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WEAKLY INCREASING ZERO-DIMINISHING SEQUENCES

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Abstract. The following problem, suggested by Laguerre's Theorem (1884), remains open: Characterize all real sequences $\{\mu_k\}_{k=0}^{\infty}$ which have the zero-diminishing property; that is, if $p(x) = \sum_{k=0}^{n} a_k x^k$ is any real polynomial, then $\sum_{k=0}^{n} \mu_k a_k x^k$ has no more real zeros than $p(x)$.

In this paper this problem is solved under the additional assumption of a weak growth condition on the sequence $\{\mu_k\}_{k=0}^{\infty}$, namely $\underline{\lim}_{n\to\infty} |\mu_n|^{1/n} < \infty$. More precisely, it is established that the real sequence $\{\mu_k\}_{k\geq 0}$ is a weakly increasing zero-diminishing sequence if and only if there exists $\sigma \in \{+1, -1\}$ and an entire function

$$\Phi(z) = be^{az} \prod_{n\geq 1} \left(1 + \frac{x}{\alpha_n}\right),\ a, b \in R^1,\ b \neq 0,\ \alpha_n > 0\ \forall n \geq 1,\ \sum_{n\geq 1} \frac{1}{\alpha_n} < \infty,$$

such that $\mu_k = \frac{\sigma^k}{\Phi(k)}$, $\forall k \geq 0$.

1. Introduction. In 1914 Pólya and Schur [14] characterized those linear transformations T of the form

$$T[x^k] = \mu_k x^k,\ \mu_k \in R^1,\ \forall k \geq 0,$$

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which map any polynomial with real coefficients (i.e., real polynomial) and with all real zeros to a polynomial with all real zeros. In 1884 Laguerre's Theorem [10, p. 116] suggested the problem of characterizing those linear transformations of the above form, which do not increase the number of real roots of any real polynomial; i.e., which satisfy the inequality

$$Z_R(T[p(x)]) \leq Z_R(p(x))$$

for any real polynomial, where, in general, $Z_D(p(x))$ denotes the number of zeros of a polynomial p, lying in the subset $D \subseteq C^1$, taking into account their multiplicity (see also S.Karlin [8, p. 382]). For recent progress and results pertaining to this area of investigation, we refer to Bakan and Golub [1], Craven and Csordas [2, 3, 4], and Iserles, Nørsett and Saff [7].

In the paper this problem is completely solved for *weakly increasing sequences* $\{\mu_n\}_{n\geq 0}$ which are defined by the condition

$$\varliminf_{n\to\infty} |\mu_n|^{1/n} < \infty.$$

In Section 2 preliminary results are established, describing the properties of the functional which counts the number of zeros of a polynomial in intervals of the form $(-\infty, a]$, $a \in R^1$. The new techniques developed in Section 3 yield a complete characterization of weakly increasing zero-diminishing sequences (Theorem 2). This characterization is applied to the problem of interpolating zero-diminishing sequences (Corollary 2), raised in [3, Problem 8].

2. General properties of the functional $Z_{(-\infty,a]}(p(x))$, $a \in R^1$. Let $\mathcal{N} := \{1, 2, \ldots\}$ be the set of all positive integers, N denote a positive integer, and R^N be an N-dimensional normed space with the usual Euclidean norm $\|x\| := (\sum_{k=1}^{N} x_k^2)^{1/2}$, $x = (x_1, \ldots, x_N)$. Let

$$\mathcal{P}_N := \{x^N + \sum_{k=0}^{N-1} p_k x^k \,|\, p_k \in R^1,\ k = 0, \ldots, N-1\}$$

denote the set of all real polynomials of degree N with leading coefficient equal to 1. We identify every vector $p := (p_0, p_1, \ldots, p_{N-1}) \in R^N$ with the polynomial $p(x) := x^N + \sum_{k=0}^{N-1} p_k x^k \in \mathcal{P}_N$. If $T : R^N \to R^N$ is a mapping from R^N to itself, then let $Tp(x)$ denote the polynomial corresponding to the vector $Tp \in R^N$.

Using this identification, we can define our zero counting functional on R^N by

$$Z_{(-\infty,a]}(p) := Z_{(-\infty,a]}(p(x)),\ \forall p \in R^N,$$

where $a \in R^1 \cup \{+\infty\}$ and $p(x) \in \mathcal{P}_N$ is the polynomial corresponding to the vector $p \in R^N$. Since $Z_{(-\infty,a]}(p) \in \{0, 1, 2, \ldots, N\}$, the functional $Z_{(-\infty,a]}(p)$ is finite–valued and partitions the whole space R^N into $N+1$ (disjoint) sets:

$$\begin{aligned} R_n^N(a) &:= \{p \in R^N \,|\, Z_{(-\infty,a]}(p) = n\},\ 0 \le n \le N; \\ R_n^N &:= R_n^N(+\infty) = \{p \in R^N \,|\, Z_R(p) = n\},\ 0 \le n \le N. \end{aligned}$$

We write $Cl\ B$ for the closure of the set B in the normed space R^N and $int\ B := \{b \in B \,|\, \exists \varepsilon > 0 :\ U_\varepsilon(b) \subseteq B\}$ for its interior, where

$$U_\varepsilon(y) := \{x \in R^N \,|\, \|x - y\| < \varepsilon\},\ y \in R^N,\ \varepsilon > 0.$$

The following lemma is easily proved using Hurwitz's theorem [15, p. 119] and the continuous dependence of the zeros of a polynomial on its coefficients (see [13, p. 279]).

Lemma 1. For an arbitrary element $a \in R^1$, the following statements hold:
(a) $int\, R_n^N(a) \neq \emptyset,\ \ \forall\, 0 \le n \le N$;
(b) $R_n^N(a) \subseteq Cl\, int\, R_n^N(a),\ \ \forall\, 0 \le n \le N$;
(c) $int\, R_n^N(a) = \{p \in R^N \,|\, p(x)$ has n distinct zeros lying in $(-\infty, a)$ and $p(a) \neq 0\}$, $\forall\, 0 \le n \le N$;
(d) $R_0^N(a) = int\ R_0^N(a)$.

Remark 1. Lemma 1 remains true for $a = +\infty$. In this case $int\ R_n^N$ will consist of all those vectors p, corresponding to polynomials which have n real distinct zeros, $0 \le n \le N$. Moreover, if $a = +\infty$, then $R_1^N = int\ R_1^N$.

If $n = 0$ and $a \in R^1$, the set $R_n^N(a)$ is open by Lemma 1. For $1 \le n \le N$, we describe the sets $R_n^N(a) \setminus int\ R_n^N(a)$ in the following lemma.

Lemma 2. *Let $a \in R^1$ and $0 \le n \le N$. Suppose that the polynomial $p(x)$ corresponding to a vector $p \in R_n^N(a)$ has the following form:*

$$p(x) = Q(x)(x-a)^{n_0} \prod_{k=1}^{r} (x - \alpha_k)^{n_k},\ n = \sum_{k=0}^{r} n_k, \tag{1}$$

where $-\infty < \alpha_r < \alpha_{r-1} < \cdots < \alpha_1 < a,\ n_0 \ge 0,\ n_k \ge 1,\ \ 1 \le k \le r,\ r$ is a nonnegative integer, $\prod_{k=1}^{0} := 1$, and $Q \in R_0^{N-n}(a)$. Let $d(p) := \sum_{k=1}^{r} [\frac{n_k}{2}]$, where $[x]$ is the greatest integer less than or equal to $x \in R^1$. Let

$$J(p) = \begin{cases} \{n - 2m \,|\, 0 \le m \le d(p)\},\ \text{if } n_0 = 0; \\ \{n, n-1, \ldots, n - 2d(p) - n_0\},\ \text{if } n_0 \ge 1. \end{cases} \tag{2}$$

Then
(a) $U_\varepsilon(p) \cap int\, R_r^N(a) \neq \emptyset,\ \forall \varepsilon > 0,\ \forall r \in J(p)$;

(*b*) $\exists\delta = \delta(p) > 0$ *such that* $U_\varepsilon(p) \cap R_r^N(a) = \emptyset$, $\forall\varepsilon \in (0,\delta)$, $\forall r \in \{0,1,2,\ldots,N\} \setminus J(p)$.

We omit the proof of Lemma 2 which requires, *ipso facto*, some involved bookkeeping. Lemma 2 states that for sufficiently small $\varepsilon > 0$, the open neighborhood $U_\varepsilon(p)$ of p can be partitioned into subsets according to the number of zeros in the interval $(-\infty, a]$: $U_\varepsilon(p) \cap R_r^N(a)$, $r \in J(p)$. There are $d(p)+1$ such sets if $p(a) \neq 0$ and $2d(p) + n_0 + 1$ if $p(a) = 0$. The main idea in counting the number of possible zeros in $(-\infty, a]$ is that in perturbing the polynomial $p(x)$, one cannot gain zeros in the interval, can lose zeros in the interior of the interval only in nonreal pairs, and can lose zeros from the endpoint a either by singly moving them to the right of a or by forming pairs of nonreal zeros. Each of the subsets of the partition has a nonempty interior and is contained in the closure of its interior. For $n \geq 1$, $p \in R_n^N(a) \setminus int\ R_n^N(a)$ if and only if either $p(x)$ has a multiple root ($d(p) \geq 1$) or $p(a) = 0$.

