_{m=0}^{n-1} s_{k,m} z^k w^m - \sum_{k=\tilde{k}+j+1}^{\infty} \sum_{m=0}^{\tilde{m}+n} s_{k,m} z^k w^m$$

$$\left. - \sum_{k=0}^{\tilde{k}+j} \sum_{m=\tilde{m}+n+1}^{\infty} s_{k,m} z^k w^m - \sum_{k=\tilde{k}+j+1}^{\infty} \sum_{m=\tilde{m}+n+1}^{\infty} s_{k,m} z^k w^m \right\}$$

from the left.

Multiplying these equalities by the coefficients $c_{j,n}^{(N_1,N_2)}$, $j \in [0, N_1]$, $n \in [0, N_2]$, and finding the sums over j from 0 to N_1 and over n from 0 to N_2, we obtain

$$\left\langle \sum_{k=0}^{\tilde{k}} \sum_{m=0}^{\tilde{m}} z^k w^m x_{k,m}, Y_{N_1,N_2} \right\rangle$$

from the right. In view of the relations of biorthogonality (5), the coefficients of the powers $(k, m) \in ([0, N_1] \times [0, N_2]) \setminus \{(N_1, N_2)\}$ in the expansion of the quantity obtained from the right in power series in z and w are equal to zero.

From the left, we obtain

$$\sum_{j=0}^{N_1} \sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} \frac{1}{z^j w^n} \left\{ f(z,w) - \sum_{k=j}^{\tilde{k}+j} \sum_{m=0}^{n-1} s_{k,m} z^k w^m - \sum_{k=0}^{j-1} \sum_{m=n}^{\tilde{m}+n} s_{k,m} z^k w^m \right.$$

$$- \sum_{k=0}^{j-1} \sum_{m=0}^{n-1} s_{k,m} z^k w^m - \sum_{k=\tilde{k}+j+1}^{\infty} \sum_{m=0}^{\tilde{m}+n} s_{k,m} z^k w^m$$

$$\left. - \sum_{k=0}^{\tilde{k}+j} \sum_{m=\tilde{m}+n+1}^{\infty} s_{k,m} z^k w^m - \sum_{k=\tilde{k}+j+1}^{\infty} \sum_{m=\tilde{m}+n+1}^{\infty} s_{k,m} z^k w^m \right\}$$

$$= \frac{1}{z^{N_1} w^{N_2}} \left\{ f(z,w) Q_{N_1,N_2}(z,w) - \sum_{j=0}^{N_1} \sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j} w^{N_2-n} \sum_{(k,m)\in D^*} s_{k,m} z^k w^m \right\},$$

where $D^* = D_{0,0} \cup D_{0,1} \cup D_{1,0} \cup D_{0,2} \cup D_{2,0} \cup D_{1,2} \cup D_{2,1} \cup D_{2,2}$ and

$$D_{0,0} = [0, j-1] \times [0, n-1], \qquad D_{0,1} = [0, j-1] \times [n, \tilde{m}+n],$$

$$D_{1,0} = [j, \tilde{k}+j] \times [0, n-1], \qquad D_{0,2} = [0, j-1] \times [\tilde{m}+n+1, \infty],$$

$$D_{2,0} = [\tilde{k}+j+1, \infty] \times [0, n-1], \qquad D_{1,2} = [j, \tilde{k}+j] \times [\tilde{m}+n+1, \infty],$$

$$D_{2,1} = [\tilde{k}+j+1, \infty] \times [n, \tilde{m}+n], \qquad D_{2,2} = [\tilde{k}+j+1, \infty] \times [\tilde{m}+n+1, \infty]$$

(see Fig. 1).

Then

$$f(z,w) Q_{N_1,N_2}(z,w) - \sum_{j=0}^{N_1} \sum_{n=1}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j} w^{N_2-n} \sum_{k=j}^{\tilde{k}+j} \sum_{m=0}^{n-1} s_{k,m} z^k w^m$$

$$- \sum_{j=1}^{N_1} \sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j} w^{N_2-n} \sum_{k=0}^{j-1} \sum_{m=n}^{\tilde{m}+n} s_{k,m} z^k w^m$$

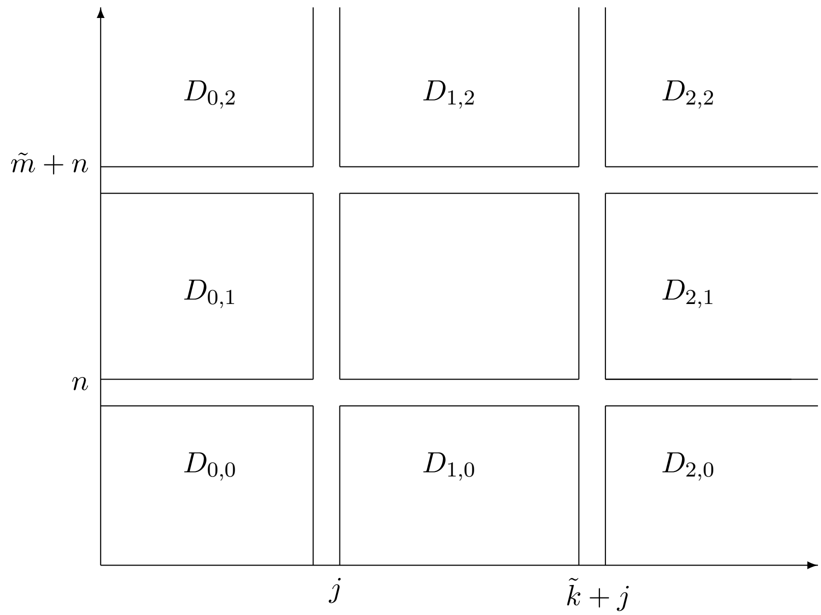

Fig. 1

$$-\sum_{j=1}^{N_1}\sum_{n=1}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j} w^{N_2-n} \sum_{k=0}^{j-1}\sum_{m=0}^{n-1} s_{k,m} z^k w^m$$

$$= O\left(w^{\tilde{m}}\right) + O\left(z^{\tilde{k}}\right) + z^{N_1} w^{N_2} \left\langle \sum_{k=0}^{\tilde{k}}\sum_{m=0}^{\tilde{m}} z^k w^m x_{k,m}, Y_{N_1,N_2} \right\rangle,$$

whence, in view of the arbitrariness of the choice of sufficiently large $\tilde{k}$ and $\tilde{m}$, we arrive at the assertion of the theorem.

Remark 1. Thus, for the Padé approximant constructed in Theorem 1, we have

$$\mathcal{D} = [0, N_1] \times [0, N_2],$$

$$\mathcal{N} = ([0, 2N_1] \times [0, 2N_2]) \setminus ([N_1, 2N_1] \times [N_2, 2N_2]),$$

$$\mathcal{E} = ([0, 2N_1] \times [0, 2N_2]) \setminus \{(2N_1, 2N_2)\}$$

(see Fig. 2; the shaded part is the domain $\mathcal{N}$ and the part bounded by the heavy contour is the domain $\mathcal{E}$).

