DZYDYK'S RESEARCH ON THE THEORY OF APPROXIMATION OF FUNCTIONS OF A COMPLEX VARIABLE

V. I. Belyi, A. P. Golub,
and I. A. Shevchuk

UDC 517.53

Dzydyk's studies in the area of mathematical analysis and its applications are well known to a broad group of specialists. In the present article, we will attempt to describe basically those of his results that belong to the constructive theory of functions of a complex variable. We are fully justified in considering Dzydyk's principal papers in this area as fundamental; in the 30 years following publication of his first articles [5-7] of the approximation of continuous functions of a complex variable in closed domains with angles, the concepts presented in these articles defined the direction of the research of the mathematical school he created as well as that of many mathematicians in the Soviet Union and abroad on the solution of the intricate and difficult problem of direct and inverse theorems in the constructive theory of functions of a complex variable.

Dzydyk obtained a number of profound and conclusive results in other areas of complex analysis, for example, analytic and harmonic transformations, limiting values of Cauchy-type integrals, Dirichlet series, Pade approximation, and the problem of moments, and also strengthened several classical results.

One monograph that has achieved appreciable renown is [1], a paper which exerted a major influence on the development of the scientific interest of specialists on the theory of approximation of functions as well as novice mathematical researchers.

I. Direct and Inverse Theorems of Polynomial Approximation of Functions on Sets in the Complex Plane. Here Dzydyk is credited with the creation of a theory that established a relationship between the structural properties of functions defined on closed sets in the complex plane and the rate of approximation of these functions by n-th order polynomials. The principal reference point for the construction of this theory were the studies of Jackson, Bernshtein, and Valleé-Poussin in the theory of approximation of periodic functions by trigonometric polynomials and their analog, the theory of approximation of functions of algebraic real polynomials continuous on the closed interval [a, b], a theory that had assumed a finished form by the end of the 1950's as a result of the studies of Nikol'skii [2], Dzydyk [3], and Timan [4]. Even for the Hölder classes H^α, $\alpha \in (0, 1)$ (a typical instance forming a constructive characteristic of these classes in the periodic case) an improvement in the approximation is observed in the nonperiodic case at the endpoints of the interval and the constructive characteristic cannot be given in terms of best approximations of $E_n(f)$.

The transition from the periodic to the nonperiodic case on $[-1, 1]$ was necessary in order to replace uniform bounds of the rate of convergence of the approximation of functions to zero expressed in terms of $1/n$, $n \to \infty$, by bounds expressed in terms of the function $\varphi(n, x) =: \sqrt{1-x^2}/n + n^{-2}$, which depend on the position of the point x in $[-1, 1]$. The necessary form of the function $\varphi(n, x)$ was found as a result of lengthy studies, though before Dzydyk's studies on polynomial approximation in the complex plane [5, 6] there had been no convincing explanation as to why it is precisely this function which is critical for deriving a constructive characteristic of the Hölder classes on $[-1, 1]$. Dzydyk [5] "deciphered" this function as a realizing order of the distance $\rho_{1+1/n}(z)$ from the point z of $[-1, 1]$ (interpreted as a set in the complex plane) to the n-th level line of the Green function of the exterior of the closed interval $[-1, 1]$ (image of a circle of radius $1 + 1/n$ under a conformal mapping of the exterior of the unit circle to the exterior of the continuum). This assertion incorporated a daring assumption, i.e., that it is precisely in terms of the function $\rho_{1+1/n}(z)$ that it is possible to create a complete constructive description of the most important classes of

Institute of Applied Mathematics and Mechanics, Academy of Sciences of the Ukrainian SSR, Donetsk. Institue of Mathematics, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 41, No. 4, pp. 441-454, April, 1989. Original article submitted December 13, 1988.

 0041-5995/89/4104-0384$12.50

functions of a complex variable analytic at interior points of the continuum* $\mathfrak{M} \subset \mathbb{C}$ and continuous on $\mathfrak{M}$, i.e., to prove the direct and inverse theorems of the constructive theory of functions in the complex plane. Note that in the narrowest formulation, where it is necessary to explain the special role of the endpoints of the interval [−1, 1] as discerned from a constructive characteristic of Hölder classes defined on them by the geometric properties of the interval interpreted as a set in $\mathbb{C}$, this problem had been posed by Nikol'skii at the Third All-Union Mathematical Congress (1956). The constructive theory of functions on closed sets of the complex plane that was created as a result of the solution of this problem extended far beyond the scope of Nikol'skii problem. Dzydyk is credited with a major contribution to the creation of the groundwork of this theory [5-12]. For its further development right through its contemporary level, we are in debt to Dzydyk and to several of his students (Alibekov, Belyi, Vorob'ev, Polyakov, Shvai, and Shevchuk) as well as Andrievskii, Lebedev, Mamedkhanov, Tamrazov, Shirokov, and others.

The scope of the present article is too narrow for a chronological presentation of all the theoretically significant concepts, results, and methods obtained by Dzydyk, and therefore we will try to identify those that are the most important according to an evaluation based on the contemporary interpretation of the processes under study and from the standpoint of the present level of this theory.

Suppose that $\mathfrak{M}$ is a continuum in a finite complex plane $\mathbb{C}$ and that $\mathfrak{M}^c$ is its complement, which is assumed to be connected; $A(\mathfrak{M})$ is the class of functions that are analytic in $\mathfrak{M}^0 = \operatorname{int}\mathfrak{M}$ and continuous on $\mathfrak{M}$, and $A^r(\mathfrak{M})$ are the subclasses of functions f from $A(\mathfrak{M})$ with r derivatives continuous on $\mathfrak{M}$; $A^0(\mathfrak{M}) = A(\mathfrak{M})$; $w = \Phi(z)$ is a conformal mapping of $\mathfrak{M}^c$ onto the exterior of the unit disc D normed by the conditions $\Phi(\infty) = \infty$ and $\lim\limits_{z\to\infty} z^{-1}\Phi(z) > 0$; $\Psi =: \Phi^{-1}(w)$; $L_{1+1/n} = \{z : |\Phi(z)| = 1 + 1/n\}$ is the n-th level line of the exterior Green function for $\mathfrak{M}$; $L = \partial\mathfrak{M}$; $\rho_{1+1/n}(z)$ is the distance of the point $z \in L$ to $L_{1+1/n}$. For every function $f \in A(\mathfrak{M})$ we let $\omega(\delta) = \omega(f;\ \delta)$ denote its modulus of continuity on $\mathfrak{M}$.

Schematically, the direct and inverse theorems of the constructive theory of functions on closed sets in $\mathbb{C}$ basically have the following form, which was rigorously confirmed after publication of [5, 6].