Remark 2. Putting $n_0 = 0$ in (1) and (2), all statements of Lemma 2 remain valid for $a = +\infty$.

Theorem 1. *Let* $a \in R^1 \cup \{+\infty\}$. *The functional* $Z_{(-\infty,a]}(p)$ *is finite valued and upper semicontinuous on the normed space* R^N. *Moreover, the functional possesses the following properties:*

(*a*) $\forall p \in R^N$, $\exists\varepsilon = \varepsilon(p) > 0$ *such that* $Z_{(-\infty,a]}(q) \leq Z_{(-\infty,a]}(p)$, $\forall q \in U_\varepsilon(p)$;

(*b*) $int[U_\varepsilon(p) \cap R^N_{Z_{(-\infty,a]}(p)}(a)] \neq \emptyset$, $\forall\varepsilon > 0$, $\forall p \in R^N$;

(*c*) *if* T *is a continuous mapping from* R^N *to* R^N, D *is an everywhere dense subset of* R^N, *and the inequality* $Z_{(-\infty,a]}(Tp) \leq Z_{(-\infty,a]}(p)$ *holds for any* $p \in D$, *then*

$$Z_{(-\infty,a]}(Tp) \leq Z_{(-\infty,a]}(p), \ \forall p \in R^N.$$

P r o o f. Lemma 2 implies part (a) as well as the weaker property of upper semicontinuity of the functional $Z_{(-\infty,a]}(p)$; that is,

$$\text{if } p_n \to p, \ n \to \infty, \text{ then } \overline{\lim_{n\to\infty}} Z_{(-\infty,a]}(p_n) \leq Z_{(-\infty,a]}(p).$$

In this connection it should be noted that part (a) is a consequence of only the properties of upper semicontinuity and finite-valuedness of the functional $Z_{(-\infty,a]}(p)$. Property (b) of the functional $Z_{(-\infty,a]}(p)$ follows from part (a) of Lemma 2 with $r = n$.

Next we prove property (c). Consider any $p \in R^N$. By property (a) of the functional $Z_{(-\infty,a]}(p)$ for $Tp \in R^N$, there exists an $\varepsilon(Tp) > 0$ such that $Z_{(-\infty,a]}(q) \leq Z_{(-\infty,a]}(Tp)$, $\forall q \in U_{\varepsilon(Tp)}(Tp)$. Due to the continuity of the mapping T, there exists a $\delta = \delta(p) > 0$ such that $T(U_\delta(p)) \subseteq U_{\varepsilon(Tp)}(Tp)$. By property (b) of the functional and the fact that the set D is everywhere dense, there exists a vector $\alpha \in D \cap [U_\delta(p) \cap R^N_{Z_{(-\infty,a]}(p)}(a)] \neq \emptyset$. But then $Z_{(-\infty,a]}(p) = Z_{(-\infty,a]}(\alpha) \leq Z_{(-\infty,a]}(T\alpha) \leq Z_{(-\infty,a]}(Tp)$. This proves the theorem. $\square$

Corollary 1. *Let $T : R^N \to R^N$ be a homeomorphism (see* [9, Ch. 1, Sec. 13, VIII]*), and for some $a \in R^1 \cup \{+\infty\}$ suppose that the inequality*

$$Z_{(-\infty,a]}(Tp(x)) \leq Z_{(-\infty,a]}(p(x)) \tag{3}$$

holds for all polynomials $p(x) \in \mathcal{P}_N$ such that both polynomials $p(x)$ and $Tp(x)$ either have no real zeros or have distinct real zeros, not belonging to some nowhere dense subset F of the real axis; that is, $Cl(R^1 \setminus Cl\ F) = R^1$ (see [9, Ch. 1, §8, I]*). Then inequality* (3) *holds for any $p(x) \in \mathcal{P}_N$.*

Proof. Without loss of generality, we shall suppose that F is a closed set. For a closed subset the property of being nowhere dense is equivalent to having an everywhere dense complement (see [9, Ch. 1, §8, I]).

We write $Z[p] := \{z \in C^1 |\ p(z) = 0\}$ for the zero set of $p(z)$ and denote

$$\mathcal{R} := \{p \in R^N |\ Z[p] \cap F \neq \emptyset\} \quad \text{and} \quad \mathcal{B} := R^N \setminus \bigcup_{n=0}^{N} int\ R_n^N.$$

By Remark 1, any vector in $R^N \setminus \mathcal{B} = \bigcup_{n=0}^{N} int\ R_n^N$ corresponds to a polynomial in $\mathcal{P}_N$, which either has no real zeros or has only distinct real zeros. Since T is a homeomorphism, we have

$$T^{-1}(\mathcal{R}) = \{T^{-1}p |\ Z[p] \cap F \neq \emptyset\} = \{p \in R^N |\ Z[Tp] \cap F \neq \emptyset\},$$

and $Tp \notin \mathcal{B}$ if and only if $p \notin T^{-1}(\mathcal{B})$. Therefore by assumption, inequality (3) holds for all $p \notin B$, where

$$B := \mathcal{B} \cup T^{-1}(\mathcal{B}) \cup \mathcal{R} \cup T^{-1}(\mathcal{R}).$$

We next prove that each of the four sets $\mathcal{B}$, $T^{-1}(\mathcal{B})$, $\mathcal{R}$ and $T^{-1}(\mathcal{R})$ is a closed nowhere dense set.

Since $\bigcup_{n=0}^{N} int\ R_n^N$ is open, $\mathcal{B}$ is closed, and from Remark 1 it follows that

$$Cl(R^N \setminus \mathcal{B}) = Cl(\bigcup_{n=0}^{N} int\ R_n^N) \supseteq \bigcup_{n=0}^{N} R_n^N = R^N.$$

And since both of these properties are invariant with respect to the homeomorphic mapping T^{-1} (see [9, Ch. 1, §13, VII, VIII]), the set $T^{-1}(\mathcal{B})$ is also a closed nowhere dense set.

Next we show that the set $\mathcal{R}$ is closed. Let $p_n \in \mathcal{R}$, $\forall n \in \mathcal{N}$, and $\lim_{n\to\infty} p_n = p \in R^N$. Since the leading coefficient of any polynomial $p_n(x) \in \mathcal{P}_N$, $n \in \mathcal{N}$, is equal to 1, the boundedness in R^N of the sequence $\{p_n\}_{n\in\mathcal{N}}$ implies the existence of $A \in (0,\infty)$ such that $Z[p_n] \subseteq V_A(0)$, $\forall n \in \mathcal{N}$. Here

$$V_\delta(b) := \{z \in C^1 |\ |z-b| < \delta\},\ b \in C^1,\ \delta > 0. \tag{4}$$

By Hurwitz's theorem [15, p. 119], $\forall \varepsilon > 0\ \exists N = N(\varepsilon)$ such that $Z[p_n] \subseteq Z[p] + V_\varepsilon(0)$, $\forall n \geq N$. Therefore, $F \cap (Z[p] + V_\varepsilon(0)) \neq \emptyset$, $\forall \varepsilon > 0$, and, hence, $F \cap Z[p] \neq \emptyset$; i.e., $p \in \mathcal{R}$. Therefore, $Cl\,\mathcal{R} = \mathcal{R}$.

Let us prove now that the set $\mathcal{R}$ is nowhere dense in R^N. As mentioned above, since $\mathcal{R}$ is closed, it suffices to show that $R^N \setminus \mathcal{R}$ is everywhere dense or, equivalently, $\mathcal{R} \subseteq Cl(R^N \setminus \mathcal{R})$. Let $p \in \mathcal{R}$. Then $p \in R_n^N$, $1 \leq n \leq N$, and at least one of its real zeros α_k, $1 \leq k \leq r$, $r \geq 1$ in the representation from equation (1) of Lemma 2 with $n_0 = 0$ and $a = +\infty$, belongs to F. Since F is nowhere dense, for any $1 \leq k \leq r$ there exists a sequence $\{\alpha_k(m)\}_{m\in\mathcal{N}} \subseteq R^1 \setminus F$, converging to α_k, i.e., $\lim_{m\to\infty} \alpha_k(m) = \alpha_k$. Therefore, by equation (1) with $n_0 = 0$ and $a = +\infty$, the sequence of polynomials

$$p_m(x) := Q(x) \prod_{k=1}^{r} (x - \alpha_k(m))^{n_k}$$

converges to $p(x)$; i.e., $\lim_{m\to\infty} p_m = p$, and $p_m \in R^N \setminus \mathcal{R}$, $\forall m \in \mathcal{N}$. This means that $\mathcal{R} \subseteq Cl(R^N \setminus \mathcal{R})$, hence $\mathcal{R}$ is a closed nowhere dense set. Therefore $T^{-1}(\mathcal{R})$ is also a closed nowhere dense set.

Finally, by the Baire category theorem (see [9, Ch. 1, §8, III, Theorem 2]), the set B, being the union of four nowhere dense sets, will itself be a closed nowhere dense set, and therefore, its complement $D := R^N \setminus B$ will be an everywhere dense set, for which inequality (3) holds. By applying part (c) of Theorem 1, the corollary is proved.

