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PADÉ-TYPE APPROXIMANTS FOR SOME CLASSES OF MULTIVARIATE FUNCTIONS

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We extend Dzyadyk's method of generalized moment representations to the multidimensional case and, on this basis, construct and investigate the Padé-type approximants for some classes of multivariate functions.

The method of Padé approximants is one of the most efficient and known methods used for the rational approximation of analytic functions. The Padé-type approximations for functions of many variables are constructed and studied for more than forty years. Numerous works are devoted to the investigation and application of these approximants (see, e.g., [1, 2] and the references in [3]).

In 1981, Dzyadyk [4] proposed a method of generalized moment representations, which enabled us to consider, from the common point of view, the problems of investigation of the Padé approximants for numerous important special functions and, in particular, for the functions that do not belong to the class of Markov functions. This approach was later developed by Holub [5, 6].

Dzyadyk's approach was generalized to the multidimensional case (see [7]). The aim of the present paper is to construct Padé-type approximants for some classes of multivariate functions of a special form.

We now give the corresponding definition:

Definition 1 [7]. *A generalized moment representation of a d-dimensional numerical sequence $\{s_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}$ on the product of linear spaces $\mathcal{X}$ and $\mathcal{Y}$ with respect to a bilinear form $\langle\,,\rangle$ given on this product is defined as a collection of equalities*

$$s_{\mathbf{k}+\mathbf{j}} = \langle x_{\mathbf{k}}, y_{\mathbf{j}}\rangle, \quad \mathbf{k},\mathbf{j}\in\mathbb{Z}_+^d, \tag{1}$$

where $\{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}\subset\mathcal{X}$ and $\{y_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}_+^d}\subset\mathcal{Y}$.

Consider a formal power series in d variables

$$f(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}_+^d} s_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}, \tag{2}$$

where

$$\mathbf{z} = (z_1, z_2, \ldots, z_d)\in\mathbb{C}^d, \qquad \mathbf{k} = (k_1, k_2, \ldots, k_d)\in\mathbb{Z}_+^d, \qquad \text{and} \qquad \mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2}\ldots z_d^{k_d}.$$

For the sake of convenience, we now introduce some notation.

For $p = 0, 1, \ldots, d,$ we denote

$$\Omega_p = \big\{\omega\subseteq\{1, 2, \ldots, d\}:\ |\omega| = p\big\}.$$

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We order elements of each set $\omega \in \Omega_p\colon\ \omega = \{l_1(\omega), l_2(\omega), \ldots, l_p(\omega)\}$ so that

$$1 \le l_1(\omega) < l_2(\omega) < \ldots < l_p(\omega) \le d.$$

We also order elements of the complement

$$\overline{\omega} = \{1, 2, \ldots, d\} \setminus \omega = \{m_1(\omega), m_2(\omega), \ldots, m_{d-p}(\omega)\} \in \Omega_{d-p}$$

so that

$$1 \le m_1(\omega) < m_2(\omega) < \ldots < m_{d-p}(\omega) \le d.$$

For each set $\omega \in \Omega_p,\ p = 1, \ldots, d,$ we introduce the notation

$$\boldsymbol{\delta}(\omega) = \big(\delta_1(\omega), \delta_2(\omega), \ldots, \delta_d(\omega)\big),$$

where

$$\delta_i(\omega) = \begin{cases} 0 & \text{for} \quad i \in \omega, \\ 1 & \text{for} \quad i \notin \omega, \end{cases}$$

$$\boldsymbol{\varepsilon}(\omega) = \big(\varepsilon_1(\omega), \varepsilon_2(\omega), \ldots, \varepsilon_d(\omega)\big).$$

Here,

$$\varepsilon_i(\omega) = \begin{cases} -1 & \text{for} \quad i \in \omega, \\ 1 & \text{for} \quad i \notin \omega, \end{cases}$$

and, hence,

$$\delta_i(\omega) = \frac{\varepsilon_i(\omega) + 1}{2}, \quad i = 1, 2, \ldots, d.$$

We also denote $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{Z}_+^d$ and $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}_+^d$. Thus,

$$\mathbf{1} = \boldsymbol{\delta}(\varnothing) \quad \text{and} \quad \mathbf{0} = \boldsymbol{\delta}\big(\{1, 2, \ldots, d\}\big).$$

For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^d,\ \mathbf{a} = (a_1, a_2, \ldots, a_d),\ \mathbf{b} = (b_1, b_2, \ldots, b_d),$ we denote their coordinatewise product by $\mathbf{a} \circ \mathbf{b}$:

$$\mathbf{a} \circ \mathbf{b} = (a_1 b_1, a_2 b_2, \ldots, a_d b_d).$$

For each vector $\mathbf{a} = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}_+^d$, we denote

$$\Delta(\mathbf{a}) = \Big\{\mathbf{j} = (j_1, j_2, \ldots, j_d) \in \mathbb{Z}_+^d\colon\ j_i \in \{0, 1, \ldots, a_i\},\ i = 1, 2, \ldots, d\Big\}.$$

