GENERALIZED MOMENT REPRESENTATIONS OF BASIS HYPERGEOMETRIC SERIES

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UDC 517.53

We present the necessary definitions.

Definition 1 (cf. [1]). Let f(z) be a formal power series of the form

$$f(z) = \sum_{k=0}^{\infty} s_k z^k. \tag{1}$$

A system of equations

$$s_{i+j} = \int_{-\infty}^{\infty} a_i(t)\, b_j(t)\, d\mu(t), \quad i, j = \overline{0, \infty}, \tag{2}$$

where μ(t) is a nondecreasing function on (−∞, ∞) and $\{a_i(t)\}_{i=0}^{\infty}$ and $\{b_j(t)\}_{j=0}^{\infty}$ are sequences of measurable functions on (−∞, ∞) such that all the integrals in (2) exist and take on finite values, is called a generalized moment representation of the series (1).

Generalized moment representations introduced by Dzyadyk in 1981 in [1] are widely used in problems of rational approximation and analytic continuation of functions (cf. [2, 3]).

In this paper generalized moment representations of basis hypergeometric series which were first considered by H. E. Heine in 1878 are constructed and analyzed (cf., e.g., [4]).

Definition 2 [4]. A basis hypergeometric series is a power series of the form

$${}_r\Phi_s\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_r; z\\ \rho_1, \rho_2, \ldots, \rho_s\end{matrix}\right] := \sum_{n=0}^{\infty} \frac{(\alpha_1)_{q,n}(\alpha_2)_{q,n}\cdot\ldots\cdot(\alpha_r)_{q,n}}{(q)_{q,n}(\rho_1)_{q,n}\cdot\ldots\cdot(\rho_s)_{q,n}} z^n, \tag{3}$$

where

$$(a)_{q,n} := (1-a)(1-aq)(1-aq^2)\cdot\ldots\cdot(1-aq^{n-1}) = (1-q)^n\left(\frac{1-a}{1-q}\right)\times$$
$$\times\left(\frac{1-a}{1-q}+a\right)\cdot\ldots\cdot\left(\frac{1-a}{1-q}+a+aq+\ldots+aq^{n-2}\right); \qquad (a)_{q,0} := 1,$$

and $\alpha_1, \ldots, \alpha_r$; $\rho_1, \ldots, \rho_s$; q are parameters with q ≠ 1.

THEOREM 1. For the function

$$f(z) = \sum_{n=0}^{\infty} s_n z^n = \sum_{n=0}^{\infty} \frac{z^n}{(\gamma+1+\rho)[\gamma+1+\rho(1+q)]\cdot\ldots\cdot[\gamma+1+\rho\times(1+q+\ldots+q^n)]} =$$
$$= \left(\frac{1-\rho}{1-q}\right) z^{-2}\left\{{}_1\Phi_1\left[\begin{matrix}q; (1-q)z\\ \rho\end{matrix}\right] - 1 - \frac{z(1-q)}{1-\rho}\right\}, \tag{4}$$

provided only $\gamma := \frac{q-\rho}{1-q} > -1$; $\rho, q > 0$; $q \neq 1$, there exists a generalized moment representation of the form

$$s_{i+j} = \int_0^1 a_i(t)\, b_j(t)\, dt, \quad i, j = \overline{0, \infty}, \tag{5}$$

Institute of Simulation and Power Engineering Problems, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 41, No. 6, pp. 803-808, June, 1989. Original article submitted September 30, 1986.

 0041-5995/89/4106-0690$12.50

where

$$a_i(t) = \frac{t^{\rho\lambda_{i+1}(q)}}{\prod_{r=1}^{i}\left(\gamma + 1 + \rho\frac{q^r - 1}{q - 1}\right)}, \quad i = \overline{0, \infty}; \tag{6}$$

$$b_j(t) = \frac{t^\nu (q-1)^j}{\prod_{r=1}^{j}(q^r - 1)} \sum_{m=0}^{j} (-1)^m q^{\frac{m(m-1)}{2}} \left[\sum_{r=1}^{m} \frac{(q^{j-r+1} - 1)}{(q^r - 1)}\right] t^{\tilde{\lambda}_m(q)}, \quad j = \overline{0, \infty};$$

$$\lambda_i(q) := \frac{q^i - 1}{q - 1}, \quad i = \overline{1, \infty}; \qquad \tilde{\lambda}_m(q) := \frac{q^m - 1}{(q-1)q^m}, \quad m = \overline{0, \infty}. \tag{7}$$

Proof. Note that a linear bounded operator A:C[0, 1] → C[0, 1] of the form

$$(A\varphi)(t) = t^\rho \int_0^1 \varphi(t^q u)\, u^\nu \mathrm{d}u \tag{8}$$

possesses the following properties:

1)

$$(Aa_i)(t) = a_{i+1}(t), \quad i = \overline{0, \infty}, \tag{9}$$

where the functions $a_i(t)$ are defined by the formulas (6);

2) for an arbitrary function ψ(t) integrable on [0, 1] and a function $\varphi(t)$ continuous on [0, 1] the following is valid:

$$\int_0^1 (A\varphi)(t)\,\psi(t)\,\mathrm{d}t = \int_0^1 \varphi(t)\,(B\psi)(t)\,\mathrm{d}t, \tag{10}$$

where the operator $B:L^1[0, 1] \to L^1[0, 1]$ is of the form

$$(B\psi)(t) = \frac{1}{q} t^\nu \int_t^1 \psi(v^{1/q})\, v^{\frac{\nu+1-\gamma q-2q}{q}} \mathrm{d}v. \tag{11}$$

The validity of Eq. (10) is verified directly, utilizing change of variables and integration by parts;

3) the k-th powers of the operators A and B are, respectively, of the form

$$(A^k\varphi)(t) = \frac{t^{\rho\lambda_k(q)}(q-1)^{k-1}}{\prod_{r=1}^{k-1}(q^r - 1)} \int_0^1 \varphi(t^{q^k} u)\, u^\nu \sum_{m=0}^{k-1} (-1)^m q^{\frac{m(m-1)}{2}} \left[\prod_{l=1}^{m}\left(\frac{q^{k-l} - 1}{q^l - 1}\right)\right] u^{\tilde{\lambda}_m(q)} \mathrm{d}u, \quad k \geqslant 1, \tag{12}$$

$$(B^k\psi)(t) = \frac{t^\nu (q-1)^{k-1}}{q^k \prod_{r=1}^{k-1}(q^r - 1)} \int_t^1 \sum_{m=0}^{k-1} (-1)^m \left[\prod_{l=1}^{m} \frac{(q^{k-l} - 1)}{(q^l - 1)}\right] \times$$

$$\times\, q^{m(m-1)/2} \left(\frac{t}{v}\right)^{\tilde{\lambda}_m(q)} v^{\rho\tilde{\lambda}_k(q) + 1/q^k - (\gamma+2)} \psi(v^{1/q^k})\,\mathrm{d}v, \quad k \geqslant 1. \tag{13}$$

Formula (12) is verified with the aid of (8) by an induction argument. Formula (13) is then deduced from (12) by utilizing the equality

$$\int_0^1 (A^k\varphi)(t)\,\psi(t)\,\mathrm{d}t = \int_0^1 \varphi(t)\,(B^k\psi)(t)\,\mathrm{d}t, \tag{14}$$

which follows from (10).

Substituting the function $b_0(t) = t^\nu$ into (13) in place of ψ(t) and integrating, we arrive at formula (7). The proof of the theorem is thus completed.

THEOREM 2. Padé polynomials of the order [N − 1/N], N = $\overline{1, \infty}$, which are nondegenerate, exist for the function f(z) of the form (4) under the conditions of Theorem 1 (i.e., j > − 1; ρ, q > 0, q ≠ 1).

Proof. Formulas (6) and (7) imply that the system of functions $\{t^{-\rho}a_i(t)\}_{i=0}^{N}$ and $\{t^{-\gamma}b_j(t)\}_{j=0}^{N}$ for each $N = \overline{0, \infty}$ are Chebyshev system (cf. [5]). In view of Lemma 1 in [6] there exists in this case for all $N = \overline{0, \infty}$ a generalized polynomial

$$A_N(t) = \sum_{i=0}^{N} c_i^{(N)} a_i(t), \tag{15}$$

possessing the biorthogonality property:

$$\int_0^1 A_N(t)\, b_j(t)\, dt = \delta_{N,j}, \quad j = \overline{0, N}. \tag{16}$$

Moreover, $t^{-\rho}A_N(t)$ has exactly N distinct roots on (0, 1); it thus follows in particular that $c_0^{(N)}$ and $c_N^{(N)}$ do not vanish for each N. In that case, in view of Theorem 2.1 in [7] the Padé polynomials $[N-1/N]_f(z)$ can be written in the form

$$[N-1/N]_f(z) = \frac{\sum_{i=1}^{N} c_i^{(N)} z^{N-i} T_{i-1}(f; z)}{\sum_{i=0}^{N} c_i^{(N)} z^{N-i}}, \tag{17}$$

where $T_i(f; z)$ are partial sums of the series (4) of order i. Moreover, the following integral representation is valid for the approximation error:

$$f(z) - [N-1/N]_f(z) = \frac{z^N}{Q_N(z)} \int_0^1 A_N(t)\, B(z, t)\, dt, \tag{18}$$

where $Q_N(z) := \sum_{i=0}^{N} c_i^{(N)} z^{N-i}$; $\quad B(z, t) := \sum_{j=0}^{\infty} z^j b_j(t)$.