Indeed, the generalized polynomial Y_{N_1,N_2} in Theorem 1 can be chosen from the conditions of biorthogonality to elements $x_{k,m}$ not for

$$(k, m) \in ([0, N_1] \times [0, N_2]) \setminus \{(N_1, N_2)\}$$

but for $(k, m) \in \mathcal{H}$, where $\mathcal{H}$ is a set from $\mathbb{Z}_+^2$ bounded by a curve $\rho = \rho(\varphi)$, $\varphi \in [0, \pi/2]$ containing $(N_1 + 1)(N_2 + 1) - 1$ points. In this case, as $\mathcal{N}$, we can choose any set from $\mathbb{Z}_+^2 \setminus ([N_1, \infty) \times [N_2, \infty))$ obtained as the union of the square $[0, N_1 - 1] \times [0, N_2 - 1]$ with sets of the form $\{(k, m)\colon k \in [0, N_1 - 1],$

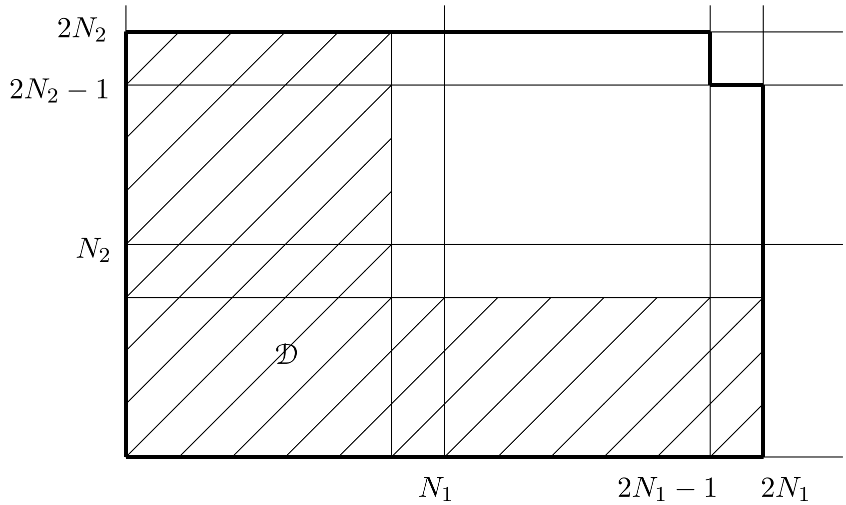

Fig. 2

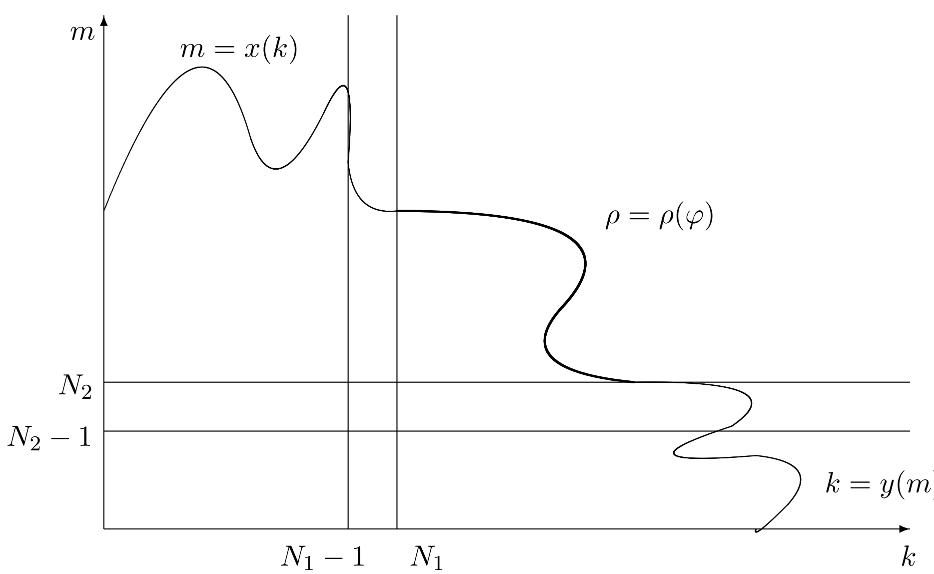

Fig. 3

$m \in [N_2, x(k)]\}$ and $\{(k,m)\colon m \in [0, N_2{-}1],\ k \in [N_1, y(m)]\}$, where $x(k)$ and $y(m)$ are functions from $\mathbb{Z}_+$ into $\mathbb{Z}_+$ such that $x(k) \geqslant N_2$ and $y(m) \geqslant N_1$ for all k and m (see Fig. 3). Then the set $\mathcal{E}$ has the form $\mathcal{N} \cup \{\mathcal{H} + (N_1, N_2)\}$, where $\mathcal{H} + (N_1, N_2)$ is the set obtained by the parallel displacement of the set $\mathcal{H}$ in which the point $(0,0)$ is mapped into the point (N_1, N_2).

Thus, the following generalization of Theorem 1 is true:

Theorem 1′. *Assume that, under the conditions of Theorem 1, for some* $N_1, N_2 \in \mathbb{N}$, *there exists a nontrivial generalized polynomial*

$$Y_{N_1,N_2} = \sum_{j=0}^{N_1} \sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} y_{j,n}$$

such that the conditions of biorthogonality

$$\langle x_{k,m}, Y_{N_1,N_2}\rangle = 0$$

are satisfied for $(k,m) \in \mathcal{H}$, *where the domain* $\mathcal{H} \subset \mathbb{Z}_+^2$ *is bounded by the graph of a function* $\rho = \rho(\varphi)$, $\varphi \in \left[0, \frac{\pi}{2}\right]$, *and contains* $(N_1+1)(N_2+1) - 1$ *points and, in addition,* $c_{N_1,N_2}^{(N_1,N_2)} \neq 0$. *Then the rational function*

$$\frac{1}{Q_{N_1,N_2}(z,w)}\left\{\sum_{k=0}^{N_1-1}\sum_{m=0}^{N_2-1} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{m} c_{N_1-j,N_2-n}^{(N_1,N_2)} s_{k-j,m-n}\right.$$

$$+ z^{N_1}\sum_{m=0}^{N_2-1}\sum_{k=0}^{y(m)-N_1} z^k w^m \sum_{j=0}^{N_1}\sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

$$\left.+ w^{N_2}\sum_{k=0}^{N_1-1}\sum_{m=0}^{x(k)-N_2} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n}\right\},$$

where

$$Q_{N_1,N_2}(z,w) = \sum_{j=0}^{N_1}\sum_{n=0}^{N_2} c_{N_1-j,N_2-n}^{(N_1,N_2)} z^j w^n,$$

admits an expansion in a power series whose coefficients coincide with the coefficients of series (3) for all $(j,n) \in \mathcal{E}$.

Proof. In the proof of Theorem 1, we have deduced the equality

$$\left\langle \sum_{k=0}^{\tilde{k}}\sum_{m=0}^{\tilde{m}} z^k w^m x_{k,m}, Y_{N_1,N_2}\right\rangle$$

$$= \frac{1}{z^{N_1}w^{N_2}}\left\{f(z,w)Q_{N_1,N_2}(z,w) - \sum_{j=0}^{N_1}\sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j}w^{N_2-n}\sum_{(k,m)\in D^*} s_{k,m}z^k w^m\right\}.$$

The sums corresponding to the domains $D_{0,2}$, $D_{1,2}$, $D_{2,2}$, $D_{2,1}$, and $D_{2,0}$ for sufficiently large $\tilde{k}$ and $\tilde{m}$ contain only $z^k w^m$ for $(k,m) \notin \mathcal{E}$.

Consider

$$\sum_{j=0}^{N_1}\sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j}w^{N_2-n}\sum_{(k,m)\in D_{0,0}} s_{k,m}z^k w^m$$

$$= z^{N_1}w^{N_2}\sum_{j=1}^{N_1}\sum_{n=1}^{N_2} c_{j,n}^{(N_1,N_2)}\sum_{k=0}^{j-1}\sum_{m=0}^{n-1} s_{k,m}z^{k-j}w^{m-n}$$

$$= \sum_{k=0}^{N_1-1}\sum_{m=0}^{N_2-1} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{m} c_{N_1-j,N_2-n}^{(N_1,N_2)} s_{k-j,m-n}.$$

Further, we have

$$\sum_{j=0}^{N_1}\sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j} w^{N_2-n} \sum_{(k,m)\in D_{0,1}} s_{k,m} z^k w^m$$

$$= z^{N_1} w^{N_2} \sum_{j=1}^{N_1}\sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} \sum_{k=0}^{j-1}\sum_{m=n}^{\tilde{m}+n} s_{k,m} z^{k-j} w^{m-n}$$

$$= w^{N_2} \sum_{k=0}^{N_1-1}\sum_{m=0}^{\tilde{m}} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n}.$$

Similarly, for the domain $D_{1,0}$, we get

$$\sum_{j=0}^{N_1}\sum_{n=0}^{N_2} c_{j,n}^{(N_1,N_2)} z^{N_1-j} w^{N_2-n} \sum_{(k,m)\in D_{1,0}} s_{k,m} z^k w^m$$

$$= z^{N_1} \sum_{k=0}^{\tilde{k}}\sum_{m=0}^{N_2-1} z^k w^m \sum_{j=0}^{N_1}\sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}.$$

We form the numerator of the two-dimensional Padé approximant as follows: We take the first sum completely and the following part of the second sum:

$$w^{N_2} \sum_{k=0}^{N_1-1}\sum_{m=0}^{x(k)-N_2} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n},$$

whereas the remaining part

$$w^{N_2} \sum_{k=0}^{N_1-1}\sum_{m=x(k)-N_2+1}^{\tilde{m}} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n}$$

belongs to the remainder.