For fixed $\mathbf{N} \in \mathbb{Z}_+^d$, we consider a continuous function $\Phi_{\mathbf{N}}\colon \mathbb{R}_+^d \to \mathbb{R}$ with the following properties:

(i) the set $\mathcal{D}_{\Phi_{\mathbf{N}}} = \{\mathbf{x} \in \mathbb{R}_+^d \mid \Phi_{\mathbf{N}}(\mathbf{x}) \le 0\}$ is bounded in $\mathbb{R}_+^d$;

(ii) the cardinality of $\mathcal{D}_{\Phi_{\mathbf{N}}} \bigcap \left\{ \mathbf{x} \in \mathbb{Z}_+^d \mid x_i \geq N_i,\ i = 1, 2, \ldots, d \right\}$ is equal to $\prod_{i=1}^{d} (N_i + 1) - 1;$

(iii) for all $i = 1, 2, \ldots, d,$ there exist uniquely defined functions

$$x_i = \varphi_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$$

with

$$(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in D_i := \Big\{ (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{R}_+^{d-1} \mid \\ \exists\, x_i \in \mathbb{R}_+ \colon \Phi_{\mathbf{N}}(\mathbf{x}) \leq 0 \Big\}$$

such that

$$\Phi_{\mathbf{N}}\big(x_1, x_2, \ldots, x_{i-1},\ \varphi_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d),\ x_{i+1}, \ldots, x_d\big) \equiv 0;$$

(iv) for each $i = 1, 2, \ldots, d,$

$$\varphi_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \geq N_i \quad \forall (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in D_i.$$

By using this notation, we establish the following result, which enables one to construct d-dimensional Padé-type approximants for series of the form (2) whose coefficients admit representations of the form (1):

Theorem 1 [7]. *Suppose that the coefficients of a formal power series of the form (2) satisfy the generalized moment representation of the form (1). If, for some $\mathbf{N} \in \mathbb{N}^d$, there exists a generalized polynomial of the form*

$$Y_{\mathbf{N}} = \sum_{\mathbf{j} \in \Delta(\mathbf{N})} c_{\mathbf{j}}^{(\mathbf{N})} y_{\mathbf{j}}$$

such that

$$c_{\mathbf{N}}^{(\mathbf{N})} \neq 0$$

and, for $\mathbf{k} \in \left\{ \mathbf{k} \in \mathbb{Z}_+^d \colon \mathbf{k} + \mathbf{N} \in \mathcal{D}_{\Phi_{\mathbf{N}}} \right\}$, the conditions of biorthogonality

$$\langle x_{\mathbf{k}}, Y_{\mathbf{N}} \rangle = 0$$

are satisfied, then the rational function

$$[\mathcal{M}/\mathcal{N}]_f(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(z)},$$

where

$$Q(z) = \sum_{\mathbf{j} \in \Delta(\mathbf{N})} c_{\mathbf{N}-\mathbf{j}}^{(\mathbf{N})} \mathbf{z}^{\mathbf{j}}$$

and

$$P(\mathbf{z}) = \sum_{p=0}^{d-1} \sum_{\omega\in\Omega_p} \prod_{r=1}^{p} z_{l_r(\omega)}^{N_{l_r(\omega)}} \sum_{\substack{0\le k_{m_i(\omega)}\le N_{m_i(\omega)}-1, i=1,2,\ldots,d-p \\ \Phi_{\mathbf{N}}(\mathbf{k})\le 0}}$$

$$\times\, \mathbf{z}^{\mathbf{k}} \sum_{\mathbf{j}\in\Delta(\boldsymbol{\delta}(\overline{\omega})\circ\mathbf{N}+\boldsymbol{\delta}(\omega)\circ\mathbf{k})} c_{\boldsymbol{\delta}(\omega)\circ\mathbf{N}+\boldsymbol{\varepsilon}(\omega)\circ\mathbf{j}}\, s_{\mathbf{k}+\boldsymbol{\varepsilon}(\omega)\circ\mathbf{j}},$$

admits an expansion in a power series whose coefficients coincide with the coefficients of series (2) for all $\mathbf{k} \in \mathcal{D}_{\Phi_{\mathbf{N}}} \cap \mathbb{Z}_+^d$ *and, hence, this rational function is a* d*-dimensional Padé-type approximant for series (2) of order* $[\mathcal{M}/\mathcal{N}]$, *where*

$$\mathcal{M} = \mathcal{D}_{\Phi_{\mathbf{N}}} \cap \mathbb{Z}_+^d \setminus \left\{\mathbf{x} \in \mathbb{Z}_+^d \colon x_i \ge N_i,\ i = 1, 2, \ldots, d\right\}$$

and $\mathcal{N} = \Delta(\mathbf{N})$.

