# A SYSTEM OF BIORTHOGONAL POLYNOMIALS AND ITS APPLICATIONS

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UDC 517.53

In [1] there were constructed generalized moment representations for basic hypergeometric series. There arose the problem of the biorthogonalization of the sequences of functions $\{\alpha_i(t) = t^{\rho\lambda_{i+1}(q)}\}_{i=0}^{\infty}$ and $\{\beta_j(t) = t^{\gamma+\tilde{\lambda}_j(q)}\}_{j=0}^{\infty}$, where $\lambda_{i+1}(q) = \dfrac{q^{i+1}-1}{q-1}$, $i=\overline{0,\infty}$; $\tilde{\lambda}_j(q) = \dfrac{q^j-1}{(q-1)q^j}$, $j=\overline{0,\infty}$, $\gamma = \dfrac{q-\rho}{1-q} > -1$, $\rho, q > 0$, $q \neq 1$, i.e., the determination of generalized polynomials

$$A_M(t) = \sum_{i=0}^{M} c_i^{(M)} \alpha_i(t), \quad c_M^{(M)} \neq 0, \quad M = \overline{0,\infty}, \tag{1}$$

and

$$B_N(t) = \sum_{j=0}^{N} d_j^{(N)} \beta_j(t), \quad d_N^{(N)} \neq 0, \quad N = \overline{0,\infty}, \tag{1'}$$

such that

$$\int_0^1 A_M(t) B_N(t)\, dt = 0, \quad M \neq N. \tag{2}$$

We will prove the following result.

THEOREM 1. The polynomials $A_M(t)$, $M = \overline{0, \infty}$, and $B_N(t)$, $N = \overline{0, \infty}$, defined by (1), (1'), and (2) can be represented in the form

$$A_M(t) = \sum_{m=0}^{M} (-1)^m q^{\frac{m(m-1)}{2}} \prod_{r=1}^{m} \frac{q^{M-r+1}-1}{q^r-1} \times \prod_{l=1}^{M} \frac{\rho q^{2M-m-l+1}-1}{q-1}\, t^{\rho\lambda_{M-m+1}}, \quad M = \overline{0,\infty},$$

$$B_N(t) = \sum_{n=0}^{N} (-1)^n q^{\frac{n(n-1)}{2}} \prod_{r=1}^{n} \frac{q^{N-r+1}-1}{q^r-1} \prod_{k=1}^{2N-n} \frac{\rho q^k-1}{q-1}\, b_{N-n}(t), \tag{3}$$

$$b_j(t) = \frac{(q-1)^j}{\prod_{r=1}^{j}(q^r-1)} \sum_{m=0}^{j} (-1)^m q^{\frac{m(m-1)}{2}} \prod_{r=1}^{m} \frac{q^{j-r+1}-1}{q^r-1} \beta_m(t), \quad j = \overline{0, N}.$$

Proof. It is obvious that if we define an operator $A: C[0, 1] \to C[0, 1]$ by the formula

$$(A\varphi)(t) = t^{\rho} \int_0^1 \varphi(t^q u)\, u^{\gamma} du$$

and a polynomial $A_M(t)$ by (3), then the relations $(A^k A_M)(1) = 0$, $k = \overline{1, M}$, $M = \overline{1, \infty}$, will hold. Therefore if we take into account the equality

$$\int_0^1 (A\varphi)(t)\psi(t)\, dt = \int_0^1 \varphi(t)(B\psi)(t)\, dt, \tag{4}$$

where the operator $B: L_1[0, 1] \to L_1[0, 1)$ has the form

$$(B\psi)(t) = \frac{1}{q} t^{\gamma} \int_t^1 \psi(v^{1/q})\, v^{\frac{\rho+1-\gamma q-2q}{q}} dv, \tag{5}$$

Institute of Problem Simulation in Energetics, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 41, No. 7, pp. 961-965, July, 1989. Original article submitted October 30, 1986.

and the relations

$$b_j(t)=\frac{(q-1)^j}{\prod_{r=1}^{j}(q^r-1)}\sum_{m=0}^{j}(-1)^m q^{\frac{m(m-1)}{2}}\prod_{r=1}^{m}\frac{q^{j-r+1}-1}{q^r-1}\beta_m(t)=(B^j b_0)(t),\quad j=\overline{0,\infty}, \tag{6}$$

where $b_0(t)=\beta_0(t)=t^{\gamma}$ (regarding (4)-(6) see [1]), we obtain

$$\int_0^1 A_M(t)\,b_j(t)\,dt=\int_0^1 A_M(t)(B^j b_0)(t)\,dt=\int_0^1 (A^j A_M)(t)\,b_0(t)\,dt=$$

$$=\int_0^1 (A^j A_M)(t)\,t^{\gamma}dt=[A(A^j A_M)](1)=(A^{j+1}A_M)(1)=0,\quad j=\overline{0,\ M-1}.$$

Thus the polynomials $A_M(t)$ are orthogonal to the functions $b_j(t)$, $j=\overline{0,\ M-1}$, and hence to the functions $\beta_j(t)$, $j=\overline{0,\ M-1}$, which are linear combinations of them:

$$\int_0^1 A_M(t)\,\beta_j(t)\,dt=0,\quad j=\overline{0,\ M-1}. \tag{7}$$

It is then easy to show that

$$\int_0^1 \alpha_i(t)\,B_N(t)\,dt=0,\quad i=\overline{0,\ N-1}. \tag{8}$$

Combining (7) and (8), we obtain (2). The theorem is proved.