Moreover, we take the following part of the third sum:

$$z^{N_1} \sum_{m=0}^{N_2-1}\sum_{k=0}^{y(m)-N_1} z^k w^m \sum_{j=0}^{N_1}\sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

and the remaining terms

$$z^{N_1}\sum_{m=0}^{N_2-1}\sum_{k=y(m)-N_1+1}^{\tilde{k}} z^k w^m \sum_{j=0}^{N_1}\sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

also belong to the remainder.

In view of the conditions of biorthogonality imposed on the generalized polynomial Y_{N_1,N_2}, we conclude that the assertion of the theorem is true.

As in the case of one-dimensional generalized moment representations, the problem of two-dimensional generalized moment representations can be formulated in the operator form. Namely, assume that the spaces $\mathcal{X}$ and $\mathcal{Y}$ are normed and, in the space $\mathcal{X}$, there exist commuting bounded operators $A, B\colon \mathcal{X} \to \mathcal{X}$ such that

$$Ax_{k,m} = x_{k+1,m},$$

$$Bx_{k,m} = x_{k,m+1}$$

for all $k, m \in \mathbb{Z}_+$. Assume that, in the space $\mathcal{Y}$, there are bounded operators $A^\star, B^\star\colon \mathcal{Y} \to \mathcal{Y}$ adjoint to the operators A and B with respect to the bilinear form $\langle .,. \rangle$ in a sense that, for any $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$\langle Ax, y\rangle = \langle x, A^\star y\rangle,$$

$$\langle Bx, y\rangle = \langle x, B^\star y\rangle.$$

Thus, representation (2) can be rewritten in the form

$$s_{k,m} = \left\langle A^k B^m x_{0,0}, y_{0,0}\right\rangle, \quad k,\, m \in \mathbb{Z}_+,$$

and series (3) converges in the vicinity of the origin to an analytic function admitting the representation

$$f(z,w) = \left\langle \widehat{R_z}(A)\widehat{R_w}(B)x_{0,0}, y_{0,0}\right\rangle,$$

where the resolvent function $\widehat{R_z}(A)$ is specified by the equality $\widehat{R_z}(A) = (I - zA)^{-1}$.

In this case, under the conditions of Theorem 1, we get the following formula for the approximation error:

$$f(z,w) - \frac{P_{\mathbb{N}}(z,w)}{Q_{N_1,N_2}(z,w)} = \frac{1}{Q_{N_1,N_2}(z,w)}\Bigg\{ z^{N_1}w^{N_2}\langle \widehat{R_z}(A)\widehat{R_w}(B)x_{0,0}, Y_{N_1,N_2}\rangle$$

$$+ z^{N_1}\sum_{m=0}^{N_2-1}\sum_{k=N_1+1}^{\infty} z^k w^m \sum_{j=0}^{N_1}\sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

$$+ w^{N_2}\sum_{k=0}^{N_1-1}\sum_{m=N_2+1}^{\infty} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n}\Bigg\}.$$

Under the conditions of Theorem $1'$, this relation takes the form

$$f(z,w) - \frac{P_{\mathcal{N}}(z,w)}{Q_{N_1,N_2}(z,w)} = \frac{1}{Q_{N_1,N_2}(z,w)} \Bigg\{ z^{N_1} w^{N_2} \langle \widehat{R_z}(A) \widehat{R_w}(B) x_{0,0}, Y_{N_1,N_2} \rangle$$

$$+ z^{N_1} \sum_{m=0}^{N_2-1} \sum_{k=y(m)-N_1+1}^{\infty} z^k w^m \sum_{j=0}^{N_1} \sum_{n=0}^{m} c_{j,N_2-n}^{(N_1,N_2)} s_{k+j,m-n}$$

$$+ w^{N_2} \sum_{k=0}^{N_1-1} \sum_{m=x(k)-N_2+1}^{\infty} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{N_2} c_{N_1-j,n}^{(N_1,N_2)} s_{k-j,m+n} \Bigg\}.$$

We consider some examples of representations of the form (2) and use them for the construction of rational approximations.

Let $\mathcal{X} = \mathcal{Y} = L_2([0,1], d\mu)$ for some measure specified by a nondecreasing function $\mu(t)$ with infinitely many points of increase on $[0,1]$. In the space $\mathcal{X}$, we define two operators

$$(A\varphi)(t) = (B\varphi)(t) = t\varphi(t).$$

The resolvent functions of these operators have the form

$$\left(\widehat{R}_z(A)\varphi\right)(t) = \frac{\varphi(t)}{1-zt},$$

$$\left(\widehat{R}_w(B)\varphi\right)(t) = \frac{\varphi(t)}{1-wt}.$$

Thus,

$$f(z,w) = \int_0^1 \frac{d\mu(t)}{(1-zt)(1-wt)} = \frac{wg(w) - zg(z)}{w-z}, \tag{6}$$

where

$$g(z) = \int_0^1 \frac{d\mu(t)}{1-zt}.$$

Hence, for $\mu(t) = t,$ we have

$$g(z) = -\frac{\ln(1-z)}{z}$$

and, therefore,

$$f(z,w) = \frac{\ln \dfrac{1-z}{1-w}}{w-z}.$$

In this case, the functions $x_{k,m}(t)$ have the form

$$x_{k,m}(t) = t^{k+m}$$

and, thus,

$$s_{k,m} = \int_0^1 t^{k+m} d\mu(t). \tag{7}$$

For

$$d\mu(t) = t^{\nu}(1-t)^{\sigma} dt, \quad \nu, \sigma > -1, \tag{8}$$

we find

$$s_{k,m} = \int_0^1 t^{k+m+\nu}(1-t)^{\sigma} dt = \frac{\Gamma(k+m+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+m+\nu+\sigma+2)}.$$

Therefore, the obtained function

$$f(z,w) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{\Gamma(k+m+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+m+\nu+\sigma+2)} z^k w^m, \tag{9}$$

coincides, to within a constant factor, with the hypergeometric Appell series

$$F_1(\alpha,\beta,\beta',\gamma,z,w) = \sum_{k,m=0}^{\infty} \frac{(\alpha)_{k+m}(\beta)_k(\beta')_m}{(\gamma)_{k+m} k! m!} z^k w^m$$

[see [13, p. 219], relation (6)] for $\alpha = \nu + 1,\ \beta = 1,\ \beta' = 1,$ and $\gamma = \nu + \sigma + 2.$

Since the function $f(z,w)$ of the form (6) is symmetric in its variables, it makes sense to approximate it by symmetric approximants. We restrict ourselves to the case $N_1 = N_2 = N$. To determine the Padé approximant for $f(z,w)$ of the form (6), according to Theorems 1 and $1'$, it is necessary to construct a generalized polynomial of the form (4) satisfying the biorthogonality conditions (5). In the analyzed case, $Y_{N,N}(t)$ is an algebraic polynomial of degree $2N$ orthogonal to polynomials of degree $\leqslant 2N-1$. Hence, to within a constant factor, it coincides with a polynomial orthonormal on $[0,1]$ with respect to the measure $d\mu(t)$ and, in the case of measure (8), with the orthonormal Jacobi polynomial shifted by $[0,1]$ (see [14, p. 268]).

