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GENERALIZED MOMENT REPRESENTATIONS AND INVARIANCE PROPERTIES OF PADÉ APPROXIMANTS

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By the method of generalized moment representations, we generalize the well-known invariance properties of Padé approximants under linear-fractional transformations of approximated functions.

Let us first introduce necessary definitions and notation.

Definition 1 [1, p. 36]. *Assume that a function* $f(z)$ *can be expanded in a power series of the form*

$$f(z) = \sum_{k=0}^{\infty} s_k z^k \tag{1}$$

in a neighborhood of the point $z = 0$.

Then the rational function

$$[M/N]_f(z) = \frac{P_M(z)}{Q_N(z)}, \tag{2}$$

where $P_M(z)$ *and* $Q_N(z)$ *are algebraic polynomials of degrees* $\leq M$ *and* $\leq N$, *respectively, is called a Padé approximant of order* $[M/N]$ *for the function* $f(z)$, *provided that*

$$f(z) - [M/N]_f(z) = O(z^{M+N+1}) \quad \text{as} \quad z \to 0.$$

As a tool for the approximation of analytic functions, Padé approximants have significant advantages over polynomial approximations and are extensively used in numerical mathematics, number theory, and theoretical physics. There are many papers devoted to the investigation of the general properties of these approximants and, in particular, of their invariance properties under various transformations of approximated functions. The following result is presented in [1]:

Theorem 1 [1, p. 44]. *Assume that, for an analytic function of the form (1), there exists a Padé approximant of order* $[N/N]$, $N \in \mathbb{N} \cup \{0\}$. *Then, for all* $a, b, c, d \in \mathbb{R}$ *such that* $c + ds_0 \neq 0$, *one can also construct a Padé approximant of the function* $\tilde{f}(z) = (a + bf(z))(c + df(z))^{-1}$ *and, moreover, if* $[N/N]_f(z) =: P_N(z)/Q_N(z)$, *then* $[N/N]_{\tilde{f}(z)} = \tilde{P}_N(z)/\tilde{Q}_N(z)$, *where*

$$\tilde{P}_N(z) = aQ_N(z) + bP_N(z) \quad \text{and} \quad \tilde{Q}_N(z) = cQ_N(z) + dP_N(z).$$

In 1981, Dzyadyk suggested a new approach to the investigation of Padé approximants based on the method of generalized moment representations [2].

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Definition 2 [2]. *The generalized moment representation of a number sequence* $\{s_k\}_{k=0}^{\infty}$ *in the Banach space* X *is defined as a collection of equalities*

$$s_{k+j} = \langle x_k, y_j \rangle, \quad k, j = 0, \dots, \infty, \tag{3}$$

where $x_k \in X$, $k = 0, \dots, \infty$, $y_j \in X^*$, $j = 0, \dots, \infty$, *and* $\langle x, y \rangle$ *denotes the value of a functional* $y \in X^*$ *on an element* $x \in X$.

By applying and developing this approach, we managed to establish numerous properties and analyze the behavior of Padé approximants and their generalizations for many special functions [3–7]. It was also discovered that the application of generalized moment representations enables one to obtain relevant generalizations of the theorems on invariance of Padé approximants.

Theorem 2. *Assume that, for an analytic function of the form (1), there exists a Padé approximant of order* $[N-1/N]$, $N \in \mathbb{N}$,

$$[N-1/N]_f(z) =: P_{N-1}(z)/Q_N(z).$$

Then, for the function

$$\tilde{f}(z) = \{f(z)[z(1+\xi_{11}s_1) - s_0\xi_{11}] + s_0^2\xi_{11}\}\{f(z)[z^2(\xi_{01}\xi_{10}s_1 - \xi_{00} - \xi_{00}\xi_{11}s_1)$$
$$+ z(-\xi_{10} + \xi_{00}\xi_{11}s_0 - \xi_{01} - \xi_{10}\xi_{01}s_0) - \xi_{11}]$$
$$+ [z(1 + \xi_{10}s_0 + \xi_{11}s_0 - \xi_{00}\xi_{11}s_0^2 + \xi_{01}s_0 + \xi_{10}\xi_{01}s_0^2) + \xi_{11}s_0]\}^{-1}, \tag{4}$$