<u>Direct Theorem.</u> Suppose that $f \in A^r(\mathfrak{M})$, $r \geqslant 0$, where $\mathfrak{M}$ is a continuum in $\mathbb{C}$ that satisfies some set of conditions (D_1). Then $\forall n \in \mathbb{N}$ an algebraic n-th order polynomial $P_n(z)$ may be found such that for all $z \in \partial\mathfrak{M}$, the following inequalities are satisfied:†

$$|f(z) - P_n(z)| \preccurlyeq [\rho_{1+\frac{1}{n}}(z)]^r\, \omega(f^{(r)};\ \rho_{1+\frac{1}{n}}(z)). \tag{1}$$

<u>Universe Theorem.</u> Suppose that $\mathfrak{M}$ is a continuum in $\mathbb{C}$ that satisfies some set of conditions (D_2). If a function f(z) defined on $\mathfrak{M}$ is the uniform limit of a sequence of polynomials $\{P_n(z)\}$ such that for all $z \in \partial\mathfrak{M}$,

$$|f(z) - P_n(z)| \preccurlyeq [\rho_{1+\frac{1}{n}}(z)]^{r+\alpha}, \quad r \in \mathbb{N},\ \alpha \in (0, 1), \tag{2}$$

then $f \in A^r(\mathfrak{M})$ and $\omega(f^{(r)};\ \delta) \preccurlyeq \delta^\alpha$.

If constraints (D_1) and (D_2) coincide on the continuum, the direct and inverse theorems together yield a constructive characteristic of the Hölder class H^α, $\alpha \in (0, 1)$.

To prove the inverse theorems, Dzydyk in 1959 investigated the behavior of the function $\rho_{1+1/n}(z)$ on the boundary $\partial\mathfrak{M}$ for a rather broad class of closed sets with piecewise smooth boundaries, proving that for any $s \in \mathbb{R}$,

$$\|P_n'(\zeta)\, \rho^{1-s}_{1+\frac{1}{n}}(\zeta)\|_{C_{\partial\mathfrak{M}}} \preccurlyeq \|P_n(\zeta)\, \rho^{-s}_{1+\frac{1}{n}}(\zeta)\|_{C_{\partial\mathfrak{M}}}, \tag{3}$$

an inequality that enabled him to establish an inverse theorem on these sets.

In 1962-63, Dzydyk [6, 7] proved a direct theorem under the constraints (D_1), according to which $\mathfrak{M}$ was a closed domain with piecewise smooth boundary consisting of arcs of continuous curvature formed at the junction of external angles $\geqslant \pi/2$ and satisfying certain

---

*Here and below it will be assumed that the continuum is bounded and does not degenerate to a point.

†Here and below $\preccurlyeq$ will denote ordinal inequality, and $\asymp$ weak equivalence, i.e., ordinal equality.

additional constraints on the boundary of the domain. Together with the inverse theorems, this result enabled Dzydyk to obtain the first constructive characteristic of the Hölder classes W^rH^α on sets with piecewise smooth boundary.

It was an extraordinarily difficult and intricate problem to obtain the direct theorems, requiring the application of techniques of conformal mappings and the creation of a new analytic apparatus for the construction of approximating polynomials. As the constraints (D_1) were gradually weakened, this apparatus had to be essentially redeveloped and improved. A number of Dzydyk's concepts and methods that were the basis for the technique of proving the direct theorems proved fundamental for the entire further development of the theory; the definition and use of the generalized rotation $\zeta_t =: \Psi[\Phi(\zeta)e^{-it}]$ and generalized dilation $\zeta =: \Psi[R\Phi(\zeta)]$, $R \geqslant 1$; the construction of polynomial kernels by means of a generalized convolution he introduced for Faber series that ensure an optimal approximation of the Cauchy kernel, and others. Let us discuss this result in some detail.

Suppose that $\mathfrak{M}$ is an arbitrary nondegenerate continuum in $\mathbb{C}$ with connected complement having a rectifiable boundary Γ and suppose that two functions $f(\zeta)$ and $K(\zeta)$ are defined on Γ such that the 2π-periodic functions $\hat{f}(t) =: f[\Psi(e^{it})]$ and $\hat{K}(t) =: K[\Psi(e^{it})]$ induced by them on the unit circle are summable on $[0, 2\pi]$.

Dzydyk's convolution theorem (1961-67) confirms the following assertion [6]:

1) if the functions $f(z)$ and $K(z)$ may be expanded in Faber series of the form

$$f(z) \sim \sum_0^\infty c_k F_k(z) \;\; u \;\; K(z) \sim \sum_0^\infty \lambda_k F_k(z) \tag{4}$$

and

2) if at least one of the functions $\hat{f}(t)$, $\hat{K}(t)$, and $\Psi(e^{it})$ has bounded variation on $[0, 2\pi]$, while the other two are Lebesgue integrable, then

$$K*f =: \frac{1}{4\pi^2 i} \int_{-\pi}^{\pi} \hat{K}(t)\,dt \int_\Gamma f(\zeta_t)(\zeta - z)^{-1}\,d\zeta = \sum_{k=0}^\infty \lambda_k c_k F_k(z). \tag{5}$$

This theorem yields a convenient integral representation for any method of summation of Faber series generated by an even summable 2π-periodic function $\hat{K}(t)$. Selecting periodic kernels (Dirichlet, Fejér, Jackson, etc. kernels) as the $\hat{K}(t)$ yields analogues of Fourier and Fejér sums, Jackson polynomials, etc. Note that after a series of transformations of the integral in (5), it was found that (4), (5) in fact gives an approximation of the Cauchy kernel $(\zeta - z)^{-1}$ by polynomial kernels. Transformations of harmonic functions may be constructed in an analogous fashion (cf. [12]).

A generalization of the results obtained was undertaken in succeeding years.

In 1968-69, the inequality for the derivative of polynomials of the form of (3) and an inverse theorem were generalized by Lebedev and Tamrazov [13, 14] to very broad classes of sets, more precisely, all compacta that are regular in the Dirichlet problem.

Results obtained in four interdependent directions played a major role in the development of studies on the direct theorems: 1) improvement of methods for the constructive creation of approximating polynomials; 2) development of an exact local theory of distance distortions in conformal mapping; 3) effective continuation of functions and the improvement of averaging methods for continued functions; 4) development of the theory of high-order smoothness moduli to sets in $\mathbb{C}$. Dzydyk made a major contribution to the development of all these research trends, though his most important contribution was to the first; the construction of universal polynomial kernels for the approximation of Cauchy kernels.