Note that, in this case, the polynomial $Y_{N,N}(t) = P_{2N}(t)$ is orthogonal not only to $x_{k,m}(t),\ (k,m) \in ([0,N]\times[0,N]) \setminus \{(N,N)\}$ but also to $x_{k,m}(t)$ for $(k,m) \in \big\{(k,m) \in \mathbb{Z}_+,\ k+m \leqslant 2N-1\big\}$. For this reason, in the construction of the Padé approximant for functions of the form (6), it makes sense to take the coefficients in the numerator not from the set

$$\mathcal{N} = ([0,2N]\times[0,2N]) \setminus ([N,2N]\times[N,2N]),$$

as proposed in Theorem 1, but from the set (see Fig. 4)

$$\mathcal{N}_1 = \big\{(k,m)\colon k+m \leqslant 4N-1\}\backslash\{(k,m)\colon k,m \geqslant N\big\}.$$

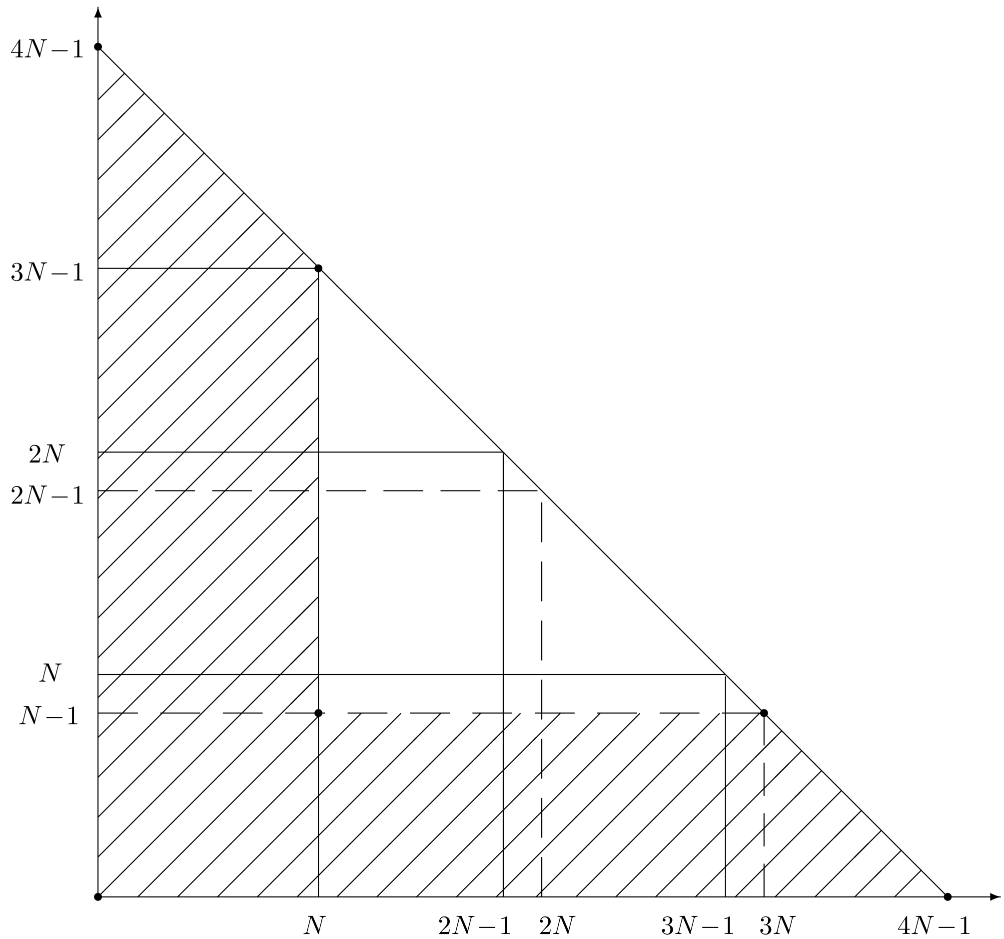

Fig. 4

Assume that the function $f(z,w)$ has the form (6). Then

$$Y_{N,N}(t) = P_{2N}(t),$$

where $P_{2N}(t)$ is a polynomial of degree $2N$ orthogonal on $[0,1]$ with respect to the measure $d\mu(t)$.

We represent it in the form

$$P_{2N}(t) = \sum_{j=0}^{2N} p_j^{(2N)} t^j.$$

Thus, we get

$$\sum_{k=0}^{N}\sum_{m=0}^{N} c_{k,m}^{(N,N)} t^{k+m} = \sum_{j=0}^{2N} p_j^{(2N)} t^j.$$

This equality enables us to find the coefficients $c_{k,m}^{(N,N)},\ k,m=\overline{0,N}$ by using various methods. Since the function $f(z,w)$ is symmetric, we are interested only in symmetric solutions. We select the following three methods:

Method (a). We choose coefficients $c_{k,m}^{(N,N)}$ such that the equalities

$$c_{k,m}^{(N,N)} = c_{k_1,m_1}^{(N,N)}$$

are true , for $k + m = k_1 + m_1$. In this case, the orthogonal polynomial $P_{2N}(t)$ can be expanded as follows:

$$\begin{aligned} P_{2N}(t) &= \sum_{j=0}^{2N} p_j^{(2N)} t^j = \sum_{k=0}^{N} \sum_{m=0}^{N} c_{k,m}^{(N,N)} t^{k+m} \\ &= \sum_{k=0}^{N} \sum_{m=0}^{N-k} c_{k,m}^{(N,N)} t^{k+m} + \sum_{k=1}^{N} \sum_{m=N-k+1}^{N} c_{k,m}^{(N,N)} t^{k+m} \\ &= \sum_{m=0}^{N} t^m \sum_{k=0}^{m} c_{k,m-k}^{(N,N)} + t^{N+1} \sum_{m=0}^{N-1} t^m \sum_{k=0}^{N-m-1} c_{N-k,m+k+1}^{(N,N)}. \end{aligned}$$

As a result, we obtain the relations

$$c_{k,m}^{(N,N)} = \begin{cases} \dfrac{1}{k+m+1} p_{k+m}^{(2N)} & \text{for } k+m \leqslant N, \\ \dfrac{1}{2N-k-m+1} p_{k+m}^{(2N)} & \text{for } k+m > N. \end{cases}$$

Method (b). We choose coefficients $c_{k,m}^{(N,N)}$ such that the coefficients with numbers lying strictly inside the square $[0, N] \times [0, N]$ are equal to zero, i.e.,

$$\sum_{j=0}^{2N} p_j^{(2N)} t^j = \sum_{k=0}^{N-1} c_{k,0}^{(N,N)} t^k + t^N \sum_{m=0}^{N} c_{N,m}^{(N,N)} t^m + \sum_{m=1}^{N-1} c_{0,m}^{(N,N)} t^m + t^N \sum_{k=0}^{N-1} c_{k,N}^{(N,N)} t^k.$$

Thus, we get

$$c_{0,0}^{(N,N)} = p_0^{(2N)}, \qquad c_{N,N}^{(N,N)} = p_{2N}^{(2N)}$$

and, for the other coefficients, we find

$$c_{k,0}^{(N,N)} = \frac{1}{2} p_k^{(2N)}, \quad k = \overline{1, N-1},$$

$$c_{N,m}^{(N,N)} = \frac{1}{2} p_{N+m}^{(2N)}, \quad m = \overline{0, N-1},$$

$$c_{0,m}^{(N,N)} = \frac{1}{2} p_m^{(2N)}, \quad m = \overline{1, N-1},$$

$$c_{k,N}^{(N,N)} = \frac{1}{2} p_{k+N}^{(2N)}, \quad k = \overline{0, N-1}.$$

Method (c). This method differs from the method (*a*) by the fact that the coefficients on the segments $k+m=p$ from the square $[0,N]\times[0,N]$ are not equal. Moreover, they are proportional to the binomial coefficients, namely,

$$c_{k,m}^{(N,N)} = \begin{cases} \dfrac{1}{2^{k+m}}\dbinom{k+m}{k} p_{k+m}^{(2N)} & \text{for } k+m \leqslant N, \\[2ex] \dfrac{1}{2^{2N-k-m}}\dbinom{2N-k-m}{N-k} p_{k+m}^{(2N)} & \text{for } k+m > N. \end{cases}$$

We construct approximants of the indicated types for functions of the form (6). Note that, for the chosen configuration of the domain $\mathcal{N}_1$, in Theorem $1'$, we must set $x(k) = 4N-1-k$ and $y(m) = 4N-1-m$.