3. Characterization of weakly increasing zero-diminishing sequences. We begin this section with some notation and definitions. Let $Z_+ := \{0, 1, 2, \ldots\}$. Let $C[a,b]$ denote the space of all continuous real-valued functions on $[a,b]$, $-\infty \leq a < b \leq +\infty$. Let $\mathcal{P}$ denote the space of all real polynomials. Let $Z_D(p(x))$, $D \subseteq R^1$ denote the functional defined in the introduction, where we shall assume that $Z_D(0) = 0$. The set of all real sequences

$$\mu := \{\mu_n\}_{n\in Z_+} := \{\mu_n\}_{n\geq 0} := (\mu_0, \mu_1, \mu_2, \ldots, \mu_n, \ldots),\ \mu_n \in R^1,\ \forall n \in Z_+,$$

will be denoted by R^∞. Set $R_+^\infty := \{\mu \in R^\infty |\ \mu_n > 0,\ \forall n \in Z_+\}$. With the help of the sequence $\mu \in R^\infty$, we define, on the linear space $\mathcal{P}$, the linear transformation T_μ by the formula

$$T_\mu x^n = \mu_n x^n,\ \ \forall n \in Z_+. \tag{5}$$

Let τ denotes the set of all nontrivial, zero-diminishing real sequences, that is

$$\tau := \{\mu \in R^\infty | \ Z_R(T_\mu p(x)) \le Z_R(p(x)) \ \ \forall p(x) \in \mathcal{P}\} \setminus k_\tau, \tag{6}$$

where $k_\tau := \bigcup\limits_{a \in R^1} \{(a, 0, 0, \ldots, 0, \ldots)\}$ (cf. [6, p. 121]). Let $\tau_+ := \tau \cap R^\infty_+$, the set of sequences with only positive terms. For any $C \in R^1 \setminus \{0\}$, we define the class

$$\tau_C := \{\mu \in R^\infty | \ Z_{[0,AC]}(T_\mu p(x)) \le Z_{[0,A]}(p(x)) \ \ \forall A \ge 0, \ \forall p(x) \in \mathcal{P}\} \setminus k_\tau, \tag{7}$$

where for $a > 0$, $[0, -a] := [-a, 0]$. We define the linear operator $M_C : R^\infty \to R^\infty$ by the formula $M_C \mu = \{C^n \mu_n\}_{n \ge 0}$, $\mu \in R^\infty$, $C \in R^1 \setminus \{0\}$. Then evidently

$$Z_{[0,AC]}(T_\mu p(x)) = Z_{[0,A]}(T_{M_C \mu} p(x)), \ \forall A \ge 0, \ \forall C \in R^1 \setminus \{0\}, \ \forall p(x) \in \mathcal{P},$$

and therefore,

$$\tau_C = M_C(\tau_1) := \{M_C \mu | \ \mu \in \tau_1\}, \ \ \forall C \in R^1 \setminus \{0\}. \tag{8}$$

For any $a \in R^1$, $\mu \in R^\infty$ we denote $a\mu := \{a\mu_n\}_{n \ge 0}$. Finally, the set of all weakly increasing sequences will be denoted by

$$W := \{\mu \in R^\infty | \ \varlimsup_{n \to \infty} |\mu_n|^{1/n} < \infty\}. \tag{9}$$

In the proof of the next lemma, we shall show that if a term of a zero-diminishing sequence τ vanishes, then all subsequent terms of τ must also vanish. From this it will follow that the terms of a zero-diminishing sequence must all be of the same sign or they must alternate in sign. In addition, using the result of [1, Lemma 1], we shall show that weakly increasing zero-diminishing sequences $\mu = \{\mu_n\}_{n=0}^\infty$, $\mu_n > 0$, $\forall n \in Z_+$, are moment sequences of a bounded nondecreasing function $\mu(x)$ on $[0, Q]$, where $Q = \varlimsup_{n \to \infty} \mu_n^{1/n} < \infty$, so that the support of the measure $d\mu$ is $[0, Q]$.

Lemma 3. *The following statements hold:*
(a) $\forall \mu \in \tau$, $\exists \sigma_1 = \sigma_1(\mu)$, $\sigma_2 = \sigma_2(\mu) \in \{+1, -1\}$ *such that* $\sigma_1 M_{\sigma_2} \mu \in \tau_+$;
(b) Let $\mu \in \tau_+$. *Then* $\mu \in W$ *if and only if the limit* $Q := \lim_{n \to \infty} \mu_n^{1/n} \in (0, +\infty)$ *exists and there exists a bounded nondecreasing function* $\mu(x)$ *on* $[0, Q]$ *such that*

$$\mu_n = \int_0^Q t^n \, d\mu(t) \ \ \forall n \in Z_+. \tag{10}$$

Proof. (a) We show that for any sequence $\mu \in \tau$, if there exists $m \in Z_+$ such that $\mu_m = 0$, then $\mu_n = 0 \;\; \forall n \geq m$. By definition of τ (cf. (6)), for any positive integer k the following relations hold: $m = Z_R(x^m(1+x^{2k})) = Z_R(x^m + x^{m+2k}) \geq Z_R(\mu_m x^m + \mu_{m+2k} x^{m+2k}) = Z_R(\mu_{m+2k} x^{m+2k})$. Since $2k+m \geq m+2$, the inequality above holds only in the case $\mu_{m+2k} = 0, \;\; \forall k \in \mathcal{N}$. But then for any $k \in Z_+$, we have

$$\begin{aligned} m &= Z_R(x^m(1+(1+x^{2k+1})^2)) = Z_R(2x^m + 2x^{m+2k+1} + x^{m+4k+2}) \\ &\geq Z_R(2\mu_m x^m + 2\mu_{m+2k+1} x^{m+2k+1} + \mu_{m+4k+2} x^{m+4k+2}) \\ &= Z_R(2\mu_{m+2k+1} x^{m+2k+1}). \end{aligned}$$

Since $m+2k+1 \geq m+1$, the inequality above is possible only in the case $\mu_{m+2k+1} = 0, \;\; \forall k \in Z_+$. Therefore, $\mu_n = 0, \;\; \forall n \geq m$, as was to be proved.

Thus $\mu_0 = 0$ implies $\mu = (0,0,\ldots,0,\ldots) \notin \tau$, and $\mu_0 \neq 0, \;\; \mu_1 = 0$ implies $\mu = (\mu_0, 0, 0, \ldots, 0, \ldots) \notin \tau$. Therefore, $\mu \in \tau$ implies $\mu_0 \neq 0$, $\mu_1 \neq 0$. It was proved in [1, Lemma 1] that for every such sequence there exists a bounded nondecreasing function $\mu(x)$ on $[0, +\infty)$ such that

$$\mu_n = \sigma_1 \sigma_2^n \int_0^\infty t^n \, d\mu(t), \quad \forall n \in Z_+, \tag{11}$$

where $\sigma_1 = \operatorname{sign}(\mu_0)$, $\sigma_2 = \sigma_1 \operatorname{sign}(\mu_1)$ and

$$\operatorname{sign}(x) := \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0. \end{cases}$$

Therefore,

$$\sigma_1 M_{\sigma_2} \mu = \left\{ \int_0^\infty t^n \, d\mu(t) \right\}_{n \geq 0}.$$

But $\mu_1 \neq 0$ implies the existence of $\rho > 0$ such that $\mu(+\infty) > \mu(\rho)$. In fact, assuming the contrary; i.e., $\mu(+\infty) = \mu(\varepsilon) \; \forall \varepsilon > 0$, we obtain

$$\mu(0+0) := \lim_{\varepsilon \downarrow 0} \mu(\varepsilon) = \mu(+\infty), \tag{12}$$

and hence,

$$\sigma_1 \sigma_2 \mu_1 = \int_0^\infty t \, d\mu(t) = 0[\mu(0+0) - \mu(0)] + \int_0^\infty t \, d\mu(0+0) = 0,$$

leading to a contradiction. Now we have $\int_0^\infty t^n \, d\mu(t) \geq \rho^n[\mu(+\infty) - \mu(\rho)] > 0, \; \forall n \in Z_+$; i.e., $\sigma_1 M_{\sigma_2} \mu \in \tau_+$, as was to be proved.