In 1967 (cf. [15] and, for a detailed discussion, [16-18]), Dzydyk proposed a general construction of such kernels in the form

$$K_{m,n}(\zeta, z) = \frac{1 - [1 - (\zeta - z)\pi_n(\zeta, z)]^m}{\zeta - z}, \tag{6}$$

where $m \in \mathbb{N}$ is a parameter and $\pi_n(\zeta, z)$ an elementary kernel in the form of a polynomial in z of order n with coefficients meromorphic in ζ (i.e., a blending) that approximates the Cauchy kernel on $\mathfrak{M}$.

In 1967-72, Dzydyk explained ([19]; cf., too, [17]) that it is best to present the elementary kernels $\pi_n(\zeta, z)$ in the form

$$\tilde{\pi}_n(\zeta, z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{J_{nk}(t)}{\tilde{\zeta}_t - z}\, dt = \sum_{\nu=0}^{k(n-1)} \frac{j_\nu}{\left(1+\frac{1}{n}\right)^\nu [\Phi(\xi)]^\nu} \Pi_\nu(z), \tag{7}$$

where $\Pi_\nu(z)$ are generalized Faber polynomials determined from the expansion

$$\frac{1}{\zeta - z} = \sum_{\nu=0}^{\infty} \frac{\Pi_\nu(z)}{[\Phi(\xi)]^\nu}, \quad z \in \mathfrak{M},\ \zeta \in \mathfrak{M}^c,$$

and $J_{n,k}(t)$ are generalized Jackson trigonometric kernels. As result, kernels of the following form were obtained:

$$K_{m,n}(\zeta, z) = \frac{1 - [1 - (\zeta - z)\tilde{\pi}_n(\zeta, z)]^m}{\zeta - z} = \frac{1}{\zeta - z}\left[1 - \left(1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} J_{nk}(t)\frac{\zeta - z}{\tilde{\zeta}_t - z}\, dt\right)^m\right]. \tag{8}$$

These kernels possess such profound properties that since the time they were first constructed around 20 years ago, they have been regularly applied for obtaining increasingly more exact direct theorems for the approximation of analytic functions by algebraic polynomials and of harmonic functions by harmonic polynomials on increasingly more complicated continua. Often Dzydyk kernels may be conveniently applied in the "permuted" form

$$\tilde{K}_{m,n}(\zeta, z) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{J_{nk}(t)}{\zeta - t}\left[1 - \left(1 - \frac{\zeta - z}{\tilde{\zeta}_t - z}\right)^m\right] dt \tag{8'}$$

(cf. [19, 20]). Note, too, the following: 1) Shevchuk recently [1] proved that, even when the continuum $\mathfrak{M}$ is the closed interval [−1, 1], these kernels are essentially novel and reduce to the form

$$K^0_{m,n}(x, y) = \frac{1}{(m-1)!}\frac{\partial^m}{\partial x^m}\left\{(x - y)^{m-1}\int_{\beta-\alpha}^{\beta+\alpha} J_n(t)\, dt\right\}, \tag{9}$$

where m and n are natural numbers; $x, y \in [-1, 1]$; $\beta = \arccos x$; $\alpha = \arccos y$. In particular, they may be effectively applied to problems of coapproximation on an open interval (cf. Shevchuk's article in this number); 2) if the Jackson kernel $J_{nk}(t)$ in (8) is replaced by some other even trigonometric kernel $K_n(t) = \sum_0^n \lambda_k \cos kt$, for example, a Poisson, Dirichlet, Faber, Rogozinskii, etc. kernel, we obtain algebraic polynomial blendings that are good analogues for $\mathfrak{M}$ of the corresponding trigonometric kernels.

The integrals $\tilde{K}_{m,n}(\zeta, z)$

$$\frac{1}{2\pi i}\int_{\partial\mathfrak{M}} f(\zeta)\tilde{K}_{m,n}(\zeta, z)\, d\zeta \text{ for } \frac{1}{2\pi i}\int_{\partial\mathfrak{M}} f(\zeta) K_{m,n}(\zeta, z)\, d\zeta$$

are approximating polynomials by means of which it is possible to prove the direct theorems for the type (B) domains that had been introduced by Dzydyk in 1972. The conditions that define domains in the class (B) played an important role as a guide post for the further development of research on direct theorems, and are of the nature of axiomatics. For this reason we will discuss them in some detail.

A finite closed simply connected domain $\bar{G}$ belongs to the class (B) if [19]:

(1) the boundary L = ∂G is rectifiable;

(2) there exists a natural number k such that for all $z \in L$, $\zeta \in L$, n = 1, 2,... and $t \in [-\pi, \pi]$, the following cases hold: a) $|\tilde{\zeta}_t - z|(1 + n|t|)^k \geqslant |\tilde{\zeta} - z|$, that is, also $|\tilde{\zeta}_t - \zeta| \leqslant (1 + n|t|)^k |\tilde{\zeta} - \zeta|$; b) $|\tilde{\zeta} - \zeta|^k \leqslant |\tilde{\zeta} - z|^{k-1}|\tilde{z} - z|$; c) $|\tilde{z}_t - z| \asymp \rho_{1+1/n+|t|}(z)$;

(3) $\forall z \in L$ and $\zeta \in L$, the inequality $s(\zeta, z) \leqslant |\tilde{\zeta} - z|$ holds, where s(ζ, z) is a linear measure of the set $L \cap \{\xi : |\xi - \zeta| \leqslant |z - \zeta|\}$.

The class (B) is rather general, though the axiomatic structure of its definition posed two important questions: what sort of geometric properties must G possess in order to belong to (B), and is such a set of conditions minimal for a domain and does not ensure the validity of the direct theorems.

A complete geometric characterization of sets belonging to class (B) was obtained in 1976-83 by Belyi and Andrievskii through an extension basically of the second and third of the above lines of reasoning.

In 1970-72 (in repeated discussions of direct theorems with Belyi), Dzydyk suggested to Belyi that, in view of the high level of the Donets school on the theory of mappings, he undertake a thorough investigation of the extension of the direct theorems to more general levels that may be described in geometric terms of the continuum. Thus was created in Donets a new scientific line of research, together with a mathematical school that extended and extensively applied methods from the theory of conformal invariants and quasiconformal mappings for the solution of problems of approximation and other problems of complex analysis.

In 1974 Belyi and Miklyukov [22], applying for the first time methods from the theory of quasiconformal mappings for the purpose of obtaining direct theorems, obtained direct theorems in domains with quasiconformal boundaries that satisfy conditions (1) and (3) and two geometric-type constraints, including the case in which $\mathfrak{M}$ is an arbitrary closed convex domain. These constraints were simultaneously eliminated by Dzydyk [23] and Belyi [24] in 1975.