For the method (*a*), we obtain

$$Q_{N,N}(z,w) = \sum_{j=0}^{N}\sum_{n=0}^{N} c_{N-j,N-n}^{(N,N)} z^j w^n$$

$$= \sum_{j=0}^{N}\sum_{n=N-j}^{N} \frac{1}{2N-j-n+1} p_{2N-j-n}^{(2N)} z^j w^n + \sum_{j=0}^{N-1}\sum_{n=0}^{N-j-1} \frac{1}{j+n+1} p_{2N-j-n}^{(2N)} z^j w^n$$

$$= \sum_{m=0}^{N} \frac{1}{m+1} p_m^{(2N)} \sum_{j=0}^{m} z^{N-j} w^{N-(m-j)} + \sum_{m=0}^{N-1} \frac{1}{m+1} p_{2N-m}^{(2N)} \sum_{j=0}^{m} z^j w^{m-j}.$$

Further, we determine the numerator

$$P_{\mathcal{N}_1}(z,w) = \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{m} c_{N-j,N-n}^{(N,N)} s_{k-j,m-n}$$

$$+ z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N}\sum_{n=0}^{m} c_{j,N-n}^{(N,N)} s_{k+j,m-n}$$

$$+ w^N \sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N} c_{N-j,n}^{(N,N)} s_{k-j,m+n}$$

$$= \sum_{m=0}^{N-1}\sum_{k=N-m}^{N-1} z^k w^m \sum_{j=N-k}^{N}\sum_{n=N-m}^{N-j} \frac{p_{j+n}^{(2N)}}{j+n+1} s_{k+j-N,m+n-N}$$

$$+ \sum_{k=1}^{N-1}\sum_{m=0}^{N-1} z^k w^m \sum_{j=N-k}^{N-1}\sum_{n=N-j+1}^{N} \frac{p_{j+n}^{(2N)}}{2N-j-n+1} s_{k+j-N,m+n-N}$$

$$+ z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N}\sum_{n=N-m}^{N-j} \frac{p_{j+n}^{(2N)}}{j+n+1} s_{k+j,m+n-N}$$

$$+ z^N \sum_{m=0}^{N-1} \sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N-1} \sum_{n=N-j+1}^{N} \frac{p_{j+n}^{(2N)}}{2N-j-n+1} s_{k+j,m+n-N}$$

$$+ w^N \sum_{k=0}^{N-1} \sum_{m=0}^{3N-1-k} z^k w^m \sum_{n=0}^{k} \sum_{j=N-k}^{N-n} \frac{p_{j+n}^{(2N)}}{j+n+1} s_{k+j-N,m+n}$$

$$+ w^N \sum_{k=0}^{N-1} \sum_{m=0}^{3N-1-k} z^k w^m \sum_{n=0}^{N-1} \sum_{j=N-n+1}^{N} \frac{p_{j+n}^{(2N)}}{2N-j-n+1} s_{k+j-N,m+n}.$$

For the method (b), we obtain

$$Q_{N,N}(z,w) = \sum_{j=0}^{N} \sum_{n=0}^{N} c_{N-j,N-n}^{(N,N)} z^j w^n = \sum_{j=0}^{N} \sum_{n=0}^{N} c_{j,n}^{(N,N)} z^{N-j} w^{N-n}$$

$$= c_{0,0}^{(N,N)} z^N w^N + c_{N,N}^{(N,N)} + z^N \sum_{n=1}^{N} c_{0,n}^{(N,N)} w^{N-n} + w^N \sum_{j=1}^{N} c_{j,0}^{(N,N)} z^{N-j}$$

$$+ \sum_{n=0}^{N-1} c_{N,n}^{(N,N)} w^{N-n} + \sum_{j=0}^{N-1} c_{j,N}^{(N,N)} z^{N-j}$$

$$= p_0^{(2N)} z^N w^N + p_{2N}^{(2N)} + \frac{1}{2} z^N \sum_{n=1}^{N} p_n^{(2N)} w^{N-n} + \frac{1}{2} w^N \sum_{j=1}^{N} p_j^{(2N)} z^{N-j}$$

$$+ \frac{1}{2} \sum_{n=0}^{N-1} p_{N+n}^{(2N)} w^{N-n} + \frac{1}{2} \sum_{j=0}^{N-1} p_{N+j}^{(2N)} z^{N-j}.$$

We now determine the numerator

$$P_{\mathrm{N}_1}(z,w) = \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{m} c_{N-j,N-n}^{(N,N)} s_{k-j,m-n}$$

$$+ w^N \sum_{k=0}^{N-1} \sum_{m=0}^{3N-1-k} z^k w^m \sum_{j=0}^{k} \sum_{n=0}^{N} c_{N-j,n}^{(N,N)} s_{k-j,m+n}$$

$$+ z^N \sum_{m=0}^{N-1} \sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N} \sum_{n=0}^{m} c_{j,N-n}^{(N,N)} s_{k+j,m-n}$$

$$= p_{2N}^{(2N)} \left(\sum_{k=0}^{N-1} \sum_{m=0}^{N-1} z^k w^m s_{k,m} + w^N \sum_{k=0}^{N-1} \sum_{m=0}^{3N-1-k} z^k w^m s_{k,m+N} + z^N \sum_{m=0}^{N-1} \sum_{k=0}^{3N-1-m} z^k w^m s_{k+N,m} \right)$$

$$+\frac{1}{2}\left\{\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}z^k w^m\sum_{n=1}^{m}p_{2N-n}^{(2N)}s_{k,m-n}+z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m}z^k w^m\sum_{n=1}^{m}p_{2N-n}^{(2N)}s_{k+N,m-n}\right.$$

$$+w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k}z^k w^m\sum_{n=1}^{N-1}p_{N+n}^{(2N)}s_{k,m+n}+\sum_{k=0}^{N-1}\sum_{m=0}^{N-1}z^k w^m\sum_{j=1}^{k}p_{2N-j}^{(2N)}s_{k-j,m}$$

$$\left.+z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m}z^k w^m\sum_{j=1}^{N}p_{j+N}^{(2N)}s_{k+j,m}+w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k}z^k w^m\sum_{j=1}^{k}p_{2N-j}^{(2N)}s_{k-j,m+N}\right\}.$$

As for the method (*a*), for the method (*c*), we get

$$Q_{N,N}(z,w)=\sum_{j=0}^{N}\sum_{n=N-j}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p_{2N-j-n}^{(2N)}z^j w^n$$

$$+\sum_{j=0}^{N-1}\sum_{n=0}^{N-j-1}\frac{1}{2^{j+n}}\binom{j+n}{n}p_{2N-j-n}^{(2N)}z^j w^n,$$

$$P_{\mathbb{N}_1}(z,w)=\sum_{m=0}^{N-1}\sum_{k=N-m}^{N-1}z^k w^m\sum_{j=N-k}^{m}\sum_{n=N-m}^{N-j}\frac{1}{2^{j+n}}\binom{j+n}{n}p_{j+n}^{(2N)}s_{k+j-N,m+n-N}$$

$$+\sum_{k=1}^{N-1}\sum_{m=0}^{N-1}z^k w^m\sum_{j=N-k}^{N-1}\sum_{n=N-j+1}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p_{j+n}^{(2N)}s_{k+j-N,m+n-N}$$

$$+z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m}z^k w^m\sum_{j=0}^{m}\sum_{n=N-m}^{N-j}\frac{1}{2^{j+n}}\binom{j+n}{n}p_{j+n}^{(2N)}s_{k+j,m+n-N}$$

$$+z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m}z^k w^m\sum_{j=0}^{N-1}\sum_{n=N-j+1}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p_{j+n}^{(2N)}s_{k+j,m+n-N}$$

$$+w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k}z^k w^m\sum_{n=0}^{k}\sum_{j=N-k}^{N-n}\frac{1}{2^{j+n}}\binom{j+n}{n}p_{j+n}^{(2N)}s_{k+j-N,m+n}$$

$$+w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k}z^k w^m\sum_{n=0}^{N-1}\sum_{j=N-n+1}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p_{j+n}^{(2N)}s_{k+j-N,m+n}.$$

Thus, we have established the following result:

Theorem 2. *For an analytic function* $f(z,w)$ *admitting the integral representation (7) and any* $N \in \mathbb{N}$, *the rational functions*

$$\pi_{\mathcal{N}_1,\mathcal{D}}(z,w) = \frac{P_{\mathcal{N}_1}(z,w)}{Q_{N,N}(z,w)}$$

such that

$$Q_{N,N}(z,w) = \sum_{j=0}^{N}\sum_{n=0}^{N} c_{N-j,N-n}^{(N,N)} z^j w^n,$$

$$P_{\mathcal{N}_1}(z,w) = \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{m} c_{N-j,N-n}^{(N,N)} s_{k-j,m-n}$$

$$+ z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N}\sum_{n=0}^{m} c_{j,N-n}^{(N,N)} s_{k+j,m-n}$$

$$+ w^N \sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k} z^k w^m \sum_{j=0}^{k}\sum_{n=0}^{N} c_{N-j,n}^{(N,N)} s_{k-j,m+n},$$

with coefficients $c_{k,m}^{(N,N)}$, $k,m = \overline{0,N}$, *satisfy the equalities*

$$\sum_{k=0}^{N}\sum_{m=0}^{N} c_{k,m}^{(N,N)} t^{k+m} = \sum_{j=0}^{2N} p_j^{(2N)} t^j,$$

where $p_j^{(2N)}$ *are the coefficients of the algebraic polynomial* $P_{2N}(t)$ *orthogonal on* $[0,1]$ *with weight* $d\mu(t)$, *have expansions in power series whose coefficients coincide with coefficients of series (3) for function (6) for all* $(j,n) \in \mathcal{E} = \{(j,n) \in \mathbb{Z}_+^2,\ j+n \leqslant 4N-1\}$. *In particular, this is true for the following rational functions:*

$$\pi_{\mathcal{N}_1,\mathcal{D}}^{(a)}(z,w) = \frac{P_{\mathcal{N}_1}^{(a)}(z,w)}{Q_{N,N}^{(a)}(z,w)},$$

where

$$Q_{N,N}^{(a)}(z,w) = \sum_{j=0}^{N-1}\sum_{n=0}^{N-j-1} \frac{1}{j+n+1} p_{2N-j-n}^{(2N)} z^j w^n + \sum_{j=0}^{N}\sum_{n=N-j}^{N} \frac{1}{2N-j-n+1} p_{2N-j-n}^{(2N)} z^j w^n,$$

$$P_{\mathcal{N}_1}^{(a)}(z,w) = \sum_{m=0}^{N-1}\sum_{k=N-m}^{N-1} z^k w^m \sum_{j=N-k}^{m}\sum_{n=N-m}^{N-j} \frac{p_{j+n}^{(2N)}}{j+n+1} s_{k+j-N,m+n-N}$$

$$+\sum_{k=1}^{N-1}\sum_{m=0}^{N-1} z^k w^m \sum_{j=N-k}^{N-1}\sum_{n=N-j+1}^{N} \frac{p_{j+n}^{(2N)}}{2N-j-n+1} s_{k+j-N,m+n-N}$$

$$+ z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-m-1} z^k w^m \sum_{j=0}^{m}\sum_{n=N-m}^{N-j} \frac{p_{j+n}^{(2N)}}{j+n+1} s_{k+j,m+n-N}$$

$$+ z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-m-1} z^k w^m \sum_{j=0}^{N-1}\sum_{n=N-j+1}^{N} \frac{p_{j+n}^{(2N)}}{2N-j-n+1} s_{k+j,m+n-N}$$

$$+ w^N \sum_{k=0}^{N-1}\sum_{m=0}^{3N-k-1} z^k w^m \sum_{n=0}^{k}\sum_{j=N-k}^{N-n} \frac{p_{j+n}^{(2N)}}{j+n+1} s_{k+j-N,m+n}$$

$$+ w^N \sum_{k=0}^{N-1}\sum_{m=0}^{3N-k-1} z^k w^m \sum_{n=0}^{N-1}\sum_{j=N-n+1}^{N} \frac{p_{j+n}^{(2N)}}{2N-j-n+1} s_{k+j-N,m+n},$$

$$\pi_{\mathcal{N}_1,\mathcal{D}}^{(b)}(z,w) = \frac{P_{\mathcal{N}_1}^{(b)}(z,w)}{Q_{N,N}^{(b)}(z,w)}.$$

Here, in turn,

$$Q_{N,N}^{(b)}(z,w) = \frac{1}{2}\sum_{n=0}^{N}(z^n + w^n)\left(p_n^{(2N)} z^{N-n} w^{N-n} + p_{2N-n}^{(2N)}\right),$$

$$P_{\mathcal{N}_1}^{(b)}(z,w) = p_{2N}^{(2N)}\left(\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} z^k w^m s_{k,m} + w^N \sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k} z^k w^m s_{k,m+N}\right.$$

$$\left. + z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m s_{k+N,m}\right)$$

$$+\frac{1}{2}\left\{\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} z^k w^m \sum_{n=1}^{m} p_{2N-n}^{(2N)} s_{k,m-n} + z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m \sum_{n=1}^{m} p_{2N-n}^{(2N)} s_{k+N,m-n}\right.$$

$$+ w^N \sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k} z^k w^m \sum_{n=1}^{N-1} p_{N+n}^{(2N)} s_{k,m+n} + \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} z^k w^m \sum_{j=1}^{k} p_{2N-j}^{(2N)} s_{k-j,m}$$

$$\left. + z^N \sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=1}^{N} p_{j+N}^{(2N)} s_{k+j,m} + w^N \sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k} z^k w^m \sum_{j=1}^{k} p_{2N-j}^{(2N)} s_{k-j,m+N}\right\},$$

$$\pi^{(c)}_{\mathcal{N}_1,\mathcal{D}}(z,w)=\frac{P^{(c)}_{\mathcal{N}_1}(z,w)}{Q^{(c)}_{N,N}(z,w)},$$

where

$$Q^{(c)}_{N,N}(z,w)=\sum_{j=0}^{N}\sum_{n=N-j}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p^{(2N)}_{2N-j-n}z^jw^n$$

$$+\sum_{j=0}^{N-1}\sum_{n=0}^{N-j-1}\frac{1}{2^{j+n}}\binom{j+n}{n}p^{(2N)}_{2N-j-n}z^jw^n,$$

$$P^{(c)}_{\mathcal{N}_1}(z,w)=\sum_{m=0}^{N-1}\sum_{k=N-m}^{N-1}z^kw^m\sum_{j=N-k}^{m}\sum_{n=N-m}^{N-j}\frac{1}{2^{j+n}}\binom{j+n}{n}p^{(2N)}_{j+n}s_{k+j-N,m+n-N}$$

$$+\sum_{k=1}^{N-1}\sum_{m=0}^{N-1}z^kw^m\sum_{j=N-k}^{N-1}\sum_{n=N-j+1}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p^{(2N)}_{j+n}s_{k+j-N,m+n-N}$$

$$+z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m}z^kw^m\sum_{j=0}^{m}\sum_{n=N-m}^{N-j}\frac{1}{2^{j+n}}\binom{j+n}{n}p^{(2N)}_{j+n}s_{k+j,m+n-N}$$

$$+z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m}z^kw^m\sum_{j=0}^{N-1}\sum_{n=N-j+1}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p^{(2N)}_{j+n}s_{k+j,m+n-N}$$

$$+w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k}z^kw^m\sum_{n=0}^{k}\sum_{j=N-k}^{N-n}\frac{1}{2^{j+n}}\binom{j+n}{n}p^{(2N)}_{j+n}s_{k+j-N,m+n}$$

$$+w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k}z^kw^m\sum_{n=0}^{N-1}\sum_{j=N-n+1}^{N}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}p^{(2N)}_{j+n}s_{k+j-N,m+n}.$$

Remark 2. For the obtained approximations, we get

$$\dim\mathcal{D}^{(a)}=\dim\mathcal{D}^{(c)}=(N+1)^2,$$

$$\dim\mathcal{D}^{(b)}=4N,$$

$$\dim\mathcal{N}_1=\frac{2N(6N+1)}{2},$$

$$\dim\mathcal{E}=\frac{4N(4N+1)}{2}.$$

It is clear that

$$\dim \mathcal{E} - (\dim \mathcal{N}_1 + \dim \mathcal{D}^{(a)} - 1) = N(N-1),$$

$$\dim \mathcal{E} - (\dim \mathcal{N}_1 + \dim \mathcal{D}^{(b)} - 1) = 2(N-1)\left(N - \frac{1}{2}\right).$$

For $N > 1$, these quantities are strictly greater than 0. It is obvious that this is caused by the fact that functions of the form (6) can be represented in the form of linear combinations of functions of one variable (see, e.g., [15]).