the Padé approximant of order $[N-1/N]$ *exists, provided that* $1 + \xi_{10}s_0 + \xi_{11}s_1 \neq 0$. *Furthermore, the denominator* $\tilde{Q}_N(z)$ *of this approximant can be represented in the form*

$$\tilde{Q}_N(z) = \frac{1}{1+\xi_{10}s_0 + \xi_{11}s_1}\left\{Q_N(z)\left[1 + (\xi_{10} + \xi_{01})s_0 + \xi_{11}s_1 + \right.\right.$$
$$\left. + \frac{1}{z}\xi_{11}s_0 - (\xi_{00}\xi_{11}s_0 - \xi_{01}\xi_{10}s_1)s_0\right]$$
$$+ P_{N-1}(z)\left[-(\xi_{10} + \xi_{01}) + (\xi_{00} + \xi_{11}s_1)\right.$$
$$\left. - \frac{1}{z}\xi_{11} + (\xi_{01}\xi_{10}s_1 - \xi_{00}\xi_{11}s_1 - \xi_{00})z\right] - z^N \sum_{j=2}^{N} c_j^{(N)}(\xi_{10}s_j + \xi_{11}s_{j+1})$$
$$\left. + z^{N+1}\left[(\xi_{00}\xi_{11}s_0 - \xi_{01}\xi_{10}s_1 - \xi_{01})z - \xi_{11}\right]\sum_{j=0}^{N} c_j^{(N)}s_j\right\}, \tag{5}$$

where $c_j^{(N)}, j = 0, \dots, N$, *are coefficients of the polynomial*

$$Q_N(z) = \sum_{j=0}^{N} c_j^{(N)} z^{N-j}.$$

Proof. Since the function $f(z)$ possesses the Padé approximant of order $[N-1/N]$, the Hankel determinant is not equal to zero, i.e., $H_N =: \det \| s_{k+j} \|_{k,j=0}^{\infty} \neq 0$. Therefore, according to [3], for the sequence $\{s_k\}_{k=0}^{\infty}$, we can construct a generalized moment representation of the form (3) in a Hilbert space X. Consider a linear transformation $A: X \Rightarrow X$ such that $Ax_k = x_{k+1}$, $k = 0, \ldots, \infty$, and the conjugate transformation $A^* y_j = y_{j+1}$, $j = 0, \ldots, \infty$. Then we can rewrite relation (3) as $s_k = \langle A^k x_0, y_0 \rangle$, $k = 0, \ldots, \infty$. We can now introduce a transformation $\tilde{A}: X \Rightarrow X$ of the form

$$\tilde{A}x = Ax + x_0 \langle x, \xi_{00} y_0 + \xi_{01} y_1 \rangle + x_1 \langle x, \xi_{10} y_0 + \xi_{11} y_1 \rangle.$$

Denote $\tilde{x}_k = \tilde{A}^k x_0$, $k = 0, \ldots, \infty$ and $\tilde{y}_j = \tilde{A}^{*j} y_0$, $j = 0, \ldots, \infty$. We seek $\tilde{x}_k$, $k = 0, \ldots, \infty$, in the form

$$\tilde{x}_k = \sum_{m=1}^{k} d_{k-m} x_m + e_k x_0, \qquad k = 0, \ldots, \infty \tag{6}$$

(for $k < 1$, we set $\sum_{m=1}^{k} = 0$). By applying the operator $\tilde{A}$ to (6), we obtain

$$\begin{aligned} \tilde{x}_{k+1} = \tilde{A}\tilde{x}_k &= \sum_{m=1}^{k} d_{k-m} x_{m+1} + e_k x_1 \\ &\quad + x_0 \left(\sum_{m=1}^{k} d_{k-m} (\xi_{00} s_m + \xi_{01} s_{m+1}) + e_k (\xi_{00} s_0 + \xi_{01} s_1) \right) \\ &\quad + x_1 \left(\sum_{m=1}^{k} d_{k-m} (\xi_{10} s_m + \xi_{11} s_{m+1}) + e_k (\xi_{10} s_0 + \xi_{11} s_1) \right) \\ &= \sum_{m=1}^{k+1} d_{k+1-m} x_m + e_{k+1} x_0. \end{aligned}$$