In 1975 Dzydyk [25] generalized the direct theorems to continua belonging to the class (B^k) whose boundary consists of a finite number of rectifiable curves and for each of which conditions (1)-(3) hold, with condition (2.c) replaced by the weaker condition $|\tilde{z}-z| \asymp \rho_{1+1/n}(z)$. Thus was the remarkable potention of the Dzydyk kernels (6) and (8) completely completely revealed in [25], as well as in subsequent studies by Belyi [26, 27], Shevchuk [28, 29], Andrievskii [30-33], and others.

In 1976, based on the local theory of distance distortions under conformal mappings he had developed and the method of conformal invariants and the theory of quasiconformal mappings, Belyi proved the direct theorems for an arbitrary finite domain G having a quasiconformal boundary [26, 27, 24]. He also proved here that i) conditions (1) and (3) are not necessary for the direct theorems to hold; ii) condition (2.a) holds for an arbitrary continuum even when k = 4; and applied the Dzydyk kernels (8) and (6) in a generalized convolution with area integral. These studies were further extended in studies completed in 1980-85 by Andrievskii [30-33], who obtained direct theorems for the approximation of functions over a broad class of continua with piecewise quasiconformal boundary and also applied the kernels (6). In particular, he developed a technique for deriving various geometric relations from information on the constructive properties of functions belonging to these classes and found necessary conditions (D_1) that were very similar to the sufficient conditions under which direct theorems expressed in terms of the function $\rho_{1+1/n}(z)$ hold.

Because of space limitations, the present survey will not discuss results on direct and inverse problems that relate to the approximation of functions with given majorant function of the k-th modulus of smoothness, the simultaneous approximation of functions and their derivatives by polynomials and by their corresponding derivatives, approximation by rational functions with fixed poles, approximation in integral metrics and a number of other problems directly and conceptually related to the direct and inverse theorems.

Interest in the proof of direct and inverse theorems in Dzydyk's formulation served as an impetus for å number of profound investigations:

- theory of finite difference smoothness and related polynomial approximation in the complex plane (Tamrazov, Shevchuk); investigation of the approximating and structural properties of functions and their derivatives belonging (on the sets $\mathfrak{M} \subset \mathbb{C}$) to classes determined by high-order smoothness moduli (Dzydyk, Tamrazov, Shevchuk, Shvai, Belyi, Andrievskii, and others);
- exact theory of local distance distortions under conformal mapping (Belyi); development of method of conformal invariants and quasiconformal mappings in approximation problems (Belyi, Andrievskii);
- investigation of the properties of Cauchy-type integrals and its generalizations (Dzydyk, Shevchuk, Andrievskii, Belyi, and others);

- description of classes of functions characterized by a given rate of decrease of their optimal uniform approximations on sets of the complex plane (Dzydyk and Alibekov, Volkov, Galan, Antonyuk, Dyn'kin, Andrievskii, and others);
- constructive description of classes of functions on continua for which the direct theorems expressed in terms of $\rho_{1+1/n}(z)$ is theoretically impossible (Andrievskii);
- approximation of harmonic functions by means of harmonic polynomials [direct and inverse theorems expressed in terms of $\rho_{1+1/n}(z)$ in $\mathbb{R}^2$ (Andrievskii)].

This far from complete list shows that Dzydyk and his students and followers created a trend of research in the theory of approximations that has proven to be among the most fruitful and most productive [Usp. Mat. Nauk, 34, No. 4, 233 (1979)].

II. Investigations on the Theory of Rational Approximation. A number of important and clever concepts were proposed by Dzydyk in the field of rational approximation. Note, above all, the interrelation between bi-orthogonality and Padé approximation Dzydyk found. Dzydyk was the first to use this interrelation to investigate the asymptotics of the errors of Padé diagonal approximations of the functions $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ [37]. Note in this connection that Perron had been the first to study the far simpler case of Padé approximation of the function e^z and that the functions $\ln(1 + z)$ and $(1 + z)^\alpha$ were studied by Luke and independently (using a different method) Dzydyk and Filozof (cf. [36]). The interrelation noted above served as a starting point for the generalization of the classical problem of moments and thereby made it possible to formulate in much broader terms many important results in the theory of Padé approximation. The extension of the a-method for the approximate solution of linear differential equations with polynomial coefficients to the case of rational approximation [35-38] served as the starting point for these studies. To illustrate this concept, consider the integral equation for e^z:

$$y(\zeta) = 1 + \int_0^\zeta y(\xi)\, d\xi, \quad \zeta \in [0, z]. \tag{10}$$

The technique of the a-method presupposes that this equation be replaced by the operator-valued equation

$$y_{n-1}(\zeta) = \int_0^\zeta y_{n-1}(\xi)\, d\xi + 1 - \tau(z) P_n\left(\frac{\zeta}{z}\right), \tag{11}$$

in which $P_n(t)$ is some fized polynomial and $y_{n-1}(\zeta) = \sum_{k=0}^{n-1} c_k \zeta^k$. By setting the coefficients of powers of ζ equal, Eq. (11) reduces to a system of linear algebraic equations in the unknowns $c_0, c_1, \ldots, c_{n-1}, \tau$, where $c_j = c_j(z)$; $\tau = \tau_n(z)$; $j = 1, 2, \ldots$. Solving this system and setting $\zeta = z$, we obtain a rational function $R_{n,n}(z) = y_{n-1}(z) + \tau(z) P_n(1)$ that approximates the exact solution. The form and properties of this approximation depend on the choice of the polynomial $P_n(t)$. Dzydyk remarked that for this example mixed orthonormal Legendre polynomials $L_n^*(t)$ must be taken as the $P_n(t)$, and ultimately found diagonal Padé approximations for e^z [36]. It is possible to construct and investigate diagonal Padé polynomials for the functions $\ln(1 + z)$, $\arctan z$, and $(1 + z)^\alpha$ in a similar way. However, even in the case of the functions $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ that satisfy second-order differential equations, it was necessary to resort to essential novel concepts. Consider the integral equation for $\sin z$:

$$y(\zeta) = \zeta - \int_0^\zeta (\zeta - \xi)\, y(\xi)\, d\xi, \quad \xi \in [0, z]. \tag{12}$$

The operator-valued equation corresponding to it has the form

$$y_{2n-1}(\zeta) = \zeta - \int_0^\zeta (\zeta - \xi)\, y_{2n-1}(\xi)\, d\xi - \tau(z) A_{2n+1}\left(\frac{\zeta}{z}\right), \tag{13}$$

where the polynomials $y_{2n-1}(\zeta)$ and $A_{2n+1}(t)$ and the function $\tau(z)$ are unknown. We construct rational polynomials $R_{2n+1,2n}(z) = y_{2n-1}(z) + A_{2n+1}(1)\tau(z)$ by means of the a-method and obtain the following representation for the error:

$$y(z) - R_{2n+1,2n}(z) = -\tau(z)\, z \int_0^1 \sin z(1 - t)\, A_{2n+1}(t)\, dt. \tag{14}$$

Since $\tau(z) \sim z^n$ as $z \to 0$, we find that to obtain the required order of tangency at the point $z = 0$, it is necessary to use as $A_{2n+1}(t)$ a polynomial determined by the bi-orthogonality properties:

$$\int_0^1 (1-t)^{2i-1} A_{2n+1}(t)\,dt = 0, \quad i = 1, 2, \ldots, n. \tag{15}$$

Dzydyk introduced such polynomials and studied their properties, and subsequently investigated the behavior of the Padé polynomials for the functions $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$ [37].