We now consider the approximation of a function $f(z, w)$ of the form (6) for the weight

$$d\mu(t) = (1-t)^{\sigma} t^{\nu} dt, \quad \delta, \nu > -1.$$

In this case, as already indicated, the orthogonal polynomial appearing in Theorem 2 coincides, to within a constant factor, with the orthonormal Jacobi polynomial of degree $2N$ shifted by $[0, 1]$. The coefficients of this polynomial can be found in the explicit form (see [16, p. 581]). We have

$$P_{2N}(t; \sigma, \nu) = C_N \sum_{m=0}^{2N} (-1)^m t^m \binom{2N}{m} \frac{\Gamma(2N + \sigma + \nu + 1 + m)}{\Gamma(\sigma + 1 + m)}.$$

This yields

$$p_k^{(2N)} = (-1)^k \binom{2N}{k} \frac{\Gamma(2N + \sigma + \nu + 1 + k)}{\Gamma(\sigma + 1 + k)}.$$

By virtue of Theorem 2, this enables us to efficiently construct rational approximants of the form described above for the Appell series (9). Thus, we get the following result [presented only for the approximants obtained by the method (c)]:

Theorem 3. *For the hypergeometric Appell series (9) and any* $N \in \mathbb{N}$, *the rational function*

$$\pi^{(c)}_{\mathcal{N}_1, \mathcal{D}}(z, w) = \frac{P^{(c)}_{\mathcal{N}_1}(z, w)}{Q^{(c)}_{N,N}(z, w)},$$

where

$$Q^{(c)}_{N,N}(z, w) = \sum_{j=0}^{N} \sum_{n=N-j}^{N} (-1)^{j+n} \frac{1}{2^{2N-j-n}} \binom{2N-j-n}{N-n} \binom{2N}{2N-j-n}$$

$$\times \frac{\Gamma(4N + \sigma + \nu + 1 - j - n)}{\Gamma(2N + \sigma + 1 - j - n)} z^j w^n$$

$$+ \sum_{j=0}^{N-1} \sum_{n=0}^{N-j-1} (-1)^{j+n} \frac{1}{2^{j+n}} \binom{j+n}{n} \binom{2N}{2N-j-n} \frac{\Gamma(4N + \sigma + \nu + 1 - j - n)}{\Gamma(2N + \sigma + 1 - j - n)} z^j w^n,$$

$$P^{(c)}_{\mathcal{N}_1}(z,w) = \sum_{m=0}^{N-1}\sum_{k=N-m}^{N-1} z^k w^m \sum_{j=N-k}^{m}\sum_{n=N-m}^{N-j}(-1)^{j+n}\frac{1}{2^{j+n}}\binom{j+n}{n}\binom{2N}{j+n}$$

$$\times \frac{\Gamma(2N+\sigma+\nu+1+j+n)}{\Gamma(\sigma+1+j+n)}\frac{\Gamma(k+j+m+n-2N+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+j+m+n-2N+\nu+\sigma+2)}$$

$$+ \sum_{k=1}^{N-1}\sum_{m=0}^{N-1} z^k w^m \sum_{j=N-k}^{N-1}\sum_{n=N-j+1}^{N}(-1)^{j+n}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}\binom{2N}{j+n}$$

$$\times \frac{\Gamma(2N+\sigma+\nu+1+j+n)}{\Gamma(\sigma+1+j+n)}\frac{\Gamma(k+j+m+n-2N+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+j+m+n-2N+\nu+\sigma+2)}$$

$$+ z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{m}\sum_{n=N-m}^{N-j}(-1)^{j+n}\frac{1}{2^{j+n}}\binom{j+n}{n}\binom{2N}{j+n}$$

$$\times \frac{\Gamma(2N+\sigma+\nu+1+j+n)}{\Gamma(\sigma+1+j+n)}\frac{\Gamma(k+j+m+n-N+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+j+m+n-N+\nu+\sigma+2)}$$

$$+ z^N\sum_{m=0}^{N-1}\sum_{k=0}^{3N-1-m} z^k w^m \sum_{j=0}^{N-1}\sum_{n=N-j+1}^{N}(-1)^{j+n}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}\binom{2N}{j+n}$$

$$\times \frac{\Gamma(2N+\sigma+\nu+1+j+n)}{\Gamma(\sigma+1+j+n)}\frac{\Gamma(k+j+m+n-N+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+j+m+n-N+\nu+\sigma+2)}$$

$$+ w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k} z^k w^m \sum_{n=0}^{k}\sum_{j=N-k}^{N-n}(-1)^{j+n}\frac{1}{2^{j+n}}\binom{j+n}{n}\binom{2N}{j+n}$$

$$\times \frac{\Gamma(2N+\sigma+\nu+1+j+n)}{\Gamma(\sigma+1+j+n)}\frac{\Gamma(k+j+m+n-N+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+j+m+n-N+\nu+\sigma+2)}$$

$$+ w^N\sum_{k=0}^{N-1}\sum_{m=0}^{3N-1-k} z^k w^m \sum_{n=0}^{N-1}\sum_{j=N-n+1}^{N}(-1)^{j+n}\frac{1}{2^{2N-j-n}}\binom{2N-j-n}{N-n}\binom{2N}{j+n}$$

$$\times \frac{\Gamma(2N+\sigma+\nu+1+j+n)}{\Gamma(\sigma+1+j+n)}\frac{\Gamma(k+j+m+n-N+\nu+1)\Gamma(\sigma+1)}{\Gamma(k+j+m+n-N+\nu+\sigma+2)},$$

admits an expansion in power series whose coefficients coincide with the coefficients of series (9) for all

$$(j,n) \in \mathcal{E} = \{(j,n) \in \mathbb{Z}^2_+,\ j+n \leqslant 4N-1\}.$$

Table 1

w	z 0.0	0.2	0.4	0.6	0.8
0.0	1 1 1.000000000	1.115717756 1.115333333 1.115555556	1.277064060 1.269333333 1.273333333	1.527151220 1.474000000 1.497142857	2.011797390 1.741333333 1.826666667
0.2	1.115717756 1.115333333 1.115555556	1.25 1.247999999 1.249586777	1.438410362 1.423333333 1.433644860	1.732867952 1.653333333 1.696774194	2.310490602 1.949999999 2.088607595
0.4	1.277064060 1.269333333 1.273333333	1.438410362 1.423333333 1.433644860	1.666666667 1.623999999 1.655319149	2.027325540 1.883333333 1.975308642	2.746530722 2.213333333 2.458823529
0.6	1.527151220 1.474000000 1.497142857	1.732867952 1.653333333 1.696774194	2.027325540 1.883333333 1.975308642	2.5 2.176000000 2.382608696	3.465735903 2.543333333 3.010526316
0.8	2.011797390 1.741333333 1.826666667	2.310490602 1.949999999 2.088607595	2.746530722 2.213333333 2.458823529	3.465735903 2.543333333 3.010526316	5 2.951999999 3.886956522

To illustrate this result, consider a special case $\nu = \sigma = 0$ and a version of the approximation (c). As already indicated, the function $f(z,w)$ has the form

$$f(z,w) = \frac{\ln \dfrac{1-z}{1-w}}{w-z}. \tag{10}$$

First, we set $N = 1$. This yields the rational approximation

$$\frac{P_{\mathbb{N}_1}(z,w)}{Q_{1,1}(z,w)} = \frac{w^3 + z^3 + w^2 + z^2 + 12}{2zw - 6z - 6w + 12}.$$

We compare the values of the approximated function (10), the partial sum of the power series

$$P_3(z,w) = 1 + \frac{1}{2}(z+w) + \frac{1}{3}(z^2 + zw + w^2) + \frac{1}{4}(z^3 + z^2 w + +zw^2 + w^3),$$

and the constructed approximation at points of the square $[0.8] \times [0.8]$ (see Table 1 and Fig. 5).