This yields

$$e_{k+1} = \sum_{m=1}^{k} d_{k-m} (\xi_{00} s_m + \xi_{01} s_{m+1}) + e_k (\xi_{00} s_0 + \xi_{01} s_1),$$

$$d_k = e_k + \sum_{m=1}^{k} d_{k-m} (\xi_{10} s_m + \xi_{11} s_{m+1}) + e_k (\xi_{10} s_0 + \xi_{11} s_1).$$

Consider the generating functions

$$E(z) = \sum_{k=0}^{\infty} e_k z^k \quad \text{and} \quad D(z) = \sum_{k=0}^{\infty} d_k z^k.$$

We arrive at the following system of equations for $E(z)$ and $D(z)$:

$$E(z) = \sum_{k=0}^{\infty} e_k z^k = e_0 + \sum_{k=0}^{\infty} e_{k+1} z^{k+1}$$

$$= 1 + \sum_{k=0}^{\infty} z^{k+1} \left(\sum_{m=1}^{k} d_{k-m} (\xi_{00} s_m + \xi_{01} s_{m+1}) + e_k (\xi_{00} s_0 + \xi_{01} s_1) \right)$$

$$= 1 + D(z) \sum_{m=1}^{\infty} (\xi_{00} s_m + \xi_{01} s_{m+1}) z^{m+1} + z (\xi_{00} s_0 + \xi_{01} s_1) E(z),$$

$$D(z) = \sum_{k=0}^{\infty} d_k z^k = \sum_{k=0}^{\infty} e_k z^k + \sum_{k=1}^{\infty} z^k \sum_{m=1}^{k} d_{k-m} (\xi_{10} s_m + \xi_{11} s_{m+1}) + (\xi_{10} s_0 + \xi_{11} s_1) \sum_{k=0}^{\infty} e_k z^k$$

$$= (1 + \xi_{10} s_0 + \xi_{11} s_1) E(z) + D(z) \sum_{m=1}^{\infty} (\xi_{10} s_m + \xi_{11} s_{m+1}) z^m.$$

Its solution has the form

$$D(z) = (1 + \xi_{10} s_0 + \xi_{11} s_1) \left\{ 1 - \sum_{m=1}^{\infty} (\xi_{10} s_m + \xi_{11} s_{m+1}) z^m \right.$$
$$+ \sum_{m=0}^{\infty} z^{m+1} [(\xi_{00} s_0 + \xi_{01} s_1)(\xi_{10} s_m + \xi_{11} s_{m+1})$$
$$\left. - (1 + \xi_{10} s_0 + \xi_{11} s_1)(\xi_{00} s_m + \xi_{01} s_{m+1})] \right\}^{-1},$$

$$E(z) = \left[1 - \sum_{m=1}^{\infty} (\xi_{10} s_m + \xi_{11} s_{m+1}) z^m \right] \left\{ 1 - \sum_{m=1}^{\infty} (\xi_{10} s_m + \xi_{11} s_{m+1}) z^m \right.$$
$$+ \sum_{m=0}^{\infty} z^{m+1} [(\xi_{00} s_0 + \xi_{01} s_1)(\xi_{10} s_m + \xi_{11} s_{m+1})$$
$$\left. - (1 + \xi_{10} s_0 + \xi_{11} s_1)(\xi_{00} s_m + \xi_{01} s_{m+1})] \right\}^{-1}.$$

Denote

$$\tilde{s}_k = \left\langle \tilde{x}_k, y_0 \right\rangle, \quad k = 0, \ldots, \infty, \qquad \tilde{f}(z) = \sum_{k=0}^{\infty} \tilde{s}_k z^k.$$

Taking the relations established above into account, we obtain

$$\begin{aligned}
\tilde{f}(z) &= \sum_{k=0}^{\infty} \left\langle \tilde{x}_k, y_0 \right\rangle z^k = \sum_{k=1}^{\infty} z^k \sum_{m=1}^{k} d_{k-m} s_m + \sum_{k=0}^{\infty} e_k z^k s_0 \\
&= f(z) D(z) + s_0 (E(z) - D(z)) \\
&= \{f(z)[z(1 + \xi_{11} s_1) - s_0 \xi_{11}] + s_0^2 \xi_{11}\} \{f(z)[z^2(\xi_{01}\xi_{10} s_1 - \xi_{00} - \xi_{00}\xi_{11} s_1) \\
&\qquad + z(-\xi_{10} + \xi_{00}\xi_{11} s_0 - \xi_{01} - \xi_{10}\xi_{01} s_0) - \xi_{11}] \\
&\qquad + [z(1 + \xi_{10} s_0 + \xi_{11} s_0 - \xi_{00}\xi_{11} s_0^2 + \xi_{01} s_0 + \xi_{10}\xi_{01} s_0^2) + \xi_{11} s_0]\}^{-1},
\end{aligned}$$