Generalized moment representations of the form (1) can be also represented in the operator form. Assume that the linear spaces $\mathcal{X}$ and $\mathcal{Y}$ are normed, the bilinear form $\langle\cdot\,,\cdot\rangle$ is separately continuous, pairwise commuting bounded linear operators

$$A_i \colon \mathcal{X} \to \mathcal{X}, \quad i = 1, 2, \ldots, d,$$

such that

$$A_i x_{\mathbf{k}} = x_{\mathbf{k}+\mathbf{e}_i,}, \quad i = 1, 2, \ldots, d,$$

for each $\mathbf{k} \in \mathbb{Z}_+^d$, where

$$\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) = \mathbf{1} - \boldsymbol{\delta}(\{i\}), \quad i = 1, 2, \ldots, d,$$

are given in the space $\mathcal{X}$, and the space $\mathcal{Y}$ contains bounded linear operators $A_i^\star \colon \mathcal{Y} \to \mathcal{Y}$, $i = 1, 2, \ldots, d$, adjoint to the operators A_i, $i = 1, 2, \ldots, d$, with respect to the bilinear form $\langle\cdot\,,\cdot\rangle$.

Thus, representations (1) can be rewritten in the form

$$s_{\mathbf{k}} = \langle x_{\mathbf{k}}, y_{\mathbf{0}}\rangle = \left\langle \prod_{i=1}^{d} A_i^{k_i} x_{\mathbf{0}}, y_{\mathbf{0}} \right\rangle, \quad \mathbf{k} \in \mathbb{Z}_+^d,$$

and series (2) is convergent in the vicinity of the origin of coordinates to an analytic function with the following representation:

$$f(\mathbf{z}) = \left\langle \prod_{i=1}^{d} \mathcal{R}_{z_i}(A_i) x_{\mathbf{0}}, y_{\mathbf{0}} \right\rangle,$$

where

$$\mathcal{R}_z(A) = (I - zA)^{-1}$$

is the resolvent function of the operator A.

Let $\mathcal{X} = \mathcal{Y} = L_2([0,1], d\mu)$ for some measure defined by a nondecreasing function μ with infinitely many points of increase on $[0,1]$. On the product of the spaces $\mathcal{X} \times \mathcal{Y}$, we define a separately continuous bilinear form

$$\langle x, y \rangle = \int_0^1 x(t)y(t)\, d\mu(t).$$

In the space $\mathcal{X}$, for some fixed $d_1,\ 1 < d_1 < d$, we consider bounded pairwise commuting linear operators

$$A_1, A_2, \ldots, A_d \colon \mathcal{X} \to \mathcal{X}$$

defined as follows:

$$(A_p\varphi)(t) = t\varphi(t), \quad p = \overline{1, d_1},$$

$$(A_l\varphi)(t) = (1-t)\varphi(t), \quad l = \overline{d_1+1, d}.$$

In this case, for $x_{\mathbf{0}}(t),\ y_{\mathbf{0}}(t) \equiv 1,$ we rewrite function (2) in the form

$$f(\mathbf{z}) = \left\langle \prod_{k=1}^{d_1} \mathcal{R}_{z_k}(A_1) \prod_{m=d_1+1}^{d} \mathcal{R}_{z_m}(A_d) x_{\mathbf{0}}, y_{\mathbf{0}} \right\rangle$$

$$= \int_0^1 \frac{d\mu(t)}{\prod_{k=1}^{d_1}(1 - z_k t) \prod_{k=d_1+1}^{d}(1 - z_k(1-t))}. \tag{3}$$

It is clear that

$$\frac{1}{1 - z_m(1-t)} = \frac{1}{1 - z_m} \frac{1}{1 - \frac{z_m t}{z_m - 1}} = \frac{1}{1 - z_m} \frac{1}{1 - \widetilde{z}_m t},$$

where

$$\widetilde{z}_m = \frac{z_m}{z_m - 1}.$$

The following relation was used in [1]:

$$\frac{1}{\prod_{k=1}^{d}(1 - w_k t)} = \frac{1}{\prod_{k<j}(w_k - w_j)} \left\{ \sum_{k=1}^{d} w_k^{d-1} (-1)^{k+1} \prod_{\substack{p<q \\ p,q \neq k}} (w_p - w_q) \frac{1}{1 - w_k t} \right\}$$

$$= (-1)^{d-1} \sum_{k=1}^{d} \frac{w_k^{d-1}}{\prod_{\substack{p=1 \\ p \neq k}}^{d} (w_p - w_k)} \frac{1}{1 - w_k t}.$$