From an analysis of the results he obtained, Dzydyk proposed the following important definition [39]:

A representation of some sequence of, in general, complex numbers $\{s_k\}_{k=0}^{\infty}$, in the form of the system of equalities (in which s_m admits m distinct representations when $j + k = m$)

$$s_{j+k} = \int_0^1 a_j(t)\, b_k(t)\, d\mu(t), \quad k, j = 0, 1, \ldots, \tag{16}$$

is called the generalized moment representation of this sequence, where $\mu(t)$ is some nondecreasing function on [0, 1] and $\{a_j(t)\}_{j=0}^{\infty}$ and $\{b_k(t)\}_{k=0}^{\infty}$ are certain sequences of functions under which Eqs. (16) hold (cf. [39, 40]).

It is easily seen that in the case $a_k(t) = b_k(t) = t^k$, we obtain a formulation of the classical problem of moments for which the existence conditions for the solution are well known.

It was proved in [42] that representations of the form (16) may be constructed for a very broad class of sequences, that is, for all sequences $\{s_k\}_{k=0}^{\infty}$ whose Hankel determinants $H_N = \det \| s_{i+j} \|_{i,j=0}^{\infty}$ are nonzero (at the same time, for the classical problem of moments, all these determinants must be positive). On the other hand, by using these representations and resorting to the concept of bi-orthogonality, it is possible to construct and analyze the Padé approximations of the generating functions of these sequences. Thus, if we set $f(z) = \sum_{k=0}^{\infty} s_k z^k$, where the sequence $\{s_k\}_{k=0}^{\infty}$ may be represented in the form (16), then the Padé diagonal polynomials of order [N/N] of the function f(z) have the form [39, 40]

$$\pi_{N,N}(f; z) = \frac{P_N(z)}{Q_N(z)} = \frac{\sum_{j=0}^{N} c_j^{(N)} z^{N-j} T_j(f; z)}{\sum_{j=0}^{N} c_j^{(N)} z^{N-j}}, \tag{17}$$

where $T_j(f; z)$ are the Taylor polynomials of the function f(z) of order j, and $c_j^{(N)}$, $j = 0, 1, \ldots, n$ are the coefficients of the nontrivial polynomial $B_N(t) = \sum_{j=0}^{N} c_j^{(N)} b_j(t)$ that satisfy the bi-orthogonality conditions $\int_0^1 a_k(t) B_N(t)\, d\mu(t) = 0,\ k = 0, \ldots, N-1$. The equality

$$f(z) - \pi_{N,N}(f; z) = \frac{z^{n+1}}{Q_n(z)} \int_0^1 \sum_{k=0}^{\infty} a_k(t)\, z^k B_N(t)\, d\mu(t)$$

will hold here for the error of the approximation. Starting from the generalized moment representation, Dzydyk established [39, 40] theorems that generalize the basic results Chebyshev had obtained for the classical moment problem (cf., for example, [41], pp. 178-194]).

Generalized moment representations were also found to be effective in the derivation of a variety of new integral representations for a host of hypergeometric functions [40, 43]. These concepts were also used and extended by Golub and Chyp, two of Dzydyk's students.

III. Investigations on the Representation of Functions by Means of Series of Exponential Functions. By the 1970's Leont'ev [44-47] had completed a number of far-reaching studies devoted to the representation of functions f(z) analytic in convex domains $Q \subset \mathbb{C}$ by means of series of exponential functions of the form

$$f(z)=\sum_{n=1}^{\infty} c_n e^{\lambda_n z}. \tag{18}$$

In 1970 Dzydyk was the first to study the convergence of series of exponential functions of the form (18) on closed convex domains $\bar{Q}$.

In 1973 Leont'ev [47] and Dzydyk and Krutigolova [48] initiated the study of this topic. In 1974 Dzydyk [49] was the first to discover (for the case where Q is a polygon) a relation between series of exponential functions of the form (18) on the boundary of the polygon ∂Q and the behavior of the Fourier series of certain periodic functions generated by the boundary values of f(z) on the sides of the polygon. With this relation he was able to establish the following result:

Suppose that $\{\lambda_n\}_{n=1}^{\infty}$ is the set of roots of the entire function $\mathcal{L}(\lambda)=\sum_{k=1}^{N} d_k e^{a_k\lambda}$, where $d_k\neq 0$, $a_k\ (k=1,2,\ldots,N;\ N\geqslant 3)$ are the vertices of the polygon Q (in [47] $d_k = 1$, though this is not essential). Then in order that the series (18) converge uniformly on ∂Q, it is sufficient that the following conditions hold: 1) $\sum_k d_k f(a_k)=0$; 2) $\int_0^1 \frac{\omega(f;t)}{t}dt<\infty$ where f; t) is the modulus of continuity of f(z) on the closed polygon Q (Dini condition).

Note, too, that in [49], and in part in an earlier study [48], may be found a representation of an arbitrary function analytic in the convex polygon Q expressed in the form of a sum of analytic periodic functions (in the case where Q is a square).

These investigations of Dzydyk's were of great importance for the subsequent development of the theory of representation of analytic functions by series of exponential functions, serving as impetus (direct or indirect) for new studies by Leont'ev [50], Mel'nik [51-55], Sedletskii [54-56], and others.

IV. Strengthening Certain Classical Results. 1. Picard's theorem for the existence of the solution y(x) of Cauchy's problem of the form

$$y'=f(x,y),\ y(x_0)=y_0,\quad x\in[x_0,x_0+h]. \tag{19}$$

is well known. Dzydyk strengthened Picard's results for the case in which the function f(x, y) in (1) is analytic with respect to both variables, and thus developed what is now called the iterative-approximation method [57, 58], by means of which it became possible to approximate the solution y(x) effectively, easily, and highly accurately.

This method was also used by Dzydyk and his students to produce highly accurate solutions of boundary-value problems of the form $y''=f(x, y, y')$, $y(0)=y_0$, $y(h)=y_1$, to solve Goursat, Cauchy, and Darboux problems in the simplest types of domains for hyperbolic-type partial differential equations, and to obtain a new highly effective quadrature formula.