i.e., we arrive at function (4). According to [2], to construct its Padé approximant of order $[N-1/N]$ it is necessary first to biorthogonalize the systems $\{\tilde{x}_k\}_{k=0}^{N}$ and $\{\tilde{y}_j\}_{j=0}^{N}$, i.e., to construct a nontrivial generalized polynomial $\tilde{X}_N = \sum_{k=0}^{N} \tilde{c}_k^{(N)} \tilde{x}_k$ such that $\left\langle \tilde{x}_N, \tilde{y}_j \right\rangle = 0$, $j = 0, \ldots, N-1$. It is easy to see that, in the case where $1 + \xi_{10} s_0 + \xi_{11} s_1 \neq 0$, this polynomial coincides, to within a constant factor, with the nontrivial generalized polynomial $X_N = \sum_{k=0}^{N} c_k^{(N)} x_k$ such that $\langle X_N, y_j \rangle = 0$, $j = 0, \ldots, N-1$.

Then under the assumption that the coefficients $c_k^{(N)}$, $k = 0, \ldots, N$, are known, it is necessary to determine the coefficients $\tilde{c}_k^{(N)}$, $k = 0, \ldots, N$. For this purpose, we express the elements x_k, $k = 0, \ldots, N$, in terms of the elements $\tilde{x}_k$, $k = 0, \ldots, N$. We get

$$\tilde{x}_k = \sum_{m=1}^{k} d_{k-m} x_m + e_k x_0.$$

Denote

$$\tilde{X}(z) = \sum_{k=0}^{\infty} \tilde{x}_k z^k \quad \text{and} \quad X(z) = \sum_{k=0}^{\infty} x_k z^k.$$

Then

$$\tilde{X}(z) = \sum_{k=0}^{\infty} z^k \sum_{m=1}^{k} d_{k-m} x_m + \sum_{k=0}^{\infty} z^k e_k x_0 = X(z) D(z) + (E(z) - D(z)) x_0,$$

whence it follows that

$$X(z) = \frac{\tilde{X}(z) + (D(z) - E(z)) x_0}{D(z)}.$$

By equating the coefficients of the same powers of z, we obtain

$$x_k = \frac{1}{1+\xi_{10}s_0+\xi_{11}s_1}\left\{\tilde{x}_k + \sum_{j=0}^{k-1}\tilde{x}_j\,[-\xi_{10}s_{k-j}-\xi_{11}s_{k-j+1} + (\xi_{00}s_0+\xi_{01}s_1)(\xi_{10}s_{k-j-1}+\xi_{11}s_{k-j}) - (1+\xi_{10}s_0+\xi_{11}s_1)(\xi_{00}s_{k-j-1}+\xi_{01}s_{k-j})] + (\xi_{10}s_k+\xi_{11}s_{k+1})\tilde{x}_0\right\}.$$

Thus,

$$\tilde{X}_N = X_N = \sum_{k=0}^{N} c_k^{(N)} x_k = \frac{1}{1+\xi_{10}s_0+\xi_{11}s_1}\left\{\sum_{k=0}^{N} c_k^{(N)}\tilde{x}_k + \sum_{j=0}^{N-1}\tilde{x}_j\sum_{k=j}^{N-1} c_{k+1}^{(N)}[-\xi_{10}s_{k-j+1}-\xi_{11}s_{k-j+2} + (\xi_{00}s_0+\xi_{01}s_1)(\xi_{10}s_{k-j}+\xi_{11}s_{k-j+1}) - (1+\xi_{10}s_0+\xi_{11}s_1)(\xi_{00}s_{k-j}+\xi_{01}s_{k-j+1})] + \tilde{x}_0\sum_{k=0}^{N} c_k^{(N)}(\xi_{10}s_k+\xi_{11}s_{k+1})\right\}.$$

