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MANY-DIMENSIONAL GENERALIZED MOMENT REPRESENTATIONS AND PADÉ-TYPE APPROXIMANTS FOR FUNCTIONS OF MANY VARIABLES

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We propose an approach to the construction of multidimensional Padé-type approximants for analytic functions based on the extension of Dzyadyk's method of generalized moment representations.

In 1981, Dzyadyk proposed a method of generalized moment representations for the construction and investigation of the Padé approximants for numerous important classes of special functions [1]. The subsequent generalizations of this method made it possible to use it for the study of various generalizations of Padé approximants, e.g., multipoint Padé approximants, Padé–Chebyshev approximants, consistent Padé approximants, etc. (see [2]). In [3–5], we proposed a definition of two- and three-dimensional generalized moment representations and illustrated their applications to the construction of rational Padé-type approximants for functions of two and three variables.

In the present paper, we generalize this approach to the case of an arbitrary dimension $d \geqslant 2$.

We introduce the following definition:

Definition 1. *We say that a d-dimensional number sequence $\{s_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}$ has a generalized moment representation on the product of linear spaces $\mathscr{X}$ and $\mathscr{Y}$ for the bilinear form $\langle .,. \rangle$ on this product if a d-dimensional sequence of elements $\{x_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{Z}_+^d}$ is defined in the space $\mathscr{X}$ and a d-dimensional sequence of elements $\{y_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}_+^d}$ is defined in the space $\mathscr{Y}$ so that*

$$s_{\mathbf{k}+\mathbf{j}} = \langle x_{\mathbf{k}}, y_{\mathbf{j}} \rangle, \qquad \mathbf{k},\, \mathbf{j} \in \mathbb{Z}_+^d. \tag{1}$$

Consider a formal power series in d variables

$$f(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}_+^d} s_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \tag{2}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_d) \in \mathbb{C}^d$, $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}_+^d$, and $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2} \ldots z_d^{k_d}$.

For the sake of convenience, we introduce the following notation:

For $p = 0, 1, \ldots, d$, we introduce a set

$$\Omega_p = \{\omega \subseteq \{1, 2, \ldots, d\} \colon |\omega| = p\}.$$

Further, we arrange elements of each set $\omega \in \Omega_p$,

$$\omega = \{l_1(\omega), l_2(\omega), \ldots, l_p(\omega)\}$$

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so that

$$1 \leqslant l_1(\omega) < l_2(\omega) < \ldots < l_p(\omega) \leqslant d.$$

We also arrange elements of the complement $\overline{\omega} = \{1, 2, \ldots, d\} \setminus \omega$,

$$\overline{\omega} = \big\{m_1(\omega), m_2(\omega), \ldots, m_{d-p}(\omega)\big\} \in \Omega_{d-p},$$

so that

$$1 \leqslant m_1(\omega) < m_2(\omega) < \ldots < m_{d-p}(\omega) \leqslant d.$$

For each set $\omega \in \Omega_p, p = 0, 1, \ldots, d,$ we introduce the notation

$$\boldsymbol{\delta}(\omega) = (\delta_1(\omega), \delta_2(\omega), \ldots, \delta_d(\omega)),$$

where

$$\delta_i(\omega) = \begin{cases} 0 & \text{for} \quad i \in \omega, \\ 1 & \text{for} \quad i \notin \omega, \end{cases}$$

$$\boldsymbol{\varepsilon}(\omega) = (\varepsilon_1(\omega), \varepsilon_2(\omega), \ldots, \varepsilon_d(\omega)),$$

$$\varepsilon_i(\omega) = \begin{cases} -1 & \text{for} \quad i \in \omega, \\ 1 & \text{for} \quad i \notin \omega, \end{cases}$$

so that

$$\delta_i(\omega) = \frac{\varepsilon_i(\omega) + 1}{2}, \quad i = 1, 2, \ldots, d.$$

We also denote

$$\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{Z}_+^d, \qquad \mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}_+^d,$$

so that

$$\mathbf{1} = \boldsymbol{\delta}(\varnothing), \qquad \mathbf{0} = \boldsymbol{\delta}(\{1, 2, \ldots, d\}).$$