2. Chebyshev, Markov, and Bernshtein had successively solved (using different techniques) the following problem posed by Chebyshev; suppose that a positive polynomial $a_0(x)=x^l+c_1x^{l-1}+\ldots+c_l$ is defined on [−1, 1]. For given $n\geqslant l/2$, $n\in\mathbb{N}$, it is necessary to construct a polynomial $T_n^*(x)$ of the form $T_n^*(x)=x^n+t_1^*x^{n-1}+\ldots+t_n^*$, under which the equality

$$\left\|\frac{T_n^*(x)}{\sqrt{a_0(x)}}\right\|_{C[-1,1]}=\inf_{t_j,j=\overline{1,n}}\left\|\frac{x^n+t_1x^{n-1}+\ldots+t_{n-1}x^{n-1}}{\sqrt{a_0(x)}}\right\|_{C[-1,1]} \tag{20}$$

holds.

This problem had been previously solved by means of a rather complex method for every fixed n [59]. Dzydyk [57] proposed a new approach to this problem and strengthened the preceding results, establishing that for all $n\in\mathbb{N}$, $n\geqslant l/2$, $T_n^*(x)$ constitutes a linear combination of the classical Chebyshev polynomials, i.e., that there exist numbers $\gamma_0, \gamma_1, \ldots, \gamma_\ell$ such that for any integers $n\geqslant l/2$, the identity

$$T_n^*(x)=\sum_{j=0}^{l}\gamma_j T_{|n-j|}(x),\quad T_k(x)=\cos k\arccos x,\quad k=0,1,2,\ldots \tag{21}$$

holds. He also indicated an effective method for the construction of the numbers γ_j, $j = 0, 1, \ldots, \ell$.

3. Let us present the following theoretically important result ([60] or [38]). Suppose that in the half-open interval (a, a + h] a singular linear differential equation is defined having (for the sake of simplifying the discussion) only a single regular singular point a, i.e., an equation of the form

$$a_0(x)(x-a)^k y^{(k)} + a_1(x)(x-a)^{k-1} y^{(k-1)} + \dots + a_k(x) y = 0, \tag{22}$$

where all the $a_j(x)$ are polynomials and $a_0(x) \geqslant c > 0 \ \forall x \in [a, a+h]$.

Each solution of the fundamental system S of this equation is called a special function. Examples of such functions are the Bessel functions, hypergeometric functions, Riemann P-functions, Laplace functions, Neiman functions, Kelvin functions, Dirac functions, etc. There are now over 1500 classes of functions and individual functions used just in applied problems.

Because of investigations carried out over the past century alone, it is known that each special function y(x) determined by Eq. (20) may be represented in the form (with a = 0)

$$y(x) = x^r [\pi_j(\ln x)\varphi_1(x) + \pi_{j-1}(\ln x)\varphi_2(x) + \dots + \pi_0(\ln x)\varphi_{j+1}(x)], \tag{23}$$
$$j = 1, 2, \dots, \mathfrak{F} = \mathfrak{F}(r),$$

where r is any of the roots of the characteristic equation $\sum_{i=0}^{k} a_i(0)(r)_{k-i} = 0$; $\mathfrak{F}$ is its multiplicity; $\varphi_\nu(x) = \varphi_\nu(r; x) = \sum_{\mu=0}^{\infty} c_\mu x^\mu$ are unknown analytic functions whose coefficients $c_\mu = c_\mu(r, j, \nu)$ may be found using the method of undetermined coefficients by substituting y(x) from Eq. (23) in (22); and $\pi_0(x) \equiv 1$, $\pi_i(s) := \frac{s(s-1)\dots(s-i+1)}{i!}$, $i = 1, 2, \dots$ is a system of standard polynomials.

By means of very far-reaching and rigorous reasonings, Dzydyk strengthened this result of Fuks', establishing that each of the functions $\varphi_j(x) = \varphi_j(r; x)$ is not only analytic, but also satisfies a certain singular, in general, nonhomogeneous linear differential equation, which he obtained using a highly original method that he gave an explicit description of (cf. [60] or [38, pp. 220-228]). The first term of this equation always coincides with the first term [equal to $a_0(x)x^k y^{(k)}$] of Eq. (22). As a result, every analytic segment of $\varphi_j(x)$ is a single-valued analytic function in the Mittag–Leffler star which may be obtained by discarding rays from the complex plane each of which begins at a zero of the coefficient $a_0(z)$ in Eq. (22) and extends to infinity in such a way that the coordinate origin is situated along the extension of this ray.

The interest in this result of Dzydyk's for experts in computational mathematics and the theory of approximation is that it made it possible to replace, in the approximate computation of special functions, the ordinary method of power series by the singular analogue (proposed by Dzydyk) of the a-method he had previously developed for the effective construction of polynomials $P_n(x)$ of degree n for every natural number n. Here the following equality holds:

$$\|\varphi(x) - P_n(x)\|_{L^2_{p(x)}} = (1+\varepsilon_n) E_n(y)_{L^2_{p(x)}}, \tag{24}$$

where $\varepsilon_n \downarrow 0$ and $E_n(y)_{L^2_{p(x)}}$ is the optimal approximation of the unknown analytic part of $\varphi(x)$ in the quasi-Chebyshev metric $L^2_{p(x)}$ generated by the weight p(x) determined by the polynomial $a_0(x)$.

Equation (22) asserts that the polynomials $P_n(x)$ yield an asymptotically optimal approximation of the analytic part of $\varphi(x)$ and approximate it at least 2^n times better than the n-th order partial sums of the power series into which $\varphi(x)$ may be expanded.

LITERATURE CITED

1. V. K. Dzydyk, Introduction to the Theory of Uniform Approximation of Functions by Means of Polynomials [in Russian], Nauka, Moscow (1977).
2. S. M. Nikol'skii, "On optimal approximation of functions that satisfy a Lipschitz condition by means of polynomials," Izv. Akad. Nauk SSSR, Ser. Mat., 10, 295-317 (1946).