Thus, by setting

$$w_k = \begin{cases} z_k & \text{for} \quad k = \overline{1, d_1}, \\ \widetilde{z}_k = \dfrac{z_k}{z_k - 1} & \text{for} \quad k = \overline{d_1 + 1, d}, \end{cases}$$

we obtain

$$\frac{1}{\prod_{k=1}^{d_1}(1 - z_k t)\prod_{k=d_1+1}^{d}(1 - z_k(1-t))}$$

$$= \prod_{k=d_1+1}^{d} \frac{1}{1 - z_k} \frac{1}{\prod_{k=1}^{d_1}(1 - z_k t)\prod_{k=d_1+1}^{d}(1 - \widetilde{z}_k t)}$$

$$= \frac{1}{\prod_{k=d_1+1}^{d}(1 - z_k)}(-1)^{d-1}\left\{\sum_{k=1}^{d_1} \frac{z_k^{d-1}}{\prod_{\substack{p=1 \\ p\neq k}}^{d_1}(z_p - z_k)\prod_{p=d_1+1}^{d}(\widetilde{z}_p - z_k)} \frac{1}{1 - z_k t}\right.$$

$$\left. + \sum_{k=d_1+1}^{d} \frac{\widetilde{z}_k^{d-1}}{\prod_{p=1}^{d_1}(z_p - \widetilde{z}_k)\prod_{\substack{p=d_1+1 \\ p\neq k}}^{d}(\widetilde{z}_p - \widetilde{z}_k)} \frac{1}{1 - \widetilde{z}_k t}\right\}$$

$$= (-1)^{d-1}\left\{\sum_{k=1}^{d_1} \frac{z_k^{d-1}}{\prod_{\substack{p=1 \\ p\neq k}}^{d_1}(z_p - z_k)\prod_{p=d_1+1}^{d}(z_p + z_k - z_p z_k)} \frac{1}{1 - z_k t}\right.$$

$$\left. +(-1)^{d_1} \sum_{k=d_1+1}^{d} \frac{z_k^{d-1}}{\prod_{p=1}^{d_1}(z_p + z_k - z_p z_k)\prod_{\substack{p=d_1+1 \\ p\neq k}}^{d}(z_k - z_p)} \frac{1}{1 - z_k(1-t)}\right\}.$$

The coefficients $s_{\mathbf{k}}$ of expansion of the function f in the formal power series with d variables have the following form:

$$s_{\mathbf{k}} = \langle x_{\mathbf{k}}, y_{\mathbf{0}}\rangle = \left\langle A_1^{k_1+k_2+\ldots+k_{d_1}} A_d^{k_{d_1+1}+\ldots+k_d} x_{\mathbf{0}}, y_{\mathbf{0}}\right\rangle$$

$$= \int_0^1 t^{k_1+k_2+\ldots+k_{d_1}}(1-t)^{k_{d_1+1}+\ldots+k_d}\, d\mu(t).$$

To determine Padé-type approximants for functions (2) by Theorem 1, it is necessary to construct the polynomials

$$X_{\mathbf{N}}(t) = \sum_{k_1=0}^{N_1} \ldots \sum_{k_d=0}^{N_d} c_{k_1,k_2,\ldots,k_d}^{(N_1,\ldots,N_d)} t^{k_1+k_2+\ldots+k_{d_1}}(1-t)^{k_{d_1+1}+\ldots+k_d}$$

satisfying the conditions of biorthogonality

$$\langle X_{\mathbf{N}}, y_{\mathbf{j}} \rangle = 0$$

for

$$\mathbf{j} \in \left\{ (j_1, j_2, \ldots, j_d) \in \mathbb{Z}_+^d \mid j_i \in [0, N_i],\ i = \overline{1, d} \right\} \setminus \left\{ (N_1, N_2, \ldots, N_d) \right\}.$$

In this case, $X_{\mathbf{N}}(t)$ is an algebraic polynomial of degree $N_1 + N_2 + \ldots + N_d$ orthogonal to polynomials of degree at most $N_1 + \ldots + N_d - 1$. Hence, to within a constant factor, it coincides with the polynomial of degree $N_1 + \ldots + N_d$ orthonormal on $[0,1]$ with respect to the measure $d\mu$ (see [8, p. 268]):

$$\sum_{k_1=0}^{N_1} \cdots \sum_{k_d=0}^{N_d} c_{k_1,\ldots,k_d}^{(N_1,\ldots,N_d)} t^{k_1+k_2+\ldots+k_{d_1}} (1-t)^{k_{d_1+1}+\ldots+k_d} = P_{N_1+N_2+\ldots+N_d}(t). \tag{4}$$