(b) Since (10) immediately implies that $\mu \in W$, it remains to be shown that $\mu \in \tau_+ \cap W$ implies the properties of the sequence μ claimed in (10). Fix an arbitrary sequence $\mu \in \tau_+ \cap W$, and set $Q := \underline{\lim}_{n\to\infty} \mu_n^{1/n} \in [0, +\infty)$. Since $\mu \in \tau_+$, representation (11) of this sequence evidently follows with $\sigma_1 = \sigma_2 = 1$. Assume now that there exists $\varepsilon > 0$ such that $\mu(+\infty) > \mu(Q+\varepsilon)$. Then

$$\mu_n \geq \int_{Q+\varepsilon}^{\infty} t^n \, d\mu(t) \geq (Q+\varepsilon)^n (\mu(+\infty) - \mu(Q+\varepsilon)), \ \forall n \in Z_+,$$

and therefore, $\underline{\lim}_{n\to\infty} \mu_n^{1/n} \geq Q + \varepsilon$, contradicting the definition of Q. Hence, $\mu(Q+0) = \mu(+\infty)$. If $Q = 0$, then equality (12) holds, which implies $\mu_1 = 0$, and, hence, $\mu \notin \tau_+$. Therefore, $Q > 0$, and without changing the values of the integrals (11), we may assume that $\mu(Q) = \mu(Q+0)$. Then (11) may be written as follows:

$$\mu_n = \int_0^Q t^n \, d\mu(t), \ \ \forall n \in Z_+.$$

It follows that $\mu_n \leq Q^n(\mu(Q) - \mu(0))$, i.e., $\overline{\lim}_{n\to\infty} \mu_n^{1/n} \leq Q = \underline{\lim}_{n\to\infty} \mu_n^{1/n}$, proving the existence of the limit $\lim_{n\to\infty} \mu_n^{1/n} = Q \in (0, +\infty)$. Thus, Lemma 3 is proved. □

Expanding on our earlier notation for W, we write

$$W_Q := \{\mu \in R^\infty | \lim_{n\to\infty} |\mu_n|^{1/n} = Q\}, \ 0 < Q < \infty. \tag{13}$$

It is evident that $\mu \in W_Q$ if and only if $M_{\frac{1}{Q}}\mu \in W_1$. Following the notation of formula (8) and the obvious relation

$$M_a(\tau_+) = \tau_+, \ \forall a > 0,$$

by part (b) of Lemma 3 we have

$$\tau_+ \cap W = \bigcup_{Q>0} M_Q(\tau_+ \cap W_1). \tag{14}$$

In order to find a relationship between $\mu \in \tau_+ \cap W_1$ and the corresponding function $\mu(x)$ satisfying (10) with $Q = 1$, we introduce the following two transformations:

$$\begin{cases} T_\mu f(x) := \int_0^1 f(xt)\, d\mu(t), \ \forall f \in C[0, A], \ A > 0; \\ R_\mu p(x) := \frac{1}{\mu_1} \int_0^1 tp(x - \log \frac{1}{t})\, d\mu(t), \ \forall p(x) \in \mathcal{P}, \end{cases} \tag{15}$$

where T_μ represents an extension of the transformation T_μ introduced in (5) from the space of polynomials to the space of continuous functions on some interval $[0, A],\ A > 0$. In the next Lemma, part (a) extends the statement of [1, Lemma 3] from the set of all polynomials in $\log(x)$ to the set of all continuous functions. Part (b) of Lemma 4 is stated in [1, Lemma 3] and the proof is sketched there. For the sake of completeness, we include a detailed proof here.

Lemma 4. *Let* $\mu \in \tau_+ \cap W_1,\ A > 0$ *and* $a \in R^1$. *Then*
(a) *if* $0 < x_1 < x_2 < \ldots < x_r < A,\ r \geq 1,$ *and* $f(x) = \phi(x) \prod_{k=1}^{r} (x - x_k),\ \phi \in C[0, A],\ \phi(x) \geq 0,\ \forall x \in [0, A]$, *then*

$$S_{(0,A)}(T_\mu f(x)) \leq r, \tag{16}$$

where for any function $g : (a, b) \to R^1,\ -\infty < a < b < +\infty$, *we write*

$$S_{(a,b)}(g(x)) :=$$

$$:= \sup \{n \in \mathcal{N} |\ \exists\, a < t_0 < t_1 < \ldots < t_n < b : g(t_j)g(t_{j+1}) < 0,\ 0 \leq j \leq n - 1\}; \tag{17}$$

(b) *Also,*

$$Z_{(-\infty,a]}(R_\mu p(x)) \leq Z_{(-\infty,a]}(p(x)),\ \ \forall p(x) \in \mathcal{P}. \tag{18}$$

Proof. (a) Let $\Delta_r(x) := \prod_{k=1}^{r} (x - x_k)$. By the Weierstrass approximation theorem [15, p. 414], for any $\varepsilon \in (0, 1)$, the function $\sqrt{\phi(x)} \in C[0, A]$ can be approximated by a real polynomial $P(x)$ such that

$$\|\sqrt{\phi(x)} - P(x)\|_{C[0,A]} < \varepsilon. \tag{19}$$

Consider the polynomial

$$Q(x) = (\varepsilon + P(x)^2)\Delta_r(x).$$

It is evident that $Z_R(Q(x)) = r$. Besides that, for any $x \in [0, A]$ we obtain

$$|T_\mu f(x) - T_\mu Q(x)| = \left| \int_0^1 (f(xt) - Q(xt))\, d\mu(t) \right|$$
$$= \left| \int_0^1 (\Delta_r(xt)\phi(xt) - \Delta_r(xt)(\varepsilon + P(xt)^2))\, d\mu(t) \right|$$

$$\leq \int_0^1 |\Delta_r(xt)|\,|(\sqrt{\phi(xt)} - P(xt))(\sqrt{\phi(xt)} + P(xt)) - \varepsilon|\, d\mu(t)$$

$$\leq (2A)^r[\varepsilon\mu_0 + \int_0^1 |\sqrt{\phi(xt)} - P(xt)|\,|\sqrt{\phi(xt)} - P(xt) - 2\sqrt{\phi(xt)}|\, d\mu(t)]$$

$$\leq \varepsilon(2A)^r\mu_0[1 + \varepsilon + 2\|\sqrt{\phi(x)}\|_{C[0,A]}].$$

Therefore, $\|T_\mu f(x) - T_\mu Q(x)\|_{C[0,A]}$ can be made as small as desired. To prove (16), we suppose that

$$S_{(0,A)}(T_\mu f(x)) \geq r + 1.$$

Choose $r+2$ points $0 < t_0 < t_1 < \ldots < t_{r+1} < A$ so that

$$T_\mu f(t_j)T_\mu f(t_{j+1}) < 0,\ 0 \leq j \leq r,$$

and write $\kappa := \min_{0\leq j\leq r+1} |T_\mu f(t_j)|$. By the argument above, there exists a polynomial $P(x)$ such that

$$\|T_\mu f(x) - T_\mu Q(x)\|_{C[0,A]} < \frac{1}{2}\kappa,$$

and hence $T_\mu Q(t_j)T_\mu Q(t_{j+1}) < 0,\ 0 \leq j \leq r$. This implies that

$$Z_R(T_\mu Q(x)) \geq Z_{(0,A)}(T_\mu Q(x)) \geq r + 1 > r = Z_R(Q(x)),$$

contradicting the fact that $\mu \in \tau_+$. Therefore inequality (16) is true.

(b) Since $R_\mu p(x) \equiv 1$ for $p(x) \equiv 1$, inequality (18) holds when the polynomial $p(x)$ is a constant. Therefore, it is sufficient to establish the validity of (18) for any $p(x) \in \bigcup_{N\geq 1} \mathcal{P}_N$. In this case, for some $N \geq 1$, let $p(x) = \sum_{k=0}^{N} p_k x^k$, where $p_N = 1$. Then

$$\begin{aligned}
R_\mu p(x) &= \frac{1}{\mu_1}\sum_{k=0}^{N} p_k \int_0^1 t(x + \log t)^k \, d\mu(t) \\
&= \frac{1}{\mu_1}\sum_{k=0}^{N}\sum_{m=0}^{k} \binom{k}{m} p_k x^m \int_0^1 t(\log t)^{k-m}\, d\mu(t) \\
&= \sum_{m=0}^{N} x^m \sum_{k=0}^{N-m} \binom{k+m}{m} p_{k+m}\frac{1}{\mu_1}\int_0^1 t(\log t)^k\, d\mu(t) \\
&= x^N + \sum_{m=0}^{N-1} x^m[p_m + \sum_{k=1}^{N-m} \binom{k+m}{m} p_{k+m}\frac{1}{\mu_1}\int_0^1 t(\log t)^k\, d\mu(t)].
\end{aligned}$$

This shows that $R_\mu p(x) \in \mathcal{P}_N$. Also, the transformation R_μ, considered as a transformation of the coefficients of the polynomials from the set $\mathcal{P}_N$, is a homeomorphism of the space R^N of the form $Ax + b$, where $b, x \in R^N$ and A is a triangular matrix with diagonal elements equal to 1. Using the result of Corollary 1, it suffices to establish the validity of (18) for those polynomials $p(x) \in \mathcal{P}_N$, for which each of the polynomials $p(x)$ and $R_\mu p(x)$ either has no zeros on $(-\infty, a]$ or all its zeros on $(-\infty, a]$ are distinct and none of them equals a. If the polynomial $p(x)$ is of one sign on $(-\infty, a]$, then by (15) it is obvious that $R_\mu p(x)$ is also of the same sign on $(-\infty, a]$, and therefore, (18) will hold.