For each vector $\mathbf{a}, \mathbf{b} \in \mathbb{Z}_+^d,$ $\mathbf{a} = (a_1, a_2, \ldots, a_d),$ $\mathbf{b} = (b_1, b_2, \ldots, b_d),$ by $\mathbf{a} \circ \mathbf{b},$ we denote the coordinate-by-coordinate product of two vectors $\mathbf{a}$ and $\mathbf{b}$:

$$\mathbf{a} \circ \mathbf{b} = (a_1 b_1, a_2 b_2, \ldots, a_d b_d).$$

For each vector $\mathbf{a} \in \mathbb{Z}_+^d,$ we denote

$$\Delta(\mathbf{a}) = \Big\{\mathbf{j} = (j_1, j_2, \ldots, j_d) \in \mathbb{Z}_+^d \colon j_i \in \{0, 1, \ldots, a_i\},\ i = 1, 2, \ldots, d\Big\}.$$

By using this notation, we establish the following fact, which enables us to construct d-dimensional Padé-type approximants for series (2) whose coefficients satisfy representations of the form (1). For the dimensions $d = 2$ and $d = 3$, the corresponding results were obtained in [3–5]. A survey of the results obtained for various multidimensional analogs of Padé approximations can be found in [6, pp. 323–332].

Theorem 1. *Assume that the coefficients of a formal power series of the form (2) admit a generalized moment representation of the form (1). If, for some* $\mathbf{N} = (N_1, N_2, \ldots, N_d) \in \mathbb{N}^d$, *there exists a generalized polynomial*

$$Y_{\mathbf{N}} = \sum_{\mathbf{j}\in\Delta(\mathbf{N})} c_{\mathbf{j}}^{(\mathbf{N})} y_{\mathbf{j}} \tag{3}$$

such that $c_{\mathbf{N}}^{(\mathbf{N})} \neq 0$ *and the condition of biorthogonality*

$$\langle x_{\mathbf{k}}, Y_{\mathbf{N}} \rangle = 0 \tag{4}$$

holds for $\mathbf{k} = \Delta(\mathbf{N}) \setminus \{\mathbf{N}\}$, *then the rational function*

$$[\mathscr{M}/\mathscr{N}]_f(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(\mathbf{z})}, \tag{5}$$

where

$$Q(\mathbf{z}) = \sum_{\mathbf{j}\in\Delta(\mathbf{N})} c_{\mathbf{N}-\mathbf{j}}^{(\mathbf{N})} \mathbf{z}^{\mathbf{j}}, \tag{6}$$

and

$$P(\mathbf{z}) = \sum_{p=0}^{d-1} \sum_{\omega\in\Omega_p} \prod_{r=1}^{p} z_{l_r(\omega)}^{N_{l_r(\omega)}} \sum_{\mathbf{k}\in\Delta(\mathbf{N}-\boldsymbol{\delta}(\omega)\circ\mathbf{1})} \mathbf{z}^{\mathbf{k}} \sum_{\mathbf{j}\in\Delta(\boldsymbol{\delta}(\overline{\omega})\circ\mathbf{N}+\boldsymbol{\delta}(\omega)\circ\mathbf{k})} c_{\boldsymbol{\delta}(\omega)\circ\mathbf{N}+\boldsymbol{\varepsilon}(\omega)\circ\mathbf{j}} s_{\mathbf{k}+\boldsymbol{\varepsilon}(\omega)\circ\mathbf{j}}, \tag{7}$$

can be expanded in a power series whose coefficients coincide with the coefficients of series (2) for all $\mathbf{k} \in \Delta(2\mathbf{N}) \setminus \big\{(2N_1, 2N_2, \ldots, 2N_d)\big\}$ *and, hence, this rational function is the* d*-dimensional Padé-type approximant of series (2) of the order* $[\mathscr{M}/\mathscr{N}]$, *where*

$$\mathscr{M} = \Delta(2\mathbf{N}) \setminus \prod_{i=1}^{d} \{N_i, N_i + 1, \ldots, 2N_i\}$$

and $\mathscr{N} = \Delta(\mathbf{N})$.

