SOME NEGATIVE RESULTS ON MULTIPLIER SEQUENCES OF THE FIRST KIND

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It is proved that the composition of a polynomial with a multiplier sequence of the first kind may lead to a diminishing of the number of real roots of this polynomial and that the reciprocals of the moments of a nonnegative function on [0, 1] need not form a multiplier sequence of the first kind. On the basis of these facts one establishes the inaccuracy of the solution, obtained by T. Craven and G. Csordas, to S. Karlin's problem on the characterization of linear transformations that do not increase the number of zeros and also the incorrectness of M. Kostova's certain results, given in L. Iliev's monograph "Laguerre entire functions" (Sofia, 1987).

The sequences of real numbers $\gamma=\{\gamma_k\}_{k=0}^{\infty}$ such that for any polynomial $f(x)=\sum_{k=0}^{n} a_k x^k$, having only real zeros, the polynomial $(\gamma * f)(x)=\sum_{k=0}^{n}\gamma_k a_k x^k$ also has only real zeros are called multiplier sequences of the first kind. The set of all such sequences will be denoted by α. Multiplier sequences of the first kind have been introduced by Pólya and Schur in [1], where it has been established that a sequence $\gamma=\{\gamma_k\}_{k=0}^{\infty}$ belongs to α if and only if the series $\Phi(x)=\sum_{k=0}^{\infty}\frac{\gamma_k}{k!}x^k$ converges in the entire complex plane and the entire function $\Phi(x)$ or $\Phi(-x)$ can be represented in the form

$$ce^{\sigma x}x^m\prod_{n=1}^{\infty}\left(1+\frac{x}{x_n}\right), \tag{1}$$

where $\sigma\geqslant 0$, $0<x_n\leqslant\infty$, $c\in\mathbb{R}^1$, $\sum_{n=1}^{\infty}\frac{1}{x_n}<\infty$, while m is a nonnegative integer (see also [2, p. 439 of the Russian edition; 3, pp. 5-8]). The class of entire functions of the form (1) is denoted by L_1. It contains, in particular, all the polynomials which have only nonpositive zeros.

The interest in the study of multiplier sequences of the first kind is due to the fact that they are applied in various areas of mathematics and theoretical physics [4-7]. This paper is devoted to the refutation of two theorems, establishing new properties of sequences

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from the set α. To this end we prove a series of facts regarding the absence of inclusions between various classes of numerical sequences.

1. LEMMA 1. Assume that the sequence of real numbers $\mu=\{\mu_k\}_{k=0}^{\infty}$ can be represented in the form

$$\mu_k=\int_a^b t^k d\sigma(t),\quad k\geqslant 0,$$

where $-\infty < a < b < +\infty$ and $\sigma(t)$ is a nonmonotone function of bounded variation on [a, b]. Then there exists a natural number N_0 such that for any $N\geqslant N_0$ the quadratic form

$$\sum_{k,j=0}^{N}\mu_{k+j}x_k x_j \tag{2}$$

is alternating.

Proof. The cone K*, conjugate to the cone K of all functions in C[a, b] that are nonnegative and continuous on [a, b], is the set of all nondecreasing functions in the space of functions of bounded variation on [a, b] [8, p. 448 of the Russian edition]. By the assumption of the theorem, $\sigma\notin\pm K^*$. This means that there exist functions φ_0, φ_1, continuous and nonnegative on [a, b], such that

$$(-1)^i\int_a^b \varphi_i(t)\,d\sigma(t)>0,\quad i=0,1.$$

If we approximate on [a, b] the functions $\sqrt{\varphi_0(t)}$, $\sqrt{\varphi_1(t)}$ by the Weierstrass theorem by the polynomials $P_N(t)=\sum_{k=0}^{N}p_kt^k$, $Q_N(t)=\sum_{k=0}^{N}q_kt^k$, respectively, then we obtain that, for a sufficiently large natural N_0, for all $N\geqslant N_0$ we have the inequalities

$$\sum_{k,j=1}^{N}\mu_{k+j}p_kp_j=\int_a^b[P_N(t)]^2\,d\sigma(t)>0,$$
$$\sum_{k,j=0}^{N}\mu_{k+j}q_kq_j=\int_a^b[Q_N(t)]^2\,d\sigma(t)<0,$$

which conclude the proof of the lemma.