Next assume to the contrary that on $(-\infty, a]$ the polynomial $p(x)$ has $r \geq 1$ zeros $-\infty < u_1 < u_2 < \cdots < u_r < a$, and that the polynomial $R_\mu p(x)$ has $q > r$ zeros $-\infty < v_1 < v_2 < \cdots < v_q < a$. Then the zeros of the functions $f(x) := xp(\log x)$ and $T_\mu f(x) = \mu_1 x R_\mu(\log x) = \int_0^1 xtp(\log xt)\, d\mu(t)$, on the interval $(0, A]$, where $A := e^a$, are of the form $0 < e^{u_1} < e^{u_2} < \cdots < e^{u_r} < A$ and $0 < e^{v_1} < e^{v_2} < \cdots < e^{v_q} < A$, respectively. Therefore, $S_{(0,A)}(T_\mu f(x)) = q$, and due to the analyticity of $f(z)$ for $\Re z > 0$ and its continuity on $[0, A]$, the function

$$\phi(x) := \frac{f(x)}{\prod_{k=1}^{r} (x - e^{u_k})}$$

will be continuous on $[0, A]$, will preserve sign on $(0, A]$, and without loss of generality may be regarded to be nonnegative on $[0, A]$. But this contradicts the first statement of the lemma. Thus the validity of (18) is established and Lemma 4 is proved.

We next consider a certain subclass of the Laguerre-Pólya class [8, p. 336], namely

$$\mathcal{LP}_1 := \left\{ be^{az} \prod_{n\geq 1} (1 + \frac{z}{\alpha_n}) |\ a, b \in R^1,\ b \neq 0,\ \alpha_n > 0\ \forall n \in \mathcal{N},\ \sum_{n\geq 1} \frac{1}{\alpha_n} < \infty \right\}. \tag{20}$$

Functions in $\mathcal{LP}_1$ with fixed $a, b \in R^1$, will be denoted by $\mathcal{LP}_1(a, b)$ and

$$\mathcal{E}_1^* := \bigcup_{a\geq 0} \mathcal{LP}_1(a, 1)$$

(cf. [8, Ch. 7, §2, p. 336]). In the next lemma, we show that the reciprocals of weakly increasing zero-diminishing positive sequences can be interpolated by functions in $\mathcal{LP}_1$.

Lemma 5. *Let $\mu \in W \cap \tau_+$ and write $Q := \lim_{n\to\infty} \mu_n^{1/n}$. There exists a function $\Phi \in \mathcal{LP}_1(\log \frac{1}{Q}, \frac{1}{\mu_0})$ such that*

$$\mu_n = \frac{1}{\Phi(n)},\ \forall n \in Z_+.$$

P r o o f. Let $\mu \in W_1 \cap \tau_+$. Let $\mu(x)$ be the corresponding function given by Lemma 3. We assume that $\mu(x)$ is normalized to be a bounded nondecreasing function on $[0,1]$ (see [12, p. 128]); i.e., $\mu(0) = 0$ and $\mu(x) = \frac{1}{2}[\mu(x-0) + \mu(x+0)]\ \forall x \in (0,1)$. It is well known that this normalization does not change the values of the moments of $\mu(x)$ on $[0,1]$. The moments determine the normalized $\mu(x)$ uniquely (see [12, p. 129]). For the proof, we first eliminate the (possible) jump of $\mu(x)$ at zero

$$\mu_* := \mu(0+0) - \mu(0) \geq 0.$$

To this end, we decompose $\mu(x)$ as follows:

$$\mu(x) = \mu_* \chi_{(0,1]}(x) + \nu(x),\ x \in [0,1], \tag{21}$$

where

$$\chi_A(x) := \begin{cases} 0, & x \notin A; \\ 1, & x \in A; \end{cases} \quad A \subseteq R^1,$$

and where $\nu(x)$ is nondecreasing, bounded on $[0,1]$ and continuous at the point 0; i.e., $\nu(0) = \nu(0+0) = 0$. Here, obviously, $\nu_n = \mu_n,\ \forall n \in \mathcal{N}$, but $\mu_0 = \nu_0 + \mu_* \geq \nu_0$, where $\nu_n := \int_0^1 t^n\, d\nu(t),\ \forall n \in Z_+$. Definition (15) of the transformation R_μ immediately implies

$$R_\mu p(x) = R_\nu p(x),\ \forall p(x) \in \mathcal{P},$$

and therefore inequality (18) holds for R_ν. From (15), with $p(x) = \sum_{k=0}^{N} p_k x^k$, we obtain

$$R_\nu p(x) = \sum_{k=0}^{N} \frac{p^{(k)}(x)}{k!} \frac{1}{\nu_1} \int_0^1 t(\log t)^k\, d\nu(t) = F(D)p(x),\ D := \frac{d}{dx},$$

where $F(z) := \frac{1}{\nu_1} \int_0^1 t^{1+z}\, d\nu(t)$ is analytic in $\Re z > -1$ and $F(0) = 1$. Hence, the function

$$\Phi(z) := \frac{1}{F(z)} = \sum_{k \geq 0} g_k z^k,\ \Phi(0) = 1,$$

will also be analytic in a disk $V_\kappa(0)$ (see (4)) of some positive radius κ. Since $p(x)$ and $R_\nu p(x)$ are polynomials, the following holds:

$$\Phi(D)(R_\nu p(x)) = (\sum_{k \geq 0} g_k \frac{d^k}{dx^k})(R_\nu p(x)) = p(x),\ \forall p(x) \in \mathcal{P}.$$

The reasoning in the proof of Lemma 4 shows that $R_\nu(\mathcal{P}) = \mathcal{P}$, and, therefore, from (18) with $a = 0$, we obtain

$$Z_{(-\infty,0]}(p(x)) \le Z_{(-\infty,0]}(\Phi(D)p(x)),\ \forall p(x) \in \mathcal{P}.$$

In particular, for $p(x) = x^n$ we have $Z_{(-\infty,0]}(\Phi(D)x^n) \ge n$, $\forall n \in \mathcal{N}$. Since the degree of the polynomial $A_n(x) := \Phi(D)x^n$ does not exceed n, the polynomials $A_n(x)$, as well as the polynomials $A_n^*(x) := x^n A_n(\frac{1}{x})$ and $A_n^*(\frac{x}{n})$ for any $n \in \mathcal{N}$, have all real nonpositive zeros. But it is known [8, Ch. 7, p. 345] that the analyticity of $\Phi(z)$ in the open disk $V_\kappa(0)$ implies the uniform convergence of the polynomials $A_n^*(\frac{z}{n})$ to $\Phi(z)$ on any compact subset of the open disk $V_\kappa(0)$. By Corollary 2.2 of [8, p. 337], $\Phi(z) \in \mathcal{E}_1^*$. Hence,

$$\Phi(z) = e^{az} \prod_{n\ge1} (1 + \frac{z}{\alpha_n}),\ a \ge 0,\ \alpha_n > 0,\ \forall n \in \mathcal{N},\ \sum_{n\ge1} \frac{1}{\alpha_n} < \infty. \tag{22}$$

For $z \in V_\kappa(0)$, we have the equality of the two analytic functions

$$\frac{1}{\nu_1} \int_0^1 t^{1+z}\, d\nu(t) = \frac{1}{\Phi(z)}. \tag{23}$$

We continue $\frac{1}{\Phi(z)}$ analytically in the domain of analyticity of the left hand side, that is $\Re z > -1$. From this we deduce that the zeros of $\Phi(z)$ satisfy the (stronger than in (22)) inequality $\alpha_n > 1$, $\forall n \in \mathcal{N}$. This inequality together with convergence of the series $\sum\limits_{n\ge1} \frac{1}{\alpha_n} < \infty$ allows us to conclude that there exists a number $\delta_\mu > 0$ such that

$$\inf_{n\in\mathcal{N}} \alpha_n = \min_{n\in\mathcal{N}} \alpha_n = 1 + 2\delta_\mu.$$

In addition, by the continuity of $\nu(x)$ at zero, we have

$$0 \le \varlimsup_{\varepsilon\downarrow0} \left| \int_0^\varepsilon t^z\, d\nu(t) \right| \le \varlimsup_{\varepsilon\downarrow0} (\nu(\varepsilon) - \nu(0)) = \nu(0+0) - \nu(0) = 0,\ \forall \Re z \ge 0,$$

and, therefore, (23) remains true also for $\Re z = -1$. In particular, for $z = -1$, $\nu_1/\nu_0 = \Phi(-1) = e^{-a} \prod\limits_{n\ge1} (1 - \frac{1}{\alpha_n})$. With $\beta_n := \alpha_n - 1$, $\forall n \in \mathcal{N}$, we obtain from (22)

$$\Phi(z-1) = e^{-a} e^{az} \prod_{n\ge1} (1 - \frac{1}{\alpha_n} + \frac{z}{\alpha_n}) = \Phi(-1) e^{az} \prod_{n\ge1} (1 + \frac{z}{\beta_n}) =$$

$$= \frac{\nu_1}{\nu_0} e^{az} \prod_{n\ge1} (1 + \frac{z}{\beta_n}).$$

After substituting this expression in (23) we obtain

$$\frac{1}{\nu_0}\int_0^1 t^z\,d\nu(t) = \frac{1}{e^{az}\prod\limits_{n\geq 1}(1+\frac{z}{\beta_n})},\ \forall \Re z \geq 0,$$

where

$$a \geq 0,\ \ \beta_n \geq 2\delta_\mu > 0,\ \forall n \in \mathcal{N}, \tag{24}$$

and, obviously, $\sum\limits_{n\geq 1}\frac{1}{\beta_n} < \infty$. Since for $n \in \mathcal{N}$,

$$\mu_n = \int_0^1 t^n\,d\nu(t) = \frac{\nu_0}{e^{an}\prod\limits_{k\geq 1}(1+\frac{n}{\beta_k})} \leq \nu_0 e^{-an},$$

we have $\overline{\lim\limits_{n\to\infty}}\mu_n^{1/n} \leq e^{-a}$. Since $a \geq 0$ and $\mu \in W_1$ so that $Q = 1$, we deduce that $a = 0$. Thus,

$$\int_0^1 t^z\,d\nu(t) = \frac{\nu_0}{\Psi(z)},\ \forall \Re z \geq 0,\ \Psi(z) = \prod_{n\geq 1}(1+\frac{z}{\beta_n}) \in \mathcal{LP}_1(0,1), \tag{25}$$

from which we obtain

$$\mu_n = \nu_n = \frac{\nu_0}{\Psi(n)},\ \forall n \in \mathcal{N};\ \mu_0 = \nu_0 + \mu_* \geq \nu_0 = \frac{\nu_0}{\Psi(0)}. \tag{26}$$