Proof. We fix a vector $\mathbf{K} \in \mathbb{Z}_+^d$ with sufficiently large coordinates $K_i \in \mathbb{Z}_+,\ i = 1, 2, \ldots, d$. We multiply each equality in (1) by $\mathbf{z}^{\mathbf{k}}$ and sum these equalities over all $\mathbf{k} \in \Delta(\mathbf{K})$. On the left-hand side, we get

$$\sum_{\mathbf{k}\in\Delta(\mathbf{K})} s_{\mathbf{k}+\mathbf{j}} \mathbf{z}^{\mathbf{k}} = \mathbf{z}^{-\mathbf{j}} \sum_{\mathbf{k}-\mathbf{j}\in\Delta(\mathbf{K})} s_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}$$

$$= \mathbf{z}^{-\mathbf{j}} \left\{ f(\mathbf{z}) - \sum_{p=0}^{d-1} \sum_{\omega\in\Omega_p} \sum_{\mathbf{k}\in\prod_{i=1}^{d}[\delta_i(\overline{\omega})j_i, j_i+\delta_i(\overline{\omega})K_i-\delta_i(\omega)]} s_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} \right\} - \mathbb{E}_{\mathbf{k},\mathbf{j}}(\mathbf{z}), \tag{8}$$

where

$$\mathbb{E}_{\mathbf{k},\mathbf{j}}(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}_+^d\setminus\Delta(\mathbf{K})} e_{\mathbf{k},\mathbf{j}}\mathbf{z}^{\mathbf{k}}.$$

We now multiply the equalities obtained as a result by $c_{\mathbf{j}}^{(\mathbf{N})}$ and find their sum over $\mathbf{j}\in\Delta(\mathbf{N})$. In view of (8), we obtain

$$\sum_{\mathbf{j}\in\Delta(\mathbf{N})} c_{\mathbf{j}}^{(\mathbf{N})}\left\{\mathbf{z}^{-\mathbf{j}}\left\{f(\mathbf{z}) - \sum_{p=0}^{d-1}\sum_{\omega\in\Omega_p}\sum_{\mathbf{k}\in\prod_{i=1}^d[\delta_i(\overline{\omega})j_i,j_i+\delta_i(\overline{\omega})K_i-\delta_i(\omega)]} s_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}\right\} - \mathbb{E}_{\mathbf{k},\mathbf{j}}(\mathbf{z})\right\}$$

$$= \mathbf{z}^{-\mathbf{N}}\left\{f(\mathbf{z})\sum_{\mathbf{j}\in\Delta(\mathbf{N})} c_{\mathbf{N}-\mathbf{j}}^{(\mathbf{N})}\mathbf{z}^{\mathbf{j}} - \sum_{p=0}^{d-1}\sum_{\omega\in\Omega_p}\sum_{\mathbf{j}\in\Delta(\mathbf{N})} c_{\mathbf{j}}^{(\mathbf{N})}\mathbf{z}^{\mathbf{N}-\mathbf{j}}\sum_{\mathbf{k}\in\prod_{i=1}^d[\delta_i(\overline{\omega})j_i,j_i+\delta_i(\overline{\omega})K_i-\delta_i(\omega)]} s_{\mathbf{k}}\mathbf{z}^{\mathbf{k}}\right\} - \widetilde{\mathbb{E}}_{\mathbf{K},\mathbf{N}}(\mathbf{z}),$$

where

$$\widetilde{\mathbb{E}}_{\mathbf{K},\mathbf{N}}(\mathbf{z}) = \sum_{\mathbf{k}\in\mathbb{Z}_+^d\setminus\Delta(\mathbf{K})} \widetilde{e}_{\mathbf{K},\mathbf{N}}\mathbf{z}^{\mathbf{k}}.$$

To get the final form of numerator (7), we use the following lemma:

Lemma 1. *For any $N, K\in\mathbb{N}$, an arbitrary sequence $\{s_k\}_{k=0}^{\infty}$, any collection of numbers $\{c_j^{(N)}\}_{j=0}^{N}$, and any $z\in\mathbb{C}$, the following identities are true:*

$$\sum_{j=1}^{N} c_j z^{N-j}\sum_{k=0}^{j-1} s_k z^k = \sum_{k=0}^{N-1} z^k\sum_{j=0}^{k} c_{N-j}s_{k-j},$$

$$\sum_{j=0}^{N} c_j z^{N-j}\sum_{k=j}^{j+K} s_k z^k = z^N\sum_{k=0}^{K} z^k\sum_{j=0}^{N} c_j s_{k+j}.$$

To prove the lemma, it suffices to perform elementary changes of variables under the signs of summation and change the order of summation.