Given a sequence of numbers $\mu=\{\mu_k\}_{k=0}^{\infty}$, in the space of all polynomials $f(x)=\sum_{k=0}^{n}a_kx^k$ with real coefficients (real polynomials) we define a functional $\mu(f)=\sum_{k=0}^{n}\mu_ka_k$ and we recall [9, p. 92 of the Russian edition] that a sequence μ is said to be nonnegative if for any polynomial P, nonnegative on the entire axis, we have the inequality $\mu(P)\geqslant 0$. A sequence μ [9, p. 10 of the Russian edition] is nonnegative if and only if all the quadratic forms of the form (2) are nonnegative. Let Λ denote the class of all nonnegative sequences, let α_+ be the set of all sequences from α with positive terms, and if $a=\{\{\alpha_k\}_{k=0}^{\infty}\}$ is some set of sequences, then by $1/a$ we shall denote the set $\left\{\left\{\frac{1}{\alpha_k}\right\}_{k=0}^{\infty}\mid\{\alpha_k\}_{k=0}^{\infty}\in a\right\}$.

LEMMA 2. The relation $\frac{1}{\alpha_+}\setminus\Lambda\neq\varnothing$ is satisfied.

Proof. Since $\sum_{k=0}^{\infty}\frac{k^2+k+1}{k!}x^k=(x+1)^2e^x\in L_1$, it follows that $\{k^2+k+1\}_{k=0}^{\infty}\in\alpha_+$. At the same time the sequence $\mu=\left\{\frac{1}{k^2+k+1}\right\}_{k=0}^{\infty}$ admits the representation

$$\frac{1}{k^2+k+1}=\int_0^1 t^k\rho(t)\,dt,\quad k\geqslant 0,$$

where the function $\rho(t)=-\frac{2}{\sqrt{3}}t^{-\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\ln t\right)$ is alternating on [0, 1]. Therefore μ satisfies the assumptions of Lemma 1 and, consequently, it is not nonnegative. The lemma is proved.

We denote by $\lambda([a, b])$, $-\infty \leqslant a < b \leqslant +\infty$, the collection of all sequences of real numbers $\{\mu_k\}_{k=0}^{\infty}$ for which the classical moment problem

$$\mu_k = \int_a^b t^k d\sigma(t), \quad k \geqslant 0,$$

is solvable on [a, b] with a nondecreasing function $\sigma(t)$, having on [a, b] an infinite number of growth points. Then from $[a, b] \subset \mathbb{R}^1$ there follows that

$$\lambda([a, b]) \subset \lambda \subset \Lambda, \tag{3}$$

where $\lambda := \lambda(\mathbb{R}^1)$.

LEMMA 3. The relation $\lambda([0, 1]) \setminus \frac{1}{\alpha_+} \neq \varnothing$ holds.

Proof. We consider the sequence $\mu = \{\mu_k\}_{k=0}^{\infty}$, $\mu_k = \frac{5}{(k+1)[5+(k+1)^2]}$, $k \geqslant 0$. From the equality

$$\frac{5}{(s+1)[5+(s+1)^2]} = \int_0^1 x^s [1 - \cos\sqrt{5} \ln x]\, dx, \quad s > -1,$$

there follows that $\mu \in \lambda([0, 1])$. On the other hand,

$$\Phi(x) = \sum_{k=0}^{\infty} \frac{x}{\mu_k k!} = \frac{1}{5} e^x P(x),$$

where $P(x) = x^3 + 6(x + 1)^2$. Since $P'(x) = 3(x+2)^2$, $P'(x) \geqslant 0\ \forall x \in \mathbb{R}^1$, $P'(x) > 0\ \forall x \in [-1, 0]$, we have $P(x) \geqslant P(0) = 6\ \forall x \geqslant 0$, $P(x) \leqslant P(-1) = 1\ \forall x \leqslant -1$ and on $[-1, 0]$ $P(x)$ has exactly one real zero. Therefore, $P(x) \notin L_1$ and, consequently, $\Phi(x) \notin L_1$. This means that $\mu \notin \frac{1}{\alpha_+}$. The lemma is proved.

For a real polynomial P by $\mathbb{Z}_{\mathbb{R}}(P)$ we denote the number of real roots of P, multiplicities included. We shall say [3, p. 121] that a sequence of real numbers $\gamma = \{\gamma_k\}_{k=0}^{\infty}$ belongs to the set τ if the inequality $\mathbb{Z}_{\mathbb{R}}(\gamma * P) \leqslant \mathbb{Z}_{\mathbb{R}}(P)$ is satisfied for any real polynomial P. Since with each sequence $\{\gamma_k\}_{k=0}^{\infty}$ from τ, the set τ contains the sequence $\{-\gamma_k\}_{k=0}^{\infty}$, we shall assume, without restricting the generality of the subsequent reasoning, that τ consist only of those sequences $\{\gamma_k\}_{k=0}^{\infty}$ for which $\gamma_0 \geqslant 0$.

LEMMA 4. The inclusion $\tau \subseteq \Lambda$ holds.