For the sake of clarity of presentation, we will defer the proof of the fact that $\mu(x)$ is continuous at $x = 0$ to Lemma 6 below. If $\mu(x)$ is continuous at zero, then $\mu(x) = \nu(x),\ \forall x \in [0,1]$. This means by (25) that

$$\forall \mu \in W_1 \cap \tau_+,\ \exists \Phi \in \mathcal{LP}_1(0,\frac{1}{\mu_0}) \text{ such that } \ \mu_n = \frac{1}{\Phi(n)},\ \forall n \in Z_+. \tag{27}$$

Consider an arbitrary sequence $\mu \in W \cap \tau_+$. Then by (14) $\exists Q > 0$ such that $M_{1/Q}\mu \in W_1 \cap \tau_+$, and by (13) $\lim\limits_{n\to\infty}\mu_n^{1/n} = Q$. In view of (27), there exists a $\Phi \in \mathcal{LP}_1(0,\frac{1}{\mu_0})$ such that $Q^{-n}\mu_n = 1/\Phi(n),\ \forall n \in Z_+$, from which it follows that $\mu_n = 1/(e^{-n\log Q}\Phi(n)),\ \forall n \in Z_+$. Since $e^{-z\log Q}\Phi(z) \in \mathcal{LP}_1(\log 1/Q, 1/\mu_0)$, Lemma 5 is proved. $\square$

Lemma 6. *The function $\mu(x)$ corresponding to $\mu \in W_1 \cap \tau_+$ by Lemma* 3 *is continuous at $x = 0$.*

P r o o f. Using the representation (25) we first shall establish the following property of the function $\nu(x)$ (see also [5, Ch. 5]):

$$\nu(x) = \begin{cases} \nu_0 \chi_{\{1\}}(x), \ x \in [0,1], \ \text{if } \Psi(z) \equiv 1; \\ \nu_0 \int\limits_0^x t^{\delta_\mu - 1} g(t)\, dt, \ x \in [0,1], \ \text{if } \Psi(z) \not\equiv 1, \end{cases} \tag{28}$$

where $g(x) \in C[0,1]$. In the case $\Psi(z) \equiv 1$, (28) follows from

$$\int\limits_0^1 t^n \, d\nu(t) = \int\limits_0^1 t^n \, d(\nu_0 \chi_{\{1\}}(t)), \ \forall n \in Z_+$$

and the uniqueness theorem (see [12, Ch. 7, Theorem 2, p. 129]). Similarly, if $\Psi(z) = 1 + \frac{z}{\beta_1}$, where by (24) $\beta_1 \geq 2\delta_\mu > 0$, we obtain

$$\nu'(x) = \frac{\nu_0}{\beta_1} x^{\beta_1 - 1} = x^{\delta_\mu - 1} g(x), \ \forall x \in (0,1],$$

where $g(x) = \frac{\nu_0}{\beta_1} x^{\beta_1 - \delta_\mu} \in C[0,1]$. But if $\Psi(z)$ has at least two zeros counting multiplicities, then $1/\Psi(\sigma + it)$ as a function of t is summable and continuous on the whole real axis for all $\sigma > -2\delta_\mu$. Then the function

$$g(x) := \frac{1}{2\pi} \int\limits_{R^1} \frac{x^{-it}}{\Psi(-\delta_\mu + it)}\, dt = \frac{x^{-\delta_\mu}}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \frac{x^{-s}}{\Psi(s)}\, ds, \ \forall \sigma > -2\delta_\mu, \ x > 0,$$

is equal to zero for all $x > 1$ and is continuous on $[0,1]$ since $1/\Psi(\sigma + it)$ is summable for all $\sigma > -2\delta_\mu$ and $g(0) = 0$. Thus we can apply the inversion theorem for Mellin transforms (see [16, Ch. 1, Theorem 29, p. 46]), which gives

$$\frac{1}{\Psi(z)} = \lim_{\lambda \to +\infty} \int\limits_{1/\lambda}^{\lambda} x^{z-1} x^{\delta_\mu} g(x)\, dx = \int\limits_0^1 x^z x^{\delta_\mu - 1} g(x)\, dx, \ \forall \Re z > -\delta_\mu.$$

Hence, $\nu'(x) = \nu_0 x^{\delta_\mu - 1} g(x), \ \forall x \in (0,1]$, and this proves (28) for $\Psi(z) \not\equiv 1$. Note also that since $\nu(x)$ is nondecreasing on $[0,1]$, $g(x)$ is nonnegative on the same interval. Thus, when $\Psi(z) \not\equiv 1$,

$$\int\limits_0^1 t^z G(t)\, dt = \frac{1}{\Psi(z)}, \ \forall \Re z > -\delta_\mu, \ \Psi(0) = 1, \tag{29}$$

where the nonnegative function $G(x) := x^{\delta_\mu - 1} g(x), \ x \in (0,1]$, is summable on the interval $[0,1]$.

Preliminaries aside, we now proceed to prove that $\mu(x)$ is continuous at zero; i.e., $\mu_* = \mu(0+0) - \mu(0) = 0$. Suppose $\mu_* > 0$, and set $w := 1/\mu_*$. Then by (21), (28) and (29), we can write (see (15))

$$T_\mu f(x) = \begin{cases} \mu_* f(0) + \nu_0 f(x), \text{ if } \Psi(z) \equiv 1; \\ \mu_* f(0) + \nu_0 \int\limits_0^1 f(xt)G(t)\,dt, \text{ if } \Psi(z) \not\equiv 1, \end{cases} \tag{30}$$

where for some $A > 0$, $f(x) \in C[0, A]$.

Let $A = 4$ and let $R(x) := (x-1)(x-3) = x^2 - 4x + 3 = (x-2)^2 - 1 \geq -1$, $\forall x \in R^1$. We first consider the case $\Psi(z) \equiv 1$. We apply the following linear and continuous changes to the positive polynomial $2 + R(x)$ in the neighborhood of zero $[0, \rho]$, $\rho \in (0, 1/2)$:

$$\psi_\rho(x) := (2 + R(x))\chi_{[\rho,4]}(x) + \frac{1}{\rho}[(2 + R(\rho))x - 2w\nu_0(\rho - x)]\chi_{[0,\rho)}(x), \ x \in [0, 4].$$

The function $\psi_\rho(x)$ has the value $(-1)2w\nu_0$ at $x = 0$. Furthermore, $\psi_\rho(x)$ satisfies the hypotheses of Lemma 4 with $r = 1$. In fact, $\psi_\rho(x)$ has the unique root $y_\rho \in (0, \rho)$ on $[0, 4]$ and can be represented in the form

$$\psi_\rho(x) = (x - y_\rho)h_\rho(x), \ \forall x \in [0, 4],$$

where $h_\rho(x) \in C[0, 4]$ and $h_\rho(x) > 0$, $\forall x \in [0, 4]$. Applying the transformation T_μ to $\psi_\rho(x)$, formula (30) gives us

$$T_\mu \psi_\rho(x) = -2w\nu_0\mu_* + \nu_0(2 + R(x)) = \nu_0 R(x), \ \forall x \in [\rho, 4].$$

Therefore, for $\rho \in (0, 1/2)$, we have

$$S_{(0,4)}(T_\mu \psi_\rho(x)) \geq S_{(\rho,4)}(T_\mu \psi_\rho(x)) = S_{(\rho,4)}(R(x)) = 2 > r = 1,$$

contradicting inequality (16). This contradiction proves $\mu_* = 0$, and hence, in this case, $\mu_0 = \nu_0$ and $\mu = \mu_0(1, 1, \ldots, 1, \ldots)$.