By using Lemma 1, for sufficiently large coordinates of the vector $\mathbf{K}$, we can prove the validity of relation (7).

As in the cases $d = 2, 3$ [3–5], Theorem 1 can be generalized if we choose a generalized polynomial $Y_{\mathbf{N}}$ from the conditions of biorthogonality of the form (4) to the elements $x_{\mathbf{k}}$ not for $\mathbf{k}\in\Delta(\mathbf{N})\setminus\{\mathbf{N}\}$ but for $\mathbf{k}\in\mathscr{H}_{\mathbf{N}}$, where $\mathscr{H}_{\mathbf{N}}$ is a subset of $\mathbb{Z}_+^d$ containing exactly

$$\prod_{i=1}^{d}(N_i+1) - 1$$

elements. To formulate the corresponding assertion, we consider a continuously differentiable function

$$\Phi(\mathbf{x})\colon \mathbb{R}_+^d \to \mathbb{R}$$

with the following properties:

(i) the set $\mathscr{D}_\Phi = \{\mathbf{x} \in \mathbb{R}^d_+ \colon \Phi_{\mathbf{N}}(\mathbf{x}) \leqslant 0\}$ is bounded in $\mathbb{R}^d_+$;

(ii) the cardinality of the set $\mathscr{D}_\Phi \bigcap \{\mathbf{x} \in \mathbb{Z}^d_+ \colon x_i \geqslant N_i,\ i = 1, 2, \ldots, d\}$ is equal to

$$\prod_{i=1}^{d} (N_i + 1) - 1\,;$$

(iii) for all $i = 1, 2, \ldots, d,$ single-valued functions $x_i = \varphi_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$ exist for

$$(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in D_i := \Big\{(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{R}^{d-1}_+ \colon \exists x_i \in \mathbb{R}_+ \text{ such that } \Phi(x_1, x_2, \ldots, x_d) \leqslant 0\Big\};$$

(iv) for any $i = 1, 2, \ldots, d,$

$$\varphi_i(x_1,\, x_2,\, \ldots,\, x_{i-1},\, x_{i+1},\, \ldots,\, x_d) \geqslant N_i$$

for all values $(x_1,\, x_2,\, \ldots,\, x_{i-1},\, x_{i+1},\, \ldots,\, x_d) \in D_i.$

In this case, by using the scheme of the proof of Theorem 1, we establish the following assertion:

Theorem 1′. *Assume that, under the conditions of Theorem 1, for some $\mathbf{N} \in \mathbb{N}^d$, there exists a generalized polynomial of the form (3) such that $c^{(\mathbf{N})}_{\mathbf{N}} \neq 0$ and that the conditions of biorthogonality (4) for $\mathbf{k} \in \{\mathbf{k} \in \mathbb{Z}^d_+ \colon \mathbf{k} + \mathbf{N} \in \mathscr{D}_\Phi\}$ are satisfied. Then a rational function of the form (5), where $Q(\mathbf{z})$ has the form (6) and*

$$P(\mathbf{z}) = \sum_{p=0}^{d-1} \sum_{\omega \in \Omega_p} \prod_{r=1}^{p} z_{l_r(\omega)}^{N_{l_r(\omega)}} \sum_{\substack{0 \leqslant k_{m_i(\omega)} \leqslant N_{m_i(\omega)} - 1, \\ i = 1, 2, \ldots, d-p, \\ \Phi(\mathbf{k}) \leqslant 0}} \mathbf{z}^{\mathbf{k}} \sum_{\mathbf{j} \in \Delta(\boldsymbol{\delta}(\overline{\omega}) \circ \mathbf{N} + \boldsymbol{\delta}(\omega) \circ \mathbf{k})} c_{\boldsymbol{\delta}(\omega) \circ \mathbf{N} + \boldsymbol{\varepsilon}(\omega) \circ \mathbf{j}}\, s_{\mathbf{k} + \boldsymbol{\varepsilon}(\omega) \circ \mathbf{j}},$$

admits an expansion in a power series whose coefficients coincide with the coefficients of series (2) for all $\mathbf{k} \in \mathscr{D}_\Phi \cap \mathbb{Z}^d_+$ and, hence, this rational function is a d-dimensional approximation of the Padé type of series (2) of order $[\mathscr{M}/\mathscr{N}]$, where

$$\mathscr{M} = \mathscr{D}_\Phi \cap \mathbb{Z}^d_+ \setminus \{\mathbf{x} \in \mathbb{Z}^d_+ \colon x_i \geqslant N_i,\ i = 1, 2, \ldots, d\}$$

and $\mathscr{N} = \Delta(\mathbf{N})$.