3. V. K. Dzydyk, "On a constructive description of functions that satisfy the condition Lip α ($0 < \alpha < 1$) on a finite interval of the real axis," Izv. Akad. Nauk SSSR, Ser. Mat., 20, 623-642 (1956).
4. A. F. Timan, "Strenthening Jackson's theorem on optimal approximation of continuous functions by polynomials on a finite interval of the real axis," Dokl. Akad. Nauk SSSR, 78, 17-20 (1951).
5. V. K. Dzydyk, "Nikol'skii problem in a complex domain," Izv. Akad. Nauk SSSR, Ser. Mat., 23, No. 5, 697-736 (1959).
6. V. K. Dzydyk, "Approximation of continuous functions in closed domains with angles and Nikol'skii's problem," Izv. Akad. Nauk SSSR, Ser. Mat., 26, No. 6, 797-824 (1962).
7. V. K. Dzydyk, "Theory of approximation of analytic functions continuous in closed domains and Nikol'skii's problem," Izv. Akad. Nauk SSSR, Ser. Mat., 27, No. 5, 1135-1164 (1963).
8. V. K. Dzydyk, "Inverse theorems of the theory of approximation of functions in complex domains," Ukr. Mat. Zh., 15, No. 4, 365-375 (1963).
9. V. K. Dzydyk, "Theorems on transformation and approximation of analytic functions," Dokl. Akad. Nauk SSSR, 151, No. 2, 269-272 (1963).
10. V. K. Dzydyk, "Polynomial approximation of functions in classes H^{ω} on closed domains," in: Investigations on Modern Problems of the Theory of Functions [in Russian], Izd. Akad. Nauk AzSSR, Baku (1965), pp. 273-283.
11. V. K. Dzydyk, "Approximation of analytic functions in domains with smooth or piecewise-smooth boundary," Proceedings of the Third Summer Mathematics School, Inst. Mat. Akad. Nauk UkrSSR, Kiev (1966), pp. 29-83.
12. V. K. Dzydyk, "Analytic and harmonic transformations and the approximation of harmonic functions," Ukr. Mat. Zh., 19, No. 5, 33-57 (1967).
13. N. A. Lebedev and P. M. Tamrazov, "Inverse approximation theorems on closed sets of the complex plane," Dokl. Akad. Nauk SSSR, 179, No. 5, 1046-1049 (1968).
14. N. A. Lebedev and P. M. Tamrazov, "Inverse approximation theorems on regular compacta of the complex plane," Izv. Akad. Nauk SSSR, Ser. Mat., 34, No. 6, 1340-1390 (1970).
15. V. K. Dzydyk and A. I. Shvai, "Constructive description of functions of Hölder classes in domains with piecewise-smooth boundary and arbitrary positive external corners," Dokl. Akad. Nauk UkrSSR, Ser. A, 11, 1012-1015 (1967).
16. A. I. Shvai, "Approximation of functions of classes W^rH^{ω} defined on closed sets with arbitrary (but nonzero and not equal to 2π) angles," Author's Abstract of Candidate's Dissertation, Mathematics and Physics, Kiev (1967).
17. R. V. Polyakov, "Approximation of continuous functions defined on bounded closed sets," Author's Abstract of Candidate's Dissertation, Mathematics and Physics, Kiev (1969).
18. V. K. Dzydyk and A. I. Shvai, "Approximation of functions of Hölder classes on closed sets with exterior acute angles," Metricheskie Vopr. Teor. Funkts. Otobrazhenii, No. 2, Naukova Dumka, Kiev (1971), pp. 74-164.
19. V. K. Dzydyk, "Application of generalized Faber Polynomials to the approximation of Cauchy-type integrals and functions in the class A^r in the domains with smooth or piecewise-smooth boundary," Ukr. Mat. Zh., 24, No. 1, 3-19 (1972).
20. N. A. Lebedev and N. A. Shirokov, "Uniform approximation of functions on closed sets with a finite number of angular points with nonzero exterior angles," Izv. Akad. Nauk ArmSSR, 6, No. 4, 311-341 (1971).
21. I. A. Shevchuk, "Properties of Dzydyk polynomial kernels on the closed interval," Ukr. Mat. Zh., 41, No. 4, 524-530 (1989).
22. V. I. Belyi and V. M. Miklyukov, "Certain properties of conformal and quasiconformal mappings and direct theorems of the constructive theory of functions," Izv. Akad. Nauk SSSR, Ser. Mat., 38, No. 6, 1343-1361 (1974).
23. V. K. Dzydyk, "Application of methods from the theory of quasiconformal mappings to the proof of direct theorems for the approximation of functions," Dokl. Akad. Nauk SSSR, 225, No. 3, 491-494 (1975).
24. V. I. Belyi, "Distortion of distances under conformal mappings and direct theorems of the constructive theory of functions," in: Theory of Approximation of Functions. Proceedings of International Conference on the Theory of Approximation of Functions [in Russian], Nauka, Moscow (1977), pp. 32-37.
25. V. K. Dzydyk, "Theory of approximation of functions on closed sets of the complex plane (Nikol'skii problem)," Tr. Mat. Inst. Akad. Nauk SSSR, 134, 63-114 (1975).
26. V. I. Belyi, "Conformal mappings and approximation of functions in domains with quasiconformal boundary," Mat. Sb., 102, No. 3, 331-361 (1977).