Proof. Let $\gamma = \{\gamma_k\}_{k=0}^{\infty} \in \tau$ and $P(t) = \sum_{k=0}^{n} x_k t^k$ be an arbitrary real polynomial. Then for any $\varepsilon > 0$ the polynomial $Q(t) = \varepsilon + [P(t)]^2$ is positive on the entire real axis and, by the definition of the set τ, one must have the equality

$$0 \leqslant \mathbb{Z}_{\mathbb{R}}(\gamma * Q) = \mathbb{Z}_{\mathbb{R}}\Big(\varepsilon\gamma_0 + \sum_{k,j=0}^{n} \gamma_{k+j} x_k x_j t^{k+j}\Big) \leqslant \mathbb{Z}_{\mathbb{R}}(Q) = 0.$$

As it is known [10, p. 545], from $\gamma \in \tau$ there follows that $\gamma_k \neq 0$ for all $k \geqslant 0$. Therefore $(\gamma * Q)(0) = (\varepsilon + x_0^2)\gamma_0 > 0$ and from the obtained equality $\mathbb{Z}_{\mathbb{R}}(\gamma * Q) = 0$ we obtain directly that $(\gamma * Q)(t) > 0$ $\forall t \in \mathbb{R}^1$. In particular,

$$(\gamma * Q)(1) = \varepsilon\gamma_0 + \sum_{k,j=0}^{n} \gamma_{k+j} x_k x_j > 0,$$

from where, by the limiting process $\varepsilon \downarrow 0$, one can see the nonnegativity of the sequence γ. The lemma is proved.

From the inclusions $\frac{1}{\alpha_+} \setminus \Lambda \subseteq \frac{1}{\alpha_+} \setminus \tau$ (Lemma 4), $\frac{1}{\alpha_+} \setminus \Lambda \subseteq \frac{1}{\alpha_+} \setminus \lambda([a, b])$ [the inclusions (3)], and Lemmas 2, 3 there follows the validity of the following statement.

THEOREM 1. The following relations hold: a) $\frac{1}{\alpha_+} \setminus \tau \neq \varnothing$; b) $\lambda([0, 1]) \setminus \frac{1}{\alpha_+} \neq \varnothing$; c) $\frac{1}{\alpha_+} \setminus \lambda([a, b]) \neq \varnothing$ $\forall -\infty \leqslant a < b \leqslant +\infty$.

We consider the set τ_+ of those sequences from τ_1 which have only positive terms. Since $\frac{1}{\tau} \subseteq \alpha$ [10, p. 544], we have $\frac{1}{\tau_+} \subseteq \alpha_+$ and, by virtue of Theorem 1, the inclusion here is strict:

$$\tau_+ \subset \frac{1}{\alpha_+}. \tag{4}$$

Relation (4) gives a negative answer to Iliev's question from [3, p. 120, Problem 4.8] regarding the validity of the inclusion $\alpha \subseteqq 1/\tau$.

2. The paper [10] contains the following result.

Proposition 1 [10, p. 545, Theorem 2.1]. If $\{\gamma_k\}_{k=0}^{\infty} \in \alpha$ and $\gamma_k \neq 0$ for all $k \geqslant 0$, then for any real polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ we have

$$Z_{\mathbb{R}}\left(\sum_{k=0}^{n} \frac{a_k}{\gamma_k} x^k\right) \leqslant Z_{\mathbb{R}}(f).$$

From Proposition 1 there would follow $1/\alpha_+ = \tau_+$ and this contradicts (4). Therefore we have the following consequence of Theorem 1.

COROLLARY 1. Proposition 1 is false.

The assertion of Proposition 1 has been communicated in [10] as a solution of Karlin's problem [4, p. 382] regarding the characterization of the sequences from the set τ, and then it has been proved and used for the derivation of various consequences in [11-13]. Corollary 1 shows that Karlin's problem remains unsolved and that the fundamental results from [10-13] are false.

In [14] (see also [15, p. 190]) one formulates the following result, proved in [16].

Proposition 2 [14, p. 88, Theorem 3]. If $\{\mu_k\}_{k=0}^{\infty} \in \lambda$ and $\mu_k \neq 0$ for all $k \geqslant 0$, then $\left\{\frac{1}{\mu_k}\right\}_{k=0}^{\infty} \in \alpha$.

From Proposition 2 there would follow, in particular, that $\lambda([0, 1]) \subseteq \frac{1}{\alpha_+}$ and this contradicts the assertion of Theorem 1.

COROLLARY 2. Proposition 2 is false.

On the basis of Propositions 1, 2, in [14] and also in [3, pp. 121-122; Theorems 4.613-15] one derives a series of statements, which, in view of Corollaries 1, 2, are false.

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