In the case where the linear spaces $\mathscr{X}$ and $\mathscr{Y}$ are normed, the bilinear form $\langle\, .\,,.\, \rangle$ is separately continuous (see, e.g., [7, p. 63]), pairwise commuting bounded linear operators $A_i \colon \mathscr{X} \to \mathscr{X},\ i = 1, 2, \ldots, d,$ such that

$$A_i x_{\mathbf{k}} = x_{\mathbf{k} + \mathbf{e}_i}, \quad i = 1, 2, \ldots, d,$$

for each $\mathbf{k} \in \mathbb{Z}^d_+$, where

$$\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0) = \mathbf{1} - \boldsymbol{\delta}(\{i\}), \quad i = 1, 2, \ldots, d,$$

are defined in the space $\mathscr{X}$, and in addition, bounded linear operators $A_i^\star\colon \mathscr{Y} \to \mathscr{Y},\ i = 1, 2, \ldots, d,$ adjoint to the operators $A_i,\ i = 1, 2, \ldots, d,$ with respect to the bilinear form $\langle .,. \rangle$ exist in the space $\mathscr{Y}$ (see [2, p. 18]), under the conditions of Theorem 1, we get the following formula for the error of approximation:

$$f(\mathbf{z}) - [\mathscr{M}/\mathscr{N}]_f(\mathbf{z}) = \frac{1}{Q(\mathbf{z})} \left\{ \prod_{i=1}^{d} z_i^{N_i} \left\langle \prod_{r=1}^{d} \widehat{R}_{z_r}(A_r) x_{\mathbf{0}}, Y_{\mathbf{N}} \right\rangle \right.$$

$$\left. + \sum_{p=0}^{d-1} \sum_{\omega \in \Omega_p} \prod_{r=1}^{p} z_{l_r(\omega)}^{N_{l_r(\omega)}} \sum_{\mathbf{k} \in \pi^{(\mathbf{N})}(\omega)} \mathbf{z}^{\mathbf{k}} \sum_{\mathbf{j} \in \Delta(\boldsymbol{\delta}(\overline{\omega}) \circ \mathbf{N} + \boldsymbol{\delta}(\omega) \circ \mathbf{k})} c_{\boldsymbol{\delta}(\omega) \circ \mathbf{N} + \boldsymbol{\varepsilon}(\omega) \circ \mathbf{j}} s_{\mathbf{k} + \boldsymbol{\varepsilon}(\omega) \circ \mathbf{j}} \right\},$$

where

$$\pi^{(\mathbf{N})}(\omega) = \left(\pi_1^{(\mathbf{N})}(\omega), \pi_2^{(\mathbf{N})}(\omega), \ldots, \pi_d^{(\mathbf{N})}(\omega) \right),$$

and

$$\pi_i^{(\mathbf{N})}(\omega) = \begin{cases} [0, N_i - 1] & \text{for} \quad i \notin \omega, \\ [N_i + 1, \infty] & \text{for} \quad i \in \omega. \end{cases}$$

Under the conditions of Theorem $1'$, this relation takes the form

$$f(\mathbf{z}) - [\mathscr{M}/\mathscr{N}]_f(\mathbf{z}) = \frac{1}{Q(\mathbf{z})} \left\{ \prod_{i=1}^{d} z_i^{N_i} \left\langle \prod_{r=1}^{d} \widehat{R}_{z_r}(A_r) x_{\mathbf{0}}, Y_{\mathbf{N}} \right\rangle + \sum_{p=0}^{d-1} \sum_{\omega \in \Omega_p} \prod_{r=1}^{p} z_{l_r(\omega)}^{N_{l_r(\omega)}} \right.$$

$$\left. \times \sum_{\substack{0 \leqslant k_{m_i(\omega)} \leqslant N_{m_i(\omega)} - 1, \\ i = 1, 2, \ldots, d-p, \\ \Phi(\mathbf{k}) > 0}} \mathbf{z}^{\mathbf{k}} \sum_{\mathbf{j} \in \Delta(\boldsymbol{\delta}(\overline{\omega}) \circ \mathbf{N} + \boldsymbol{\delta}(\omega) \circ \mathbf{k})} c_{\boldsymbol{\delta}(\omega) \circ \mathbf{N} + \boldsymbol{\varepsilon}(\omega) \circ \mathbf{j}} s_{\mathbf{k} + \boldsymbol{\varepsilon}(\omega) \circ \mathbf{j}} \right\}.$$