The coefficients

$$c_{k_1,\ldots,k_d}^{(N_1,\ldots,N_d)}, \quad \mathbf{k} \in \Delta(\mathbf{N}),$$

in equality (4) can be defined in many different ways. Since functions of the form (2) are symmetric in their variables if and only if

$$d\mu(t) \equiv d\mu(1-t),$$

it is necessary to consider two cases:

Case 1. In the asymmetric case, as one of the possibility of determination of the coefficients $c_{k_1,\ldots,k_d}^{(N_1,\ldots,N_d)}$, $\mathbf{k} \in \Delta(\mathbf{N})$, we consider the following procedure:

$$\begin{aligned}
\sum_{i=0}^{N_1+\ldots+N_d} p_i^{(N_1+\ldots+N_d)} t^i = {} & \sum_{k_1=0}^{N_1-1} c_{k_1,0,\ldots,0} t^{k_1} + t^{N_1} \sum_{k_2=0}^{N_2-1} c_{N_1,k_2,0,\ldots,0} t^{k_2} \\
& + t^{N_1+N_2} \sum_{k_3=0}^{N_3-1} c_{N_1,N_2,k_3,0,\ldots,0} t^{k_3} \\
& + \ldots + t^{N_1+\ldots+N_{d_1-1}} \sum_{k_{d_1}=0}^{N_{d_1}-1} c_{N_1,N_2,\ldots,N_{d_1-1},k_{d_1},0,\ldots,0} t^{k_{d_1}} \\
& + t^{N_1+\ldots+N_{d_1}} \sum_{k_{d_1+1}=0}^{N_{d_1+1}-1} c_{N_1,\ldots,N_{d_1},k_{d_1+1},0,\ldots,0} (1-t)^{k_{d_1+1}} \\
& + t^{N_1+\ldots+N_{d_1}} (1-t)^{N_{d_1+1}} \sum_{k_{d_1+2}=0}^{N_{d_1+2}-1} c_{N_1,\ldots,N_{d_1},N_{d_1+1},k_{d_1+2},0,\ldots,0} (1-t)^{k_{d_1+2}}
\end{aligned}$$

$$+ \ldots + t^{N_1+\ldots+N_{d_1}}(1-t)^{N_{d_1+1}+\ldots+N_{d-2}} \sum_{k_{d-1}=0}^{N_{d-1}-1} c_{N_1,\ldots,N_{d_1},N_{d_1+1},\ldots,k_{d-1},0}(1-t)^{k_{d-1}}$$

$$+ t^{N_1+\ldots+N_{d_1}}(1-t)^{N_{d_1+1}+\ldots+N_{d-1}} \sum_{k_d=0}^{N_d} c_{N_1,\ldots,N_{d_1},N_{d_1+1},\ldots,N_{d-1},k_d}(1-t)^{k_d}.$$

Thus, for $k_1 = \overline{0, N_1 - 1},\ k_2 = \ldots = k_d = 0,$ we get

$$c_{k_1,\ldots,k_d}^{(N_1,\ldots,N_d)} = p_{k_1}^{(N_1+\ldots+N_d)}.$$

For $k_1 = N_1,\ k_2 = \overline{0, N_2 - 1},\ k_3 = \ldots = k_d = 0,$ we can write

$$c_{N_1,k_2,0,\ldots,0}^{(N_1,\ldots,N_d)} = p_{N_1+k_2}^{(N_1+\ldots+N_d)}$$

and, for $k_1 = N_1,\ k_2 = N_2,\ k_3 = \overline{0, N_3 - 1},\ k_4 = \ldots = k_d = 0,$ we obtain

$$c_{N_1,N_2,k_3,0,\ldots,0}^{(N_1,\ldots,N_d)} = p_{N_1+N_2+k_3}^{(N_1+\ldots+N_d)}.$$

Continuing in a similar way and using the same reasoning, we find:

for $k_1 = N_1,\ k_2 = N_2, \ldots, k_{d_1-1} = N_{d_1-1},\ k_{d_1} = \overline{0, N_{d_1} - 1},\ k_{d_1+1} = \ldots = k_d = 0,$

$$c_{N_1,N_2,\ldots,N_{d_1-1},k_{d_1},0,\ldots,0}^{(N_1,\ldots,N_d)} = p_{N_1+N_2+\ldots+N_{d_1-1}+k_{d_1}}^{(N_1+\ldots+N_d)},$$

for $k_1 = N_1,\ k_2 = N_2, \ldots, k_{d_1} = N_{d_1},\ k_{d_1+1} = \overline{0, N_{d_1+1} - 1},\ k_{d_1+2} = \ldots = k_d = 0,$

$$c_{N_1,N_2,\ldots,N_{d_1},k_{d_1+1},0,\ldots,0}^{(N_1,\ldots,N_d)} = (-1)^{k_{d_1+1}} \sum_{i=k_{d_1+1}}^{N_{d_1+1}-1} p_{i+N_1+N_2+\ldots+N_{d_1}}^{(N_1+\ldots+N_d)} \binom{i}{k_{d_1+1}},$$