If $q > 1$, then (18) will be valid for all $z \in \mathbb{C}$; if, however, $q < 1$, it is valid for $|z| < 1$.

Formula (17) together with the above-mentioned inequalities $c_0^{(N)} c_N^{(N)} \neq 0$, $N = \overline{0, \infty}$, imply that the Padé polynomials are nondegenerate. Theorem 2 is thus proved.

Remark. In [8] Padé diagonal polynomials were essentially constructed for the q-analog of an exponential function, which is a particular case of (4).

THEOREM 3. For the function

$$f(z) = \sum_{n=0}^{\infty} s_n z^n = \sum_{n=0}^{\infty} \frac{(\rho+\gamma+\sigma+1)[\rho(q+1)+\gamma+\sigma+1]\cdot\ldots\cdot[\rho(q^{n-1}+\ldots+1)+\gamma+\sigma+1]}{(\rho+\gamma+1)[\rho(q+1)+\gamma+1]\cdot\ldots\cdot[\rho(q^n+\ldots+1)+\gamma+1]} z^n =$$

$$= \frac{(1-q)\alpha}{(1-\alpha)z\rho}\left\{ {}_2\Phi_1\left[\begin{matrix} q, \alpha; & \dfrac{\xi z}{\alpha q} \\ \xi & \end{matrix}\right] - 1\right\} \tag{19}$$

[here $\alpha := \rho/\varkappa - \sigma(q-1)$; $\xi := \rho q/\varkappa$; $\varkappa := \rho - (q-1)(\gamma+1)$] the generalized moment representation of the form

$$s_{i+j} = \int_0^1 a_i(t)\, b_j(t)\, dt, \quad i, j = \overline{0, \infty}, \tag{20}$$

is valid provided

$$\gamma > -1;\ \rho, q > 0;\ q \neq 1;\ \sigma \neq \frac{\varkappa(q^r-1)}{q^r(q-1)},\ r = \overline{1, \infty},$$

where

$$a_i(t) = t^{\rho\lambda_{i+1}(q)} \prod_{r=1}^{i} \frac{\left(\gamma+\sigma+1+\rho\dfrac{q^r-1}{q-1}\right)}{\left(\gamma+1+\rho\dfrac{q^r-1}{q-1}\right)}, \quad i = \overline{0, \infty}; \tag{21}$$

$$b_j(t)=\sum_{m=0}^{j} t^{\tilde{\lambda}_m(q)+\gamma}\prod_{l=1}^{j-m}\frac{\sigma+\varkappa\left(\frac{q^{l-1}-1}{q-1}\right)}{\varkappa\left(\frac{q^{l}-1}{q-1}\right)}\prod_{r=1}^{m}\left(\frac{1}{q}-\frac{\sigma(q-1)q^{r-1}}{\varkappa(q^{r}-1)}\right),\qquad j=\overline{0,\infty}; \tag{22}$$

and as above $\lambda_i(q)=(q^i-1)/(q-1)$, $i=\overline{1,\infty}$; $\tilde{\lambda}_m(q)=(q^m-1)/(q-1)q^m$, $m=\overline{0,\infty}$.

Proof. Functions (21) can be constructed by means of successive application of the linear continuous operator $A:C[0,1]\to C[0,1]$ of the form

$$(A\varphi)(t)=\sigma t^{\rho}\int_0^1\varphi(t^q u)u^{\gamma}\mathrm{d}u+t^{\rho}\varphi(t^q) \tag{23}$$

to the function $a_0(t)=t^{\rho}$. Moreover, the equalities

$$s_i=\int_0^1 a_i(t)t^{\gamma}\mathrm{d}t,\quad i=\overline{0,\infty} \tag{24}$$

evidently hold.

Taking this into account, we construct a linear continuous operator $B:L^1[0,1]\to L^1[0,1]$ of the form

$$(B\psi)(t)=\frac{\sigma}{q}t^{\gamma}\int_t^1\psi(v^{1/q})v^{(\rho+1-\gamma q-2q)/q}\mathrm{d}v+\frac{1}{q}t^{(\rho-q+1)/q}\psi(t^{1/q}), \tag{25}$$

possessing the property

$$\int_0^1(A\varphi)(t)\psi(t)\,\mathrm{d}t=\int_0^1\varphi(t)(B\psi)(t)\,\mathrm{d}t \tag{26}$$

for an arbitrary function $\psi(t)$ integrable on [0, 1] and a function $\varphi(t)$ continuous on [0, 1].