Remark. By $\widehat{R_z}(A)$, we denote the resolvent function of the operator $A\colon \widehat{R_z}(A) = (I - zA)^{-1}$, where $I\colon \mathscr{X} \to \mathscr{X}$ is the identity operator.

We now consider the case where all operators $A_i,\ i = 1, 2, \ldots, d,$ coincide, i.e.,

$$A_1 = A_2 = \ldots = A_d = A.$$

Then the approximated function takes the form

$$f(\mathbf{z}) = \left\langle \prod_{i=1}^{d} \widehat{R}_{z_i}(A) x_{\mathbf{0}}, y_{\mathbf{0}} \right\rangle. \tag{9}$$

We have established the following result:

Lemma 2. *Any function of the form (9) admits the representation*

$$f(\mathbf{z}) = \frac{1}{\prod\limits_{s<t}(z_s - z_t)} \sum_{i=1}^{d} z_i^{d-1}(-1)^{i+1} \prod_{\substack{s<t \\ s,t\neq i}} (z_s - z_t) g(z_i), \tag{10}$$

where

$$g(z) = \langle \widehat{R}_z(A)x_{\mathbf{0}}, y_{\mathbf{0}} \rangle.$$

This lemma can be easily proved by induction.

We now assume that the operator A is an operator of multiplication by an independent variable in the space $L_2([0,1], d\mu)$, where μ is a nondecreasing function with infinitely many points of increase on $[0,1]$:

$$(A\varphi)(t) = t\varphi(t).$$

Also let $\mathbf{N} = (N, N, \ldots, N) \in \mathbb{N}^d$. Thus, in order to construct the d-dimensional Padé-type approximants for a function $f(\mathbf{z})$ of the form (10) with

$$g(z) = \int\limits_0^1 \frac{d\mu(t)}{1 - zt}$$

according to Theorems 1 and $1'$, it is necessary to construct the polynomials

$$Y_{\mathbf{N}}(t) = \sum_{\mathbf{j}\in\Delta(\mathbf{N})} c_{\mathbf{j}}^{(\mathbf{N})} t^{|\mathbf{j}|}, \qquad \text{where} \quad |\mathbf{j}| = j_1 + j_2 + \ldots + j_d, \tag{11}$$

satisfying the conditions of biorthogonality

$$\int\limits_0^1 t^{|\mathbf{k}|} Y_{\mathbf{N}}(t) d\mu(t) = 0 \tag{12}$$

for $\mathbf{k} \in \Delta(\mathbf{N}) \setminus \{\mathbf{N}\}$. This is equivalent to the equality

$$\int\limits_0^1 t^k Y_{\mathbf{N}}(t) d\mu(t) = 0 \tag{13}$$

for $k = 0, 1, \ldots, dN - 1$.

It follows from (13) that, to within a constant factor, the polynomial $Y_{\mathbf{N}}(t)$ must coincide with the algebraic polynomial

$$P_{dN}(t) = \sum_{j=0}^{dN} p_j^{(dN)} t^j \tag{14}$$

orthonormal on $[0,1]$ with the measure $d\mu(t)$.

Since condition (12) is satisfied not only for $\mathbf{k} \in \Delta(\mathbf{N}) \setminus \{\mathbf{N}\}$ but also for $\mathbf{k} \in \{\mathbf{k} \in \mathbb{Z}_+^d : |\mathbf{k}| \leqslant 2dN - 1\}$, it is necessary to choose the function $\Phi(x_1, x_2, \ldots, x_d)$ from the statement of Theorem $1'$ in the form of a function

$$\Phi(x_1, x_2, \ldots, x_d) = |\mathbf{x}| - 2dN + 1.$$

In this case, the functions $\varphi_i \colon \mathbb{R}_+^{d-1} \to \mathbb{R}$ take the form

$$\varphi_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) = 2dN - 1 - x_1 - x_2 - \ldots - x_{i-1} - x_{i+1} - \ldots - x_d.$$