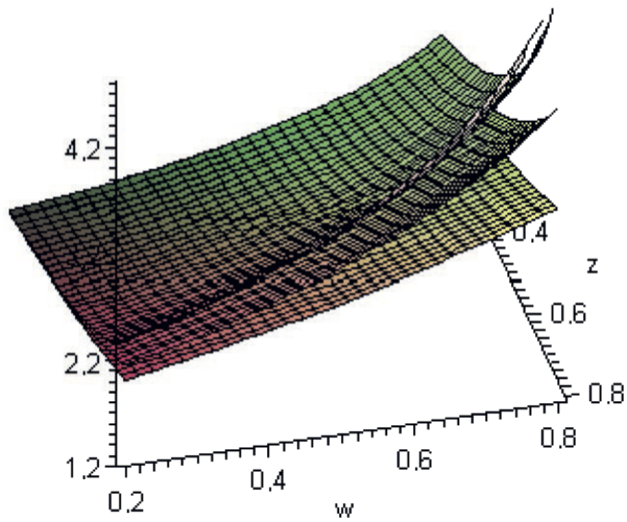

Fig. 5

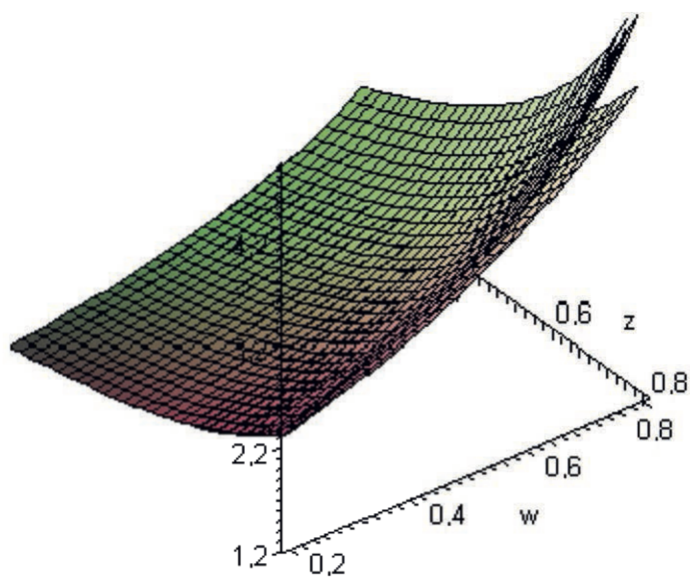

Fig. 6

We now take $N = 2.$ This yields the rational function

$$\frac{P_{\mathcal{N}_1}(z,w)}{Q_{2,2}(z,w)} = (60z^7 + 60w^7 - 48z^6w - 48zw^6 + 68z^6 + 68w^6 - 56z^5w - 56zw^5$$
$$+ 79z^5 + 79w^5 - 67z^4w - 67zw^4 + 96z^4 + 96w^4 - 84z^3w - 84zw^3 + 130z^3 + 130w^3$$
$$- 130z^2w - 130zw^2 + 260z^2 + 260w^2 - 40zw - 840z - 840w + 1680)$$
$$\times (24z^2w^2 - 240zw^2 - 240z^2w + 540z^2 + 540w^2 + 1080zw - 1680z - 1680w + 1680)^{-1}.$$

Table 2

w	z				
	0.0	0.2	0.4	0.6	0.8
0.0	1 1 1.000000000	1.115717756 1.115717409 1.115717633	1.277064060 1.276949943 1.277013333	1.527151220 1.523044343 1.524834792	2.011797390 1.941530209 1.960940469
0.2	1.115717756 1.115717409 1.115717633	1.25 1.249996798 1.249999930	1.438410362 1.438182476 1.438370451	1.732867952 1.726707809 1.730591604	2.310490602 2.216801140 2.254184731
0.4	1.277064060 1.276949943 1.277013333	1.438410362 1.438182476 1.438370451	1.666666667 1.665574402 1.666626430	2.027325540 2.015233143 2.025280391	2.746530722 2.606110476 2.684240361
0.6	1.527151220 1.523044343 1.524834792	1.732867952 1.726707809 1.730591604	2.027325540 2.015233143 2.025280391	2.5 2.458009601 2.497044927	3.465735903 3.196987809 3.397215406
0.8	2.011797390 1.941530209 1.960940469	2.310490602 2.216801140 2.254184731	2.746530722 2.606110476 2.684240361	3.465735903 3.196987809 3.397215406	5 4.161139198 4.854606766

The values of the approximated function (10), a partial sum of the power series

$$\begin{aligned} P_7(z,w) = 1 + \frac{1}{2}(z+w) + \frac{1}{3}(z^2+zw+w^2) \\ + \frac{1}{4}(z^3+z^2w+zw^2+w^3) + \frac{1}{5}(z^4+z^3w+z^2w^2+zw^3+w^4) \\ + \frac{1}{6}(z^5+z^4w+z^3w^2+z^2w^3+zw^4+w^5) \\ + \frac{1}{7}(z^6+z^5w+z^4w^2+z^3w^3+z^2w^4+zw^5+w^6) \\ + \frac{1}{8}(z^7+z^6w+z^5w^2+z^4w^3+z^3w^4+z^2w^5+zw^6+w^7) \\ = \sum_{k=0}^{7}\frac{1}{k+1}\sum_{m=0}^{k} z^m w^{k-m}, \end{aligned}$$

and the constructed approximation are presented in Table 2 and in Fig. 6.

REFERENCES

1. V. K. Dzyadyk, "On generalization of the problem of moments," *Dop. Akad. Nauk URSR*, **6**, 8–12 (1981).
2. A. P. Holub, *Generalized Moment Representations and Padé Approximations* [in Ukrainian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv (2002).
3. G. A. Baker, Jr., and P. Graves-Morris, *Padé Approximants* [Russian translation], Mir, Moscow (1986).
4. C. Alabiso and P. Butera, "N-variable rational approximants and method of moments," *J. Math. Phys.*, **16**, No. 4, 840–845 (1975).
5. J. S. Chisholm, "Rational approximants defined from double power series," *Math. Comput.*, **27**, 841–848 (1973).
6. P. R. Graves-Morris and R. Hughes Jones, "An analysis of two variable rational approximants," *J. Comput. Appl. Math.*, **2**, No. 1, 41–48 (1976).
7. A. Cuyt, "How well can the concept of Padé approximant be generalized to the multivariate case?," *J. Comput. Appl. Math.*, **105**, No. 1-2, 25–50 (1999).
8. A. Cuyt, K. Driver, J. Tan, and B. Verdonk, "Exploring multivariate Padé approximants for multiple hypergeometric series," *Adv. Comput. Math.*, **10**, No. 1, 29–49 (1999).
9. R. Hughes Jones, "General rational approximants in N variables," *J. Approxim. Theory*, **16**, 201–233 (1976).
10. C. Lutterodt, "A two-dimensional analogue of Padé approximant theory," *J. Phys. A.: Math.*, **7**, 1027–1037 (1974).
11. P. Zhou, "Explicit construction for multivariate Padé approximants," *J. Comput. Appl. Math.*, **79**, 1–17 (1997).
12. Kh. I. Kukhmins'ka, *Two-Dimensional Continued Fractions* [in Ukrainian], Institute for Applied Problems in Mechanics and Mathematics, Ukrainian National Academy of Sciences, Lviv (2010).
13. H. Bateman and A. Erdélyi, *Higher Transcendental Functions,* Vol. 1 [Russian translation], Nauka, Moscow (1973).
14. P. K. Suetin, *Classical Orthogonal Polynomials* [in Russia], Nauka, Moscow (1979).
15. A. Cuyt, J. Tan, and P. Zhou, "General order multivariate Padé approximants for pseudo-multivariate functions," *Math. Comput.*, **75**, No. 254, 727–741 (2006).
16. M. Abramowitz and I. A. Stegun (editors), *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables* [Russian translation], Nauka, Moscow (1979).