for $k_1 = N_1, \ldots, k_{d_1} = N_{d_1},\ k_{d_1+1} = N_{d_1+1},\ k_{d_1+2} = \overline{0, N_{d_1+2} - 1},\ k_{d_1+3} = \ldots = k_d = 0,$

$$c_{N_1,N_2,\ldots,N_{d_1},N_{d_1+1},k_{d_1+2},0,\ldots,0}^{(N_1,\ldots,N_d)} = (-1)^{k_{d_1+2}} \sum_{i=k_{d_1+2}}^{N_{d_1+2}-1} p_{i+N_1+N_2+\ldots+N_{d_1+1}}^{(N_1+\ldots+N_d)} \binom{i}{k_{d_1+2}},$$

for $k_1 = N_1, \ldots, k_{d_1} = N_{d_1},\ k_{d_1+1} = N_{d_1+1},\ k_{d_1+2} = \overline{0, N_{d_1+2} - 1},\ k_{d_1+3} = \ldots = k_d = 0,$

$$c_{N_1,N_2,\ldots,N_{d_1},N_{d_1+1},k_{d_1+2},0,\ldots,0}^{(N_1,\ldots,N_d)} = (-1)^{k_{d_1+2}} \sum_{i=k_{d_1+2}}^{N_{d_1+2}-1} p_{i+N_1+N_2+\ldots+N_{d_1+1}}^{(N_1+\ldots+N_d)} \binom{i}{k_{d_1+2}},$$

and so on.

We also present the last two equalities:

for $k_1 = N_1, \ldots, k_{d-2} = N_{d-2},\ k_{d-1} = \overline{0, N_{d-1} - 1},\ k_d = 0,$

$$c^{(N_1,\ldots,N_d)}_{N_1,N_2,\ldots,N_{d-2},k_{d-1},0} = (-1)^{k_{d-1}} \sum_{i=k_{d-1}}^{N_{d-1}-1} p^{(N_1+\ldots+N_d)}_{i+N_1+N_2+\ldots+N_{d-2}} \binom{i}{k_{d-1}};$$

for $k_1 = N_1, \ldots, N_{d-1},\ k_d = \overline{0, N_d - 1},$

$$c^{(N_1,\ldots,N_d)}_{N_1,N_2,\ldots,N_{d-1},k_d} = (-1)^{k_d} \sum_{i=k_d}^{N_d-1} p^{(N_1+\ldots+N_d)}_{i+N_1+N_2+\ldots+N_{d-1}} \binom{i}{k_d}.$$

Case 2. In the symmetric case, we assume that

$$N_1 = N_2 = \ldots = N_{d_1},$$

$$N_{d_1+1} = N_{d_1+2} = \ldots = N_d,$$

and, in addition,

$$N_1 + N_2 + \ldots + N_{d_1} = N_{d_1+1} + N_{d_1+2} + \ldots + N_d.$$

Let

$$N_1 = N_2 = \ldots = N_{d_1} = N \quad \text{and} \quad N_{d_1+1} = N_{d_1+2} = \ldots = N_d = M.$$

Then

$$d_1 N = (d - d_1) M$$

and

$$X_{\mathbf{N}}(t) = \sum_{k_1=0}^{N} \ldots \sum_{k_{d_1}=0}^{N} \sum_{k_{d_1+1}=0}^{M} \ldots \sum_{k_d=0}^{M} c^{(N_1,\ldots,N_d)}_{k_1,\ldots,k_d} t^{k_1+\ldots+k_{d_1}} (1-t)^{k_{d_1+1}+\ldots+k_d} = P_{2d_1N}(t).$$

We set

$$c_{k_1,k_2,\ldots,k_d} = 0 \quad \text{for} \quad k_1 + k_2 + \ldots + k_{d_1} \neq k_{d_1+1} + \ldots + k_d,$$

$$c_{k_1,k_2,\ldots,k_d} = \widetilde{c}_{|\mathbf{k}|/2} \quad \text{for} \quad k_1 + k_2 + \ldots + k_{d_1} = k_{d_1+1} + \ldots + k_d = |\mathbf{k}|/2.$$

Then

$$X_{\mathbf{N}}(t) = \sum_{m=0}^{2d_1N} \widetilde{c}_{|\mathbf{k}|/2} t^{\sum_{i=1}^{d_1} k_i} (1-t)^{\sum_{i=d_1+1}^{d} k_i}$$

$$= P_{2d_1N}(t) = \sum_{i=0}^{2d_1N} p_i^{(2d_1N)} t^i.$$

According to Lemma 3.1 (see [9]), the coefficients $\widetilde{c}_m$ have the form

$$\widetilde{c}_m = \begin{cases} p_0^{(2d_1N)} & \text{for} \quad m = 0, \\ \displaystyle\sum_{j=1}^{m} \frac{(2m-j-1)!j}{m!(m-j)!} p_j^{(2d_1N)} & \text{for} \quad m \geq 1. \end{cases} \tag{5}$$