Next, setting $b_0(t)=t^{\gamma}$, we easily obtain

$$s_{i+j}=\int_0^1(A^i a_0)(t)(B^j b_0)(t)\,\mathrm{d}t. \tag{27}$$

To complete the proof of the theorem, it remains only to show that $(B^j b_0)(t)$, $j=\overline{0,\infty}$, are expressed by formula (22). However, formula (22) is verified directly. The theorem is thus proved.

THEOREM 4. Padé polynomials of the order $[N-1/N]$, $N=\overline{1,\infty}$, which are nondegenerate, exist for the function f(x) of the form (19) under the conditions of Theorem 3 [i.e., $\gamma>-1$; $\rho, q>0$; $q\neq 1$; $\sigma\neq\varkappa\tilde{\lambda}_m(q)$, $m=\overline{1,\infty}$]. Moreover, if $\tilde{A}_N(t)$ is a generalized polynomials of the form

$$\tilde{A}_N(t)=\sum_{i=0}^{N}c_i^{(N)}a_i(t),\quad N=\overline{0,\infty}, \tag{28}$$

possessing biorthogonality properties

$$\int_0^1\tilde{A}_N(t)b_j(t)\,\mathrm{d}t=\delta_{j,N},\quad j=\overline{0,N}, \tag{29}$$

then the Padé polynomials of order $[N-1/N]$, $N=\overline{1,\infty}$, of the function f(z) can be written in the form

$$[N-1/N]_f(z)=\frac{\sum_{i=1}^{N}c_i^{(N)}z^{N-i}T_{i-1}(f;z)}{\sum_{i=0}^{N}c_i^{(N)}z^{N-i}}, \tag{30}$$

where $T_i(f;z)$ are partial sums of the series (19) of order i. Then the integral representation

$$f(z)-[N-1/N]_f(z)=\frac{z^N}{Q_N(z)}\int_0^1\tilde{A}_N(t)B(z,t)\,\mathrm{d}t \tag{31}$$

for the approximation error, where $Q_N(z) := \sum_{i=0}^{N} c_i^{(N)} z^{N-i}$; $B(z,t) := \sum_{j=0}^{\infty} z^j b_j(t)$, is valid for $|z| < 1$.

The proof is analogous to the proof of Theorem 2. Note that the biorthogonal polynomial $\tilde{A}_N(t)$ defined by formulas (28), (29) coincides up to a multiplicative constant with the polynomial $A_N(t)$ defined by the equalities (15) and (16).

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STRONG SUMMABILITY OF FOURIER SERIES OF (ψ, β)-DIFFERENTIABLE FUNCTIONS

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UDC 517.5

Let f(·) be a summable 2π-periodic function $(f \in L)$, $S[f] = a_0(f)/2 + \sum_{k=1}^{\infty}(a_k(f)\cos kx + b_k(f)\sin kx)$ its Fourier series, $S_k(f, x)$ the k-th partial sums of the series, $\rho_k(f; x) = f(x) - S_k(f, x)$, $\lambda = (\lambda_k)_{k\in N}$ and $\delta = (\delta_k)_{k\in N}$ nonnegative sequences of numbers (the numbers λ_k may also depend on a parameter m), φ a function defined and nonnegative on $[0, \infty)$.

Consider the operator

$$H_n^{\varphi}(f;\, x,\, \lambda,\, \delta) = \sum_{k=n}^{\infty} \lambda_k \varphi(\delta_k |\rho_k(f;\, x)|). \tag{1}$$

Operators of type (1), with $\varphi(u) = u^p$, $p > 0$, where first studied by Hardy and Littlewood [1, 2], who thereby laid the foundations for the modern theory of strong summability of Fourier series. Similar objects were subsequently investigated by other authors [3-5].

In this paper we derive estimates for the values of (1) in the uniform metric for the Fourier series of functions $f \in C_\beta^\psi C$. These classes of functions were first defined by Stepanets [6], as follows. Let $(\psi(k))_{k\in N}$ be a fixed sequence of numbers, β a fixed number and

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} (a_k(f)\cos(kx+\theta) + b_k(f)\sin(kx+\theta)), \quad \theta = \beta\pi/2,$$

the Fourier series of some function $f_\beta^\psi \in L$. This function is called the (ψ, β)-derivative of f. The set of functions $f \in C$ for which $f_\beta^\psi \in C$ is denoted by $C_\beta^\psi C$.

We shall assume that the numbers $\psi(k)$ are the traces on N of a function $\psi(v)$ of a continuous argument $v \geqslant 1$. assumed to be convex downward for all $v \in [1, \infty)$ and such that $\lim_{v\to\infty}\psi(v) = 0$. The set of all such functions will be denoted by $\mathfrak{M}$.

Abkhazian University, Sukhumi. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 41, No. 6, pp. 808-814, June, 1989. Original article submitted April 22, 1988.

0041-5995/89/4106-0694$12.50