This result can be applied to Padé approximation of basic hypergeometric series. Note that the continued fraction expansions of these series were considered in [2, 3].

THEOREM 2. For the function [4, pp. 195-196]

$$f(z)=\frac{{}_1\Phi_1\left[\begin{matrix}q;\ (1-q)z\\ \rho\end{matrix}\right]-1-\frac{z(1-q)}{1-\rho}}{z^2}\left(\frac{1-\rho}{1-q}\right)=$$

$$=\sum_{n=0}^{\infty}\frac{z^n}{(\gamma+1+\rho)[\gamma+1+\rho(1+q)]\cdot\ldots\cdot[\gamma+1+\rho(1+q+\ldots+q^n)]}, \tag{9}$$

where $\gamma:=\frac{q-\rho}{1-q}>-1;\ \rho,\ q>0;\ q\neq 1,$ the Padé polynomials of order $[N-1/N]$, $N=\overline{1,\ \infty}$, can be represented in accordance with the formulas

$$[N-1/N]_f(z)=\frac{P_{N-1}(z)}{Q_N(z)}=\frac{\sum_{m=0}^{M}(-1)^m q^{\frac{m(m-1)}{2}}\prod_{r=1}^{m}\frac{q^{M-r+1}-1}{q^r-1}\prod_{k=1}^{2M-m}\frac{\rho q^k-1}{q-1}z^m T_{M-m-1}(f;z)}{\sum_{m=0}^{M}(-1)^m q^{\frac{m(m-1)}{2}}\prod_{r=1}^{m}\frac{q^{M-r+1}-1}{q^r-1}\prod_{k=1}^{2M-m}\frac{\rho q^k-1}{q-1}z^m},$$

where $T_j(f;\ z)$ is the partial sum of order j of series (9). The error of the approximation can be represented in the form

$$f(z)-[N-1/N]_f(z)=\frac{z^N}{Q_N(z)}\int_0^1 A_N(t)\sum_{j=0}^{\infty}z^j b_j(t)dt.$$

THEOREM 3. For the function [4, pp. 195-196]

$$f(z)=\frac{(1-q)\alpha}{(1-\alpha)z\rho}\left[{}_2\Phi_1\left(\begin{matrix}q;\ \alpha;\ \frac{\xi z}{\alpha q}\\ \xi\end{matrix}\right)-1\right]=$$

$$=\sum_{n=0}^{\infty}\frac{(\rho+\gamma+\sigma+1)[\rho(q+1)+\gamma+\sigma+1]\ldots[\rho(q^{n-1}+\ldots+1)+\gamma+\sigma+1]z^n}{(\rho+\gamma+1)[\rho(q+1)+\gamma+1]\ldots[\rho(q^n+\ldots+1)+\gamma+1]} \tag{10}$$

[here $\alpha := \frac{\rho}{\varkappa - \sigma(q-1)}$; $\xi := \frac{\rho q}{\varkappa}$; $\varkappa := \rho - (q-1)(\gamma+1)$ ], where $\gamma := \frac{q-\rho}{1-q} > -1$; $\rho, q > 0$; $q \neq 1$; $\sigma \neq \frac{\varkappa(q^r-1)}{q^r(q-1)}$, $r = \overline{1, \infty}$, the Padé polynomials of order [N − 1/N], N = $\overline{1, \infty}$, can be represented in accordance with the formulas

$$[N-1/N]_f = \frac{P_{N-1}(z)}{Q_N(z)} = \frac{\sum_{n=0}^{N-1} (-1)^n q^{\frac{n(n-1)}{2}} \prod_{r=1}^{n} \frac{q^{N-r+1}-1}{q^r-1} \frac{\prod_{k=1}^{2N-n} (\rho q^k - 1)}{\prod_{l=1}^{N-n} (\rho q^l - 1 + \sigma(q-1))} z^n T_{N-1-n}(f; z)}{\sum_{n=0}^{N} (-1)^n q^{\frac{n(n-1)}{2}} \prod_{r=1}^{n} \frac{q^{N-r+1}-1}{q^r-1} \frac{\prod_{k=1}^{2N-n} (\rho q^k - 1)}{\prod_{l=1}^{N-n} (\rho q^l - 1 + \sigma(q-1))} z^n},$$

where $T_j(f; z)$ is the partial sum of order j of series (10). The error of the approximation can be represented in the form

$$f(z) - [N-1/N]_f(z) = \frac{z^n}{Q_N(z)} \int_0^1 A_N(t) \sum_{j=0}^{\infty} z^j \tilde{b}_j(t)\, dt,$$

where

$$\tilde{b}_j(t) = \sum_{m=0}^{j} t^{\gamma + \tilde{\lambda}_m(q)} \prod_{l=1}^{j-m} \frac{\sigma + \varkappa \frac{q^{l-1}-1}{q-1}}{\varkappa\left(\frac{q^l-1}{q-1}\right)} \prod_{r=1}^{m} \left(\frac{1}{q} - \frac{\sigma(q-1)q^{r-1}}{\varkappa(q^r-1)}\right), \quad j = \overline{0, \infty}.$$

The proofs of Theorems 2 and 3 follow easily from Theorem 1 of the present paper and Theorems 2 and 4 of [1].

Remark. Formulas for the diagonal Padé polynomials of the q-analogue of the exponential function, which is a special case of (9), were obtained in [5]. A special case of the functions (10) was considered in [6].

## LITERATURE CITED


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