This implies that

$$\tilde{c}_k^{(N)} = \frac{1}{1+\xi_{10}s_0+\xi_{11}s_1}\left\{\delta_{k,n}c_N^{(N)} + (1-\delta_{k,N})\sum_{j=k}^{N-1} c_{j+1}^{(N)}[-\xi_{10}s_{j-k+1}-\xi_{11}s_{j-k+2} + (\xi_{00}s_0+\xi_{01}s_1)(\xi_{10}s_{j-k}+\xi_{11}s_{j-k+1}) - (1+\xi_{10}s_0+\xi_{11}s_1)(\xi_{00}s_{k-j}+\xi_{01}s_{k-j+1})] + \delta_{k,0}\sum_{j=0}^{N} c_j^{(N)}(\xi_{10}s_j+\xi_{11}s_{j+1})\right\},$$

where

$$\delta_{k,j} := \begin{cases} 1 & \text{for } k=j, \\ 0 & \text{for } k\neq j. \end{cases}$$

According to [2], the denominator $\tilde{Q}_N(z)$ of the Padé approximant of order $[N-1/N]$ for the function $\tilde{f}(z)$ has the form

$$\tilde{Q}_N(z) = \sum_{k=0}^{N}\tilde{c}_k^{(N)}z^{N-k} = \frac{1}{1+\xi_{10}s_0+\xi_{11}s_1}\left\{\sum_{k=0}^{N} c_k^{(N)}z^{N-k} + \sum_{k=0}^{N-1} z^{N-k}\sum_{j=k}^{N-1} c_{j+1}^{(N)}[-\xi_{10}s_{j-k+1}-\xi_{11}s_{j-k+2} + (\xi_{00}s_0+\xi_{01}s_1)(\xi_{10}s_{j-k}+\xi_{11}s_{j-k+1}) - (1+\xi_{10}s_0+\xi_{11}s_1)(\xi_{00}s_{k-j}+\xi_{01}s_{k-j+1})] + z^N\sum_{k=0}^{N} c_k^{(N)}(\xi_{10}s_k+\xi_{11}s_{k+1})\right\},$$

whence, by virtue of the equalities

$$[N-1/N]_f(z) = \frac{P_{N-1}(z)}{Q_N(z)}, \quad Q_N(z) = \sum_{k=0}^{N} c_k^{(N)} z^{N-k},$$

$$P_{N-1}(z) = \sum_{k=0}^{N} c_k^{(N)} z^{N-k} \sum_{j=0}^{k-1} s_j z^j,$$

we arrive at relation (5).

Remark 1. The relation for the numerator of the Padé approximant of order $[N-1/N]$ for the function $\bar{f}(z)$ can be obtained in exactly the same way.

Remark 2. In the case where $\xi_{01} = \xi_{10} = \xi_{11} = 0$, Theorem 2 implies the result equivalent to the assertion of Theorem 1.

REFERENCES

1. G. A. Baker (Jr.) and P. Graves-Morris, *Padé Approximants* [Russian translation], Mir, Moscow (1986).
2. V. K. Dzyadyk, "Generalization of the moment problem," *Dokl. Akad. Nauk Ukr. SSR, Ser. A*, No. 6, 8–12 (1981).
3. V. K. Dzyadyk and A. P. Golub, "Generalized moment problem and Padé approximation," in: *Generalized Moment Problem and Padé Approximation* [in Russian], Preprint No. 81.58, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1981), pp. 3–15.
4. A. P. Golub, "Application of the generalized moment problem to Padé approximation of some functions," in: *Generalized Moment Problem and Padé Approximation* [in Russian], Preprint No. 81.58, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1981), pp. 16–56.
5. A. P. Golub, *Generalized Moment Representations and Rational Approximations* [in Russian], Preprint No. 87.25, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1987).
6. A. P. Golub, "Some properties of biorthogonal polynomials and their applications to Padé approximants," *Ukr. Mat. Zh.*, **46**, No. 8, 977–984 (1994).
7. A. P. Golub, "Generalized moment representations, biorthogonal polynomials, and Padé approximants," *Ukr. Mat. Zh.*, **46**, No. 10, 1328–1335 (1994).