27. V. I. Belyi, Approximation of functions of classes $A^r(\overline{G})$ in finite domains with quasiconformal boundary," Theory of Functions and Mappings [in Russian], Naukova Dumka, Kiev (1979), pp. 37-46.
28. I. A. Shevchuk, "Classes of functions $H^{\varphi(t)}_{k,\mathfrak{M}}$ on sets $\mathfrak{M}$ of the complex plane," in: Theory of Approximation of Functions. Proceedings of International Conference on the Theory of Approximation of Functions, Nauka, Moscow (1977), pp. 409-412.
29. I. A. Shevchuk, "Constructive description of functions continuous on the set $\mathfrak{M} \subset \mathbb{C}$ for the k-th modulus of continuity," Mat. Zametki, 25, No. 2, 225-247 (1979).
30. V. V. Andrievskii, "Direct theorems of the theory of approximation on quasiconformal arcs," Izv. Akad. Nauk SSSR SSSR, Ser. Mat., 44, No. 2, 243-261 (1980).
31. V. V. Andrievskii, "Geometric properties of Dzydyk domains," Ukr. Mat. Zh., 33, No. 6, 723-727 (1981).
32. V. V. Andrievskii, "Approximative characterization of classes of functions on continua in the complex plane," Mat. Sb., 125, No. 1, 70-87 (1984).
33. V. V. Andrievskii, "Geometric structure of domains and direct theorems of the constructive theory of functions," Mat. Sb., 126, No. 1, 41-58 (1985).
34. V. I. Belyi, "The application of conformal invariants to the approximation problems of functions of the complex variable," Colloquia Math. Soc. Janos Bolyai, Vol. 19, Fourier Analysis and Approximation Theory, Budapest (1976), pp. 113-123.
35. V. K. Dzydyk, "Approximation method for the approximation of the solutions of linear differential equations by algebraic polynomials," Izv. Akad. Nauk SSSR, Ser. Mat., 38, No. 4, 937-967 (1974).
36. V. K. Dzydyk and L. I. Filozof, "Rate of convergence of Padé approximations for certain elementary functions," Mat. Sb., 107, No. 3, 347-363 (1978).
37. V. K. Dzydyk, "Asymptotics of diagonal Padé approximations of the functions $\sin z$, $\cos z$, $\sinh z$, $\cosh z$," Mat. Sb., 108, No. 2, 247-267 (1979).
38. V. K. Dzydyk, Approximation Methods for the Solution of Differential and Integral Equations [in Russian], Naukova Dumka, Kiev (1988).
39. V. K. Czydyk, "Generalization of the problem of moments," Dokl. Akad. Nauk UkrSSR, Ser. A, No. 6, 8-12 (1981).
40. V. K. Dzydyk, "Generalized problem of moments and Padé approximation," Ukr. Zh., 35, No. 3, 297-302 (1983).
41. P. L. Chebyshev, Selected Works [in Russian], N. I. Akhiezer (ed.), Gostekhteoretizdat, Moscow—Leningrad (1946); "Brief survey of Chebyshev's studies," 178-194.
42. V. K. Dzydyk and A. P. Golub, Generalized Problem of Moments and Padé Approximation [in Russian], Kiev (1981); Preprint 81:58 of the Institute of Mathematics, Academy of Sciences of the Ukrainian SSR.
43. M. N. Chyp, Generalized Problem of Moments and Integral Representations of Functions [in Russian], Kiev (1985); Preprint 85:49 of the Institute of Mathematics, Academy of Sciences of the Ukrainian SSR.
44. A. F. Leont'ev, "Representation of functions by a sequence of Dirichlet polynomials," Mat. Sb., 70, No. 1, 122-144 (1966).
45. A. F. Leont'ev, "Representation of functions by generalized Dirichlet series," Usp. Mat. Nauk, 24, No. 2, 97-164 (1969).
46. A. F. Leont'ev, "Representation of analytic functions by Dirichlet series," Mat. Sb., 80, No. 1, 117-150 (1969).
47. A. F. Leont'ev, "Representation of analytic functions in a closed domain by Dirichlet series," Izv. Akad. Nauk SSSR, Ser. Mat., 37, No. 3, 577-592 (1973).
48. V. K. Dzydyk and E. K. Krutigolova, "Representation of analytic functions by Dirichlet series on the boundary of the domain of convergence," Mat. Zametki, 14, No. 6, 769-780 (1973).
49. V. K. Dzydyk, "Conditions of convergence of Dirichlet series on closed polygons," Mat. Sb., 95, No. 4, 475-493 (1974).
50. A. F. Leont'ev, "Representation of analytic functions in the form of a sum of periodic functions," Mat. Sb., 93, No. 4, 512-528 (1972).
51. Yu. I. Mel'nik, "Representation of regular functions by Dirichlet series in a closed disk," Mat. Sb., 94, No. 4, 493-501 (1975).
52. Yu. I. Mel'nik, "Representation of regular functions by Dirichlet series in closed convex polygons," Ukr. Mat. Zh., 29, No. 6, 826-830 (1977).
53. Yu. I. Mel'nik, "Representation of regular functions in the form of a sum of periodic functions," Mat. Zametki, 36, No. 6, 847-856 (1984).
54. A. M. Sedletskii, "Bases formed by exponential functions in spaces E^p on convex polygons," Izv. Akad. Nauk SSSR, Ser. Mat., 42, No. 5, 1101-1119 (1978).

55. A. M. Sedletskii, "Expansions of functions in Dirichlet series on closed convex polygons," Sib. Mat. Zh., 19, No. 4, 878-887 (1978).
56. A. M. Sedletskii, "Expansion of analytic functions into a sum of periodic functions," Izv. Akad. Nauk SSSR, Ser. Mat., 48, No. 4, 833-853 (1984).
57. V. K. Dzjadyk (Dzydyk), "On a problem of Chebyshev and Markov," Anal. Math., 3, No. 3, 171-175 (1977).
58. V. K. Dzydyk, "Iterative-approximation method of approximation the solution of Cauchy's problem for ordinary differential equations by means of polynomials," Zh. Vychislit. Mat. Mat. Fiz., 26, No. 3, 357-372 (1986).
59. N. I. Akhiezer, Lectures on Approximation Theory [in Russian], Nauka, Moscow (1965).
60. V. K. Dzydyk, "Theory of special functions and their approximation," Mat. Sb., 131, No. 4, 438-464 (1986).

DZYDYK'S APPROXIMATION METHODS FOR THE SOLUTION OF DIFFERENTIAL AND INTEGRAL EQUATIONS

V. I. Bilenko, V. N. Konovalov,
I. A. Lukovskii, A. Yu. Luchka,
G. E. Pukhov, and N. I. Ronto

UDC 519:622+517.5

The theory and computational applications of approximation methods for the solution of operator-valued equations is one of the fields of mathematics in which Dzydyk obtained a number of important results.

Beginning in 1969, he undertook a far-reaching investigation of the potential and methods of application of a number of methods, concepts, and results of the Chebyshev theory of approximation of functions for the construction of novel and highly efficient (in the sense of the number of operations required and the precision achieved) methods and algorithms for the solution of differential and integral equations. In the process, he created and provided a rigorous grounding for three interdependent and highly complementary computational approximation methods:

1) linear polynomial operators;

2) approximation method (a-method) for the solution of linear differential equations with polynomial coefficients;

3) interative-approximation method (ia-method) for the solution of nonlinear differential and integral equations under analytic conditions.

The sources for the methods developed by Dzydyk can be found in the classical studies of Chebyshev, Rits, Radon, Galerkin, Bernshtein, Krylov, Bogolyubov, Kravchuk, Kantorovich, Pol'skii, Vainikko, and others. Dzydyk, one of the leading specialists in the theory of approximation of functions, the foundation of the field of computational mathematics, synthesized the most important results in the Chebyshev theory of approximation of functions and a number of computational methods for the solution of equations of mathematical physics, and with this as a starting point, constructed an elegant theory of approximation methods with an abundance of promising concepts. The basic assertions of this theory were systematically set forth by him in a monograph [1] which has most certainly served as a powerful impetus for the further development and improvement of computational methods for the solution of operator-valued equations of mathematical physics as a whole.

In the present survey, we will briefly set forth the essentials of the approximation methods, and analyze some of the more important theoretical results and numerical examples that provide a good illustration of the high degree of efficiency and constructivity of these methods.

Institute of Mathematics, Academy of Sciences of the Ukrainian SSR, Kiev. Institute of Simulation Problems in the Power Industry, Academy of Sciences of the Ukrainian SSR, Kiev.
Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 41, No. 4, pp. 454-465, April, 1989. Original article submitted December 19, 1988.

0041-5995/89/4104-0395$12.50