Equality (11) enables us to find the coefficients

$$c_{\mathbf{j}}^{(\mathbf{N})}, \quad \mathbf{j} \in \Delta(\mathbf{N}),$$

by using the coefficients $p_j^{(dN)}$, $j = 0, 1, \ldots, dN$. However, this can be done in infinitely many ways. For the sake of definiteness, we assume that $c_{\mathbf{j}}^{(\mathbf{N})} = c_{\mathbf{k}}^{(\mathbf{N})}$ for $|\mathbf{j}| = |\mathbf{k}|$.

Thus, it is possible to establish the following auxiliary result:

Lemma 3. *Let $N \in \mathbb{N}$ and $0 \leqslant j \leqslant dN$. Then the number of vectors $\mathbf{k} \in \mathbb{Z}_+^d$ such that $k_i \leqslant N$, $i = 1, 2, \ldots, d$, and $|\mathbf{k}| = j$, is equal to*

$$\gamma_j^{(N)} = \sum_{r=0}^{\left[\frac{j}{N+1}\right]} \binom{d}{r} (-1)^r \frac{(d + j - r(N+1) - 1)!}{(d-1)!(j - r(N+1))!}.$$

Proof. Consider a polynomial

$$V_{\mathbf{N}}(\mathbf{z}) = \sum_{\mathbf{k} \in \Delta(\mathbf{N})} \mathbf{z}^{\mathbf{k}}.$$

It is clear that

$$V_{\mathbf{N}}(\mathbf{z}) = \prod_{i=1}^{d} \left(\sum_{k=0}^{N} z_i^k \right) = \prod_{i=1}^{d} \frac{1 - z_i^{N+1}}{1 - z_i}.$$

Setting $z_1 = z_2 = \ldots = z_d = z$, we obtain

$$\frac{(1 - z^{N+1})^d}{(1 - z)^d} = \sum_{j=0}^{dN} \gamma_j^{(N)} z^j.$$

By using the decompositions

$$(1 - z^{N+1})^d = \sum_{m=0}^{d} \binom{d}{m} (-1)^m z^{m(N+1)}$$

and

$$\frac{1}{(1-z)^d} = \sum_{k=0}^{\infty} \frac{(d+k-1)!}{(d-1)!k!} z^k,$$

we prove the lemma.

It follows from Lemma 3 that the coefficients $c_{\mathbf{j}}^{(\mathbf{N})},\ \mathbf{j} \in \Delta(\mathbf{N}),$ of the polynomials $Y_{\mathbf{N}}$ must be chosen in the form

$$c_{\mathbf{j}}^{(\mathbf{N})} = \frac{p_{|\mathbf{j}|}^{(dN)}}{\gamma_{|\mathbf{j}|}^{(N)}} = \frac{p_{|\mathbf{j}|}^{(dN)}}{\sum_{r=0}^{\left[\frac{|\mathbf{j}|}{N+1}\right]} \binom{d}{r} (-1)^r \dfrac{\big(d + |\mathbf{j}| - r(N+1) - 1\big)!}{(d-1)!\big(|\mathbf{j}| - r(N+1)\big)!}}. \tag{15}$$

By using these arguments and Theorem $1'$, we arrive at the following assertion:

Theorem 2. *For every $N \in \mathbb{N}$, the rational function*

$$[\mathscr{M}/\mathscr{N}]_f(\mathbf{z}) = \frac{P(\mathbf{z})}{Q(\mathbf{z})},$$

where

$$Q(\mathbf{z}) = \sum_{\mathbf{j} \in \Delta(\mathbf{N})} c_{\mathbf{N}-\mathbf{j}}^{(\mathbf{N})} \mathbf{z}^{\mathbf{j}},$$