Next consider the case when $\Psi(z) \not\equiv 1$. Let

$$\begin{aligned} Q(x) &:= \Psi(2)x^2 - 4\Psi(1)x + 3\Psi(0) \\ &= \Psi(2)(x - 2\frac{\Psi(1)}{\Psi(2)})^2 + 3\Psi(0) - 4\frac{(\Psi(1))^2}{\Psi(2)} > -4\frac{(\Psi(1))^2}{\Psi(2)}, \ \forall x \in R^1. \end{aligned}$$

According to formula (25),

$$\begin{aligned} \frac{(\Psi(1))^2}{\Psi(2)} &= \prod_{n\geq 1}(\frac{1 + 2/\beta_n + 1/\beta_n^2}{1 + 2/\beta_n}) = \prod_{n\geq 1}(1 + \frac{1}{\beta_n^2}\frac{1}{1 + 2/\beta_n}) \\ &\leq \prod_{n\geq 1}(1 + \frac{1}{\beta_n^2}) \leq \exp\left(\sum_{n\geq 1} 1/\beta_n^2\right), \end{aligned}$$

from which we obtain $Q(x)+d>0,\ \forall x\in R^1$, where $d:=4\exp(\sum_{n\geq 1}1/\beta_n^2)$. In addition, (29) implies that

$$\int_0^1 G(t)\,dt=1,\ \int_0^1 Q(xt)G(t)\,dt=R(x),\ \forall x\in R^1. \tag{31}$$

We apply the following linear and continuous changes to the positive polynomial $Q(x)+d$ in the neighborhood of zero $[0,\rho]$, $\rho\in(0,1/2)$:

$$f_\rho(x)=(d+Q(x))\chi_{[\rho,4]}(x)+\frac{1}{\rho}[(d+Q(\rho))x-d\nu_0w(\rho-x)]\chi_{[0,\rho)}(x),\ x\in[0,4]. \tag{32}$$

Now $f_\rho(0)=(-1)dw\nu_0$ and $f_\rho(x)$ satisfies the hypotheses of Lemma 4 with $r=1$. That is, on $[0,4]$, $f_\rho(x)$ has unique root

$$x_\rho=\rho\frac{d\nu_0w}{d\nu_0w+d+Q(\rho)}\in(0,\rho),$$

and is represented in the form $f_\rho(x)=(x-x_\rho)\phi_\rho(x),\ \forall x\in[0,4]$, where the function

$$\phi_\rho(x)=\frac{d+Q(x)}{x-x_\rho}\chi_{[\rho,4]}(x)+\frac{1}{\rho}[d+d\nu_0w+Q(\rho)]\chi_{[0,\rho)}(x),\ x\in[0,4],$$

is positive and continuous. Now for $x\geq 1/2$, we apply the transformation T_μ to $f_\rho(x)$ using formula (30). By (31), (32) and the fact that $t\geq 2\rho$ so that $xt\geq\rho$,

$$\begin{aligned}T_\mu f_\rho(x) &= -d\nu_0w\mu_*+\nu_0\int_0^{2\rho}f_\rho(xt)G(t)\,dt+\nu_0\int_{2\rho}^1 f_\rho(xt)G(t)\,dt\\ &= -d\nu_0+\nu_0\int_0^1(Q(xt)+d)G(t)\,dt+\nu_0\int_0^{2\rho}(f_\rho(xt)-Q(xt)-d)G(t)\,dt\\ &= \nu_0R(x)+\nu_0\int_0^{2\rho}(f_\rho(xt)-Q(xt)-d)G(t)\,dt,\ x\in[1/2,4].\end{aligned}$$

By definition (32), $\|f_\rho(x)\|_{C[0,4]}$ is bounded by the constant

$$\max\{d\nu_0w,\|d+Q(x)\|_{C[0,4]}\},$$

which is independent of ρ. As $G(x)$ is summable on $[0,1]$, the equality above implies

$$\lim_{\rho\downarrow 0}\|T_\mu f_\rho(x)-\nu_0R(x)\|_{C[1/2,4]}=0.$$

Therefore, there exists $\rho \in (0, 1/2)$ such that

$$S_{(1/2,4)}(T_\mu f_\rho(x)) \geq S_{(1/2,4)}(R(x)) = 2.$$

But then

$$S_{(0,4)}(T_\mu f_\rho(x)) \geq S_{(1/2,4)}(T_\mu f_\rho(x)) \geq 2 > r = 1,$$

contradicting inequality (16), which must hold for any $\mu \in \tau_+ \cap W_1$. Thus continuity of $\mu(x)$ at zero is proved. □

Theorem 2. *For the sequence of real numbers* $\mu = \{\mu_n\}_{n\geq 0}$ *the following statements are equivalent:*

(*a*) $\mu \in \tau$ *and* $\varliminf_{n\to\infty} |\mu_n|^{1/n} < \infty$;

(*b*) $\exists \Phi \in \mathcal{LP}_1$ *such that* $\mu_n = 1/\Phi(n),\ \forall n \geq 0$ *or* $\mu_n = (-1)^n/\Phi(n),\ \forall n \geq 0$;

(*c*) $\mu \in \bigcup_{C \in R^1\setminus\{0\}} \tau_C$ (*see* (7)).

P r o o f. First note that by the definition (9), condition (a) of the Theorem means that $\mu \in \tau \cap W$.

(a)⇒(b). By statement (a) of Lemma 3 for the sequence $\mu \in \tau \cap W$, we can find $\sigma_1, \sigma_2 \in \{+1, -1\}$, such that $\sigma_1 M_{\sigma_2}\mu \in \tau_+ \cap W$. Then by Lemma 5, there is a function $\Phi \in \mathcal{LP}_1$ such that $\sigma_1\sigma_2^n\mu_n = 1/\Phi(n),\ \forall n \geq 0$. Hence (b) holds.

(b)⇒(c). Let $\Phi \in \mathcal{LP}_1(a, b),\ a \in R^1,\ b \in R^1\setminus\{0\}$ and $\mu_n = \sigma^n/\Phi(n),\ \forall n \in Z_+$, where $\sigma \in \{+1, -1\}$. Then, for the sequence $\nu := bM_{\sigma e^a}\mu$, we have

$$\nu_n = b(\sigma e^a)^n\mu_n = \frac{1}{\Psi(n)},\ \forall n \in Z_+,$$

where $\Psi(z) = \frac{1}{b}e^{-az}\Phi(z) \in \mathcal{LP}_1(0, 1)$. For the operator H defined by the formula $Hx^n = \frac{1}{\nu_n}x^n,\ \forall n \in Z_+$, it is clear that

$$Z_{\{0\}}(Hp(x)) = Z_{\{0\}}(p(x)),\ \forall p(x) \in \mathcal{P}. \tag{33}$$

Now consider an arbitrary polynomial $p(x) \in \mathcal{P}$. Following the proof of Lemma 7.4.2 from [6, Ch. 7, pp. 167-168] we will prove that

$$Z_{[0,A]}(p(x)) \leq Z_{[0,A]}(Hp(x)), \forall A > 0. \tag{34}$$

If the polynomial $p(x)$ is constant, then (34) is obvious. Suppose $p(x)$ has degree at least 1. If $p(x)$ has no positive roots, then (34) holds by (33) and

$$Z_{[0,A]}(p(x)) = Z_{\{0\}}(p(x)) = Z_{\{0\}}(Hp(x)) \leq Z_{[0,A]}(Hp(x)).$$

Let $0 < x_1 < x_2 < \cdots < x_r < \infty,\ r \geq 1$, be the positive roots of $p(x)$. Then $p(x)$ has the following representation

$$p(x) = q(x)x^{m_0}\prod_{k=1}^{r}(x - x_k)^{m_k},\ q(x) > 0,\ \forall x \geq 0,$$

$$m_0 \in Z_+,\ m_k \in \mathcal{N},\ 1 \le k \le r.$$

For $\alpha > 0$, let $f(x) := x^\alpha p(x)$. By Rolle's theorem we have

$$f'(x) = \alpha x^{\alpha-1}(p(x) + \frac{x}{\alpha}p'(x)) = \alpha x^{\alpha-1}[Q(x)x^{m_0}\prod_{k=1}^{r}[(x-y_k)(x-x_k)^{m_k-1}]],$$

where $Q(x) \ge 0,\ \forall x \ge 0,\ 0 < y_1 < x_1 < y_2 < x_2 < \cdots < y_r < x_r$. Hence,

$$Z_{[0,A]}(p(x) + \frac{x}{\alpha}p'(x)) \ge Z_{[0,A]}(p(x)),\ \forall A, \alpha > 0.$$

Successive application of this operator with $\alpha = \alpha_n,\ 1 \le n \le N$, where α_n are the zeros of $\Psi(z)$ (see (20)), gives

$$Z_{[0,A]}(H_N p(x)) \ge Z_{[0,A]}(p(x)),\ \forall A > 0,\ \forall N \ge 1,$$

where $H_N x^n = x^n \prod_{k=1}^{N}(1 + \frac{n}{\alpha_k}),\ \forall n \in Z_+$. In these inequalities, using the compactness of the segment $[0, A]$ for every $A > 0$, it is possible to pass to the limit as $N \to \infty$ and obtain the required inequalities (34). This, together with (33), means $bM_{\sigma e^a}\mu = \nu \in \tau_1$, and hence by formula (8) we have $\mu \in \tau_{\sigma e^{-a}}$.