For the measure

$$d\mu(t) = t^{\nu}(1-t)^{\sigma}dt,$$

the coefficients of the power expansion of function (3) have the form

$$s_{\mathbf{k}} = \frac{\Gamma(k_1 + k_2 + \ldots + k_{d_1} + \nu + 1)\Gamma(k_{d_1+1} + \ldots + k_d + \sigma + 1)}{\Gamma(|\mathbf{k}| + \nu + \sigma + 2)}$$

$$= \frac{\Gamma\left(\sum_{i=1}^{d_1} k_i + \nu + 1\right)\Gamma\left(\sum_{i=d_1+1}^{d} k_i + \sigma + 1\right)}{\Gamma(|\mathbf{k}| + \nu + \sigma + 2)}. \tag{6}$$

Hence, we obtain the function

$$f(\mathbf{z}) = \sum_{k_1,k_2,\ldots,k_d=0}^{\infty} \frac{\Gamma\left(\sum_{i=1}^{d_1} k_i + \nu + 1\right)\Gamma\left(\sum_{i=d_1+1}^{d} k_i + \sigma + 1\right)}{\Gamma(|\mathbf{k}| + \nu + \sigma + 2)} z_1^{k_1} \ldots z_d^{k_d}$$

in the form of a hypergeometric series of the second order (see [10]).

In this case, the polynomial $X_{\mathbf{N}}(t)$ coincides, to within a constant factor, with the orthonormal Jacobi polynomial $P_{|\mathbf{N}|}^{(\nu,\sigma)*}(t)$ of degree $|\mathbf{N}|$ shifted to $[0, 1]$.

We now write the explicit expression for the coefficients of orthogonal Jacobi polynomials (see [11, p. 581], Sec. 22.3.3; for the sake of convenience, we set the constant equal to 1):

$$P_{N_1+\ldots+N_d}^{(\nu,\sigma)*}(t) = \sum_{m=0}^{N_1+\ldots+N_d} (-1)^m \binom{N_1 + \ldots + N_d}{m} \frac{\Gamma(N_1 + \ldots + N_d + \nu + \sigma + m + 1)}{\Gamma(\nu + m + 1)} t^m.$$

For $\nu = \sigma$, which corresponds to the symmetric case, the polynomial $X_{\mathbf{N}}$ coincides, to within a constant factor, with the orthonormal Gegenbauer polynomial $C_{2d_1N}^{(\nu+1/2)}$ shifted to $[0, 1]$.

The coefficients of this polynomial can be found from the relationship connecting it with the Jacobi polynomial (see [11, p. 584], Sec. 22.5.27):

$$C_N^{(\nu)}(t) = \frac{(2\nu)_N}{\left(\nu + \frac{1}{2}\right)_N} P_N^{(\nu-1/2,\nu-1/2)}(t).$$

Thus, we get

$$p_i^{(2d_1N)} = (-1)^i \frac{(2\nu+1)_{2d_1N}}{(\nu+1)_{2d_1N}} \binom{2d_1N}{i} \frac{\Gamma(2d_1N + 2\nu + 1 + i)}{\Gamma(\nu + 1 + i)}. \tag{7}$$

Substituting (7) in (5), we obtain

$$\widetilde{c}_m = \begin{cases} \dfrac{\Gamma^2(2d_1N+2\nu+1)}{\Gamma(2d_1N+\nu+1)\Gamma(2\nu+1)}, & m=0, \\ \displaystyle\sum_{j=1}^{m}(-1)^j\binom{2d_1N}{j}\frac{(2\nu+1)_{2d_1N}}{(\nu+1)_{2d_1N}}\frac{(2m-j-1)!j}{m!(m-j)!}\frac{\Gamma(2d_1N+2\nu+1+j)}{\Gamma(\nu+1+j)}, & m\geq 1. \end{cases} \tag{8}$$

Hence, for the multidimensional hypergeometric series of the second kind given by the formula

$$f(\mathbf{z}) = \sum_{k_1,k_2,\ldots,k_d=0}^{\infty} \frac{\Gamma\left(\sum_{i=1}^{d_1} k_i+\nu+1\right)\Gamma\left(\sum_{i=d_1+1}^{d} k_i+\nu+1\right)}{\Gamma(|\mathbf{k}|+2\nu+2)} z_1^{k_1}\ldots z_d^{k_d}, \tag{9}$$

it is possible to construct Padé-type approximants by using Theorem 1, namely, the following assertion is true:

Theorem 2. *For any $\mathbf{N}=(N,\ldots,N,M,\ldots,M)\in\mathbb{N}^d$, the rational function*

$$[\mathcal{M}/\mathcal{N}]_f(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(\mathbf{z})},$$

where

$$Q(\mathbf{z}) = \sum_{\mathbf{j}\in\Delta(\mathbf{N})} c^{(\mathbf{N})}_{\mathbf{N}-\mathbf{j}}\mathbf{z}^{\mathbf{j}},$$

$$P(\mathbf{z}) = \sum_{p=0}^{d-1}\sum_{\omega\in\Omega_p}\prod_{r=1}^{p} z_{l_r(\omega)}^{N_{l_r(\omega)}} \sum_{\substack{0\leq k_{m_i(\omega)}\leq N_{m_i(\omega)}-1, i=1,2,\ldots,d-p \\ \Phi_{\mathbf{N}}(\mathbf{k})\leq 0}} \mathbf{z}^{\mathbf{k}}$$

$$\times \sum_{\mathbf{j}\in\Delta(\boldsymbol{\delta}(\overline{\omega})\circ\mathbf{N}+\boldsymbol{\delta}(\omega)\circ\mathbf{k})} c_{\boldsymbol{\delta}(\omega)\circ\mathbf{N}+\boldsymbol{\varepsilon}(\omega)\circ\mathbf{j}}\, s_{\mathbf{k}+\boldsymbol{\varepsilon}(\omega)\circ\mathbf{j}},$$

$$\Phi_{\mathbf{N}}(\mathbf{k}) = k_1+k_2+\ldots+k_d-2d_1N+1,$$

the coefficients $c^{(\mathbf{N})}_{\mathbf{j}}$ are determined by relations (8), and the quantities $s_{\mathbf{k}}$ are given by relations (6), admits an expansion in a power series whose coefficients coincide with the coefficients of the Taylor–Maclaurin series for a function of the form (9) for all

$$\mathbf{k}\in\left\{\mathbf{k}\in\mathbb{Z}_+^d:\ |\mathbf{k}|\leq 4d_1N-1\right\}$$

and, hence, this rational function is a d-dimensional Padé-type approximant of function (9) of the order $[\mathcal{M}/\mathcal{N}]$, where

$$\mathcal{M} = \left\{\mathbf{k}\in\mathbb{Z}_+^d:\ |\mathbf{k}|\leq 4d_1N-1\right\}\setminus\left\{\mathbf{k}\in\mathbb{Z}_+^d:\ k_1\geq N_1,\ldots,k_d\geq N_d\right\}$$

and

$$\mathcal{N} = \Delta(\mathbf{N}).$$

REFERENCES

1. G. A. Baker, Jr., and P. Graves-Morris, *Padé Approximants*, Addison-Wesley, London (1981).
2. A. Cuyt, *Padé Approximants for Operators: Theory and Applications*, Springer, Berlin (1984).
3. A. P. Holub and L. O. Chernets'ka, "Two-dimensional generalized moment representations and rational approximations of functions of two variables," *Ukr. Mat. Zh.*, **65**, No. 8, 1035–1058 (2013); ***English translation:*** *Ukr. Math. J.*, **65**, No. 8, 1155–1179 (2014).
4. V. K. Dzyadyk, "On a generalization of the problem of moments," *Dop. Akad. Nauk URSR*, **6**, 8–12 (1981).
5. A. P. Holub, *Generalized Moment Representations and Padé Approximations* [in Ukrainian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv (2002).
6. A. P. Holub, "Method of generalized moment representations in the theory of rational approximation (a survey)," *Ukr. Mat. Zh.*, **55**, No. 3, 307–359 (2003); ***English translation:*** *Ukr. Math. J.*, **55**, No. 3, 377–433 (2003).
7. A. P. Holub and L. O. Chernets'ka, "Many-dimensional generalized moment representations and Padé-type approximants for functions of many variables," *Ukr. Mat. Zh.*, **66**, No. 9, 1166–1174 (2014); ***English translation:*** *Ukr. Math. J.*, **66**, No. 9, 1302–1311 (2015).
8. P. K. Suetin, *Classical Orthogonal Polynomials* [in Russia], Nauka, Moscow (1979).
9. A. P. Holub and L. O. Chernets'ka, "Construction of Padé approximations for some hypergeometric Appell series by the method of generalized moment representations," in: *Approximation Theory and Related Problems* [in Ukrainian], Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, **11**, No. 3 (2014), pp. 78–103.
10. T. M. Sadykov and A. K. Tsikh, *Hypergeometric and Algebraic Functions of Many Variables* [in Russian], Nauka, Moscow (2014).
11. M. Abramowitz and I. A. Stegun (editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* [Russian translation], Nauka, Moscow (1979).