$$P(\mathbf{z}) = \sum_{p=0}^{d-1} \sum_{\omega \in \Omega_p} \prod_{r=1}^{p} z_{l_r(\omega)}^{N_{l_r(\omega)}} \sum_{\substack{0 \leqslant k_{m_i(\omega)} \leqslant N-1, \\ i=1,2,\ldots,d-p, \\ |\mathbf{k}| \leqslant 2dN-1}} \mathbf{z}^{\mathbf{k}} \sum_{\mathbf{j} \in \Delta(\boldsymbol{\delta}(\overline{\omega}) \circ \mathbf{N} + \boldsymbol{\delta}(\omega) \circ \mathbf{k})} c_{\boldsymbol{\delta}(\omega) \circ \mathbf{N} + \boldsymbol{\varepsilon}(\omega) \circ \mathbf{j}}\, s_{\sum_{i=1}^{d-p} j_{m_i} + |\mathbf{k}| - \sum_{i=1}^{p} j_{l_i}},$$

$$\mathbf{N} = (N, N, \ldots, N),$$

the coefficients $c_{\mathbf{j}}^{(\mathbf{N})}$ are determined by relations (15), and $s_k = \int_0^1 t^k d\mu(t)$, can be expanded in a power series whose coefficients coincide with the coefficients of the Taylor–Maclaurin series for a function $f(\mathbf{z})$ of the form (10) for all $\mathbf{k} \in \big\{\mathbf{k} \in \mathbb{Z}_+^d \colon |\mathbf{k}| \leqslant 2dN - 1\big\}$ and, hence, this rational function plays the role of d-dimensional Padé-type approximant for a function of the form (10) of order $[\mathscr{M}/\mathscr{N}]$, where $\mathscr{M} = \big\{\mathbf{k} \in \mathbb{Z}_+^d \colon |\mathbf{k}| \leqslant 2dN - 1\big\}$ and $\mathscr{N} = \Delta(\mathbf{N})$.

If $d\mu(t) = t^\nu (1-t)^\sigma dt,\ \nu, \sigma > -1,$ then function (10) is a special case of the d-dimensional Lauricella function

$$F_D(a, b_1, \ldots, b_d; c; z_1, \ldots, z_d) = \sum_{\mathbf{k} \in \mathbb{Z}_+^d} \frac{(a)_{|\mathbf{k}|} (b_1)_{k_1} \ldots (b_d)_{k_d}}{c_{|\mathbf{k}|} k_1! \ldots k_d!} {z_1}^{k_1} \ldots {z_d}^{k_d}$$

for $a = \nu + 1,\ b_1 = b_2 = \ldots = b_d = 1,$ and $c = \rho + \nu + 2$ (see [8; 9, p. 33]).

In this case, polynomials (14) are shifted Jacobi polynomials orthonormal on $[0, 1]$ with weight $t^\nu (1-t)^\sigma$ and their coefficients can be found in the explicit form (see [10, p. 581]). Hence, by Theorem 2, we obtain the explicit form of the d-dimensional Padé-type approximations for the corresponding functions.

REFERENCES

1. V. K. Dzyadyk, "On a generalization of the problem of moments," *Dop. Akad. Nauk Ukr. RSR*, **6**, 8–12 (1981).
2. A. P. Holub, *Generalized Moment Representations and Padé Approximations* [in Ukrainian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv (2002).

3. A. P. Holub and L. O. Chernets'ka, "Two-dimensional generalized moment representations and rational approximations of functions of two variables," *Ukr. Mat. Zh.*, **65**, No. 8, 1035–1058 (2013); ***English translation:*** *Ukr. Math. J.*, **65**, No. 8, 1155–1179 (2014).
4. L. O. Chernets'ka, "Three-dimensional generalized moment representations and Padé-type approximations for functions of three variables," *Dop. Nats. Akad. Nauk Ukr.*, No. 7, 36–42 (2014).
5. A. P. Holub and L. O. Chernets'ka, "Construction of Padé approximations for some hypergeometric Lauricella series by the method of generalized moment representations," *Proc. of the Institute of Mathematics, Ukrainian National Academy of Sciences* [in Ukrainian], Kyiv, **11**, No. 3 (2014), pp. 78–103.
6. G. A. Baker, Jr., and P. Graves-Morris, *Padé Approximants* [Russian translation], Mir, Moscow (1986).
7. W. Rudin, *Functional Analysis* [Russian translation], Nauka, Moscow (1975).
8. G. Lauricella, "Sulle funzioni ipergeometriche a piú variabili," *Rend. Circ. Mat. Palermo*, No. 7, 111–158 (1983).
9. H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Wiley, New York (1985).
10. M. Abramowitz and I. A. Stegun (editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* [Russian translation], Nauka, Moscow (1979).