(c)⇒(a). If $\mu \in \tau_C,\ C \in R^1 \setminus \{0\}$, then by formula (8), $\nu := M_{C^{-1}}\mu \in \tau_1$, and according to definition (7) for the transformation T_ν, the following inequalities hold:

$$Z_{[0,A]}(T_\nu p(x)) \le Z_{[0,A]}(p(x)),\ \forall A \ge 0,\ \forall p(x) \in \mathcal{P}. \tag{35}$$

The change of variables in these inequalities $p(x)$ to $p^*(x) := p(-x)$ taking into account

$$Z_{[-A,0]}(T_\nu p(x)) = Z_{[0,A]}(T_\nu p(-x)) = Z_{[0,A]}(T_\nu p^*(x)),$$

leads to the following relations:

$$Z_{[-A,0]}(T_\nu p(x)) \le Z_{[-A,0]}(p(x)),\ \forall A \ge 0,\ \forall p(x) \in \mathcal{P}. \tag{36}$$

In addition, inequality (35) with $A = 0$, together with

$$Z_{\{0\}}(T_\mu p(x)) \ge Z_{\{0\}}(p(x)),\ \forall p(x) \in \mathcal{P}\ \forall \mu \in R^\infty,$$

means that

$$Z_{\{0\}}(T_\nu p(x)) = Z_{\{0\}}(p(x)),\ \forall p(x) \in \mathcal{P}. \tag{37}$$

Now, using (35), (36) and (37), for arbitrary $A, B > 0$ and $p(x) \in \mathcal{P}$, we obtain

$$Z_{[-B,A]}(T_\nu p(x)) = Z_{[-B,0]}(T_\nu p(x)) + Z_{[0,A]}(T_\nu p(x)) - Z_{\{0\}}(T_\nu p(x))$$

$$\leq Z_{[-B,0]}(p(x)) + Z_{[0,A]}(p(x)) - Z_{\{0\}}(p(x)) = Z_{[A,B]}(p(x)).$$

Choosing A and B greater than the radius of that disk of the complex domain with center at zero which contains in its interior all roots of the polynomials $p(x)$ and $T_\nu p(x)$, we obtain $Z_R(T_\nu p(x)) \leq Z_R(p(x))$. Since $p(x) \in \mathcal{P}$ is arbitrary, we conclude that $\nu \in \tau$, and hence $\mu \in M_C(\tau) = \tau$. Now by part (a) of Lemma 3, for $\nu \in \tau$ we can find $\sigma_1,\ \sigma_2 \in \{+1, -1\}$ such that $\sigma_1 M_{\sigma_2}\nu \in \tau_+$. In particular, this means $\nu_0 \neq 0$ and $\nu_1 \neq 0$. Then inequality (36) with $p(x) = x - 1$ gives

$$0 \leq Z_{[-A,0]}(\nu_1 x - \nu_0) \leq Z_{[-A,0]}(x-1) = 0,\ \forall A \geq 0.$$

Therefore the numbers ν_1 and ν_0 have the same sign and hence $\sigma_2 = 1$. Thus by the evident equality $\sigma_1\tau_1 = \tau_1$, we have $\sigma_1\nu \in \tau_+ \cap \tau_1$ and thus by [1, Lemma 2] for the sequence $\sigma_1\nu \in \tau_+ \cap \tau_1$, it is possible to find a nondecreasing bounded function $\nu(x)$ on $[0, 1]$ such that $\sigma_1\nu_n = \int_0^1 t^n\, d\nu(t)\ \forall n \in Z_+$. Hence $\nu \in W$, and therefore $\mu = M_C\nu \in W$, proving the theorem. □

The main Theorem in [1, p. 1487] is a special case of the equivalence of (b) and (c) in Theorem 2. Since the proof in [1] was not complete it has been completed here. The three principal differences in the proof, compared to that in [1], are the detailed use of the analyticity of the Mellin transform of the measure μ , the elimination of its jump at zero and the establishment of the absolute continuity of the remainder.

Finally, as an application of the foregoing results, we can solve an open problem raised in [3, Problem 8].

Corollary 2. *Let $\Phi \in \mathcal{LP}_1$ and $p(x) \in \mathcal{P}$. Then the sequence*

$$\left\{\frac{1}{p(n)\Phi(n)}\right\}_{n\geq 0} \in \tau$$

if and only if either the polynomial $p(x)$ is a nonzero constant, or all its zeros are real and negative.

P r o o f. If $p(x)$ is a nonzero constant or all its zeros are real and negative, then $p(x)\Phi(x) \in \mathcal{LP}_1$, and the statement of corollary follows from Theorem 2.

Conversely, let

$$\mu = \{\mu_n\}_{n\geq 0} = \{1/(p(n)\Phi(n))\}_{n\geq 0} \in \tau.$$

Let $\sigma \in \{+1, -1\}$ denote the sign of the leading coefficient of the polynomial $p(x)$, and let $\Phi(x) \in \mathcal{LP}_1(a, b),\ a, b \in R^1, b \neq 0$ (see (20)). Furthermore, we assume that the zeros of Φ are enumerated in increasing order: $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le \cdots$. Then the sequence $\nu := \sigma b M_{e^a}\mu$ will also belong to τ, and

$$\nu_n = \frac{1}{\sigma p(n)\Psi(n)},\ \forall n \in Z_+,\ \Psi(z) := \frac{1}{b}e^{-az}\Phi(z) = \prod_{n\ge 1}(1+\frac{z}{\alpha_n}) \in \mathcal{LP}_1(0,1).$$

Since $\lim\limits_{n\to\infty} \sigma p(n) = +\infty,\ \Psi(n) > 0,\ \forall n \in Z_+$, and the number of sign changes in the sequence ν_n is finite, we conclude from Lemma 3 that $\nu \in \tau_+$. Since $\Psi(n) \ge 1,\ \forall n \in Z_+$, it follows that

$$\overline{\lim_{n\to\infty}}\, \nu_n^{1/n} \le \frac{1}{\underline{\lim}_{n\to\infty} (\sigma p(n))^{1/n}} = 1;$$

i.e., $\nu \in \tau_+ \cap W$.

Let $n(r),\ r \ge 0,\ r \in R^1$ denote the number of terms in the sequence $\{\alpha_n\}_{n\ge 0}$, for which $\alpha_n \le r$, where $n(r) = 0$ if $0 \le r < \alpha_1$. Then for any $R > 0$ (see [15, p. 271]),

$$\log \Psi(R) = \sum_{n\ge 1} \log(1+\frac{R}{\alpha_n}) = \int_0^\infty \log(1+\frac{R}{x})\, dn(x) = R\int_0^\infty \frac{n(x)}{x(x+R)}\, dx,$$

and thus, using the convergence of the integral $\int_0^\infty \frac{n(x)}{x^2}\, dx$ (see [11, Ch. 1, p. 10]) and the nonnegativity of the function $n(x)$, it is easy to derive that

$$\lim_{R\to\infty} \int_0^\infty \frac{n(x)}{x(x+R)}\, dx = 0.$$

Therefore,

$$\forall \varepsilon > 0,\ \exists R = R(\varepsilon) > 0 \text{ such that } |\Psi(z)| \le \Psi(|z|) \le e^{\varepsilon|z|},\ \forall |z| > R.$$

This means that $\Psi(z)$, as well as $p(z)\Psi(z)$, is an entire function of order at most one and of minimal type. Since the indicator function

$$h_{A(x)}(\phi) := \overline{\lim_{R\to\infty}} \frac{\log|A(Re^{i\phi})|}{R}$$

of an entire function $A(x)$ of exponential type does not exceed its type, we have $h_{p(x)\Psi(x)}(\phi) \le 0,\ \forall \phi \in [0, 2\pi]$. This, together with the well-known property of the indicator function

$$h_{A(x)}(\phi) + h_{A(x)}(\pi + \phi) \ge 0,\ \forall \phi \in [0, 2\pi] \tag{38}$$

(cf. [11, Sec. 16, p. 53]) gives $h_{p(x)\Psi(x)}(\phi) = 0$, $\forall \phi \in [0, 2\pi]$. Hence,

$$\underline{\lim}_{n\to\infty} \nu_n^{1/n} = \frac{1}{\overline{\lim}_{n\to\infty}[(\sigma p(n))^{1/n} e^{\frac{\log \Psi(n)}{n}}]} = 1,$$

and by Lemma 3, $\lim_{n\to\infty} \nu_n^{1/n} = 1$, i.e., $\nu \in W_1 \cap \tau_+$. Now, using Lemma 5, we can find $F \in \mathcal{LP}_1(0, \frac{1}{\nu_0})$ such that $\nu_n = \frac{1}{F(n)}$, $\forall n \in Z_+$. Then

$$F(n) = p(n)\Psi(n), \quad \forall n \in Z_+. \tag{39}$$

Since $F \in \mathcal{LP}_1(0, \frac{1}{\nu_0})$, its indicator function, as well as the indicator function of Ψ, equals 0 at every point in $[0, 2\pi]$. Now, using properties of indicator functions [11, (1.65), (1.66), pp. 51-52] and (38), we obtain that the entire function $F(x) - p(x)\Psi(x)$ has zero indicator function and is of order at most one and of minimal type. Therefore by (39), Carlson's theorem is applicable [11, p. 168], so that

$$p(z)\Psi(z) \equiv F(z),$$

and consequently, the polynomial $p(x)$ is either a nonzero constant or has only real negative roots. This completes the proof of the Corollary. □

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